\[ A_j^{\text{min}}(V; M) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V) \]

are isomorphisms for \( j \leq n \) if \( V \) is a finite dimensional \( k \)-module and so they are isomorphisms for \( j \leq n \) for any \( k \)-module \( V \).

Let \( f_V : T(V) \to T(V) \) be a functorial coalgebra morphism over \( M(V) \). Notice that, for each \( n \geq 1 \), the composite

\[ A_j^{\text{min}}(V) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V; M) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \]

is an isomorphism for \( j \leq n \) by Lemma 4.10. Thus the composite

\[ A_j^{\text{min}}(V) \xrightarrow{f_V} T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V) \]

is an isomorphism for any \( 1 \leq j \leq n \). The assertion follows. \( \Box \)

**Corollary 4.13.** Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \) and let \( B(V) \) be a natural coalgebra retract of \( T(V) \) over \( M(V) \). Then \( A_j^{\text{min}}(V; M) \) is a natural coalgebra retract of \( B(V) \) over \( M(V) \).

Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \). The coalgebra \( A_j^{\text{min}}(V; M) \) is called the **minimal natural coalgebra retract of \( T(V) \) over \( M(V) \)**. Notice that the minimal natural coalgebra retract of \( T(V) \) of \( M(V) \) is unique up to isomorphism over \( M(V) \) by the universal property in Corollary 4.13. In the special case where \( M(V) = V \), let \( A_j^{\text{min}}(V) \) denote the **minimal natural coalgebra retract of \( T(V) \) over \( V \)**.

## 5. SOME LEMMAS ON COALGEBRAS

In this section, the term quasi-Hopf algebra will be used as in [12] to refer to a Hopf algebra in which the multiplication need not be associative, although the comultiplication is assumed to be coassociative.

Let \( C \) be a connected graded \( k \)-coalgebra. Let \( P(C) \) denote the set of primitive elements in \( C \). By the proof of Proposition 3.9 in [12], one has the following lemma.

**Lemma 5.1.** Let \( f : A \to B \) be a morphism of connected graded coalgebras over a field. Then \( f : A_j \to B_j \) is a monomorphism for \( j \leq n \) if and only if \( P(f) : P(A_j) \to P(B_j) \) is a monomorphism for \( j \leq n \).

Let \( C \) be a connected graded cocommutative coalgebra and let \( B \) be a connected graded cocommutative quasi-Hopf algebra. Notice that the convolution product may not be associative in this setting. There exists a unique left conjugation

\[ \chi^l : B \to B \]
such that
$$\chi^l \ast \text{id} = \eta_B \circ \epsilon_B.$$  

The map $\chi^l : B \to B$ is a morphism of graded coalgebras which is given inductively by:

1. $\chi^l : B_0 \to B_0$ is the identity;
2. $$\chi^l(x) = -x - \sum \chi(x')x'',$$
   when $x \in B_n$ with $n > 0$ and $\psi(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$.  

Similarly, there exists a unique right conjugation $\chi^r : B \to B$ such that
$$\text{id} \ast \chi^r = \epsilon_B \circ \eta_B.$$  

**Lemma 5.2.** Let $p : B \to C$ and $s : C \to B$ be morphisms of connected graded cocommutative coalgebras such that
$$p \circ s : C \to C$$

is the identity map. Suppose that $B$ is a connected graded cocommutative quasi-Hopf algebra of finite type. Let
$$\phi : B \to \text{colim}_{(s \otimes \Delta)^{\text{op}}} B$$

be the canonical map to the colimit. Then the composite
$$B \xrightarrow{\psi} B \otimes B \xrightarrow{\phi \otimes p} \text{colim}_{(s \otimes \Delta)^{\text{op}}} B \otimes C$$

is an isomorphism of graded coalgebras.

**Proof.** Let $f$ denote the convolution product $(s \circ p \circ \chi^l) \ast \text{id}$ and let $\alpha \in P(B)$. Then
$$f(\alpha) = \alpha - s \circ p(\alpha).$$

Notice that the self map
$$s \circ p : B \to B$$

is idempotent. Thus the canonical map
$$P(\text{Im}(f : B \to B)) \to P(\text{colim}_f B)$$

is a monomorphism and so the canonical coalgebra map
$$\text{Im}(f : B \to B) \to \text{colim}_f B$$

is a monomorphism. By the proof of Theorem 4.5 the canonical map
$$\phi : B \to \text{colim}_f B$$

is a split epimorphism of coalgebras. Thus the canonical map
$$\text{Im}(f : B \to B) \to \text{colim}_f B$$
is an isomorphism. Notice that

\[ C \cong \text{Im}(s \circ p : B \to B). \]

Let \( g \) denote the composite

\[ B \xrightarrow{\psi} B \otimes B \xrightarrow{f \otimes (s \circ p)} \text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B). \]

Let \( \alpha \in P(B) \). Then

\[ g(\alpha) = (\alpha - s \circ p(\alpha), s \circ p(\alpha)) \in P(\text{Im}(f : B \to B)) \oplus P(\text{Im}(s \circ p : B \to B)). \]

Thus

\[ g : B \to \text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B) \]

is a monomorphism and so the composite

\[ B \xrightarrow{\psi} B \otimes B \xrightarrow{\phi \otimes p} \text{colim}_f B \otimes C \]

is a monomorphism.

Let \( h \) denote the composite

\[ \text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B) \xrightarrow{\psi} B \otimes B \xrightarrow{f \otimes (s \circ p)} \text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B). \]

Notice that

\[ P(\text{Im}(f : B \to B)) = \text{Im}(id - s \circ p : P(B) \to P(B)), \]

\[ P(\text{Im}(s \circ p : B \to B)) = \text{Im}(s \circ p : P(B) \to P(B)). \]

Let

\[ (\alpha - s \circ p(\alpha), s \circ p(\beta)) \in P(\text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B)) \]

\[ = P(\text{Im}(f : B \to B)) \oplus P(\text{Im}(s \circ p : B \to B)), \]

where \( \alpha, \beta \in P(B) \). Then

\[ h(\alpha - s \circ p(\alpha), s \circ p(\beta)) = (\alpha - s \circ p(\alpha), s \circ p(\beta)). \]

Thus the coalgebra map \( h \) is a monomorphism and so the composite

\[ \text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B) \xrightarrow{\psi} B \otimes B \xrightarrow{\mu} B \]

is a monomorphism. In particular, the Poincaré series

\[ \chi(\text{colim}_f B \otimes C) = \chi(\text{Im}(f : B \to B) \otimes \text{Im}(s \circ p : B \to B)) \leq \chi(B). \]

The assertion follows.
Let $B$ and $C$ be connected graded cocommutative coalgebras and let $p: B \to C$ be a morphism of graded coalgebras. $B$ is a $C$-comodule via the map $p$ and the cotensor product $k \square B$ is a subcoalgebra of $B$.

**Lemma 5.3.** Let $p: B \to C$ and let $s: C \to B$ be morphisms of connected graded cocommutative coalgebras such that

$$p \circ s: C \to C$$

is the identity map. Let $A = k \square B$ be the cotensor product and let $j: A \to B$ be the canonical inclusion. Suppose that $B$ is a connected graded cocommutative quasi-Hopf algebra of finite type. Then the composite

$$A \otimes C \xrightarrow{j \otimes s} B \otimes B \xrightarrow{\mu} B$$

is an isomorphism of graded coalgebras.

**Proof.** Let $f$ denote the convolution product $(s \circ p \circ \chi') * id: B \to B$ and let $\theta$ denote the composite

$$B \xrightarrow{\psi} B \otimes B \xrightarrow{p \otimes \phi} C \otimes \text{colim}_f B.$$ 

Notice that there is a commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\theta} & C \otimes \text{colim}_f B \\
| & & | \\
p & id_C \otimes \epsilon_{\text{colim}_f B} & \downarrow \\
C & \xrightarrow{id} & C.
\end{array}
$$

Thus there is a commutative diagram

$$
\begin{array}{ccc}
A = k \square B & \xleftarrow{\phi} & B & \xrightarrow{id} & \text{colim}_f B \\
| & & \theta & \downarrow & | \\
& & & \text{colim}_f B & \xrightarrow{id} \text{colim}_f B & C.
\end{array}
$$

Hence the induced map

$$A \to \text{colim}_f B$$
is a monomorphism. By Lemma 5.2, the map $\theta$ is an isomorphism. Thus there is a commutative diagram

$$
\begin{array}{ccc}
A & \xleftarrow{\theta^{-1}} & B \\
\downarrow & & \downarrow \\
\text{colim}_f B & \xleftarrow{\times} & C \\
\end{array}
\quad
\begin{array}{ccc}
C \otimes \text{colim}_f B & \xrightarrow{\text{id}_C \otimes \epsilon_{\text{colim}_f B}} & C.
\end{array}
$$

Therefore the induced map

$$
\text{colim}_f B \rightarrow A
$$

is a monomorphism. Notice that both $\text{colim}_f B$ and $A$ are of finite type. Thus

$$
A \cong \text{colim}_f B.
$$

By examining the restriction to primitives we see that the composite

$$
A \otimes C \xrightarrow{j \otimes s} B \otimes B \xrightarrow{\mu} B
$$

is a monomorphism. However we have equality

$$
\chi(A \otimes C) = \chi(B)
$$

of Poincaré series and so the assertion follows.

\[ \Box \]

