ON HOMOTOPY GROUPS OF THE SUSPENDED CLASSIFYING SPACES

ROMAN MIKHAILOV AND JIE WU

Abstract. In this paper, we determine the homotopy groups $\pi_4(\Sigma K(G, 1))$, $\pi_5(\Sigma K(G, 1))$ and $\pi_5(\Sigma^2 K(G, 1))$ for different groups $G$ by using different facts and methods from group theory and homotopy theory: derived functors, the Carlsson simplicial construction, the Baues-Goerss spectral sequence, homotopy decompositions and the methods of algebraic K-theory.

1. Introduction

It is well-known that the suspension functor applied to a topological space shifts homology groups, but "chaotically" changes homotopy groups. For example, one can take a circle $S^1$, whose homotopy type is very simple. Its suspension $\Sigma S^1 = S^2$ has obvious homology groups, however the problem of investigating the homotopy groups of $S^2$ is one of the deepest problems of algebraic topology. Consider the following functors from the category of groups to the category of abelian groups:

$$\pi_n(\Sigma^m K(-, 1)) : \text{Gr} \rightarrow \text{Ab}, \ n \geq 1, \ m \geq 1$$

defined by

$$A \mapsto \pi_n(\Sigma^m K(A, 1)),$$

where $\Sigma^m$ is the $m$-fold suspension. It is clear that $\pi_n(\Sigma^m K(\mathbb{Z}, 1)) = \pi_n(S^{m+1})$, that is the homotopy groups of spheres appear as the simplest case of a general theory of homotopy groups of suspensions of classifying spaces.

For the case $m = 1, 2$ and $n = 3, 4$ there is the following natural commutative diagram with exact rows [5]:

$$
\begin{array}{cccccc}
0 & \rightarrow & \pi_3(\Sigma K(G, 1)) & \rightarrow & G \otimes G & \rightarrow & [G, G] & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \pi_4(\Sigma^2 K(G, 1)) & \rightarrow & G\widehat{\otimes} G & \rightarrow & [G, G] & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H_2(G) & \rightarrow & G \wedge G & \rightarrow & [G, G] & \rightarrow & 1
\end{array}
$$
where $G \otimes G$ is the non-abelian square of $G$ in the sense of Brown-Loday [5], $G \tilde{\otimes} G$ (resp. $G \wedge G$) is the quotient of $G \otimes G$ by the normal subgroup generated by elements $g \otimes h + h \otimes g$ (resp. $g \otimes g, g \in G$). In particular, for an abelian group $A$, there are natural isomorphisms
\[
\pi_3(\Sigma K(A, 1)) \simeq A \otimes A
\]
\[
\pi_4(\Sigma^2 K(A, 1)) \simeq \pi_5^S K(A, 1) \simeq A \tilde{\otimes} A.
\]

In this paper we study homotopy groups $\pi_4(\Sigma K(G, 1))$, $\pi_5(\Sigma K(G, 1))$ and $\pi_5(\Sigma^2 K(G, 1))$ for different groups $G$. In order to investigate the structure of these homotopy groups, we use different facts and methods of group theory and homotopy theory: derived functors, the Carlsson simplicial construction, the Baues-Goerss spectral sequence, homotopy decompositions and the methods of algebraic $K$-theory. The combination of these different methods provides an effective way for determining these homotopy groups. As reader will see, some our computations use commutator tricks in simplicial groups.

In section 2 we recall certain facts from the homotopy theory, such as the Whitehead exact sequence, the Carlsson simplicial construction and describe a spectral sequence (2.8), which converges to $\pi_*(\Sigma^m K(A, 1))$ for any abelian group $A$, with $E^2$-terms are given by the derived functors of certain polynomial functors. We illustrate how it works in section 5, where we give a new proof of the well-known fact that $\pi_5(\Sigma^2 \mathbb{R}P^\infty) = \mathbb{Z}/8$. The proof is also based on the computations of the derived functors of the antisymmetric square $\tilde{\otimes}^2$.

It is natural to compare the homotopy groups of suspensions of classifying spaces with homotopy groups of Moore spaces. Given an abelian group $A$, let $M(A, 1)$ be a 1-dimensional Moore space. The natural map $M(A, 1) \to K(A, 1)$ induces the maps
\[
M(A, m + 1) \simeq \Sigma^m M(A, 1) \to \Sigma^m K(A, 1), \; m \geq 1
\]
and corresponding maps
\[
\mu_{n,m} : \pi_n(M(A, m + 1)) \to \pi_n(\Sigma^m K(A, 1)).
\]
Observe that, for and $n = m + 1$, the map $\mu_{n,m}$ is the identity isomorphism. For $n = m + 2$, the map $\mu_{n,m}$ is in general neither a monomorphism nor an epimorphism. In this case we have the following natural map:
\[
\pi_3(M(A, 2)) \to \pi_3(K(A, 1))
\]
\[
\Gamma_2(A) \to A \otimes A
\]
where $\Gamma_2$ is the quadratic functor due to Whitehead. In particular, for $A = \mathbb{Z}/2$, the map
\[
\mathbb{Z}/4 = \pi_3(\Sigma \mathbb{R}P^2) = \pi_3(M(\mathbb{Z}/2, 2)) \to \pi_3(\Sigma K(\mathbb{Z}/2, 1)) = \pi_3(\Sigma \mathbb{R}P^\infty) = \mathbb{Z}/2
\]
is an epimorphism. For higher homotopy groups, the maps $\mu_{n,m}$ are much more difficult. The map $\mathbb{Z}/4 = \pi_4(\Sigma \mathbb{R}P^2) \to \pi_4(\Sigma \mathbb{R}P^\infty) = \mathbb{Z}/4$ is an isomorphism, but the map
\[
(Z/2)^{\otimes 3} = \pi_5(\Sigma \mathbb{R}P^2) \to \pi_5(\Sigma \mathbb{R}P^\infty) = (Z/2)^{\otimes 2}
\]
is not an epimorphism (see section 4). In sections 3 and 4 we compute homotopy groups $\pi_i(\Sigma K(\mathbb{Z}/p, 1))$ for $i = 4, 5$. The results of Baues-Goerss from [4] give a natural method for extending these computations to the case of symmetric group $\Sigma_3$. 

In this case we have the following natural map:
\[
\pi_3(M(A, 2)) \to \pi_3(K(A, 1))
\]
\[
\Gamma_2(A) \to A \otimes A
\]
There is a natural relation between the problem considered and algebraic K-theory. Since the plus-construction $K(G, 1) \to K(G, 1)^+$ is a homological equivalence, there is a natural weak homotopy equivalence

$$\Sigma K(G, 1) \to \Sigma(K(G, 1)^+)$$

This defines the natural suspension map:

$$\pi_n(K(G, 1)^+) \to \pi_{n+1}(\Sigma(K(G, 1)^+)) = \pi_{n+1}(\Sigma K(G, 1))$$

for $n \geq 1$. This map was studied in [3] in the case of a perfect group $G$. We consider the case $G = E(R)$, i.e. the group of elementary matrices over a ring $R$. In this case the natural map

$$K_3(R) = \pi_3(K(E(R), 1)^+) \to \pi_4(\Sigma K(E(R), 1))$$

is an isomorphism (see section 6). The natural relation to K-theory gives a way how to compute homotopy groups $\pi_i(\Sigma K(E(R), 1))$ for $i = 4, 5$ for some rings. For example, the case $G = SL(\mathbb{Z})$ is considered.

Some computations given in the paper are summarized in the following table:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\pi_3(\Sigma K(G, 1))$</th>
<th>$\pi_4(\Sigma K(G, 1))$</th>
<th>$\pi_5(\Sigma K(G, 1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/4$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$\mathbb{Z}/p$ ($p \neq 2$)</td>
<td>$\mathbb{Z}/p$</td>
<td>$\mathbb{Z}/p$</td>
<td>$\mathbb{Z}/p \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$\Sigma_3$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/12$</td>
<td>$\mathbb{Z}/2 \oplus \mathbb{Z}/2$</td>
</tr>
<tr>
<td>$SL(\mathbb{Z})$</td>
<td>$\mathbb{Z}/2$</td>
<td>$\mathbb{Z}/48$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
</tbody>
</table>

2. Basic facts from homotopy theory

2.1. Whitehead exact sequence. Let $X$ be a $(r - 1)$-connected CW-complex, $r \geq 2$. There is the following long exact sequence of abelian groups [13]:

$$\cdots \to H_{n+1}(X) \to \Gamma_n(X) \to \pi_n(X) \xrightarrow{h_n} H_n(X) \to \Gamma_{n-1}(X) \to \cdots$$  (2.1)

where $\Gamma_n(X) = im\{\pi_n(sk_{n-1}(X)) \to \pi_n(sk_n(X))\}$ (here $sk_i(X)$ is the $i$-th skeleton of $X$), $h_n$ is the $n$th Hurewicz homomorphism. The Hurewicz theorem is equivalent to the statement $\Gamma_i(X) = 0$, $i \leq r$. J.H.C. Whitehead computed the term $\Gamma_{r+1}(X)$ (see [13]):

$$\Gamma_{r+1}(X) = \begin{cases} 
\Gamma_2(\pi_2(X)), & r = 2 \\
\pi_r(X) \otimes \mathbb{Z}/2, & r > 2
\end{cases}$$

where $\Gamma_2 : Ab \to Ab$ is the universal quadratic functor.

