REPRESENTATIONS AND HOMOTOPY THEORY

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Let $X = \Sigma Y$ be a suspension. A question in homotopy theory is how to decompose the n-fold self smash product $X^{(n)}$ into a wedge of spaces.

Consider the set of homotopy classes $[X^{(n)}, X^{(n)}]$. Let the symmetric group S_n act on $X^{(n)}$ by permuting positions. So for each $\sigma \in S_n$ there is a map $\sigma \colon X^{(n)} \to X^{(n)}$. This gives a function $\theta \colon S_n \to [X^{(n)}, X^{(n)}]$. By assuming that X is a suspension, $[X^{(n)}, X^{(n)}]$ is an abelian group and so there is an extension $\theta \colon \mathbb{Z}(S_n) \to [X^{(n)}, X^{(n)}]$. Furthermore, assume that X is a p-local suspension, the map θ extends to a map $\theta \colon Z_{(p)}(S_n) \to [X^{(n)}, X^{(n)}]$, where $Z_{(p)}$ is the set of p-local integers, that is the rational numbers a/b with $b \not\equiv 0 \bmod p$. (Note: $[X^{(n)}, X^{(n)}]$ is semi-ring, that is, $h \circ (f+g) = h \circ f + h \circ g$ holds but $(f+g) \circ h \neq f \circ h + g \circ h$ in general.) The map θ is a morphism of semi-rings.

Now let $1 = \sum_\alpha e_\alpha$ is a decomposition of the identity into orthogonal idempotents in $\mathbb{Z}_{(p)}(S_n)$. Now consider the map $e_\alpha \colon X^{(n)} \to X^{(n)}$. We have $e_\alpha \circ e_\alpha \simeq e_\alpha \colon X^{(n)} \to X^{(n)}$. Define a space $e_\alpha(X^{(n)})$ by the homotopy colimit

$$e_\alpha(X^{(n)}) = \operatorname{hocolim}_{e_\alpha} X^{(n)},$$

where the homotopy colimit of a self map $f \colon Y \to Y$ is the union of the following sequence of spaces

$$Y_1 \xhookrightarrow{f_1} Y_2 \xhookrightarrow{f_2} Y_3 \hookrightarrow \cdots,$$

where $Y_i \simeq Y$ for each i, $f_i\colon Y_i \to Y_{i+1}$ is a cofibration and there is a homotopy commutative diagram

$$\begin{array}{ccc} Y_i & \xrightarrow{f_i} & Y_{i+1} \\ \Big\downarrow \simeq & & \Big\downarrow \simeq \\ Y & \xrightarrow{f} & Y \end{array}$$

for each i. The homology of $\mathrm{hocolim}_f Y$ is the stable fixed set of $f_*\colon H_*(Y) \to H_*(Y)$. Now one can easily show that the composite

$$X^{(n)} \xrightarrow{\text{comulti}} \bigvee_{\alpha} X^{(n)} \to \bigvee_{\alpha} e_\alpha(X^{(n)}),$$

is a homotopy equivalence. Thus we have

> decompositions of the identity in $\mathbb{Z}_{(p)}(S_n) \Rightarrow$ (functorial) decompositions of $X^{(n)}$ for p-local suspensions X.

Now let's consider this question from a different point of view. Let $\bar{H}_*(X)$ be the reduced mod p homology of X. Then $\bar{H}_*(X)$ is a module over the Steenrod algebra. Suppose that $X^{(n)} \simeq Y \vee Z$ is a decomposition. Then there is a decomposition of modules over the Steenrod algebra $\bar{H}_*(X^{(n)}) = (\bar{H}_*(X))^{\otimes n} \cong \bar{H}_*(Y) \oplus \bar{H}_*(Z)$. Thus we have

> decompositions of $X^{(n)} \Rightarrow$ decompositions of $(\bar{H}_*(X))^{\otimes n}$ over the Steenrod algebra.

Now let's consider the special case where $X = \Sigma^{k-2}\mathbb{R}P^2$ with $k \geq 3$ (in this case $p = 2$). (Note: The decompositions below actually holds for any two cell complexes localized at 2.) The reduced (mod 2) homology of X has a basis v and u with $\dim(v) = k$, $\dim(u) = k - 1$ and $Sq^1_*(v) = u$. Let $V = \bar{H}_*(X)$. Then the Steenrod operations on $V^{\otimes n}$ is uniquely determined by the Cartan formula that

$$Sq_*^m(a\otimes b)=\sum_{i+j=m} Sq_*^i(a)\otimes Sq_*^j(b).$$

(Note: Given a module M over the Steenrod algebra. As a module over the Steenrod algebra $M^{\otimes n}$ is uniquely given since the Cartan formula determines the Steenrod operations on tensor products.) Algebraic arguments show that

> decompositions of $V^{\otimes n}$ as graded modules over the Steenrod algebra $\Leftrightarrow$ decompositions of $V^{\otimes n}$ as ungraded modules over so-called 'hyperalgebra' $\mathcal{U}_V$.