6. **Functorial Version of the Poincaré-Birkhoff-Witt Theorem**

Given an $k$-module $V$, let $L^m(V)$ denote the free restricted Lie algebra generated by $V$ if the characteristic of $k$ is greater than 0 and let it denote $L(V)$, the free Lie algebra generated by $V$, if $k$ is of characteristic 0. Recall that $A^\min(V; M)$ denotes the minimal natural coalgebra retract of $T(V)$ over $M(V)$. Thus $A^\min(-; M)$ is a functor from $k$-modules to connected graded quasi-Hopf algebras, where $A^\min(V; M)$ is graded by assigning dimension 2 to elements of $V$ and the multiplication in $A^\min(V; M)$ is given by the composite

$$
A^\min(V; M) \otimes A^\min(V; M) \xrightarrow{S_{A^\min} \otimes S_{A^\min}} T(V) \otimes T(V) \xrightarrow{\mu} T(V) \xrightarrow{r_{A^\min}} A^\min(V; M).
$$

Suppose $n \geq 1$ and let

$$
M^n(V) = \bigoplus_{1 \leq j \leq n} L^m_j(V) \subseteq T(V).
$$

Let

$$
p: T(V) \rightarrow A^\min(V; M^n)
$$

be the $k$-linear map defined by:

1. $p: T_0(V) \rightarrow A^\min_0(V; M^n)$ is the identity;
(2).
\[ p(x_1 \cdots x_m) = (\cdots ((x_1 \cdot x_2) \cdot x_3) \cdots \cdot x_m) \]
for \( m \geq 1 \) and \( x_j \in V \), where \( \cdot \) is the multiplication in \( A_{\text{min}}(V; M^n) \).

**Proposition 6.1.** Let \( p: T(V) \to A_{\text{min}}(V; M^n) \) be defined as above. Then:

1. The map \( p: T(V) \to A_{\text{min}}(V; M^n) \) is a functorial morphism of coalgebras.
2. The map \( p: T_j(V) \to A_{\text{min}}(V; M^n) \) is an isomorphism for \( j \leq n \).
3. The composite
   \[ A_{\text{min}}(V; M^n) \xrightarrow{S_{A_{\text{min}}}} T(V) \xrightarrow{p} A_{\text{min}}(V; M^n) \]
   is a functorial isomorphism of coalgebras.
4. Let \( I \) be the kernel of \( p: T(V) \to A_{\text{min}}(V; M^n) \). Then \( I \) is a natural right ideal of \( T(V) \).
5. Let \( B_{\text{max}}(V; M^n) = k \square_{\text{min}} T(V) \), where \( p \) is used to define the \( A_{\text{min}}(V; M^n) \)-comodule structure on \( T(V) \). Then \( B \) is a natural subHopf algebra of \( T(V) \).
6. There is a functorial isomorphism of coalgebras
   \[ k \otimes B_{\text{max}}(V; M^n) T(V) \cong A_{\text{min}}(V; M^n). \]

**Proof.** (1) Let \( C(V) \) be the connected graded coalgebra defined by:
   
   (i) \( C(V)_0 = k \);
   (ii) \( C(V)_2 = V \);
   (iii) \( P(C(V)) = V \).

Then the inclusion map \( C(V) \to A_{\text{min}}(V; M^n) \) is a natural map of graded coalgebras. Thus the composite

\[ C(V)^{\otimes m} \xhookrightarrow{} (A_{\text{min}}(V; M^n))^{\otimes m} \xrightarrow{\phi} A_{\text{min}}(V; M^n) \]

is a natural morphism of graded coalgebras, where \( \phi \) is \( m \)-fold multiplication from left to right. Notice that \( T(V) \) is the colimit of the diagram given by

\[ C(V)^{\otimes n} \xrightarrow{t_j} C(V)^{\otimes (n+1)}, \]

for \( n \geq 0 \) and \( 1 \leq j \leq n + 1 \), where \( t_j: C(V)^{\otimes n} \to C(V)^{\otimes (n+1)} \) is given by

\[ C(V)^{\otimes n} \xrightarrow{id_{C(V)} \otimes \cdots \otimes id_{C(V)}} (\eta_{C(V)} \circ \epsilon_{C(V)}) \otimes id_{C(V)} \otimes \cdots \otimes id_{C(V)} \xrightarrow{C^{\otimes (n+1)}(V)}. \]

Assertion (1) follows.

(2) Notice that

\[ L_j^{\text{res}}(V) \subseteq A_{\text{min}}(V; M^n) \]
for $1 \leq j \leq n$. Thus, by Lemma 5.1,

$$p: T_j(V) \to A_j^\min(V)$$

is a monomorphism for $j \leq n$. Notice that $A_j^\min(V; M^n)$ is a retract of $T(V)$. Thus for finitely dimensional $V$

$$p: T_j(V) \to A_j^\min(V)$$

is an isomorphism for $j \leq n$. Notice that every $k$-module is a colimit of finitely dimensional $k$-modules. Assertion (2) follows.

(3) By assertion (1), the map $p: T(V) \to A^\min(V; M^n)$ is a functorial map of coalgebras over $M^n$. By Theorem 4.12, the composite

$$A^\min(V; M^n) \xrightarrow{S_{A^\min}} T(V) \xrightarrow{P} A^\min(V; M^n) \xrightarrow{T_{A^\min}} T(V) \xrightarrow{T_{A^\min}} A^\min(V; M^n)$$

is an isomorphism. Assertion (3) follows.

(4) Let $\alpha \in T_k(V)$ be such that

$$p(\alpha) = 0$$

and let $x_j \in V$. Then

$$p(\alpha x_j) = p(\alpha) \cdot x_j = 0.$$ 

Assertion (4) follows.

(5) Let $\alpha, \beta \in B^\max(V; M^n)$ be positive dimensional (homogeneous) elements. Let

$$\psi(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum \alpha' \otimes \alpha'',$$

$$\psi(\beta) = \beta \otimes 1 + 1 \otimes \beta + \sum \beta' \otimes \beta''.$$ 

Then, in $T(V)$,

$$\psi(\alpha \beta) = \alpha \beta \otimes 1 + 1 \otimes \alpha \beta + \alpha \otimes \beta + (-1)^{\|\beta\|} \beta \otimes \alpha + \sum((-1)^{\|\beta'\|} \alpha' \beta \otimes \alpha'' + \alpha' \otimes \alpha'' \beta) + \sum(\alpha \beta' \otimes \beta'' + (-1)^{\|\beta'\|} \beta' \otimes \alpha \beta'') + \sum(-1)^{\|\beta''\|} \alpha \beta' \otimes \alpha'' \beta''.$$ 

Notice that because of cocommutativity

$$\alpha, \alpha', \alpha'', \beta, \beta', \beta'' \in I(B) \subseteq I = \ker(p).$$ 

By assertion (4), $I$ is a right ideal. Thus

$$(p \otimes \text{id}_{T(V)}) \circ \psi(\alpha \beta) = 1 \otimes \alpha \beta$$

in $C \otimes T(V)$ and so $\alpha \beta \in B^\max(V; M^n)$. Assertion (5) follows.

(6) By assertions (4) and (5) the map

$$p: T(V) \to A^\min(V; M^n)$$
factors through $k \otimes B_{\text{max}}(V; M^n) T(V)$. Let
\[
\bar{p} : k \otimes B_{\text{max}}(V; M^n) T(V) \to A_{\text{min}}(V; M^n)
\]
denote the resulting epimorphism. If $V$ is finitely dimensional, then, by Lemma 5.3,
the Poincaré series
\[
\chi(A_{\text{min}}(V; M^n)) = \chi(T(V))/\chi(B_{\text{max}}(V; M^n)).
\]
By Theorem 4.4 in [12], we have
\[
\chi(k \otimes B_{\text{max}}(V; M^n) T(V)) = \chi(T(V))/\chi(B_{\text{max}}(V; M^n)).
\]
Thus $\bar{p} : k \otimes B_{\text{max}}(V; M^n) T(V) \to A_{\text{min}}(V; M^n)$ is an isomorphism if $V$ is finitely dimensional. Notice that every $k$-module is a colimit of finitely dimensional $k$-modules. Assertion (6) follows.

By Lemma 5.3, the sub-Hopf algebra $B_{\text{max}}(V; M^n)$ is a functorial coalgebra retract of $T(V)$ and thus, by Lemma 4.2, it is a functorial graded coalgebra retract of $T(V)$. Notice that
\[
B_{\text{max}}^j(V; M^n) = 0
\]
for $0 < j < n + 1$. Thus
\[
B_{n+1}^\text{max}(V; M^n) = P(B_{n+1}^\text{max}(V; M^n)) \subseteq P(T_{n+1}(V)) = L_{n+1}^\text{max}(V).
\]
Let $L_{n+1}^\text{max}(V)$ denote $B_{n+1}^\text{max}(V; M^n)$ for $n \geq 1$. Recall that $k(S_n)$ acts functorially on the right on $V^\otimes n$ by permuting positions. For $\lambda \in k(S_n)$, we also use $\lambda$ to denote the functorial map $V^\otimes n \to V^\otimes n$ given by the action of $\lambda$.

Define $\beta_n : V^\otimes n \to V^\otimes n$ by $\beta_n(x_1 \otimes \cdots \otimes x_n) = [[x_1, x_2, \ldots, x_n]]$.

**Lemma 6.2.** Suppose $n \geq 2$ and let $L_n^\text{max}(V)$ be defined as above. Then:
1. $L_n^\text{max}(V)$ is a functorial retract of $L_n(V)$;
2. there exists an element $\lambda_n^\text{max} \in k(S_n)$ such that:
   (i) $\beta_n \circ \lambda_n^\text{max} \circ \beta_n \circ \lambda_n^\text{max} = \beta_n \circ \lambda_n^\text{max}$;
   (ii) there is a functorial isomorphism
   \[
   L_n^\text{max}(V) \cong \text{colim}_{\beta_n, \lambda_n^\text{max}} V^\otimes n = \text{Im}(\beta_n \circ \lambda_n^\text{max} : V^\otimes n \to V^\otimes n)
   \]
   for $k$-modules $V$.