Recall the description of the functors $\Gamma_{r+2}(X)$ due to H.-J. Baues [2]. Consider the third super-Lie functor

$$\mathcal{L}_s^3 : Ab \to Ab$$

defined as

$$\mathcal{L}_s^3(A) = im\{A \otimes A \otimes A \xrightarrow{i} A \otimes A \otimes A\}$$

where

$$l(a \otimes b \otimes c) = \{a, b, c\} := a \otimes b \otimes c + b \otimes a \otimes c - c \otimes a \otimes b - c \otimes b \otimes a, \ a, b, c \in A.$$
Observe that $L^3_s(A) = \ker\{A \otimes \Lambda^3(A) \xrightarrow{r} \Lambda^3(A)\}$, where $\Lambda^i(A)$ is the $i$th exterior power of $A$ and the map $r$ is given as
\[
r(a \otimes b \wedge c) = a \wedge b \wedge c, \quad a, b, c \in A.
\]

Let $r = 2$ and let the complex $X$ be simply connected. The map $\eta_1 : \Gamma_2(\pi_2(X)) \to \pi_3(X)$ is induced by the Hopf map $S^3 \to S^2$. Given an abelian group $A$, define the map
\[
q : \Gamma_2(A) \otimes A \to L^3_s(A) \oplus \Gamma_2(A) \otimes \mathbb{Z}/2
\]
by setting
\[
q(\gamma_2(a) \otimes b) = -\{b, a, a\} + (\gamma_2(a + b) - \gamma_2(a) - \gamma_2(b)) \otimes 1, \quad a, b \in A.
\]
Define the group $\Gamma^2_2X = \Gamma^2_2(\Gamma_2(\pi_2X) \to \pi_3X)$ as the pushout:
\[
\begin{align*}
\Gamma_2(\pi_2(X)) \otimes (\pi_2(X) \oplus \mathbb{Z}/2) & \xrightarrow{q \otimes id} L^3_s(\pi_2(X)) \oplus \Gamma_2(\pi_2(X)) \otimes \mathbb{Z}/2 \\
\pi_3(X) \otimes (\pi_2(X) \oplus \mathbb{Z}/2) & \xrightarrow{\eta_1 \otimes id} \Gamma^2_2(X)
\end{align*}
\]

There is the following natural short exact sequence [2]:
\[
0 \to \Gamma^2_2(X) \to \Gamma_4(X) \to R_2(\pi_2(X)) \to 0,
\]
where $R_2 : \text{Ab} \to \text{Ab}$ is the functor defined by Eilenberg and MacLance [7], which is the first derived functor of $\Gamma_2$ and has the property $H_5(K(A,2)) = R_2(A)$.

For $r = 3$, there is a natural exact sequence
\[
0 \to \pi_4(X) \otimes \mathbb{Z}/2 \oplus \Lambda^2(\pi_3(X)) \to \Gamma_5(X) \to \text{Tor}(\pi_3(X), \mathbb{Z}/2) \to 0. \tag{2.3}
\]
For $r \geq 4$, the situation is much easier: there is a natural exact sequence
\[
0 \to \pi_{r+1}(X) \otimes \mathbb{Z}/2 \to \Gamma_{r+2}(X) \to \text{Tor}(\pi_r(X), \mathbb{Z}/2) \to 0.
\]
For the computation of the higher terms $\Gamma_r(X)$ one can use the homotopy operation spectral sequence described in [4].

**Remark.** Observe that for every abelian group $A$, the Hurewicz homomorphism induces a natural epimorphism
\[
\pi_{n+1}(\Sigma K(A, 1)) \to \Lambda^n(A) (\subseteq H_n(A)).
\]
This follows from the fact that this is true for a free abelian $A$, by the decomposition
\[
\Sigma K(A \oplus \mathbb{Z}, 1) \simeq S^2 \vee \Sigma K(A, 1) \vee \Sigma^2 K(A, 1).
\]

2.2. **Carlsson construction.** Let $G_*$ be a simplicial group and $X$ a pointed simplicial set with a base point $\ast$. Consider the simplicial group $F^{G_*}(X)$ defined as
\[
F^H(X)_n = \prod_{x \in X_n} (G_n)_x,
\]
i.e. in each degree $F^{G_*}(X)_n$ is the free product of groups $G_n$ numerated by elements of $X_n$ modulo $(G_n)_\ast$, with the canonical choice of face and degeneracy morphisms. It is proved

\[
\text{Tor}(\pi_3(X), \mathbb{Z}/2) \to 0.
\]
in [6] that the geometric realization $|F^{G^*}(X)|$ is homotopy equivalent to the loop space $\Omega(|X| \wedge B|G|)$. The main example we will consider is the simplicial circle $X = S^1$ with $$S_0^1 = \{\ast\}, \ S_1^1 = \{\ast, \sigma\}, \ S_2^1 = \{\ast, s_0\sigma, s_1\sigma\}, \ldots, \ S_n^1 = \{\ast, x_0, \ldots, x_n\},$$ where $x_i = s_n \ldots s_i s_0\sigma$ and the simplicial group $G_s$ with $G_n = G$ for a given group $G$, with identity homomorphisms as all face and degeneracy maps. In this case we use the notation $F^G(X) = F^{G^*}(X)$. One has a homotopy equivalence $$|F^G(S^1)| \simeq \Omega \Sigma K(G, 1).$$

The group $F^G(S^1)_n$ is the $n$-fold free product of $G$: $$F^G(S^1)_1 = G, \ F^G(S^1)_2 = G * G, \ F^G(S^1)_3 = G * G * G, \ldots$$

We can formally identify $G * G$ with $s_0G * s_1G, G * G * G$ with $s_1s_0G * s_2s_0G * s_2s_1G$, etc., and to define naturally the face and degeneracy maps:

$$F^{G^*}(S^1) : \ldots \longrightarrow G * G * G \longrightarrow G * G \longrightarrow G.$$

**Remark.** Consider the second term $F^G(S^1)_2 = G * G$ and face morphisms $d_0, d_1, d_2 : G * G = s_0(G) * s_1(G) \longrightarrow G$ defined as

$$d_0 : \begin{cases} s_0(g) \mapsto g \\ s_1(g) \mapsto 1 \end{cases}, \ d_1 : \begin{cases} s_0(g) \mapsto g \\ s_1(g) \mapsto g \end{cases}, \ d_2 : \begin{cases} s_0(g) \mapsto 1 \\ s_1(g) \mapsto g \end{cases}.$$

There is a natural commutative diagram

$$\begin{array}{cccccc}
\pi_3(\Sigma K(G, 1)) & \hookrightarrow & G \otimes G & \longrightarrow & G & \longrightarrow & G_{ab} \\
\downarrow \simeq & & \downarrow f & & \downarrow \simeq & \\
\pi_3(\Sigma K(G, 1)) & \hookrightarrow & (\ker(d_1) \cap \ker(d_2))/B_2 & \longrightarrow & G & \longrightarrow & G_{ab}
\end{array}$$

where $B_2$ is the 2-boundary subgroup of $G * G$ and the map $f$ is defined as

$$f(g \otimes h) = [s_0(g)s_1(g)^{-1}, s_0(h)].B_2.$$

There is a natural description of the 2-boundary (see [8], for example):

$$B_2 = [\ker(d_0), \ker(d_1) \cap \ker(d_2)] [\ker(d_1), \ker(d_2) \cap \ker(d_0)] [\ker(d_2), \ker(d_0) \cap \ker(d_1)].$$

Diagram (2.4) implies that $f$ is a natural isomorphism.

In the case $G = \mathbb{Z}$, the simplicial group $F^G(S^1)$ is identical to the Milnor construction $F(S^1)$, with $F(S^1)_n$ a free group of rank $n$, for $n \geq 1$:

$$F(S^1) : \ldots \longrightarrow F_3 \longrightarrow F_2 \longrightarrow \mathbb{Z}.$$}

In this case there is a homotopy equivalence

$$|F(S^1)| \simeq \Omega S^2.$$

---

1We use the standard commutator relations: $[g, h] = g^{-1}h^{-1}gh$
and the construction \(F(S^1)\) provides a combinatorial model for the computation of homotopy groups of the 2-sphere \(S^2\). The construction \(F(S^1)\) was studied from the group-theoretical point of view in [14]. It is easy to find the simplicial generators of the homotopy classes of \(\pi_i(F(S^1)) = \pi_i(S^2)\) for \(i = 3, 4, 5\). In order to find these simplicial generators, consider the sequence of maps between Milnor simplicial constructions \(F(S^1) \to F(S^2) \to F(S^2) \to F(S^1)\) such that the induced homomorphisms \(\mathbb{Z} = \pi_2(F(S^1)) \to \pi_2(F(S^1)) = \mathbb{Z}\) and \(\mathbb{Z} = \pi_3(F(S^1)) \to \pi_3(F(S^1)) = \mathbb{Z}/2\) are epimorphisms and define the homotopy classes of \(\pi_3(S^2)\) and \(\pi_4(S^3)\) respectively.

\[
\begin{array}{c}
F(S^3)_4 \xrightarrow{\eta^2} \mathbb{Z} \\
\downarrow \\
F(S^2)_4 \xrightarrow{\eta} \mathbb{Z} \\
\downarrow \\
F(S^1)_4 \xrightarrow{\eta} \mathbb{Z}
\end{array}
\]