(Note: This is a special property. For instance, let X be a suspension of $\mathbb{R}P^n$ with $n>2$ and $V=\bar{H}_*(X)$. It is still not clear whether this statement is still true.)

The hyperalgebra $\mathcal{U}_V=\Gamma(\mathrm{End}(V))=(\Lambda(\mathrm{End}(V)^*)^*$, where $\Lambda(M)$ is the free commutative algebra generated by M. $\mathcal{U}_V$ action on $V^{\otimes n}$ is given by the Cartan formula. (Note: By making V to be a graded module. A difference between the hyperalgebra and the Steenrod algebra is that Steenrod operations have positive degree on cohomology, that is, go upward and go downward on homology, but operations from the hyperalgebra go either upward or downward. In other word, the hyperalgebra use both $Sq^1\colon u\mapsto v$ and $Sq^1_*\colon v\mapsto u$.) The Schur algebra $\mathcal{S}_n(V)$ is a quotient algebra of $\mathcal{U}_V$ and it is known that $\mathcal{S}_n(V)\cong \mathrm{End}_{\mathbf{k}(S_n)}(V^{\otimes n})$. It is also known that decompositions $W^{\otimes n}$ over $\mathcal{U}_W$ $\Leftrightarrow$ decompositions of $W^{\otimes n}\otimes\mathbf{K}$ over the general linear group $GL_m(\mathbf{K})$, where $m=\dim W$ and $\mathbf{K}$ is any extension field with sufficiently large order.

From the representation theory, we have

> Let $f\colon V^{\otimes n}\to V^{\otimes n}$ be a map over the Hyperalgebra. Then f is a linear combination $f=\sum_{\sigma\in S_n}k_\sigma\sigma$ for some $k_\sigma\in\mathbb{Z}/2$, where $\sigma\colon V^{\otimes n}\to V^{\otimes n}$ permutes

positions. In other words, $\mathbb{Z}/2(S_n) \twoheadrightarrow \mathrm{End}_{\mathcal{U}_V}(V^{\otimes n})$. Furthermore if f is an idempotent, then there exists an idempotent $g = \sum_{\sigma \in S_n} k'_\sigma \sigma \in \mathbb{Z}_{(2)}(S_n)$ such that $g = f\colon V^{\otimes n} \to V^{\otimes n} \bmod 2$.

(Note: This statement holds for any V, that is we do not need to assume that $\dim V = 2$. Also this statement holds in odd primary cases as well.)

Now assume that $f\colon V^{\otimes n} \to V^{\otimes n}$ is graded map over the Steenrod algebra. We have an element $g \in \mathbb{Z}_{(2)}(S_n)$ such that $g = f \bmod 2$ and so a map $g\colon X^{(n)} \to X^{(n)}$ such that $g_* = f\colon \bar{H}_*(X^{(n)}) \to \bar{H}_*(X^{(n)})$. This gives the following result:

> decompositions of $\bar{H}_*(X^{(n)})$ over the Steenrod algebra $\Rightarrow$ functorial decompositions of $Y^{(n)}$ for any two-cell complexes localized at 2 $\Rightarrow$ individual space decompositions of $X^{(n)}$ $\Rightarrow$ decompositions of $\bar{H}_*(X^{(n)})$ over the Steenrod algebra.

Now we want to see how to give decompositions of $V^{\otimes n}$ over the Steenrod algebra. If $n = 1$, V is indecomposable because $Sq^1_*(v) = u$. When $n = 2$, we have $Sq^1_*(v^2) = [u, v] = uv - vu$, $Sq^2_*(v^2) = u^2$ and $Sq^1_*(uv) = u^2$. Thus $V^{\otimes 2}$ is indecomposable. $V^{\otimes 3}$ is decomposable as follows:

$$V^{\otimes 3} \cong L_2(V) \otimes V \oplus L_2(V) \otimes V \oplus P_2(V) \otimes V = (2L_2(V) \oplus P_2(V)) \otimes V$$

as modules over the Steenrod algebra, where $L_n(W) = \{[[a_1, a_2], \cdots, a_n] \in W^{\otimes n}\}$ is the set of n-fold Lie elements and $P_{2^s}(W) = \{a^{2^s} | a \in W\}$.