**Proof.** Let
\[
j : B_{\text{max}}^\text{max}(V; M^{n-1}) \to T(V)
\]
be the canonical functorial injection and let
\[
r : T(V) \to B_{\text{max}}^\text{max}(V; M^{n-1})
\]
be a functorial coalgebra retraction. Let \( f \) denote the composite \( j \circ r : T(V) \to T(V) \). Notice that \( f \) is idempotent and that

\[
E^\text{max}(V; M^{n-1}) = \operatorname{colim}_f T(V).
\]

By Lemma 2.1, there exists an element \( \theta \in k(S_n) \) such that

\[
\theta = f|_{T_n(V)} : T_n(V) = V^\otimes n \to V^\otimes n.
\]

Let \( \tilde{V} \) be the \( k \)-module with basis \( \{x_1, \ldots, x_n\} \). Let \( \gamma_n \) denote the submodule of \( T_n(\tilde{V}) = \tilde{V}^\otimes n \) spanned by the elements

\[
x_{\sigma(1)} \cdots x_{\sigma(n)}
\]
as \( \sigma \) runs through \( S_n \). Consider the map \( \theta : \tilde{V}^\otimes n \to \tilde{V}^\otimes n \). Notice that

\[
\operatorname{Im}(\theta : \tilde{V}^\otimes n \to \tilde{V}^\otimes n) = \operatorname{Im}(f : T_n(\tilde{V}) \to T_n(\tilde{V})) = L^\text{max}_n(\tilde{V}) \subseteq L^\text{max}_n(\tilde{V}).
\]

Thus

\[
\theta(x_1 \cdots x_n) \in \gamma_n \cap L^\text{max}_n(\tilde{V}) = \gamma_n \cap L_n(\tilde{V}) = \text{Lie}(n).
\]

Thus

\[
\theta(x_1 \cdots x_n) = \sum_{\tau \in S_{n-1}} k_{\tau}[x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)}]
\]
for some \( k_{\tau} \in k \), where \( S_{n-1} \) acts on \( \{2, 3, \ldots, n\} \). For each \( \tau \in S_{n-1} \), let \( \bar{\tau} \in S_n \) be defined by

\[
\bar{\tau}(1) = 1, \bar{\tau}(j) = \tau(j)
\]
for \( 2 \leq j \leq n \). Now let

\[
\lambda^\text{max}_n = \sum_{\tau \in S_{n-1}} k_{\tau} \bar{\tau} \in k(S_n).
\]

Let

\[
\theta' = \beta_n \circ \lambda^\text{max}_n : V^\otimes n \to V^\otimes n.
\]

In the case \( V = \tilde{V} \),

\[
\theta(x_1 \cdots x_n) = \theta'(x_1 \cdots x_n).
\]

Let \( e : k(S_n) \to \text{Hom}_k(V^\otimes n, V^\otimes n) \) be the representation map of the \( k(S_n) \)-action on \( V^\otimes n \). Then

\[
e(\theta) = e(\theta')
\]
in \( \text{Hom}_k(V^\otimes n, V^\otimes n) \). Notice that the representation map \( e : k(S_n) \to \text{Hom}_k(V^\otimes n, V^\otimes n) \) is faithful. Thus

\[
\theta = \theta' = \beta_n \circ \lambda^\text{max}_n : V^\otimes n \to V^\otimes n.
\]

Notice that

\[
\theta \circ \theta = f \circ f = f = \theta.
\]
The assertions follow. \( \square \)
Lemma 6.3. Suppose $n \geq 2$. Then there is a functorial isomorphism of coalgebras

$$A^\min(V; M^n) \cong A^\min(V; L_n^\max) \otimes A^\min(V; M^{n-1})$$

for any $k$-module $V$.

Proof. Consider the commutative diagram of short exact sequences of coalgebras

$$
\begin{array}{ccc}
B^\max(V; M^{n-1}) & \xrightarrow{j} & T(V) & \xrightarrow{p} & A^\min(V; M^n) \\
\downarrow{S_{A^\min(V; M^n)}} & & & & \downarrow{p \circ S_{A^\min(V; M^n)}} \\
\mathbb{k}[\max_{i=1}^{n-1}(V,M^{n-1})] & \xrightarrow{j} & A^\min(V; M^n) & \xrightarrow{p \circ S_{A^\min(V; M^n)}} & A^\min(V; M^n) \\
\end{array}
$$

Notice that the composite

$$A^\min(V; M^{n-1}) \xrightarrow{S_{A^\min(V; M^n)}} T(V) \xrightarrow{p \circ S_{A^\min(V; M^n)}} A^\min(V; M^n)$$

is a natural map of coalgebra over $M^{n-1}(V) = \bigoplus_{1 \leq j \leq n-1} L_j^\max(V)$. By Theorem 4.12, this composite is a functorial isomorphism of graded coalgebras. Thus there is a cross-section map of coalgebras

$$\tilde{s}: A^\min(V; M^{n-1}) \to A^\min(V; M^n)$$

such that the composite

$$p \circ S_{A^\min(V; M^n)} \circ \tilde{s}: A^\min(V; M^{n-1}) \to A^\min(V; M^{n-1})$$

is the identity map. Let $D(V)$ denote the cotensor product $\mathbb{k}[\max_{i=1}^{n-1}(V,M^{n-1})] \otimes A^\min(V; M^n)$. By Lemma 5.3, $D(V)$ is a natural coalgebra retract of $A^\min(V; M^n)$ if $V$ is finite dimensional. Using a colimit argument, we conclude that $D(V)$ is a natural coalgebra retract of $A^\min(V; M^n)$ for any $k$-module $V$ and so $D(V)$ is a natural coalgebra retract of $T(V)$ for any $k$-module $V$. Notice that because of Proposition 6.1(2)

$$D_n(V) = B_n^\max(V; M^{n-1}) = L_n^\max(V).$$

Thus $D(V)$ is a natural coalgebra retract of $T(V)$ over $L_n^\max(V)$ and so $A^\min(V; L_n^\max)$ is a natural coalgebra retract of $D(V)$. More precisely, there exist functorial maps of coalgebras $f: A^\min(V; L_n^\max) \to D(V)$ and $g: D(V) \to A^\min(V; L_n^\max)$ such that

1. $f|_{A^\min(V; L_n^\max)}: A^\min(V; L_n^\max) = L_n^\max(V) \to D_n(V) = L_n^\max(V)$ and $g|_{D_n(V)}: D_n(V) = L_n^\max(V) \to A^\min(V; L_n^\max)$ are the identity maps;

2. the composite $g \circ f: A^\min(V; L_n^\max) \to A^\min(V; L_n^\max)$ is the identity map.
By Lemmas 5.2 and 5.3, there is a functorial coalgebra retraction
\[ \tilde{r}: A^{\text{min}}(V; M^n) \rightarrow D(V) \]
such that the composite
\[
A^{\text{min}}(V; M^n) \xrightarrow{\tilde{r}} A^{\text{min}}(V; M^n) \otimes A^{\text{min}}(V; M^n) \xrightarrow{\tilde{r} \otimes (p \circ s_{A^{\text{min}}(V; M^n)})} D(V) \otimes A^{\text{min}}(V; M^n) \]
is a functorial isomorphism of coalgebras for any finitely dimensional \( k \)-module \( V \) and so for any \( k \)-module \( V \). Let \( \phi: T(V) \rightarrow T(V) \) be the composite
\[
T(V) \xrightarrow{id \otimes \tilde{r}} A^{\text{min}}(V; M^n) \xrightarrow{\tilde{r} \otimes (p \circ s_{A^{\text{min}}(V; M^n)}) \circ \psi} D(V) \otimes A^{\text{min}}(V; M^n) \]
and
\[
T(V) \xrightarrow{\mu \circ (j \otimes \tilde{s})} A^{\text{min}}(V; M^n) \xrightarrow{\tilde{r}} A^{\text{min}}(V; M^n) \]
Then \( \phi|_{T_j(V)}: T_j(V) \rightarrow T_j(V) \) is a natural isomorphism for \( j \leq n \). Thus
\[
\text{colim}_\phi T(V)
\]
is a natural coalgebra retract of \( T(V) \) over \( M^n \) and so \( A^{\text{min}}(V; M^n) \) is a natural coalgebra retract of \( \text{colim}_\phi T(V) \). Notice that the map \( \phi: T(V) \rightarrow T(V) \) factors through \( A^{\text{min}}(V; L_{n}^{\text{max}}) \otimes A^{\text{min}}(V; M^{n-1}) \). Thus \( \text{colim}_\phi T(V) \) is a coalgebra retract of \( A^{\text{min}}(V; L_{n}^{\text{max}}) \otimes A^{\text{min}}(V; M^{n-1}) \) and so \( A^{\text{min}}(V; M^n) \) is a natural coalgebra retract of \( A^{\text{min}}(V; L_{n}^{\text{max}}) \otimes A^{\text{min}}(V; M^{n-1}) \). In particular, the Poincaré series of primitive submodules satisfy
\[
\chi(P(A^{\text{min}}(V; M^n))) = \chi(P(D(V))) + \chi(P(A^{\text{min}}(V; M^{n-1})))
\]
for any finitely dimensional \( k \)-module \( V \) and so
\[
\chi(P(D(V))) \leq \chi(P(A^{\text{min}}(V; L_{n}^{\text{max}})))
\]
for any finitely dimensional \( k \)-module \( V \). Notice that
\[
f: P(A^{\text{min}}(V; L_{n}^{\text{max}})) \rightarrow P(D(V))
\]
is a monomorphism. Thus \( f: P(A^{\text{min}}(V; L_{n}^{\text{max}})) \rightarrow P(D(V)) \) is a functorial isomorphism for any finitely dimensional \( k \)-module \( V \). Recall that
\[
g \circ f: A^{\text{min}}(V; L_{n}^{\text{max}}) \rightarrow A^{\text{min}}(V; L_{n}^{\text{max}})
\]
is the identity map and so
\[
g: P(D(V)) \rightarrow P(A^{\text{min}}(V; L_{n}^{\text{max}}))
is an epimorphism. Thus $g: P(D(V)) \rightarrow P(A^\text{min}(V; L_n^{\text{max}}))$ is a natural isomorphism for any finitely dimensional $k$-module $V$. Therefore