For \(n \geq 3\), the homotopy class of \(\pi_n(S^{n-1})\) defined as \(\pi_{n-1}(F(S^{n-2}))\) is generated by \([s_0(\sigma_{n-2}), s_1(\sigma_{n-2})]\) in \(F(S^{n-2})_{n-1}\) (see [14]), where \(\sigma_{n-2}\) is a generator of \(F(S^{n-2})_{n-2} = \mathbb{Z}\). That is, we can define the simplicial suspension maps \(\eta^i : F(S^{i+1})_{i+1} \to F(S^i)_{i+1}\) by

\[
\eta^i : \sigma_{i+1} \to [s_0(\sigma_i), s_1(\sigma_i)], \quad i \geq 1.
\]

Since the generators of \(\pi_i(S^2)\) are presented by suspensions over Hopf fibration for \(i = 3, 4, 5\), the simplicial generators of \(\pi_i(F(S^1))\), \(i = 2, 3, 4\) are given by the following elements:

\[
w_2(x_0, x_1) = [x_0, x_1] \tag{2.5}
\]

\[
w_3(x_0, x_1, x_2) = [[x_0, x_2], [x_0, x_1]] \tag{2.6}
\]

\[
w_4(x_0, x_1, x_2, x_3) = [[[x_0, x_3], [x_0, x_1]], [[x_0, x_2], [x_0, x_1]]]. \tag{2.7}
\]

Here we use the natural notations \(x_j := s_i \ldots s_j s_0(\sigma_1), \ j = 0, \ldots, i\) for the basis elements in \(F(S^1)_{i+1}\).

2.3. Spectral sequence. Consider an abelian group \(A\) and its two-step flat resolution

\[
0 \to A_1 \to A_0 \to A \to 0.
\]

By Dold-Kan correspondence, we obtain the following free abelian simplicial resolution of \(A\):

\[
N^{-1}(A_1 \leftarrow A_0) : \quad \quad \xrightarrow{\cdot} \quad A_1 \oplus s_0(A_0) \quad \xrightarrow{\cdot} A_0.
\]

One can continue the process of construction of elements \(w_{n+1}(x_0, \ldots, x_n)\) by the following law: \(w_{n+1}(x_0, \ldots, x_n) = [w_n(x_0, \ldots, x_{n-1}, x_n), w_n(x_0, \ldots, x_{n-1})]\). In this case, the 16-commutator bracket \(w_5(x_0, \ldots, x_4)\) corresponds to the element of order 2 in \(\pi_6(S^2)\), but the 32-commutator bracket \(w_6(x_0, \ldots, x_3)\) lies in the simplicial boundary subgroup \(BF(S^1)_6\) (see [7]). The construction of a simplicial generator of the 3-torsion in \(\pi_6(S^2)\) is more tricky: it is possible to find its simplicial representative which is a product of six brackets of the commutator weight six.
Applying Carlsson construction to the resolution $N^{-1}(A_1 \hookrightarrow A_0)$, we obtain the following bisimplicial group:

$$F^{N^{-1}(A_1 \hookrightarrow A_0)}_2(S^n) \quad \Rightarrow \quad N^{-1}(A_1 \hookrightarrow A_0)_2$$

Here the $m$th horizontal simplicial group is Carlsson construction $F^{N^{-1}(A_1 \hookrightarrow A_0)}_m(S^n)$. By the result of Quillen [12], we obtain the following spectral sequence:

$$E^{2}_{p,q} = \pi_q(\pi_p(\Sigma^n K(N^{-1}(A_1 \hookrightarrow A_0), 1)) \Rightarrow \pi_{p+q}(\Sigma^n K(A, 1)). \quad (2.8)$$

3. On group $\pi_4(\Sigma K(A, 1))$

For a simply-connected space $X$, the Whitehead exact sequence at the term $\pi_4$ has the following form (see [4]):

$$\Gamma_2^2(\eta^1) \quad (3.1)$$

$$H_5(X) \quad \rightarrow \quad \Gamma_4(X) \quad \rightarrow \quad \pi_4(X) \quad \rightarrow \quad H_4(X)$$

$$R_2(\pi_2 X).$$

For a free group $A$, we obtain the following description of $\pi_4(\Sigma K(A, 1))$:

$$\mathcal{L}_s^3(A) \oplus \Gamma_2(A) \otimes \mathbb{Z}/2$$

$$\Gamma_2^2(\Gamma_2(A) \rightarrow A \otimes A) \subseteq \pi_4(\Sigma K(A, 1)) \quad \rightarrow \quad \Lambda^3(A)$$

$$\Lambda^2(A) \otimes (A \otimes \mathbb{Z}/2).$$
In other words, the filtration \( \pi_4(\Sigma K(A, 1)) \) has the following natural associated grading pieces:

\[
\begin{align*}
gr^1 &\pi_4(\Sigma K(A, 1)) = \pi_4(M(A, 2)) = \mathcal{L}_s^3(A) \oplus \Gamma_2(A) \otimes \mathbb{Z}/2 \\
gr^2 &\pi_4(\Sigma K(A, 1)) = \Lambda^2(A) \otimes (A \oplus \mathbb{Z}/2) \\
gr^3 &\pi_4(\Sigma K(A, 1)) = \Lambda^3(A).
\end{align*}
\]

The definition of the functor \( \Gamma_2 \) given in (2.2) implies that for \( A = \mathbb{Z}/p \) where \( p \) is an odd prime, \( R_2(A) = 0 \) and the diagram (3.1) has the following form:

\[
\begin{array}{cccccc}
0 & \downarrow & & & & 0 \\
& 0 & \longrightarrow & \Gamma_4(\Sigma K(\mathbb{Z}/p, 1)) & \longrightarrow & \pi_4(X) & \longrightarrow & \mathbb{Z}/p. \\
& & \downarrow & & & \downarrow & & \downarrow \\
& & 0 & & \longrightarrow & \pi_4(\Sigma K(\mathbb{Z}/p, 1)) & \longrightarrow & H_4(\Sigma K(\mathbb{Z}/p, 1)). \\
\end{array}
\]

This implies the following result.

**Theorem 3.1.** Let \( p \) be an odd prime integer. Then

\[
\pi_4(\Sigma K(\mathbb{Z}/p, 1)) = \mathbb{Z}/p
\]

and the Hurewicz homomorphism

\[
\pi_4(\Sigma K(\mathbb{Z}/p, 1)) \to H_4(\Sigma K(\mathbb{Z}/p, 1))
\]

is an isomorphism.

Now consider the case \( A = \mathbb{Z}/2 \). Here we give a combinatorial proof of the following theorem due to M. Hennes [10]:

**Theorem 3.2.** \( \pi_4(\Sigma K(\mathbb{Z}/2, 1)) \simeq \mathbb{Z}/4 \).

**Proof.** Consider the sequence 2.1 for the case \( X = \Sigma K(\mathbb{Z}/2, 1) \). It is of the following form:

\[
\begin{array}{cccccc}
\Gamma_3^2(\mathbb{Z}/4 \to \mathbb{Z}/2) & \downarrow & & & & \\
\Gamma_4(\Sigma K(\mathbb{Z}/2, 1)) & \hookrightarrow & \pi_4(\Sigma K(\mathbb{Z}/2, 1)) & 0 & \longrightarrow & H_3(\mathbb{Z}/2) & \longrightarrow & \Gamma_3(\Sigma K(\mathbb{Z}/2, 1)) & \longrightarrow & \pi_3(\Sigma K(\mathbb{Z}/2, 1)) \\
R_2(\mathbb{Z}/2) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2
\end{array}
\]

(3.3)
By definition of the functor $\Gamma^2_2$, we have the following pushout diagram:

$$
\begin{array}{ccc}
\mathbb{Z}/4 \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2) & \longrightarrow & \mathbb{Z}/4 \otimes \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2) & \longrightarrow & \Gamma^2_2(\mathbb{Z}/4 \to \mathbb{Z}/2)
\end{array}
$$

It follows that $\Gamma^2_2(\mathbb{Z}/4 \to \mathbb{Z}/2) = \mathbb{Z}/2$. We have to show that the short exact sequence

$$
\begin{array}{ccc}
\mathbb{Z}/2 \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2) & \longrightarrow & \Gamma^2_2(\mathbb{Z}/4 \to \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & \pi_4(\Sigma K(\mathbb{Z}/2, 1)) \\
\end{array}
\longrightarrow \mathbb{Z}/2
$$

is not split. For that we need the following:

**Lemma 3.1.** The natural epimorphism $\mathbb{Z} \to \mathbb{Z}/2$ induces a non-trivial homomorphism $\pi_4(S^2) = \pi_4(\Sigma K(\mathbb{Z}, 1)) \to \pi_4(\Sigma K(\mathbb{Z}/2, 1))$.