(Note: 1) $\dim L_2(V) = 1$. 2) As a module over the Steenrod algebra, $P_{2^s}(V)$ has a basis v^{2^s} and u^{2^s} with $Sq^{2^s}_*(v^{2^s}) = u^{2^s}$. 3) For any module W, there is a functorial decomposition $W^{\otimes 3} \cong L_3(W) \oplus L_3(W) \oplus M_3(W)$ but $L_3(W) \cong L_2(W) \otimes W$ if and only if $\dim W \leq 2$. In other words, the general functorial decomposition of $W^{\otimes 3}$ can not be written down as the form above.)

Since $\dim P_{2^s}(V) = 2$, we have

$$P_{2^s}(V)^{\otimes 3} \cong (2L_2(P_{2^s}(V)) \oplus P_2(P_{2^s}(V))) \otimes P_{2^s}(V) = (2L_2(V)^{\otimes n} \oplus P_{2^{s+1}}(V))) \otimes P_{2^s}(V)$$

as modules over the Steenrod algebra. It turns out that this gives an algorithm to completely determine decompositions of $V^{\otimes n}$ over the Steenrod algebra. Let's look at one more example ($n = 7$). Write MN for $M \otimes N$, $M + N$ for $M \oplus N$, L_2 for $L_2(V)$ and P_{2^s} for $P_{2^s}(V)$.

$$V^7 = V^3V^3V = ((2L_2 + P_2)V)^2V = (2L_2 + P_2)^2V^3 = (2L_2 + P_2)^3V$$

$$= 8L_2^3V + 12L_2^2P_2V + 6L_2P_2^2V + P_2^3V$$

$$= 8L_2^3V + 12L_2^2P_2V + 6L_2P_2^2V + (2L_2^2 + P_4)P_2V$$

$$= 8L_2^3V + 14L_2^2P_2V + 6L_2P_2^2V + P_4P_2V.$$

This show that $X^{(7)}$ has four factors (multiplicities are 8, 14, 6 and 1). By using homotopy theory we can actually show that

$$(P^n(2))^{(7)} \simeq \mathbf{H}P^2 \wedge \mathbf{C}P^2 \wedge P^{7n-12}(2) \vee \bigvee^{6} (\mathbf{C}P^2)^{(2)} \wedge P^{7n-9}(2) \vee \bigvee^{14} \mathbf{C}P^2 \wedge P^{7n-6}(2) \vee \bigvee^{8} P^{7n-3}(2),$$

where $P^n(2) = \Sigma^{n-2}\mathbb{R}P^2$ with $n \geq 3$. (Note: This formula gives an interesting relation among diffirent projective planes.)

Let m be a non-negative integer. Let $\epsilon(m)$ be defined by $\epsilon(m) = 2$ if $m > 0$ and even, $\epsilon(m) = 1$ if m is odd, and $\epsilon(0) = 0$. A decomposition for general $V^{\otimes n}$ is as follows.

Proposition 0.1. *There is a decomposition over the Steenrod algebra*

$$V^{\otimes n} \cong \bigoplus_{\substack{0\leq i_l\leq \frac{i_{l-1}-\epsilon(i_{l-1})}{2}\\ i_{l-1}>\epsilon(i_{l-1})\\ 1\leq l\leq s\\ 0\leq i_s\leq \epsilon(i_s)}} k_I L_2^{a_I} P_{2^s}^{\epsilon(i_s)}\cdot P_{2^{s-1}}^{\epsilon(i_{s-1})}\cdots P_2^{\epsilon(i_1)} P_1^{\epsilon(n)},$$

where

$$a_I = 1/2(n-2^s\epsilon(i_s)-2^{s-1}\epsilon_(i_{s-1})-\cdots-2\epsilon(i_1)-\epsilon(n)$$

and

$$k_I=\binom{\frac{n-\epsilon(n)}{2}}{i_1}\binom{\frac{i_1-\epsilon(i_1)}{2}}{i_2}\cdots\binom{\frac{i_{s-1}-\epsilon(i_{s-1})}{2}}{i_s}2^{(n-(\epsilon(n)+i_1+\epsilon(i_1)+\cdots+i_{s-1}+\epsilon(i_{s-1})+i_s))/2}.$$

(This general formula, which looks complicated, can be elemetarily worked out by induction and using $P_{2^s}^3=(2L_2^s+P_{2^{s+1}})P_{2^s}$.)