$$g: D(V) \rightarrow A^\text{min}(V; L_n^{\text{max}})$$

is a functorial monomorphism and thus an isomorphism for any finitely dimensional $k$-module $V$. Therefore $g: D(V) \rightarrow A^\text{min}(V; L_n^{\text{max}})$ is a functorial isomorphism of coalgebras for any $k$-module $V$. The assertion follows.

**Corollary 6.4.** The functor $A^\text{min}(-; L_n^{\text{max}})$ has the property

$$A^\text{min}_j(V; L_n^{\text{max}}) = \begin{cases} 0 & \text{for } 0 < j < n; \\ L_n^{\text{max}}(V) & \text{for } j = n. \end{cases}$$

Let $L_n^{\text{max}}(V)$ denote $V$. A functorial coalgebra decomposition of tensor algebras is as follows.

**Theorem 6.5 (Functorial Poincaré-Birkhoff-Witt).** There exists a functorial isomorphism of coalgebras

$$T(V) \cong \bigotimes_{n=1}^{\infty} A^\text{min}(V; L_n^{\text{max}})$$

for any $k$-module $V$.

**Proof.** By Lemmas 6.3 and induction, there is a functorial isomorphism of coalgebras

$$\phi_n: \bigotimes_{j=1}^{n} A^\text{min}(V; L_j^{\text{max}}) \rightarrow A^\text{min}(V; M^n)$$

such that the diagram

$$\begin{array}{ccc}
\bigotimes_{j=1}^{n} A^\text{min}(V; L_j^{\text{max}}) & \xrightarrow{\phi_n} & A^\text{min}(V; M^n) \\
\uparrow & & \downarrow \tilde{s}_n \\
\bigotimes_{j=1}^{n-1} A^\text{min}(V; L_j^{\text{max}}) & \xrightarrow{\phi_{n-1}} & A^\text{min}(V; M^{n-1})
\end{array}$$

commutes, where $\tilde{s}_n: A^\text{min}(V; M^{n-1}) \rightarrow A^\text{min}(V; M^n)$ is an injection of coalgebras. Notice that the colimit of the sequence

$$A^\text{min}(V) = A^\text{min}(V; M^1) \hookrightarrow A^\text{min}(V; M^2) \hookrightarrow A^\text{min}(V; M^3) \hookrightarrow \cdots$$

is isomorphic to $T(V)$. The assertion follows. \qed
**Theorem 6.6.** Suppose \( n \geq 2 \) and let \( \lambda \in k(S_n) \) be any element. Then the colimit

\[
\text{colim}_{\beta_n \circ \lambda} V^\otimes n
\]

is a functorial retract of \( L_n^{\text{max}}(V) \).

**Proof.** Let \( M(V) \) denote \( \text{colim}_{\beta_n \circ \lambda} V^\otimes n \). Let \( p_m : V^\otimes n \to \text{colim}_{\beta_n \circ \lambda} V^\otimes n \) be the canonical map from the \( m \)-th term in the sequence

\[
\begin{array}{cccc}
V^\otimes n & \xrightarrow{\beta_n \circ \lambda} & V^\otimes n & \xrightarrow{\beta_n \circ \lambda} & V^\otimes n & \xrightarrow{\beta_n \circ \lambda} & \ldots
\end{array}
\]

to the colimit. By Lemma 4.3, the \( k \)-module \( \text{colim}_{\beta_n \circ \lambda} V^\otimes n \) is a functorial retract of \( V^\otimes n \). Thus there is a functorial submodule

\[
M'(V) \subseteq L_n(V)
\]
such that the composite

\[
M'(V) \xhookrightarrow{} L_n(V) \xrightarrow{} V^\otimes n \xrightarrow{p_2} M(V)
\]
is a functorial isomorphism.

Now the functorial injection \( j : M'(V) \subseteq V^\otimes n \) induces a unique map of algebras

\[
\phi : T(M'(V)) \to T(V)
\]
such that \( \phi|_{M'(V)} : M'(V) \to V^\otimes n \xhookrightarrow{} T(V) \) is the functorial injection. Since \( M'(V) \subseteq L_n(V) \subseteq PT(V) \) the functorial map \( \phi : T(M'(V)) \to T(V) \) is a map of Hopf algebras.

Let \( H_n : T(V) \to T(V^\otimes n) \) be the James-Hopf map \([8]\) and let

\[
\phi' : T(V) \to T(M(V))
\]
be the composite

\[
T(V) \xrightarrow{H_n} T(V^\otimes n) \xrightarrow{T(p_2)} T(M(V)).
\]

Then the map \( \phi' : T(V) \to T(M(V)) \) is a functorial map of coalgebras. Notice that the diagram

\[
\begin{array}{ccc}
T(L_n(V)) & \xhookrightarrow{} & T(V) \\
\downarrow & & \downarrow \quad H_n \\
T(V^\otimes n) & \xrightarrow{} & T(V^\otimes n)
\end{array}
\]
commutes by [8, Proposition 5.1] or [17, Corollary 1.2]. Thus there is a commutative diagram

\[
\begin{array}{ccc}
T(L_n(V)) & \overset{\varepsilon}{\longrightarrow} & T(V) \\
\downarrow & & \downarrow \\
T(M'(V)) & \overset{T(p_2) \circ j}{\longrightarrow} & T(M(V))
\end{array}
\]

Notice that \( p_2 \circ j : M'(V) \to M(V) \) is an isomorphism. Thus \( T(M(V)) \) is a natural coalgebra retract of \( T(V) \). By Lemma 5.2, there exists a natural coalgebra retract \( A(V) \) of \( T(V) \) such that there is a functorial isomorphism of graded coalgebras

\[
T(V) \cong T(M(V)) \otimes A(V)
\]

for any finite dimensional \( \mathbf{k} \)-module \( V \) and so for any \( \mathbf{k} \)-module \( V \). Notice that

\[
T_j(V) \cong A_j(V)
\]

for \( j \leq n - 1 \). Thus \( A(V) \) is a natural coalgebra retract of \( T(V) \) over \( M^{n-1}(V) = \bigoplus_{1 \leq j \leq n-1} L_j(V) \). By Corollary 4.13, \( A^{\min}(V; M^{n-1}) \) is a functorial coalgebra retract of \( A(V) \) over \( M^{n-1}(V) \). By Lemma 5.2, there exists a functorial coalgebra \( D(V) \) such that there is a functorial isomorphism of coalgebras

\[
A(V) \cong D(V) \otimes A^{\min}(V; M^{n-1})
\]

for any finite dimensional \( \mathbf{k} \)-module \( V \) and so for any \( \mathbf{k} \)-module \( V \). Thus there is a functorial isomorphism of coalgebras

\[
T(V) \cong T(M(V)) \otimes D(V) \otimes A^{\min}(V; M^{n-1})
\]

and so there is a functorial isomorphism of coalgebras

\[
\theta : B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \cong T(M(V)) \otimes D(V) \otimes A^{\min}(V; M^{n-1}).
\]

Let

\[
q = \epsilon_{T(M(V)) \otimes D(V)} \otimes id_{A^{\min}(V; M^{n-1})} : T(M(V)) \otimes D(V) \otimes A^{\min}(V; M^{n-1}) \to A^{\min}(V; M^{n-1})
\]

be the canonical projection and let \( \tilde{\mathcal{B}}(V) \) denote the cotensor product

\[
\tilde{\mathcal{B}}(V) = \mathbf{k} \square_{A^{\min}(V; M^{n-1})} (B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}))
\]

where the \( A^{\min}(V; M^{n-1}) \)-comodule structure on \( B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \) is induced by the composite of the coalgebra maps