**Proof.** The statement follows from simple analysis of the diagram

$$
\begin{array}{ccc}
\Gamma^2_2(\mathbb{Z}) & \longrightarrow & \Gamma^2_2(\mathbb{Z}/2) \\
\downarrow & & \downarrow \\
\pi_4(\Sigma K(\mathbb{Z}, 1)) & \longrightarrow & \pi_4(\Sigma K(\mathbb{Z}/2, 1))
\end{array}
$$

The definition of the functor $\Gamma^2_2$ implies that the upper horizontal map in this diagram is an isomorphism. \hfill \Box

**Lemma 3.2.** (Lemma 2.1 from [14]) Let $G_\ast$ be a simplicial group and let $n \geq 0$. Suppose that $\pi_0(G_\ast)$ acts trivially on $\pi_n(G_\ast)$. Then the homotopy group $\pi_n(G_\ast)$ is contained in the center of $G_n/BG_n$, where $BG_n$ is the $n$th simplicial boundary subgroup of $G_n$. \hfill \Box

**Continuation of the Proof of Theorem 3.2.** Now consider the natural map between the simplicial constructions $F(S^1) \to F^{\mathbb{Z}/2}(S^1)$ induced by natural map $\mathbb{Z} \to \mathbb{Z}/2$:

$$
\begin{array}{ccc}
\ldots & \longrightarrow & \mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z} \\
\downarrow & & \downarrow \\
\ldots & \longrightarrow & \mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2
\end{array}
$$

Choose generators $x_0 = s_2s_1(\sigma), x_1 = s_2s_0(\sigma), x_2 = s_1s_0(\sigma)$ of the group $F^{\mathbb{Z}/2}(S^1)_3 = \mathbb{Z}/2 \ast \mathbb{Z}/2 \ast \mathbb{Z}/2$. Lemma 3.1 implies that the simplicial cycle

$$
\bar{w}_3 = [[x_0, x_2], [x_0, x_1]] \in F^{\mathbb{Z}/2}(S^1)
$$
does not lie in the boundary $BF^{Z/2}(S^1)_3$. Observe that
\[
\tilde{w}_3 = \left[[x_0, x_2], (x_0x_1)^2\right] = \left[[x_0, x_2], x_0x_1\right]^2 = \left[[x_0, x_2], x_0x_1\right] = \left[[x_0, x_2], x_0x_1\right].
\]
Observe also that the element
\[
v_3 = \left[[x_0, x_2], x_0x_1\right]
\]
is a cycle in $F^{Z/2}(S^1)$. Lemma 3.2 implies that
\[
\tilde{w}_3 \equiv v_3^2 \mod BF^{Z/2}(S^1)_3.
\]
Hence the element $v_3$ defines an element of order 4 in $\pi_4(\Sigma K(Z/2, 1))$. Theorem 3.2 is proved.

As a corollary we have the following well-known computation of $\pi_4(M(Z/2, 2))$:

**Proposition 3.1.** The map $\Sigma \mathbb{R}P^2 \to \Sigma K(Z/2, 1)$ induces an isomorphism
\[
\pi_4(\Sigma \mathbb{R}P^2) \to \pi_4(\Sigma K(Z/2, 1))
\]
In particular, $\pi_4(M(Z/2, 2)) \cong \mathbb{Z}/4$.

**Proof.** Consider the map between the Whitehead sequences induced by the natural map $M(Z/2, 2) \to \Sigma K(Z/2, 1)$:
\[
\begin{array}{ccc}
\Gamma_4(M(Z/2, 2)) & \xrightarrow{\sim} & \pi_4(M(Z/2, 2)) \\
\downarrow & & \downarrow \\
\Gamma_4(\Sigma K(Z/2, 1)) & \xrightarrow{\sim} & \pi_4(\Sigma K(Z/2, 1))
\end{array}
\]
The lower isomorphism follows from the exact sequence (3.3). Description of the functor $\Gamma_4$ implies the following commutative diagram:
\[
\begin{array}{ccc}
\Gamma_2^{\Sigma}(Z/4 \to Z/4) & \xrightarrow{\sim} & \Gamma_2^{\Sigma}(Z/4 \to Z/2) \\
\downarrow & & \downarrow \\
\pi_4(M(Z/2, 2)) & \xrightarrow{\sim} & \pi_4(\Sigma K(Z/2, 1)) \\
\downarrow & & \downarrow \\
R_2(Z/2) & \xrightarrow{\sim} & R_2(Z/2)
\end{array}
\]
The definition of the functor $\Gamma_2^{\Sigma}$ as a pushout (2.2) implies that the upper horizontal map is an isomorphism and the proposition follows.

**Theorem 3.3.** $\pi_4(\Sigma K(\Sigma_3, 1)) = \mathbb{Z}/12$.

**Proof.** The integral homology groups of $\Sigma_3$ are 4-periodic with the following initial terms:
\[
H_1(\Sigma_3) = \mathbb{Z}/2, \ H_2(\Sigma_3) = 0, \ H_3(\Sigma_3) = \mathbb{Z}/6, \ H_4(\Sigma_3) = 0.
\]
The sequence (2.1) has the following form:

\[ \Gamma_3^2(\mathbb{Z}/4 \to \mathbb{Z}/2) \]
\[ \Gamma_4(\Sigma K(\Sigma_3, 1)) \leftarrow \pi_4(\Sigma K(\Sigma_3, 1)) \rightarrow H_3(\Sigma_3) \rightarrow \Gamma_3(\Sigma K(\Sigma_3, 1)) \rightarrow \pi_3(\Sigma K(\Sigma_3, 1)) \]
\[ R_2(\mathbb{Z}/2) \]
\[ \mathbb{Z}/6 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \]

The statement follows from Theorem 3.2. □

4. On group \( \pi_5(\Sigma K(A, 1)) \)

For a simply-connected space \( X \), the Whitehead exact sequence at the term \( \pi_5 \) has the following form (see [4]):

\[ L_2\Gamma_2^2(\eta^1) \]
\[ \Gamma_3^2(\eta^1, \eta^2) \]
\[ H_6(X) \rightarrow \Gamma_5(X) \rightarrow \pi_5(X) \rightarrow H_5(X) \]
\[ L_1\Gamma_2^2(\eta^1) \]

Recall from [4] the description of functors appearing in (4.1). The functor \( \Gamma_2^3(\eta^1, \eta^2) \) is defined as follows. Let \( \pi_2, \pi_3, \pi_4 \) be abelian groups and

\[ \eta^1 : \Gamma_2(\pi_2) \rightarrow \pi_3 \]
\[ \eta^2 : \Gamma_2^3(\eta^1) \rightarrow \pi_4 \]

homomorphisms. For \( x, y \in \pi_2 \), define

\[ \eta(x) = \eta^1(\gamma_2(x)) \in \pi_3, \ [x, y] = \eta^1(\gamma_2(x + y) - \gamma_2(x) - \gamma_2(y)) \in \pi_3 \]

Consider the composite map \( \pi_3 \otimes (\mathbb{Z}/2 \oplus \pi_2) \xrightarrow{q} \Gamma_2^2(\eta^1) \xrightarrow{\eta^2} \pi_4 \) and, for \( z \in \pi_3 \), define

\[ \eta(z) = \eta^2 q(z \otimes 1) \in \pi_4, \ [z, y] = \eta^2 q(z \otimes y) \in \pi_4. \]

Then

\[ \Gamma_2^3(\eta^1, \eta^2) = (\pi_4(X) \otimes (\mathbb{Z}/2 \oplus \pi_2(X)) \oplus \Lambda^2(\pi_3(X)))/\sim, \]
The commutative diagram

where \( \sim \) is the following equivalence relation:

\[
\eta(z) \otimes y \sim [z, y] \otimes 1 \\
g(z \otimes \eta(x)) \sim [z, x] \otimes 1 + [z, x] \otimes x \\
[z, x] \otimes y + [z, y] \otimes x \sim q(z \otimes [x, y]).
\]

The derived functors \( L_1\Gamma_2^2(\eta^1) \) live in the following long exact sequence (see p.183 [4]):

\[
0 \to L_3\Gamma_2^2(\eta^1) \to Tor(R_2(\pi_2(X)), \mathbb{Z}/2 \oplus \pi_2(X)) \to Tor(R_2(\pi_2(X)), \mathbb{Z}/2) \oplus L_2\mathcal{L}_3^2(\pi_2(X)) \\
\to L_2\Gamma_2^2(\eta^1) \to \left( \begin{array}{c} R_2(\pi_2(X)) \otimes (\mathbb{Z}/2 \oplus \pi_2(X)) \\ Tor(ker(\eta^1), \mathbb{Z}/2 \oplus \pi_2(X)) \end{array} \right) \to \left( \begin{array}{c} \pi_1 \left( \Gamma_2(\pi_2(X)) \otimes \mathbb{Z}/2 \right) \\ \oplus \left( L_1\mathcal{L}_3^2(\pi_2(X)) \right) \end{array} \right) \\
\to L_1\Gamma_2^2(\eta^1) \to \left( \begin{array}{c} \ker(\eta^1) \otimes (\mathbb{Z}/2 \oplus \pi_2(X)) \\ Tor(coker(\eta^1), \mathbb{Z}/2 \oplus \pi_2(X)) \end{array} \right) \to \Gamma_2(\pi_2(X)) \otimes \mathbb{Z}/2 \oplus \mathcal{L}_3^2(\pi_2(X)) \\
\to \Gamma_2^2(\eta^1) \to coker(\eta^1) \otimes (\mathbb{Z}/2 \oplus \pi_2(X)) \to 0 \quad (4.2)
\]

The direct sum decompositions within each of the three pairs of brackets are natural in the sense of universal coefficient theorem for homology.