Theorem 0.2. *The decomposition in Proposition 0.1 is a complete decomposition over the Steenrod algebra, that is, each factor is indecomposable over the Steenrod algebra.*

Note: The formula in Proposition 0.1 is a (functorial) decomposition for any two-cell complex X localized at 2. If X is a suspension of $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ or $\mathbb{K}P^2$, then this is a complete decomposition because it is a complete decomposition over the Steenrod algebra. For a general two-cell complex (say X is a wedge of two spheres), it could be decomposed further. However it looks a good question whether this gives a complete decomposition when X is an indecomposable 2-local two-cell complex. (I guess that one may look at general homology theory instead of mod 2 homology if

X is indecomposable but with trivial Steenrod operations.) The homology of each indecomposable factor is just a suspension of the following module:

$$P_{2^k}P_{2^{k-1}}\cdots P_1P_{2^{a_1}}P_{2^{a_2}}\cdots P_{2^{a_s}}$$

for some sequence $0\leq a_1<a_2<\cdots<a_s\leq k$ which is a submodule of $\Lambda_{2^{k+1}-1}(V)\otimes\Lambda_{2^{k+1}-1}(V)$. (Note: It looks a good question whether these are all geometrically realizable submodules of $\Lambda_{2^{k+1}-1}(V)^{\otimes 2}$. Geometrical realization of modules over the Steenrod algebra is one of very interesting topics in homotopy theory.) We can explain more about this module over the Steenrod algebra:

Let $M_k(X)=P_{2^k}P_{2^{k-1}}\cdots P_1$. If X is a suspension of $\mathbb{R}P^2$, then $M_k(X)$ is the exterior algebra generated by $Sq^1,Sq^2,Sq^4,\cdots,Sq^{2^k}$. If X is a suspension of $\mathbb{K}P^2$, then $M_k(X)$ is the exterior algebra generated by $Sq^8,Sq^{16},\cdots,Sq^{2^{k+3}}$.

A geometric realization of $M_k(X)$ can be taken the smallest retract of $X^{(2^{k+1}-1)}$ which contains the bottom cell. (Note: it is possible that there are different spaces realizing $M_k(X)$. For instance, X is a suspension of $\mathbb{R}P^2$ and $k=15$, a suspension of $\mathbb{K}P^2\wedge\mathbb{H}P^2\wedge\mathbb{C}P^2\wedge\mathbb{R}P^2$ is a different realizing space.) The canonical inclusion $\Sigma^{?}X^{(2^k-1)}\to X^{(2^{k+1}-1)}$ induces a spectrum which is a geometrical realization of the abelianizer of the Steenrod algebra if X is a suspension of $\mathbb{R}P^2$. This spectrum is different from $\mathbb{R}P^\infty$. When X is a suspension of $\mathbb{K}P^2$, we get a spectrum which cohomology is the exterior algebra generated by $Sq^8,Sq^{16},\cdots$ and so one-cell in each dimension of $8n$. (Note: all factors in $X^{(n)}$ are self-dual and so possibly we can construct spectrum which have cells from $-\infty$ to $+\infty$ using these factors.) It looks that we can construct 'new' spectra in this way, but the question to me is whether we can get any special properties for these spectra.

Note: In stable homotopy category, 2-local two-cell complexes $X \leftrightarrow$ 2-primary elements in the stable homotopy group $\pi_*^s(S^0)$ by the relation that X is taken as the homotopy cofiber of a map $\alpha\colon S^k \to S^0$. The decompositions above give that for each $\alpha \in \pi_*^s(S^0)$ there is a family of family of spaces with specific information in homology.

Now we give some remarks about this topic. One may like to see whether the decompositions for two-cell complexes can be extended to general case. This is tightly related to modular representation theory of the symmetric group. Roughly, we can figure out these decompositions because we can understand primitive projective modules over $\mathbb{Z}/2(S_n)$ corresponding to two-row Young diagrams. It is known that

> functorial decompositions of self smashes $\Leftrightarrow$ modular representation theory of symmetric groups $\Leftrightarrow$ functorial decompositions of tensor products.

Since modular representation theory of symmetric groups is still quite unknown, functorial decompositions of self smashes are unknown in general.