\[
q \circ \theta : B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \to A^{\min}(V; M^{n-1}).
\]
Let
\[ i = \eta_{B^{\max}(V; M^{n-1})} \otimes \text{id}_{A^{\min}(V; M^{n-1})} : A^{\min}(V; M^{n-1}) \to B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \]
be the canonical inclusion map and let \( \theta' : A^{\min}(V; M^{n-1}) \to A^{\min}(V; M^{n-1}) \) denote the composite
\[ \theta' = q \circ \theta \circ i : A^{\min}(V; M^{n-1}) \to A^{\min}(V; M^{n-1}). \]
Then
\[ \theta' : A_j^{\min}(V; M^{n-1}) \to A_j^{\min}(V; M^{n-1}) \]
is an isomorphism for \( j \leq n - 1 \). Thus, by Theorem 4.12, the functorial coalgebra map \( \theta' : A^{\min}(V; M^{n-1}) \to A^{\min}(V; M^{n-1}) \) is an isomorphism. Thus the short exact sequence of coalgebras
\[ \tilde{B}(V) \hookrightarrow B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \xrightarrow{q \circ \theta} A^{\min}(V; M^{n-1}) \]
splits functorially by Lemma 5.3 and so
\[ B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \cong \tilde{B}(V) \otimes A^{\min}(V; M^{n-1}) \]
as coalgebras. By the commutative diagram of short exact sequences of coalgebras
\[
\begin{array}{ccc}
T(M(V)) \otimes D(V) & \hookrightarrow & T(M(V)) \otimes D(V) \otimes A^{\min}(V; M^{n-1}) \\
\downarrow & & \downarrow q \\
\tilde{B}(V) & \xrightarrow{i'} & B^{\max}(V; M^{n-1}) \otimes A^{\min}(V; M^{n-1}) \\
\downarrow & & \downarrow q \circ \theta \\
& & A^{\min}(V; M^{n-1}),
\end{array}
\]
the resulting functorial coalgebra map \( \tilde{B}(V) \to T(M(V)) \otimes D(V) \) is a monomorphism. Notice that both \( B(V) \) and \( T(M(V)) \otimes D(V) \) have the same Poincaré series if \( V \) is finite dimensional. Thus there is a functorial isomorphism of coalgebras
\[ B(V) \cong T(M(V)) \otimes D(V) \]
for any finite dimensional \( k \)-module \( V \) and so for any \( k \)-module \( V \). The assertion follows.

\begin{corollary}
Suppose the ground field \( k \) is of characteristic \( p \) and let \( n \geq 1 \) be an integer such that \( n \) is not divisible by \( p \). Then
\[ I_n^{\max}(V) = I_n(V). \]
If \( \text{char}(k) = 0 \), then \( I_n^{\max}(V) = I_n(V) \) for all \( n \geq 1 \).
\end{corollary}
Proof. Notice that $L_n^{\max}(V) \subseteq L_n(V)$ and $L_n(V) \cong \text{colim}_\beta V^\otimes n$ by the Dynkin-Specht-Weber formula

$$\beta_n \circ \beta_n = n\beta_n.$$ The assertion follows.

Further functorial coalgebra decomposition of $A^{\text{min}}(V; L_n^{\max})$ can be given by using the following proposition.

**Proposition 6.8.** Let $M$ be a subfunctor of $L_n$ from $k$-modules to $k$-modules. Suppose that

1) $M$ is a retract of the functor $L_n^{\max}$;
2) there are functors $M'$ and $M''$ from $k$-modules to $k$-modules such that there is a functorial isomorphism of $k$-modules

$$M(V) \cong M'(V) \oplus M''(V).$$ Then there is a functorial isomorphism of coalgebras

$$A^{\text{min}}(V; M) \cong A^{\text{min}}(V; M') \otimes A^{\text{min}}(V; M'').$$

**Proof.** One may assume that $M'$ is a subfunctor of $M$ from $k$-modules to $k$-modules. Notice that

$$M'(V) \subseteq M(V) \subseteq A^{\text{min}}(V; M)$$

and $A^{\text{min}}(V; M)$ is a natural coalgebra retract of $T(V)$. Thus $A^{\text{min}}(V; M')$ is a natural coalgebra retract of $A^{\text{min}}(V; M)$ by Corollary 4.13 and so there is a functorial coalgebra decomposition

$$A^{\text{min}}(V; M) \cong A^{\text{min}}(V; M') \otimes B(V)$$

for any finite dimensional $k$-module $V$ by Lemma 5.3, where

$$B(V) = k \boxtimes_{A^{\text{min}}(V; M')} A^{\text{min}}(V; M)$$

and $A^{\text{min}}(V; M)$ is an $A^{\text{min}}(V; M')$-comodule via a given (functorial) coalgebra retraction from $A^{\text{min}}(V; M)$ to $A^{\text{min}}(V; M')$. By using colimit arguments, this decomposition holds for any $k$-module $V$.

Let $\iota_{L_n^{\max}}: V^\otimes n \to L_n^{\max}(V)$ and $r_M: L_n^{\max}(V) \to M(V)$ be functorial retractions. Notice that the composite of functorial coalgebra maps

$$T(M(V)) \xrightarrow{\text{H}_n} T(V^\otimes n) \xrightarrow{T(\iota_{L_n^{\max}})} T(L_n^{\max}(V)) \xrightarrow{T(r_M)} T(M(V)).$$

is multiplicative, where $H_n$ is the James-Hopf map. Thus this composite is determined by the restriction to $M(V)$ and so it is an isomorphism. Thus $T(M(V))$ is a functorial
coalg. retract of $T(V)$ over $M(V)$ and so $A^{\text{min}}(V; M)$ is a functorial coalgebra retract of $T(M(V))$. Thus $A^{\text{min}}_j(V; M) = 0$ for $0 < j < n$ and $A^{\text{min}}_n(V; M) = M(V)$. Similarly, $A^{\text{min}}_j(V; M') = 0$ for $0 < j < n$ and $A^{\text{min}}_n(V; M') = M'(V)$. Thus $B_j(V) = 0$ for $0 < j < n$ and $B_n(V) \cong M''(V)$. By Corollary 4.13, $A^{\text{min}}(V; M'')$ is a natural coalgebra retract of $B(V)$. Thus $A^{\text{min}}(V; M') \otimes A^{\text{min}}(V; M'')$ is a natural coalgebra retract of $A^{\text{min}}(V; M)$. So $A^{\text{min}}(V; M') \otimes A^{\text{min}}(V; M'')$ is also a natural coalgebra retract of $T(V)$ over $M(V) \cong M'(V) \oplus M''(V)$. By Corollary 4.13, $A^{\text{min}}(V; M)$ is a natural coalgebra retract of $A^{\text{min}}(V; M') \otimes A^{\text{min}}(V; M'')$. The assertion follows. \qed

**Corollary 6.9.** Suppose that there are functors $M_{\alpha, \gamma}$ from $k$-modules to $k$-modules for $\alpha \in I_n$ and $n \geq 1$ such that there is a functorial isomorphism of $k$-modules

$$L_{n}^{\text{max}}(V) \cong \bigoplus_{\alpha \in I_n} M_{\alpha, \gamma}(V)$$

for $n \geq 1$. Then there is a functorial isomorphism of coalgebras

$$T(V) \cong \bigotimes_{n=1}^{\infty} \bigotimes_{\alpha \in I_n} A^{\text{min}}(V; M_{\alpha, \gamma}).$$

Let $M(V)$ be a subfunctor of $T(V)$ from $k$-modules to $k$-modules. The notations $A^{\text{min}}(V; M)$ and $A^{\text{min}}(M(V))$ have different meanings. The first one means the ‘smallest’ functorial coalgebra retract of $T(V)$ that contains $M(V)$ and the second one means the functor $A^{\text{min}}(-)$ evaluated on the $k$-module $M(V)$ (thus the ‘smallest’ functorial retract of $T(M(V))$ containing $M(V)$, where functorial means functorial in $M(V)$ rather than in $V$). A relation between $A^{\text{min}}(V; M)$ and $A^{\text{min}}(M(V))$ for some special functors $M$ is as follows.

**Proposition 6.10.** Let $M$ be a subfunctor of $L_n$ from $k$-modules to $k$-modules such that $M$ is a retract of the functor $L_n^{\text{max}}$. Then $A^{\text{min}}(V; M)$ is a natural coalgebra retract of $A^{\text{min}}(M(V))$ over $M(V)$.

Proof. Let $W$ denote $M(V)$. Notice that $A^{\text{min}}(W)$ is a natural coalgebra retract of $T(W)$, where the inclusion and the retraction are functorial in $W$ and so in $V$. By the proof of Proposition 6.8, $T(M(V))$ is a natural coalgebra retract of $T(V)$. Thus $A^{\text{min}}(M(V))$ is a natural coalgebra retract of $T(V)$ over $M(V)$. The assertion follows from Corollary 4.13. \qed

$A^{\text{min}}(V; M)$ can be a proper coalgebra retract of $A^{\text{min}}(M(V))$ as seen by the following example.
Example 6.11. Let the ground field $k$ be of characteristic 2. Let $\phi: T(V) \to T(V)$ be the composite
\[ T(V) \xrightarrow{H_6} T(V^6) \xrightarrow{T(\tau_{2,4})} T(V^6) \xrightarrow{T([\beta_3, \beta_3])} T([L_3(V), L_3(V)]) \xrightarrow{j} T(V), \]
where $j$ is the inclusion, $\tau_{2,4}(a_1 a_2 \cdots a_6) = a_1 a_4 a_3 a_2 a_5 a_6$ given by interchanging positions 2 and 4 and
\[ [\beta_3, \beta_3](a_1 a_2 \cdots a_6) = [[[a_1, a_2], a_3], [[[a_4, a_5], a_6]]. \]
Then we have
\[ \phi([[x, y], [y, z], [z, y], [x, z], [z, x], [x, y], [y, z]]) = ([[x, y], [y, z], [z, y], [x, z], [z, x], [x, y]] \oplus [[x, y], [y, z], [z, y], [x, z], [z, x], [x, y]]) \]
for $x, y, z \in V$. Let $B(V) = \text{colim}_\phi T(V)$. Then $B_6(V)$ is non-trivial in general because $\phi$ has nonzero fixed point when $\dim(V) \geq 3$. By Corollary 4.4, $B(V)$ is a natural coalgebra retract of $T(V)$. Notice that the inclusion $T([L_3(V), L_3(V)]) \to T(V)$ factors through the subHopf algebra $T(L_3(V))$ of $T(V)$. Therefore the map $\phi$ factors through $T(L_3(V))$. Thus $B(V)$ is a natural coalgebra retract of $T(L_3(V))$ and so there is a natural coalgebra decomposition
\[ T(L_3(V)) \cong B(V) \otimes A(V) \]
for some functor $A$ from $k$-modules to coalgebras. Notice that $B_j(V) = 0$ for $0 < j < 6$ and so $L_3(V) \subseteq A(V)$. Thus $A^{\text{min}}(V; L_3)$ is a natural retract of $A(V)$. Recall that $B_6(V)$ is non-trivial in general. Thus $A_6(V)$ is not functorially isomorphic to $L_3(V) \otimes L_3(V)$ and so $A_6^{\text{min}}(V; L_3)$ is not functorially isomorphic to $L_3(V) \otimes L_3(V)$.