**Proposition 4.1.** The group \( \Gamma_5(\Sigma \mathbb{R}P^\infty) \) is of order 8.

**Proof.** Using the same trick as in ([4], example 5.16), i.e. comparing the sequences (4.2) for spaces \( \Sigma \mathbb{R}P^2, \Sigma \mathbb{R}P^\infty \) and \( K(\mathbb{Z}/2, 2) \), we conclude that

\[
L_1\Gamma_2^2(\Gamma_2(\pi_2(\Sigma \mathbb{R}P^\infty))) \to \pi_3(\Sigma \mathbb{R}P^\infty)) = L_1\Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
L_1\Gamma_2^2(\Gamma_2(\pi_2(\Sigma \mathbb{R}P^2))) \to \pi_3(\Sigma \mathbb{R}P^2)) = L_1\Gamma_2^2(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4) = \mathbb{Z}/2 \oplus \mathbb{Z}/2.
\]

The commutative diagram

\[
\begin{array}{ccc}
L_2\Gamma_2^2(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4) & \longrightarrow & L_2\Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2) \\
\downarrow & & \downarrow \\
\Gamma_2^3(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4, \mathbb{Z}/2 \to \mathbb{Z}/4) & \longrightarrow & \Gamma_2^3(\mathbb{Z}/4 \to \mathbb{Z}/2, \mathbb{Z}/2 \to \mathbb{Z}/4) \\
\downarrow & & \downarrow \\
\Gamma_5(\Sigma \mathbb{R}P^2) & \longrightarrow & \Gamma_5(\Sigma \mathbb{R}P^\infty) \\
\downarrow & & \downarrow \\
L_1\Gamma_2^2(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4) & \longrightarrow & L_1\Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2)
\end{array}
\]
has the following structure:

\[
\begin{array}{c}
\mathbb{Z}/2 \\
\downarrow \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\downarrow \\
\Gamma_5(\Sigma \mathbb{R}P^2) \\
\downarrow \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\downarrow \\
0
\end{array} \xrightarrow{\phi} \begin{array}{c}
\mathbb{Z}/2 \\
\downarrow \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \\
\downarrow \\
\Gamma_5(\Sigma \mathbb{R}P^\infty) \\
\downarrow \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2
\end{array}
\]

Since \( \Gamma_5(\Sigma \mathbb{R}P^2) = \pi_5(\Sigma \mathbb{R}P^2) = \mathbb{Z}/2^3 \) (see [15]), we conclude that the map \( L_2 \Gamma_2^3(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4) \to \Gamma_2^3(\mathbb{Z}/4 \xrightarrow{\sim} \mathbb{Z}/4, \mathbb{Z}/2 \to \mathbb{Z}/4) \) is a monomorphism and the needed statement follows.

\[\square\]

**Theorem 4.1.** Let \( p \) be a prime with \( p > 2 \) and let \( r \geq 1 \). Then

\( \pi_5(\Sigma K(\mathbb{Z}/p^r, 1)) = \mathbb{Z}/p^r \oplus \mathbb{Z}/p^r. \)

**Proof.** From the Hopf fibration

\[
\Sigma K(\mathbb{Z}/p^r, 1) \wedge K(\mathbb{Z}/p^r, 1) \to \Sigma K(\mathbb{Z}/p^r, 1) \to K(\mathbb{Z}/p^r, 2),
\]

we have

\( \pi_5(\Sigma K(\mathbb{Z}/p^r, 1)) \cong \pi_5(\Sigma K(\mathbb{Z}/p^r, 1) \wedge K(\mathbb{Z}/p^r, 1)). \)

Recall that there is a canonical decomposition [9]

\[
\Sigma K(\mathbb{Z}/p^r, 1) \cong X_1 \vee \cdots \vee X_{p-1},
\]

where \( \tilde{H}_q(X_i; \mathbb{Z}) \neq 0 \) if and only if \( q \equiv 2i \mod 2p - 2 \). Let \( P^n(p^r) = M(\mathbb{Z}/p^r, n-1) \). By taking 5-skeleton to the above decomposition, we have

\[\text{sk}_5(\Sigma K(\mathbb{Z}/p^r, 1)) \cong P^3(p^r) \vee P^5(p^r)\]

and so

\[
\text{sk}_6(\Sigma K(\mathbb{Z}/p^r, 1) \wedge K(\mathbb{Z}/p^r, 1)) \cong \text{sk}_6(\Sigma K(\mathbb{Z}/p^r, 1) \wedge K(\mathbb{Z}/p^r, 1)) \\
\cong \text{sk}_6((P^3(p^r) \vee P^5(p^r)) \wedge K(\mathbb{Z}/p^r, 1)) \\
\cong \text{sk}_6((P^2(p^r) \vee P^4(p^r)) \wedge \Sigma K(\mathbb{Z}/p^r, 1)) \\
\cong \text{sk}_6((P^2(p^r) \vee P^4(p^r)) \wedge \Sigma K(\mathbb{Z}/p^r, 1)) \\
\cong \text{sk}_6((P^2(p^r) \vee P^4(p^r)) \wedge (P^3(p^r) \vee P^5(p^r))).
\]

From the fact that

\[
P^2(p^r) \wedge P^n(p^r) \cong P^{n+2}(p^r) \vee P^{n+1}(p^r)
\]

for \( n \geq 3 \) and \( p^r > 2 \), we have

\[
\pi_5(\Sigma K(\mathbb{Z}/p^r, 1)) \cong \pi_5(\Sigma K(\mathbb{Z}/p^r, 1) \wedge K(\mathbb{Z}/p^r, 1)) \\
\cong \pi_5(P^4(p^r) \vee P^5(p^r) \vee 2P^6(p^r) \vee 2P^7(p^r) \vee P^8(p^r) \vee P^9(p^r)) \\
\cong \pi_5(P^4(p^r)) \oplus \pi_5(P^5(p^r)) \oplus \pi_5(P^6(p^r)) \oplus \pi_5(P^7(p^r)) \\
\cong \mathbb{Z}/p^r \oplus \mathbb{Z}/p^r
\]
since $\pi_5(P^6(p^r)) = \mathbb{Z}/p^r$ and $\pi_5(P^4(p^r)) = \pi_5(P^5(p^r)) = 0$. □

**Remark 4.1.** From the decomposition

$$\Sigma K(\mathbb{Z}/p^n, 1) \simeq X_1 \vee \cdots \vee X_{p-1},$$

we have

$$\text{sk}_{2p-1}(\Sigma K(\mathbb{Z}/p^n, 1)) \simeq P^3(p^r) \vee P^5(p^r) \vee \cdots \vee P^{2p-1}(p^r).$$

Thus the Hurewicz homomorphism

$$h_2: \pi_2(\Sigma K(\mathbb{Z}/p^n, 1)) \to H_2(\Sigma K(\mathbb{Z}/p^n, 1))$$

is onto for $i \leq p - 1$.

Let us now compute $\pi_5(\Sigma \mathbb{R}P^\infty)$. From the fibration

$$\Sigma \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty \to \Sigma \mathbb{R}P^\infty \to K(\mathbb{Z}/2, 2),$$

we have $\pi_5(\Sigma \mathbb{R}P^\infty) \cong \pi_5(\Sigma \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty)$. We need some lemmas in order to analyze the 6-skeleton of $\Sigma \mathbb{R}P^\infty \wedge \mathbb{R}P^\infty$.

**Lemma 4.1.** [11, Theorem 4.6] There is a homotopy decomposition

$$\Sigma^2 \mathbb{R}P^3 \simeq S^5 \vee P^4(2).$$

□

**Lemma 4.2.** Let $[2]: \Sigma^2 \mathbb{R}P^3 \to \Sigma^2 \mathbb{R}P^3$ be the degree 2 map. Then

$$[2]_* (\alpha) = 2\alpha$$

for $\alpha \in \pi_5(\Sigma^2 \mathbb{R}P^3)$.

**Proof.** Let $J(\Sigma \mathbb{R}P^3)$ be the James construction on $\Sigma \mathbb{R}P^3$. Note that $\Omega \Sigma^2 \mathbb{R}P^3 \simeq J(\Sigma \mathbb{R}P^3)$ by computations using the Cohen group [16] for the distributivity law, the map

$$\Omega[2]: \Omega \Sigma^2 \mathbb{R}P^3 \to \Omega \Sigma^2 \mathbb{R}P^3$$

restricted to $J_2(\Sigma \mathbb{R}P^3)$ is homotopic to the product of the power map

$$2: \Omega \Sigma^2 \mathbb{R}P^3 \to \Omega \Sigma^2 \mathbb{R}P^3$$

and a composite

$$f: J_2(\Sigma \mathbb{R}P^3) \xrightarrow{\text{pinch}} (\Sigma \mathbb{R}P^3)\wedge^2 \xrightarrow{\tau} (\Sigma \mathbb{R}P^3)\wedge^2 \xrightarrow{S_2} \Omega \Sigma^2 \mathbb{R}P^3,$$

where $\tau(x \wedge y) = y \wedge x$ and $S_2$ is the Samelson product. Consider the following diagram, whose commutativity is a consequence of the naturality of the Samelson product

$$\begin{array}{ccc}
(\Sigma \mathbb{R}P^3)\wedge^2 & \xrightarrow{S_2} & \Omega \Sigma^2 \mathbb{R}P^3 \\
\downarrow i \wedge i & & \downarrow i \\
S^2 \wedge S^2 & \xrightarrow{S_2} & \Omega S^3,
\end{array}$$

where $S_2: S^2 \wedge S^2 \to \Omega S^3$ is null homotopic since $S^3$ is an $H$-space. We obtain that the map

$$S_2_*: \pi_4((\Sigma \mathbb{R}P^3)\wedge^2) \to \pi_4(\Omega \Sigma^2 \mathbb{R}P^3)$$

is indeed homotopy equivalent to the identity.
is zero. It follows that
\[ \Omega[2](\alpha) = 2\alpha + f_*(\alpha) = 2\alpha \]
for \( \alpha \in \pi_4(J_2(\Sigma\mathbb{R}P^3)) \). The assertion now follows from the fact that \( J_2(\Sigma\mathbb{R}P^3) \) is the 5-skeleton of \( \Omega\Sigma^2\mathbb{R}P^3 \).