History Remarks: These decomposition formulas are given in one of my joint papers with Paul Selick. (It is still a preprint.) In 1980s, Mike Hopkins, Doug Ravenel, Jeff Smith and others have been to study some particular retracts of self smashes using representation theory. These retracts play a key role in Hopkins' nilpotence theorem. Fred Cohen and I have been to apply the formula $P^n(2)^{(3)} \simeq P^{3n-1}(2) \vee P^{3n-1}(2) \vee \Sigma^{3n-6}\mathbb{C}P^2 \wedge \mathbb{R}P^2$ to show that there are infinitely many $\mathbb{Z}/8$-summands in $\pi_*(P^n(2))$. (Barratt conjecture states that $8\pi_*(P^n(2)) = 0$. So if the Barratt conjecture is true (it is still quite open), 8 is the best possible exponent for $\pi_*(P^n(2))$.) The specific geometrical decomposition of $P^n(2)^{(7)}$ was given in my thesis using classical homotopy theory. When I met Jeff Smith in 1993 or so, he said that he is interested in the smallest factor of $P^n(2)^{(k)}$ that contains the bottom cell. (Somehow now we can tell all what he wanted.) There are several people have been to study splittings of $P^n(2)^{(k)}$ using representation theory. (So our methods or ideas are not new, but the result is new.) The multiplicity k_I in Proposition 0.1 is the multiplicity of primitive projective $\mathbb{Z}/2(S_n)$-modules corresponding to two-row Young diagrams. It turns out that this multiplicity was known in representation theory. Furthermore the multiplicity formula in odd prime cases is also known. This suggested that it should have a specific decomposition formula for $X^{(n)}$ when X is a p-local two-cell suspension for $p > 2$.

Another interesting question in homotopy theory is how to decompose a loop space into a product of other spaces. A paticular case is how to decompose $\Omega\Sigma X$. It turns out this question is related to modular representation theory of symmetric groups as well.

Let $\bar{V}$ be an n-dimensional $\mathbf{k}$-module generated by letters $x_1, \cdots, x_n$. Let Lie(n) be the submodule of $\bar{V}^{\otimes n}$ spanned by $\{[[x_{\sigma(1)}, x_{\sigma(2)}], \cdots, x_{\sigma(n)}] \in \bar{V}^{\otimes n} | \sigma \in S_n\}$. Roughly speaking, Lie$(n)$ is given by the set of general n-fold commutators. The symmetric group S_n acts on Lie(n) by permuting letters. So Lie(n) is a module over S_n. We have

functorial decompositions of $\Omega\Sigma X \Leftrightarrow$ modular representation theory of Lie(n)

$\Leftrightarrow$ functorial coalgebra decompositions of the tensor algebra $T(V)$.

It is known in homotopy theory that there is a tight relation between $\Omega\Sigma X$ and self-smashes. For instance,

$$[\Omega\Sigma X, \Omega\Sigma X] \cong [\Sigma\Omega\Sigma X, \Sigma X] \cong [\Sigma \bigvee_{n=1}^{\infty} X^{(n)}, \Sigma X],$$

where the last isomorphism is NOT a group isomorphism.

Question: Let X be a 2-local two-cell suspension with a nontrivial Steenrod operation. Is it true that

decompositions of $\Omega\Sigma X \Leftrightarrow$ coalgbera decompositions of $H_*(\Omega\Sigma X) = T(\bar{H}_*(X))$ over the Steenrod algebra?

If so, can we work out a specific complete decomposition.

For general X, we have the following question.

Question: How to determine the maximal (largest) projective S_n-submodule, denoted by $\mathrm{Lie}^{\max}(n)$, of $\mathrm{Lie}(n)$?

$\mathrm{Lie}^{\max}(n)$ is unique up to isomorphism. $\mathrm{Lie}(n)$ itself is projective if and only if $n \not\equiv 0 \bmod p$. In this case, $\mathrm{Lie}^{\max}(n) = \mathrm{Lie}(n)$. The most interesting case to us is $\mathrm{Lie}^{\max}(p^r)$. So far we only know $\mathrm{Lie}^{\max}(p^r)$ for $r \leq 3$ when $p = 2$ and $r = 1$ when $p > 2$. In other words $\mathrm{Lie}^{\max}(p^2)$ is still unknown for $p > 2$. For $p = 2$, it took a lot of calculations to determine $\mathrm{Lie}^{\max}(8)$.

Now we end this talk with following picture

- functorial decompositions of $X^{(n)} \Leftrightarrow$ representation theory of symmetric groups.
- restricted on individual spaces $\Leftrightarrow$ representation theory of $V^{\otimes n}$ over the general linear group.
- functorial decompositions of loop suspensions $\Leftrightarrow$ representation theory of $\mathrm{Lie}(n)$.
- restricted to individual spaces $\overset{?}{\Leftrightarrow}$ representation theory of $L_n(V)$ over the general linear group.

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