In contrast, we show that
\[ A_2^{\text{min}}(V) \cong T_2(V) = V \otimes V \]
for any $k$-module $V$ and so $A_6^{\text{min}}(L_3(V))$ is functorially isomorphic to $L_3(V) \otimes L_3(V)$. To check that $A_2^{\text{min}}(V)$ is isomorphic to $V^{\otimes 2}$, let $j_V: A^{\text{min}}(V) \to T(V)$ be the functorial inclusion and let $r_V: T(V) \to A^{\text{min}}(V)$ be a functorial coalgebra retraction. Let $f_V = j_V \circ r_V: T(V) \to T(V)$ be the composite. Notice that $f_V$ is a map of coalgebras and $f_V(x) = x$ for $x \in V$. There is an element $\zeta \in k$ such that
\[ f_V(xy) = (1 + \zeta)xy - \zeta yx \]
for any $x, y \in V$. Notice that $f_V$ is idempotent. Thus
\[ f_V(xy) = f_V \circ f_V(xy) = ((1 + \zeta)^2 + \zeta^2)xy - 2\zeta(1 + \zeta)yx = xy \]
for any $x, y \in V$ and so $A_2^{\text{min}}(V) = \text{colim}_{f_V} V^{\otimes 2}$ is isomorphic to $V^{\otimes 2}$.

If the field $k$ is of characteristic 0, then the natural coalgebra decomposition of tensor algebras of Theorem 6.5 can be described explicitly. Let $S(V)$ denote the symmetric algebra generated by $V$. 


Proposition 6.12. If $k$ is of characteristic 0, then there is functorial isomorphism of coalgebras
\[ A^{\min}(V; L_n^{\max}) \cong S(L_n(V)). \]
Thus there is a functorial isomorphism of coalgebras
\[ T(V) \cong \bigotimes_{n=1}^{\infty} S(L_n(V)). \]

Proof. First we show that $A^{\min}(V)$ is naturally isomorphic to $S(V)$.

Let $p : T(V) \to A^{\min}(V)$ be the canonical map induced by the multiplication in $A^{\min}(V)$. (See Proposition 6.1.) Let $B^{\max}(V) = k \square A^{\min}(V) T(V)$. By Proposition 6.1, $B^{\max}(V)$ is a sub-Hopf algebra of $T(V)$. Notice that $L_n(V) \subseteq B^{\max}(V)$ for $n \geq 2$. Let $\tilde{B}(V)$ denote the sub-Hopf algebra of $T(V)$ generated by $L_n(V)$ with $n \geq 2$. Then $\tilde{B}(V)$ is a normal sub-Hopf algebra of $T(V)$ with $k \otimes_{B(V)} T(V) \cong S(V)$ because all commutators are in $\tilde{B}(V)$. Notice that the map $p : T(V) \to A^{\min}(V)$ factors through $k \otimes_{\tilde{B}(V)} T(V) \cong S(V)$. The resulting coalgebra map
\[ \tilde{p} : S(V) \to A^{\min}(V) \]
is an epimorphism. Notice that
\[ \tilde{p} : V = P(S(V)) \to P(A^{\min}(V)) = V \]
is an isomorphism. Thus $\tilde{p} : S(V) \to A^{\min}(V)$ is a monomorphism and so $\tilde{p} : S(V) \to A^{\min}(V)$ is an isomorphism of coalgebras. Since $L_1^{\max}(V) = V$, this shows the first statement for $n = 1$.

Now consider the case where $n \geq 2$. Notice that $L_n^{\max}(V) = L_n(V)$ according to Corollary 6.7. By the first step applied with $L_n(V)$ replacing $V$, there is an isomorphism
\[ A^{\min}(L_n(V)) \cong S(L_n(V)) \]
which is functorial in $L_n(V)$ and thus functorial in $V$. Therefore, by Proposition 6.10, $A^{\min}(V; L_n^{\max})$ is a natural coalgebra retract of $S(L_n(V))$ over $L_n(V)$. Consider the retraction map $r : S(L_n(V)) \to A^{\min}(V; L_n^{\max})$. Notice that
\[ r : P(S(L_n(V))) = L_n(V) \to P(A^{\min}(V; L_n^{\max})) \]
is a monomorphism. Thus the retraction map $r : S(L_n(V)) \to A^{\min}(V; L_n^{\max})$ is a monomorphism. The assertion follows. \hfill \Box

7. Projective $k(S_n)$-Submodules of $\text{Lie}(n)$

In this section we will sometimes work with an extension field $K$ of the ground field $k$. In this case, the tensor product of two $K$-modules $A$ and $B$ over $K$ will be denoted by $A \otimes_K B$ and the $n$-th fold self tensor product of a $K$-modules $A$ over $K$ will be denoted by $A^{\otimes_K n}$. 


Let $V$ be the $k$-module with basis $\{x_1, \ldots, x_n\}$ and let $\gamma_n$ denote the sub $k$-module of $V^\otimes n$ generated by the elements $x_{\sigma(1)} \cdots x_{\sigma(n)}$, where $\sigma$ runs through all elements in the symmetric group $S_n$. The $k$-module $\text{Lie}(n)$ is defined to be the intersection 

$$\text{Lie}(n) = \gamma_n \cap L_n(V).$$

There are two $k(S_n)$-actions on $\gamma_n$. One is induced by the canonical representation 

$$k(S_n) \to \text{End}(V),$$

and is called the internal $k(S_n)$-action. Another (right) $k(S_n)$-action on $\gamma_n$, called the position action, is given by permuting factors as described earlier. With these actions, $\gamma_n$ becomes isomorphic as a $k(S_n)$-bimodule to the group algebra $k(S_n)$ itself. Given $\lambda \in k(S_n)$ we also use $\lambda$ to denote the $k(S_n)$-module map $\gamma_n \to \gamma_n$ induced by the position action of $\lambda$. Observe that $\text{Lie}(n)$ is the image of the map $\beta_n: \gamma_n \to \gamma_n$. Notice that the two $k(S_n)$-actions on $\gamma_n$ commute. It follows that $\beta_n$ is a homomorphism with respect to the internal action. Thus the internal $k(S_n)$-action on $\gamma_n$ induces a $k(S_n)$-action on $\text{Lie}(n) = \text{Im}(\beta_n)$. One can consider $\text{Lie}(n)$ either as an internal $k(S_n)$-submodule of $\gamma_n$ using the inclusion 

$$\text{Lie}(n) \subseteq \gamma_n$$

or as an internal $k(S_n)$ quotient module of $\gamma_n$ under the identification 

$$\text{Lie}(n) = \text{Im}(\beta_n: \gamma_n \to \gamma_n).$$

From here on, unless stated otherwise the phrase “$k(S_n)$-module” will mean a module with respect to the internal action.

Note: $L_n(V)$ and $\text{Lie}(n)$ are not closed under the position action.

**Definition 7.1.** A $k(S_n)$-module $P$ is called a Lie projective module if

1. $P$ is a projective $k(S_n)$-module;
2. there exists a morphism of $k(S_n)$-modules $\phi: \gamma_n \to P$ such that the composite 

$$\gamma_n \xrightarrow{\beta_n} \gamma_n \xrightarrow{\phi} P$$

is an epimorphism of $k(S_n)$-modules.

We will show that $P$ is a Lie projective module if and only if it is a projective $k(S_n)$-module which is a submodule of $\text{Lie}(n)$.

**Lemma 7.2.** Let $P$ be a $k(S_n)$-module. Then $P$ is a Lie projective module if and only if there exists an element $\lambda \in k(S_n)$ such that there is an isomorphism of $k(S_n)$-modules 

$$P \cong \text{colim}_{\beta_n \circ \lambda} \gamma_n.$$
Proof. Suppose that $P$ is a Lie projective module. Choose $\phi: \gamma_n \to P$ as in the definition. Let $s: P \to \gamma_n$ be a $k(S_n)$-cross section map of the surjection

$$\gamma_n \xrightarrow{\beta_n} \gamma_n \xrightarrow{\phi} P.$$ 

Then

$$\text{colim}_{\beta_n \circ (s \circ \phi)} \gamma_n \cong \text{colim}_{s \circ \phi \circ \beta_n} \gamma_n \cong \text{Im}(s \circ \phi: \gamma_n \to \gamma_n) \cong P$$

and so one direction of the assertion follows. The converse is easy. \hfill \Box

Lemma 7.3. Let $P$ be any projective $k(S_n)$-submodule of $\text{Lie}(n)$. Then $P$ is a Lie projective module.