Proposition 4.2. There is a homotopy decomposition
\[ \Sigma\mathbb{R}P^2 \land \mathbb{R}P^3 \simeq P^6(2) \lor \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2. \]

Proof. Let \( f: S^5 \to \Sigma^2\mathbb{R}P^3 \) be a map inducing isomorphism on \( H_5 \). By Lemma 4.2, there is a homotopy commutative diagram of cofibre sequences
\[
\begin{array}{cccc}
S^5 & \xrightarrow{[2]} & S^5 & \xrightarrow{f} & P^6(2) & \xrightarrow{j} & S^6 \\
\downarrow{f} & & \downarrow{f} & & \downarrow{j} & & \downarrow{\Sigma f} \\
\Sigma^2\mathbb{R}P^3 & \xrightarrow{[2]} & \Sigma^2\mathbb{R}P^3 & \xrightarrow{\Sigma f} & \Sigma\mathbb{R}P^2 \land \mathbb{R}P^3 & \xrightarrow{\Sigma j} & \Sigma^3\mathbb{R}P^3.
\end{array}
\]

Let \( j: \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2 \to \Sigma\mathbb{R}P^2 \land \mathbb{R}P^3 \) be the inclusion. Then the map
\( (f, j): P^6(2) \lor \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2 \to \Sigma\mathbb{R}P^2 \land \mathbb{R}P^3 \)
induces an isomorphism on mod 2 homology so that the result follows.

Lemma 4.3. \( \pi_5(\Sigma\mathbb{R}P^2 \land \mathbb{R}P^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)

Proof. Let \( F \) be the homotopy fibre of the pinch map \( \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2 \to P^5(2) \). Then there is a homotopy commutative diagram
\[
\begin{array}{cccc}
\Omega P^5(2) & \xrightarrow{E} & F & \xrightarrow{j} & \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2 & \xrightarrow{P^5(2)} \\
\downarrow{P^4(2)} & & \downarrow{j} & & \downarrow{P^4(2)} & & \downarrow{P^4(2)} & & \downarrow{P^4(2)}
\end{array}
\]
where the top row is a fibre sequence and the bottom row is a cofibre sequence. By applying the Serre spectral sequence to the fibration
\[ \Omega P^5(2) \to F \to \Sigma\mathbb{R}P^2 \land \mathbb{R}P^2, \]
we have
\[ j_*: H_q(P^4(2); \mathbb{Z}/2) \to H_q(F; \mathbb{Z}/2) \]
is an isomorphism for \( q \leq 5 \) and so
\[ j_*: \pi_5(P^4(2)) \to \pi_5(F) \]
is an epimorphism. Note that the stabilization \( P^4(2) \to \Omega^\infty\Sigma^\infty P^4(2) \) factors through \( j: P^4(2) \to F \). There is a commutative diagram
\[
\begin{array}{ccc}
\pi_5(P^4(2)) & \xrightarrow{j_*} & \pi_5(F) \\
\downarrow{E^\infty} & & \downarrow{\pi_5(F)} \\
\pi_5^S(P^4(2)) & & \pi_5^S(P^4(2))
\end{array}
\]
From [15],
\[ E^\infty : \pi_5(P^4(2)) = \mathbb{Z}/4 \longrightarrow \pi_5^*(P^4(2)) = \mathbb{Z}/4 \]
is an isomorphism. Thus
\[ j_* : \pi_5(P^4(2)) \longrightarrow \pi_5(F) \]
is an isomorphism. Now consider the commutative diagram
\[
\begin{array}{ccc}
\pi_5(\Omega P^5(2)) & \longrightarrow & \pi_5(F) \\
\downarrow \cong & & \downarrow \cong \\
\pi_5(P^4(2)) & \longrightarrow & \pi_5(\Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^2) \\
\end{array}
\]
where the top row is the long exact sequence associated to the fibration. There is a short exact sequence
\[ \mathbb{Z}/2 = \pi_5(P^4(2))/2 \longrightarrow \pi_5(\Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^2) \longrightarrow \pi_5(P^5(2)) = \mathbb{Z}/2. \]
By applying \( \pi_5(\cdot) \) to the commutative diagram
\[
\begin{array}{ccc}
S^4 & \longrightarrow & \Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^2 \\
\downarrow i & & \downarrow \text{pinch} \\
& P^5(2) \\
\end{array}
\]
and using the fact that \( i : \pi_5(S^4) = \mathbb{Z}/2 \rightarrow \pi_5(P^5(2)) \) is an isomorphism, it follows that the epimorphism
\[ \pi_5(\Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^2) \longrightarrow \pi_5(P^5(2)) \]
admits a cross-section. Thus \( \pi_5(\Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)

**Theorem 4.2.** \( \pi_5(\Sigma \mathbb{RP}^{\infty}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)

**Proof.** Let \( x_1 \) (and \( y_i \)) denote the basis for \( H_4(\mathbb{R}P^{\infty}; \mathbb{Z}/2) \). Then \( s^{-1}H_k(\Sigma \mathbb{RP}^{\infty} \wedge \mathbb{RP}^{\infty}; \mathbb{Z}/2) \) with \( k \leq 6 \) has a basis given by the table

\[
\begin{pmatrix}
k = 6 & x_3y_2 & x_2y_3 & a = x_4y_1 + x_3y_2 & b = x_4y_1 + x_3y_2 + x_1y_4 + x_2y_3 \\
5 & x_3y_1 & x_1y_3 & x_2y_2 & \\
4 & x_1y_1 & x_2y_1 & & \\
3 & & & x_2y_1 + x_1y_2 & x_1y_1
\end{pmatrix}
\]

with the Steenrod operations

\[
\begin{align*}
S^1_q(x_3y_2) &= x_3y_1 \\
S^1_q(x_2y_3) &= x_1y_3 \\
S^1_q(a) &= 0 \\
S^1_q(b) &= 0 \\
S^2_q(x_3y_2) &= 0 \\
S^2_q(x_2y_3) &= 0 \\
S^2_q(a) &= x_2y_1 \\
S^2_q(b) &= x_2y_1 + x_1y_2.
\end{align*}
\]

From the above Steenrod operations, there are no spherical elements in \( H_6(\Sigma \mathbb{RP}^{\infty} \wedge \mathbb{RP}^{\infty}) \) and the Hurewicz homomorphism

\[ \pi_6(\Sigma \mathbb{RP}^{\infty} \wedge \mathbb{RP}^{\infty}) \rightarrow H_6(\Sigma \mathbb{RP}^{\infty} \wedge \mathbb{RP}^{\infty}) \text{ is } 0. \] (4.3)

Let \( \phi_1 \) and \( \phi_2 \) be the composites given by

\[ \phi_1 : P^6(2) \xrightarrow{f} \Sigma \mathbb{RP}^2 \wedge \mathbb{RP}^4 \longrightarrow \Sigma \mathbb{RP}^{\infty} \wedge \mathbb{RP}^{\infty}, \]
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\[ \phi_2: P^6(2) \xrightarrow{f'} \Sigma \mathbb{RP}^3 \land \mathbb{RP}^2 \cong \Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty, \]

where \( f \) and \( f' \) are the maps inducing isomorphism on \( H_6(-; \mathbb{Z}/2) \). Let

\[ \phi_3: \Sigma \mathbb{RP}^2 \land \mathbb{RP}^2 \to \Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty \]

be the inclusion. Consider the map

\[ \phi: X = P^6(2) \vee P^6(2) \vee \Sigma \mathbb{RP}^2 \land \mathbb{RP}^2 \xrightarrow{(\phi_1, \phi_2, \phi_3)} \Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty. \]

Then \( \phi \) induces a monomorphism on \( \text{mod} \, 2 \) homology and its image contains all the basis elements of \( H_k(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty; \mathbb{Z}/2) \) for \( k \leq 6 \) except for \( a \) and \( b \). Thus there is a cofibre sequence

\[ S^5 \vee S^5 \xrightarrow{\theta} X \to \text{sk}_6(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty). \]

and so a right exact sequence

\[ \pi_5(S^5 \vee S^5) \xrightarrow{\theta_*} \pi_5(X) \to \pi_5(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty). \]

Observe that

\[ \pi_5(X) \cong \pi_5(P^6(2)) \oplus \pi_5(P^6(2)) \oplus \pi_5(\Sigma \mathbb{RP}^2 \land \mathbb{RP}^2) = \mathbb{Z}/2^{34}. \]

The image

\[ \text{Im}(\theta_*: \pi_5(S^5 \vee S^5) \to \pi_5(X)) \]

is \( 0, \mathbb{Z}/2 \) or \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \).