Proof. Let $P$ be a projective $k(S_n)$-submodule of $\text{Lie}(n)$. Let $j: P \to \gamma_n$ denote the inclusion $P \subseteq \text{Lie}(n) \subseteq \gamma_n$. Notice that $k(S_n)$ is a Fröbenius algebra. (See [1, 10, 13].) Thus $P$ is an injective $k(S_n)$-module and so there exists a morphism of $k(S_n)$-modules

$$r: \gamma_n \to P$$

such that the composite $r \circ j: P \to P$ is the identity map.

Let $\phi: \gamma_n \to \gamma_n$ denote the composite

$$\gamma_n \xrightarrow{r} P \xrightarrow{\iota} \text{Lie}(n) \xrightarrow{\iota} \gamma_n.$$ 

Then $\phi(x_1 \cdots x_n) \in \text{Lie}(n)$ and so there exists $k_\tau \in k$ (for $\tau \in S_{n-1}$) such that

$$\phi(x_1 \cdots x_n) = \sum_{\tau \in S_{n-1}} k_\tau [x_1, x_{\tau(2)}, \cdots, x_{\tau(n)}],$$

where $S_{n-1}$ acts on $\{2, \ldots, n\}$. Let $\lambda: \gamma_n \to \gamma_n$ be the morphism of internal $k(S_n)$-modules determined by

$$\lambda(x_1 \cdots x_n) = \sum_{\tau \in S_{n-1}} k_\tau x_1 x_{\tau(2)} \cdots x_{\tau(n)}.$$ 

Then

$$\phi(x_1 \cdots x_n) = \beta_n \circ \lambda(x_1 \cdots x_n).$$

Since $\gamma_n$ is the cyclic $k(S_n)$-module generated by $x_1 \cdots x_n$ this implies $\phi = \beta_n \circ \lambda$. Notice that $\phi \circ \phi = \phi: \gamma_n \to \gamma_n$ and

$$P \cong \text{colim}_\phi \gamma_n.$$ 

The assertion follows. \hfill \Box
Recall that there exists an element $\lambda_n \in k(S_n)$ such that there is a natural isomorphism

$$\text{colim}_{\beta_n \circ \lambda_n} V^\otimes_n \cong L^\text{max}_n(V)$$

for any $k$-module $V$. (See Lemma 6.2.) Let $\text{Lie}^\text{max}(n)$ be defined as the colimit

$$\text{Lie}^\text{max}(n) \cong \text{colim}_{\beta_n \circ \lambda_n} \gamma_n.$$  

Notice that $\text{Lie}^\text{max}(n)$ is a Lie projective $k(S_n)$-module.

Let $P$ be a Lie projective $k(S_n)$-module and let $\lambda \in k(S_n)$ be such that

$$P \cong \text{colim}_{\beta_n \circ \lambda} \gamma_n.$$  

We may assume that $P$ is a $k(S_n)$-submodule of $\gamma_n$. Let $V$ be any $k$-module. Let the functor $L_n(-; P)$ be defined by

$$L_n(V; P) = \text{colim}_{\beta_n \circ \lambda} V^\otimes_n.$$  

**Theorem 7.4.** The $k(S_n)$-module $\text{Lie}^\text{max}(n)$ is the maximum Lie projective $k(S_n)$-module in the following sense:

Let $P$ be any Lie projective $k(S_n)$-module. Then $P$ is a $k(S_n)$-retract of $\text{Lie}^\text{max}(n)$.

**Proof.** Let $\tilde{V}$ be the $n$-dimensional $k$-module with basis $\{x_1, \ldots, x_n\}$. Let $d_j : \tilde{V} \to \tilde{V}$ be the map defined by

$$d_j(x_i) = \begin{cases} x_i & \text{if } i \neq j; \\ 0 & \text{if } i = j \end{cases}$$

for $1 \leq j \leq n$. Notice that

$$\gamma_n = \bigcap_{j=1}^n \text{Ker}(d_j : \tilde{V}^\otimes_n \to \tilde{V}^\otimes_n).$$

Thus there is an isomorphism of $k(S_n)$-modules

$$P \cong \bigcap_{j=1}^n \text{Ker}(L_n(-; P)(d_j) : L_n(\tilde{V}; P) \to L_n(\tilde{V}; P)).$$

The assertion follows from Proposition 6.6. \qed

**Lemma 7.5.** Let $\lambda, \mu \in k(S_n)$. Suppose that $\text{colim}_{\beta_n \circ \lambda} \gamma_n$ is a $k(S_n)$-retract of $\text{colim}_{\beta_n \circ \mu} \gamma_n$. Then $\text{colim}_{\beta_n \circ \lambda} V^\otimes_n$ is a functorial retract of $\text{colim}_{\beta_n \circ \mu} V^\otimes_n$ for any $k$-module $V$.

**Proof.** By Corollary 4.4, one may assume that

1. $\beta_n \circ \lambda \circ \beta_n \circ \lambda = \beta_n \circ \lambda : \gamma_n \to \gamma_n$.
2. $\beta_n \circ \mu \circ \beta_n \circ \mu = \beta \circ \mu : \gamma_n \to \gamma_n$.
Let $M(V)$ and $N(V)$ denote the images

\[ M(V) = \text{Im}(\beta_n \circ \lambda : V^\otimes n \to V^\otimes n), \]
\[ N(V) = \text{Im}(\beta_n \circ \mu : V^\otimes n \to V^\otimes n). \]

Then $M(V)$ and $N(V)$ are functorially equivalent to $\text{colim}_{\beta_n \circ \lambda} V^\otimes n$ and $\text{colim}_{\beta_n \circ \mu} V^\otimes n$ respectively as $k$-modules.

Let $P$ and $Q$ denote the images

\[ P = \text{Im}(\beta_n \circ \lambda : \gamma_n \to \gamma_n), \]
\[ Q = \text{Im}(\beta_n \circ \mu : \gamma_n \to \gamma_n). \]

Then $P$ is a $k(S_n)$-retract of $Q$. Let $f : P \to Q$ and let $g : Q \to P$ be morphisms of $k(S_n)$-modules such that $g \circ f : P \to P$ is the identity map. Let $\theta(f), \theta(g) : \gamma_n \to \gamma_n$ denote the composites

\[ \gamma_n \xrightarrow{r_P} P \xrightarrow{f} Q \xleftarrow{c} \gamma_n, \]
\[ \gamma_n \xrightarrow{r_Q} Q \xrightarrow{g} P \xleftarrow{c} \gamma_n, \]

respectively, where $r_P : \gamma_n \to P$ and $r_Q : \gamma_n \to Q$ are retraction maps. The $k(S_n)$-maps $\theta'(f), \theta'(g) : \gamma_n \to \gamma_n$ given respectively by the composites

\[ \gamma_n \xrightarrow{\beta_n \circ \lambda} P \xrightarrow{f} Q \xleftarrow{c} \gamma_n, \]
\[ \gamma_n \xrightarrow{\beta_n \circ \mu} Q \xrightarrow{g} P \xleftarrow{c} \gamma_n, \]

make the diagram
commute. Let $\phi_f, \phi_f', \phi_g, \phi_g' \in k(S_n)$ denote the elements
\[
\phi_f = \theta(f)(x_1 \cdots x_n), \quad \phi_f' = \theta'(f)(x_1 \cdots x_n),
\]
\[
\phi_g = \theta(g)(x_1 \cdots x_n), \quad \phi_g' = \theta'(g)(x_1 \cdots x_n).
\]
The commutativity of the preceding diagram yields two equalities within $k(S_n)$ which in turn imply that the diagram
\[
\begin{array}{ccc}
\gamma_n & \xrightarrow{\beta_n \circ \lambda} & \gamma_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n
\end{array}
\]
commutes naturally for any connected graded $k$-module $V$.

Notice that there is a commutative diagram
\[
\begin{array}{ccc}
\gamma_n & \xrightarrow{\beta_n \circ \lambda} & \gamma_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n
\end{array}
\]
Thus
\[
\phi_g |_{N(V)} \circ \phi_f |_{M(V)} : M(V) \to M(V)
\]
is the identity map. The assertion follows.

Recall that the notation $\widetilde{V}$ means the $n$-dimensional $k$-module with basis $\{x_1, \ldots, x_n\}$. Let $d_j : \widetilde{V} \to \widetilde{V}$ be the $j$-th projection map. (See the proof of Theorem 7.4.) Let $M$ be a functor from $k$-modules to $k$-modules. Let $\gamma(M) \subseteq M(\widetilde{V})$ be defined by
\[
\gamma(M) = \bigcap_{j=1}^n \ker(M(d_j) : M(\widetilde{V}) \to M(\widetilde{V})).
\]
Notice that the (internal) $k(S_n)$-action on $\widetilde{V}$ induces a $k(S_n)$-action on $\gamma(M)$. The $k(S_n)$-module $\gamma(M)$ is called the **associated $k(S_n)$-module** of the functor $M$.  

\[
\begin{array}{ccc}
\gamma_n & \xrightarrow{\beta_n \circ \lambda} & \gamma_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n \\
V^\otimes_n & \xrightarrow{\phi_f} & V^\otimes_n
\end{array}
\]
Proposition 7.6. Let $M$ be a subfunctor of the Lie functor $L_n$ from $k$-modules to $k$-modules. Suppose that the associated $k(S_n)$-module $\gamma(M)$ is a projective $k(S_n)$-module. Then there exists an element $\lambda \in k(S_n)$ such that:

1. $\beta_n \circ \lambda \circ \beta_n \circ \lambda = \beta_n \circ \lambda : V^\otimes_n \to V^\otimes_n$ for any $k$-module $V$;
2. there is a functorial isomorphism

\[
M(V) = \text{Im}((\beta_n \circ \lambda : V^\otimes_n \to V^\otimes_n) \cong \text{colim}_{\beta_n \circ \lambda} V^\otimes_n.
\]

Proof. Notice that

\[
\gamma(M) = M(\tilde{V}) \cap \gamma_n = M(\tilde{V}) \cap \text{Lie}(n).
\]

By the proof of Lemma 7.3, there exists an element $\lambda \in k(S_n)$ such that:

1. $\beta_n \circ \lambda \circ \beta_n \circ \lambda = \beta_n \circ \lambda : V^\otimes_n \to V^\otimes_n$ for any connected graded $k$-module $V$;
2. $\gamma(M) = \text{Im}(\beta_n \circ \lambda : \gamma_n \to \gamma_n)$.