Suppose that \( \text{Im}(\theta_*: \pi_5(S^5 \vee S^5) \to \pi_5(X)) = 0 \) or \( \mathbb{Z}/2 \). Then there exists a map

\[ g: S^5 \to S^5 \vee S^5 \]

such that \([g]\) generates a \( \mathbb{Z}\)-summand of \( \pi_5(S^5 \vee S^5) = \mathbb{Z} \oplus \mathbb{Z} \) and

\[ \theta \circ g: S^5 \to X \]

is null homotopic. From the homotopy commutative diagram of cofibre sequences

\[ \begin{diagram}
\node{S^5} \arrow{e} \node{\ast} \arrow{s,l}{g} \node{S^6} \arrow{s,r}{\theta}
\node{S^5 \vee S^5} \arrow{e} \node{X} \arrow{s} \node{\text{sk}_6(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty)}
\node{S^5} \arrow{e} \node{X}
\end{diagram} \]

it follows that the homology \( H_6(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty; \mathbb{Z}/2) \) has a spherical class. This contradicts to the fact that the Hurewicz homomorphism is zero on \( \pi_6 \), as observed in (4.3).

Hence \( \text{Im}(\theta_*: \pi_5(S^5 \vee S^5) \to \pi_5(X)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) and so

\[ \pi_5(\Sigma \mathbb{RP}^\infty \land \mathbb{RP}^\infty) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \]

\[ \square \]

**Theorem 4.3.** \( \pi_5(\Sigma K(\Sigma_3, 1)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2. \)
Proof. This follows from the analysis of the map between the sequences (2.1) induced by the natural map \( \mathbb{Z}/2 \hookrightarrow \Sigma_3 \):

\[
\begin{array}{c}
H_5(\mathbb{Z}/2) \xrightarrow{\cong} \Gamma_5(\Sigma \mathbb{RP}^\infty) \xrightarrow{\cong} \pi_5(\Sigma \mathbb{RP}^\infty) \xrightarrow{\cong} H_4(\mathbb{Z}/2) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H_5(\Sigma_3) \xrightarrow{\cong} \Gamma_5(\Sigma K(\Sigma_3, 1)) \xrightarrow{\cong} \pi_5(\Sigma K(\Sigma_3, 1)) \xrightarrow{\cong} H_4(\Sigma_3)
\end{array}
\]

Here the natural isomorphism \( \Gamma_5(\Sigma \mathbb{RP}^\infty) \to \Gamma_5(\Sigma K(\Sigma_3, 1)) \) follows from the diagram

\[
\begin{array}{c}
L_2 \Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2) \xrightarrow{\cong} L_2 \Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2) \\
\downarrow \\
\Gamma_2^3(\mathbb{Z}/4 \to \mathbb{Z}/2, \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4) \xrightarrow{\cong} \Gamma_2^3(\mathbb{Z}/4 \to \mathbb{Z}/2, \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/12) \\
\downarrow \\
\Gamma_5(\Sigma \mathbb{RP}^\infty) \xrightarrow{\cong} \Gamma_5(\Sigma K(\Sigma_3, 1)) \\
\downarrow \\
L_1 \Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2) \xrightarrow{\cong} L_1 \Gamma_2^2(\mathbb{Z}/4 \to \mathbb{Z}/2)
\end{array}
\]

\[\square\]

5. **On group \( \pi_5(\Sigma^2 K(A, 1)) \)**

Let \( A \) be an abelian group and

\[0 \to A_1 \xrightarrow{\delta} A_0 \to A \to 0\]

a two-step flat resolution of \( A \), i.e. \( A_0 \) is a free abelian group. The diagram (1.1) implies that there is a natural isomorphism

\[\pi_4(\Sigma^2 K(A, 1)) \cong A \tilde{\otimes} A,\]

where \( \tilde{\otimes}^2 \):

\[\tilde{\otimes}^2(A) = A \tilde{\otimes} A := A \otimes A/(a \otimes b + b \otimes a, a, b \in A).\]

Given a free abelian group \( \tilde{A} \), the sequence (2.3) implies the following natural exact sequence:

\[
\begin{array}{c}
\Gamma_5(\Sigma^2 K(\tilde{A}, 1)) \xleftarrow{\cong} \pi_5(\Sigma^2 K(A, 1)) \xrightarrow{\cong} H_5(\Sigma^2 K(A, 1)) \\
\downarrow \quad \downarrow \quad \downarrow \\
\tilde{A} \tilde{\otimes} \tilde{A} \otimes \mathbb{Z}/2 \oplus \Lambda^2(\tilde{A}) \xleftarrow{\cong} \pi_5(\Sigma^2 K(\tilde{A}, 1)) \xrightarrow{\cong} \Lambda^3(\tilde{A})
\end{array}
\]
The spectral sequence (2.8) for \( n = 2 \), gives the following diagram of exact sequences:

\[
\begin{array}{cccccc}
& L_1A^3(A) & \\
& \downarrow & \\
A\hat{\otimes}A \otimes \mathbb{Z}/2 \oplus \Lambda^2(A) & \triangleleft \pi_5(\Sigma^2K(A, 1)) & \rightarrow & L_1\tilde{\otimes}^2(A) \\
& \downarrow & \\
\pi_0(\pi_5(\Sigma^2K(N^{-1}(A_1 \xrightarrow{\delta} A_0), 1))) & \triangleleft \pi_5(\Sigma^2K(A, 1)) & \rightarrow & \pi_1(\pi_4\Sigma^2K(N^{-1}(A_1 \xrightarrow{\delta} A_0), 1)) \\
& \downarrow & \\
\Lambda^3(A) & \\
\end{array}
\]

Consider the first derived functor of the functor \( \tilde{\otimes}^2 \). The short exact sequence

\[ LSP^2(A) \rightarrow L \otimes^2 (A) \rightarrow L\tilde{\otimes}^2(A) \]

in the derived category has the following model:

\[
\begin{array}{cccccc}
& \Lambda^2(A_1) \triangleleft A_1 \otimes A_0 & \triangleleft SP^2(A_0) \\
& \downarrow & \downarrow & \\
A_1 \otimes A_1 & \triangleleft (A_1 \otimes A_0) \oplus (A_0 \otimes A_1) & \triangleleft A_0 \otimes A_0 \\
& \downarrow & \downarrow & \\
SP^2(A_1) & \triangleleft A_1 \otimes A_0 & \triangleleft A_0\tilde{\otimes}A_0 \\
\end{array}
\]

with

\[
\begin{align*}
\delta_2(a_1 \wedge a'_1) &= a_1 \otimes \delta(a'_1) - a'_1 \otimes \delta(a_1) \\
\delta_1(a_1 \otimes a_0) &= a_0\delta(a_1) \\
\delta'_2(a_1 \otimes a'_1) &= (a_1 \otimes \delta(a'_1), -a'_1 \otimes \delta(a_1)) \\
\delta'_1(a_1 \otimes a_0, a'_1 \otimes a'_0) &= \delta(a_1) \otimes a_0 + \delta(a'_1) \otimes a'_0 \\
\delta''_2(a_1 a'_1) &= a_1 \otimes \delta(a'_1) + a'_1 \otimes \delta(a_1) \\
\delta''_1(a_1 \otimes a_0) &= \partial(a_1)\tilde{\otimes}a_0
\end{align*}
\]

for \( a_0, a'_0 \in A_0, a_1, a'_1 \in A_1 \). For \( n \geq 2 \), looking at the resolution \( \mathbb{Z} \xrightarrow{n} \mathbb{Z} \) of the cyclic group \( \mathbb{Z}/n \), we obtain the following representative of the element \( L\tilde{\otimes}^2(\mathbb{Z}/n) \) in the derived category:

\[ \mathbb{Z} \xrightarrow{2^n} \mathbb{Z} \xrightarrow{n} \mathbb{Z}/2 \]
In particular,

\[ L_1 \widetilde{\boxtimes}^3(\mathbb{Z}/2^k) = \mathbb{Z}/2^{k+1}, \quad k \geq 1 \]  
\[ L_1 \widetilde{\boxtimes}^3(\mathbb{Z}/p^k) = \mathbb{Z}/p^k, \quad (p \text{ prime } \neq 2), k \geq 1. \]  

Diagram (5.1) implies the following result:

**Theorem 5.1.** Let \( p \) be an odd prime. Then

\[ \pi_5(\Sigma^2 K(\mathbb{Z}/p^k, 1)) = \mathbb{Z}/p^k. \]

For the case \( p = 2 \), we give a new proof of the following well-known result:

**Theorem 5.2.** \( \pi_5(\Sigma^2 \mathbb{R} P^\infty) = \mathbb{Z}/8. \)

**Proof.** The natural epimorphism \( \mathbb{Z} \to \mathbb{Z}/2 \) induces the homomorphisms

\[ \pi_n(S^3) = \pi_n(\Sigma^2 K(\mathbb{Z}, 1)) \to \pi_n(\Sigma^2 K(\mathbb{Z}/2, 1)) = \pi_n(\Sigma^2 \mathbb{R} P^\infty), \quad n \geq 1. \]

The diagram (5.1) together with (5.2) implies the following short exact sequences:

\[
\begin{array}{cccc}
\mathbb{Z}/2 & \longrightarrow & \pi_5(S^3) & \longrightarrow \\
\downarrow & & \downarrow & \\
\mathbb{Z}/2 & \longrightarrow & \pi_5(\Sigma^2 \mathbb{R} P^\infty) & \longrightarrow & \mathbb{Z}/4
\end{array}
\]  