Let $M$ be the functor from $k$-modules to $k$-modules defined by

\[
M(V) = \text{Im}((\beta_n \circ \lambda : V^\otimes_n \to V^\otimes_n)
\]

for any $k$-module $V$. Notice that $M(\tilde{V}) \cap \gamma_n = M(\tilde{V}) \cap \gamma_n$.

Let $V$ be an $k$-module and let $a_1, \ldots, a_n \in V$. Let $f: \tilde{V} \to V$ be the morphism of $k$-modules determined by

\[
f(x_j) = a_j
\]

for $1 \leq j \leq n$. Notice that

\[
\beta_n \circ \lambda(x_1 \cdots x_n) \in M(\tilde{V}) \cap \gamma_n = M(\tilde{V}) \cap \gamma_n.
\]

Observe that there is a commutative diagram

\[
\begin{array}{ccc}
M(V) & \xrightarrow{\gamma} & V^\otimes_n \\
\downarrow M(f) & & \downarrow f^\otimes_n \\
M(\tilde{V}) & \xrightarrow{\gamma} & \tilde{V}^\otimes_n.
\end{array}
\]

Thus

\[
\beta_n \circ \lambda(a_1 \cdots a_n) = f^\otimes_n(\beta_n \circ \lambda(x_1 \cdots x_n)) \in M(V)
\]

for any $a_1, \ldots, a_n \in V$ and so one has a natural inclusion

\[
\tilde{M}(V) \subseteq M(V).
\]

Now let $y \in M(V) \subseteq V^\otimes_n$. Then there are elements $a_{ij} \in V$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

\[
y = \sum_{i=1}^{m} k_i a_{i1} \cdots a_{in}
\]
for some coefficients \( k_1, \ldots, k_m \in k \). Let \( f_i : \tilde{V} \to V \) be the morphism of \( k \)-modules such that

\[ f_i(x_j) = a_{ij} \]

for \( 1 \leq j \leq n \). Notice that there is a commutative diagram

\[
\begin{CD}
\tilde{M}(V) @>>> M(V) \\
\tilde{M}(f_i) @VVV \quad \tilde{M}(f_i) @VVV \\
\tilde{M}(\tilde{V}) @>>> M(\tilde{V})
\end{CD}
\]

for each \( 1 \leq i \leq m \). Thus one has a commutative diagram

\[
\begin{CD}
\tilde{M}(V) @>>> M(V) \\
\sum_{i=1}^m \tilde{M}(f_i) @VVV \quad \sum_{i=1}^m M(f_i) @VVV \\
\bigoplus_{i=1}^m \tilde{M}(\tilde{V}) \cap \gamma_n @>>> \bigoplus_{i=1}^m M(\tilde{V}) \cap \gamma_n.
\end{CD}
\]

Notice that

\[ y \in \text{Im} \left( \sum_{i=1}^m M(\tilde{V}) \cap \gamma_n \to M(V) \right). \]

Thus \( y \in \tilde{M}(V) \) and the assertion follows.

Let \( K \) be an extension field of \( k \). For a \( K(S_n) \)-module \( M \) the canonical inclusion \( k(S_n) \subseteq K(S_n) \) induces a \( k(S_n) \)-action on \( M \), and we write \( U(M) \) for the resulting \( k(S_n) \)-module.

For a \( k \)-module \( M \) we let \( M^K \) denote the \( K \)-module \( K \otimes_k M \). If \( M \) is a \( k(S_n) \)-module, then \( M^K \) is a \( K(S_n) \)-module. This gives a functor from the category of \( k(S_n) \)-modules to the category of \( K(S_n) \)-modules. The inclusion \( k \to K \) induces a natural transformation

\[ i: M \to U(M^K) \]

for \( k(S_n) \)-modules \( M \).
Let $M$ be a $k(S_n)$-module and let $N$ be a $K(S_n)$-module. Let $f: M \to U(N)$ be a morphism of $k(S_n)$-modules. Let $e(f): M^k \to N$ be a morphism of $K(S_n)$-modules defined by
\[
f(a \otimes x) = af(x)
\]
for $a \in K$ and $x \in M$. Thus for any $k(S_n)$-module $M$ and $K(S_n)$-module $N$ one has a natural function

\[
e: \text{Hom}_{k(S_n)}(M, U(N)) \to \text{Hom}_{K(S_n)}(M^K, N)
\]
which is an isomorphism of $K$-modules. Thus, as is well known, the above associations form a pair of adjoint functors. (See, for example, [1, 11]).

**Lemma 7.7.** Let $K$ be an extension field of the ground field $k$. Let $M$ be a subfunctor of the Lie functor $L_n$ from $k$-modules to $k$-modules. Suppose that $K \otimes_k \gamma(M)$ is a projective $K(S_n)$-module. Then there exists an element $\lambda \in k(S_n)$ such that

1. $\beta_n \circ \lambda \circ \beta_n \circ \lambda = \beta_n \circ \lambda: V^{\otimes n} \to V^{\otimes n}$ for any connected graded $k$-module $V$.
2. there is a natural isomorphism
\[
M(V) = \text{Im}(\beta_n \circ \lambda: V^{\otimes n} \to V^{\otimes n}) \cong \text{colim}_{\lambda \circ \beta_n} V^{\otimes n}
\]
for any $k$-module $V$.

That is, $M(V)$ is a Lie projective $k(S_n)$ module.

**Proof.** Notice that $\gamma(M)$ is a $k(S_n)$-retract of $U(K \otimes_k \gamma(M))$. Notice that the functor $U$ sends projective $K(S_n)$-modules to projective $k(S_n)$-modules. (See, for example, [11].) The assertion follows. \qed

**Lemma 7.8.** Let $K$ be an extension field of the ground field $k$. Let

\[
f_V: T(V) \to T(V)
\]
be a functorial map of coalgebras over the ground field $k$. Then there exists a functorial map of coalgebras over $K$, $\tilde{f}_W: T(W) \to T(W)$ for $K$-modules $W$ such that $\tilde{f}$ is an extension of $f$ in the following sense:

Let $V$ be any $k$-module. Then

\[
\tilde{f}_V = K \otimes_k f: T(V^K) = K \otimes_k T(V) \to T(V^K) = K \otimes_k T(V).
\]

**Proof.** There exists a sequence of elements $\alpha_n \in k(S_n)$ for $n \geq 0$ such that

\[
f_V = \alpha_n: T_n(V) = V^{\otimes n} \to T_n(V) = V^{\otimes n}.
\]

Notice that $\alpha_n \in k(S_n) \subseteq K(S_n)$. Let $W$ be any $K$-module. Let

\[
\tilde{f}_W: T(W) \to T(W)
\]
be the functorial map defined by
\[ \tilde{f}_W = \alpha_n: T_n(W) = W^\otimes n \rightarrow T_n(W) = W^\otimes n \]
for \( n \geq 0 \). Then one can check that:

1) \( \tilde{f} \) is a functorial map of coalgebras over \( K \);
2) \( \tilde{f} \) is an extension of \( f \).

The assertion follows. \( \square \)

8. The Functor \( A^\text{min} \) over a Field of Characteristic \( p > 0 \)

In this section, the ground field \( k \) is of characteristic \( p > 0 \).

8.1. An upper bound on the size of \( A^\text{min}(V) \).

**Lemma 8.1.** Let \( m > 1 \) such that \( (m, p) = 1 \). Suppose that the polynomial
\[ x^m - 1 \]
splits in \( k[x] \). Then there exists a functorial map of coalgebras
\[ \phi_V: T(V) \rightarrow T(V) \]
for any \( k \)-module \( V \) such that:

1) \( \phi_V \circ \phi_V = \phi_V: T(V) \rightarrow T(V) \);
2) Let \( \alpha \in P(T_n(V)) \) be a primitive element of tensor length \( n \). Then
\[ \phi_V(\alpha) = \begin{cases} 
\alpha & \text{if } n = m, 2m, 3m, \ldots ; \\
0 & \text{otherwise.}
\end{cases} \]

**Proof.** Let \( \zeta \in k \) be a primitive \( m \)th root of 1. Let
\[ T(\zeta): T(V) \rightarrow T(V) \]
be the algebra map determined by
\[ T(\zeta)(v) = \zeta v \]
for \( x \in V \). Then \( T(\zeta): T(V) \rightarrow T(V) \) is a functorial map of Hopf algebras. Let \( \chi: T(V) \rightarrow T(V) \) be the conjugation and let
\[ f_V = T(\zeta) \ast \chi: T(V) \rightarrow T(V) \]
be the convolution product. Then \( f_V: T(V) \rightarrow T(V) \) is a functorial map of coalgebras.

Let \( \alpha \in P(T_n(V)) \) be a primitive element of tensor length \( n \). Then
\[ f_V(\alpha) = (\zeta^n - 1)\alpha. \]