Consider this map simplicially, at the level of the natural map between the Carlsson constructions \( F(S^2) = F^\mathbb{Z}(S^2) \to F^\mathbb{Z}/2(S^2) : \)

\[
\begin{array}{cccc}
F(S^2)_4 & \longrightarrow & F(S^2)_3 & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
F^\mathbb{Z}/2(S^2)_4 & \longrightarrow & F^\mathbb{Z}/2(S^2)_3 & \longrightarrow & F^\mathbb{Z}/2(S^2)_2
\end{array}
\]

Here \( F^\mathbb{Z}/2(S^2)_k \) is the free product of \( \binom{k}{2} \) copies of \( \mathbb{Z}/2 \). In particular

\[ F^\mathbb{Z}/2(S^2)_4 = \langle s_j s_i(\sigma) \mid 0 \leq i < j \leq 3, (s_j s_i(\sigma))^2 = 1 \rangle \]

Using the description of the element (2.6), we see that the simplicial cycle which defines the image of \( \pi_5(S^3) \) in \( \pi_5(\Sigma^2 \mathbb{R} P^\infty) \) can be chosen of the form

\[ [[s_2 s_1(\sigma), s_1 s_0(\sigma)], [s_2 s_1(\sigma), s_2 s_0(\sigma)]] \in F^\mathbb{Z}/2(S^2)_4 \]

With the help of lemma 3.2, we have

\[ [s_2 s_1(\sigma), s_1 s_0(\sigma)], [s_2 s_1(\sigma), s_2 s_0(\sigma)] = [[[s_2 s_1(\sigma), s_1 s_0(\sigma)], (s_2 s_1(\sigma) s_2 s_0(\sigma))^2] \equiv \\
[[s_2 s_1(\sigma), s_1 s_0(\sigma)], (s_2 s_1(\sigma) s_2 s_0(\sigma))^2] \mod B^\mathbb{Z}/2(S^2)_4 \]

Since \( [[s_2 s_1(\sigma), s_1 s_0(\sigma)], s_2 s_1(\sigma) s_2 s_0(\sigma)] \) is a cycle in \( F^\mathbb{Z}/2(S^2) \). That is, the image of the element \( \pi_5(S^3) \) is divisible by 2 in \( \pi_5(\Sigma^2 \mathbb{R} P^\infty) \). The diagram (5.4) implies the result. \( \square \)
6. Relation to K-theory

As we mentioned in the introduction, there is a natural relation between the problem considered and algebraic K-theory. Since the plus-construction $K(G, 1) \to K(G, 1)^+$ is a homological equivalence, there is a natural weak homotopy equivalence

$$\Sigma K(G, 1) \to \Sigma(K(G, 1)^+)$$

This defines the natural suspension map:

$$\pi_n(K(G, 1)^+) \to \pi_{n+1}(\Sigma(K(G, 1)^+)) = \pi_{n+1}(\Sigma K(G, 1))$$

for $n \geq 1$. The Whitehead exact sequences form the following commutative diagram:

$$
\begin{array}{ccc}
H_4(G) & \to & \Gamma_2(H_2(G)) \to \pi_3(K(G, 1)^+) \to H_3(G) \\
\downarrow & & \downarrow \\
H_4(G) & \to & H_2(G) \otimes \mathbb{Z}/2 \to \pi_4(\Sigma K(G, 1)) \to H_3(G)
\end{array}
$$

(6.1)

Let $R$ be a ring and $G = E(R)$, the group of elementary matrices. The group $E(R)$ is perfect and the plus-construction $K(E(R), 1)^+$ also denoted $\tilde{K}(R)$, defines the algebraic K-theory of $R$: $K_n(R) = \pi_n(K(E(R), 1)^+)$, $n \geq 2$. In this case, one has the natural homomorphisms:

$$K_n(R) \to \pi_{n+1}(\Sigma K(E(R), 1)), \quad n \geq 2.$$ 

For $n = 2$, clearly one has the natural isomorphism:

$$K_2(R) \simeq H_2(E(R)) \simeq \pi_3(\Sigma K(E(R), 1)).$$

(6.2)

It is shown in [1] that the map $\Gamma_2(K_2(R)) \to K_3(R)$ factors as

$$\Gamma_2(K_2(R)) \to K_2(R) \otimes K_1(\mathbb{Z}) \xrightarrow{\star} K_3(R),$$

where $\star$ is the product in algebraic K-theory: $\star : K_i(S) \otimes K_j(T) \to K_{i+j}(S \otimes T)$. Hence the diagram (6.1) has the following form:

$$
\begin{array}{ccc}
H_4(E(R)) & \to & \Gamma_2(K_2(R)) \to K_3(R) \to H_3(E(R)) \\
\downarrow & & \downarrow \star \\
H_4(E(R)) & \to & K_2(R) \otimes K_1(\mathbb{Z}) \to \pi_4(\Sigma K(E(R), 1)) \to H_3(E(R))
\end{array}
$$

(6.3)

and the natural map

$$K_3(R) \to \pi_4(\Sigma K(E(R), 1))$$

(6.4)

is an isomorphism. From equations (6.2) and (6.4) together with the fact that $SL(\mathbb{Z}) = E(\mathbb{Z})$, we have the following:

**Theorem 6.1.** The natural homomorphism

$$K_n(R) \to \pi_{n+1}(\Sigma K(E(R), 1))$$

is an isomorphism for $n = 2, 3$. In particular,

$$\pi_3(\Sigma K(SL(\mathbb{Z}), 1)) \cong K_2(\mathbb{Z}) \cong \mathbb{Z}/2 \text{ and }$$

$$\pi_4(\Sigma K(SL(\mathbb{Z}), 1)) \cong K_3(\mathbb{Z}) \cong \mathbb{Z}/48.$$
Remark 6.1. The isomorphism (6.4) and Carlsson construction $F^{E(R)}(S^1)$ gives a way, for an element of $K_3(R)$, to associate an element from $F^{E(R)}(S^1) = E(R) * E(R) * E(R)$ (uniquely modulo $BF^{E(R)}(S^1)$):

$$K_3(R) \xrightarrow{\cong} E(R) * E(R) * E(R) \xrightarrow{\pi_5(E(R))} \frac{Z^{F^{E(R)}(S^1)_{S1}}}{BF^{E(R)}(S^1)_{S1}}.$$ 

It is interesting to represent in this way known elements from $K_3(R)$ for different rings. For $R = \mathbb{Z}$, $x \in SL(\mathbb{Z}) = E(\mathbb{Z})$, denote by $x^{(1)}, x^{(2)}, x^{(3)}$ the correspondent elements in the free cube $SL(\mathbb{Z}) * SL(\mathbb{Z}) * SL(\mathbb{Z})$. Take the following commuting elements of $SL(\mathbb{Z})$:

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

The structure of the element (2.6), diagram (6.3) and well-known facts about structure of $K_2(\mathbb{Z})$ imply that, using the above notations, the element

$$[[u^{(2)}, v^{(3)}], [u^{(1)}, v^{(3)}]]$$

corresponds to the element of order 2 in $K_3(\mathbb{Z})$. It would be interesting to see an element of $SL(\mathbb{Z}) * SL(\mathbb{Z}) * SL(\mathbb{Z})$ which corresponds to the generator of $K_3(\mathbb{Z}) = \mathbb{Z}/48$. □

Consider the case $R = \mathbb{Z}$ and $n = 5$. In this case, $E(\mathbb{Z}) = SL(\mathbb{Z})$ and we have the following commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccc}
\mathbb{Z} \oplus (\mathbb{Z}/2)^2 & \rightarrow & (\mathbb{Z}/2)^3 & \rightarrow & 0 & \rightarrow & \mathbb{Z}/2 \\
H_5SL(\mathbb{Z}) & \rightarrow & \Gamma_4(\tilde{K}(\mathbb{Z})) & \rightarrow & K_4(\mathbb{Z}) & \rightarrow & \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \\
H_5(SL(\mathbb{Z})) & \rightarrow & \Gamma_5(\Sigma K(SL(\mathbb{Z}), 1)) & \rightarrow & \pi_5(\Sigma K(SL(\mathbb{Z}), 1)) & \rightarrow & \mathbb{Z}/2 \rightarrow 0 \rightarrow \mathbb{Z}/2
\end{array}$$
and the following commutative diagram:

\[ \begin{array}{ccc}
(Z/2)^2 & \rightarrow & (Z/2)^3 \\
\downarrow & & \downarrow \\
\Gamma_2^3(K_2(Z)) & \rightarrow & \Gamma_4(K(Z)) \\
\downarrow & & \downarrow \\
\pi_4(\Sigma K(SL(Z)), 1) \otimes \mathbb{Z}/2 & \rightarrow & \Gamma_5(\Sigma K(SL(Z)), 1) \\
\downarrow & & \downarrow \\
Z/2 & \rightarrow & (Z/2)^2 \\
\end{array} \]

Simple analysis shows that the suspension map \( \Gamma_4(K(Z)) \rightarrow \Gamma_5(\Sigma K(SL(Z)), 1) \) is an epimorphism and therefore we have the following theorem:

**Theorem 6.2.** The Hurewicz homomorphism

\[ \pi_5(\Sigma K(SL(Z), 1)) \rightarrow H_4(SL(Z)) = \mathbb{Z}/2 \]

is an isomorphism. \( \square \)

**Remark.** Since \( K_4(Z) = 0 \), we see that the natural homomorphism

\[ K_4(Z) \rightarrow \pi_5(\Sigma K(SL(Z), 1)) \]

is not an isomorphism.

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