# REPRESENTATIONS AND HOMOTOPY THEORY

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This talk is intended to give some general ideas how to do representation in homotopy theory. One of general ideas in mathematics to classifying certain mathematical objects is 1) decompose an object into 'smallest factors' (indecomposable factors) and 2) study these smallest objects. Now given a space, say $X$, a natural question in homotopy theory is how to decompose $X$ as $Y \vee Z$ or $Y \times Z$, that is whether $X$ is homotopy equivalent to $Y \vee Z$ or $Y \times Z$ for some $Y$ and $Z$, where $Y \vee Z$ is the wedge of $Y$ and $Z$, that is the quotient of the disjoint union of $Y$ and $Z$ by identifying the base-points of $Y$ and $Z$. So there are two types of decompositions in homotopy theory. For instance $M_g \setminus \{ \text{ a point}\} \simeq \vee^g S^1$, where $M_g$ is the Riemann surface of genius $g$. (Note: In homotopy theory, which type of decompositions we want usually depends on which kind of spaces we want to look at. For instance, for $H$-spaces (e.g. topological groups), we ask Cartesian product decompositions. For co-$H$-spaces (e.g. $X = \Sigma Y = S^1 \wedge Y = S^1 \times Y/(S^1 \vee Y)$), we ask wedge decompositions.)

Let's consider wedge decompositions. Assume that $X \simeq Y \vee Z$. Then we have a map $f_Y : X \xrightarrow{\text{projection}} Y \xrightarrow{\text{inclusion}} X$. The map $f_Y$ is a homotopy idempotent, that is $f_Y \circ f_Y \simeq f_Y$. (Note: $f_Y \circ f_Y \neq f_Y$ in general) In other word the homotopy class $[f_Y]$ is an idempotent in $[X, X]$. Conversely, given a map $f \colon X \to X$ such that $f \circ f \simeq f$, one can construct a space $Y = \operatorname{hocolim}_f X$. Roughly speaking $\operatorname{hocolim}_f X$ is the space of 'stable fixed set' of the map $f$. The space $Y$ occurs to a homotopy retract of $X$ under a very weak assumption. This observation shows that homotopy retracts of $X \leftrightarrow$ idempotents in $[X, X]$. In other words one needs to study the set $[X, X]$.

Now let's consider the $n$-fold self smash product $X^{(n)}$. Two natural questions:

**1.** Can we find a functorial decomposition formula for $X^{(n)}$? In other words, decompose $X^{(n)}$ into $A(X) \vee B(X) \vee \cdots$, where $A$, $B$ are functors on $X$.

**2.** Given a particular $X$, what is the homology of $A(X)$, $B(X)$, $\cdots$?

Let's keep doing observations. Let the symmetric group $S_n$ act on $X^{(n)}$ by permuting positions. So for each $\sigma \in S_n$ there is a map $\sigma\colon X^{(n)} \to X^{(n)}$. This gives a function $\theta\colon S_n \to [X^{(n)}, X^{(n)}]$. *Suppose that $X = \Sigma Y$ is a suspension.* Then $[X^{(n)}, X^{(n)}]$ is an abelian group for $n \geq 2$, where the addition is given by $X^{(n)} \to X^{(n)} \vee X^{(n)} \xrightarrow{f \vee g} X^{(n)} \vee X^{(n)} \xrightarrow{\text{fold}} X^{(n)}$. So there is an extension $\theta\colon \mathbb{Z}(S_n) \to [X^{(n)}, X^{(n)}]$. Furthermore, *assume that $X$ is a $p$-local suspension*, that is, roughly speaking, $1/m\colon X \to X$ is defined for $m \not\equiv 0 \bmod p$. There is a further extension $\theta\colon Z_{(p)}(S_n) \to [X^{(n)}, X^{(n)}]$, where $Z_{(p)}$ is the set of $p$-local integers, that is the rational numbers $a/b$ with $b \not\equiv 0 \bmod p$. (Note: $[X^{(n)}, X^{(n)}]$ is semi-ring, that is, $h \circ (f+g) = h \circ f + h \circ g$ holds but $(f+g) \circ h \neq f \circ h + g \circ h$ in general.) The map $\theta$ is a morphism of semi-rings. In other words,

*$\mathbb{Z}_{(p)}(S_n)$ is the universal ring for $[X^{(n)}, X^{(n)}]$ where $X$ runs over all $p$-local suspensions.*

Now let $1 = \sum_\alpha e_\alpha$ is a decomposition of the identity into orthogonal idempotents in $\mathbb{Z}_{(p)}(S_n)$. For each $\alpha$, we have a map $e_\alpha\colon X^{(n)} \to X^{(n)}$ and so a space $e_\alpha(X) =$

$\mathrm{hocolim}_{e_\alpha} X^{(n)}$. Standard arguments in homotopy theory show that the composite

$$X^{(n)} \xrightarrow{\text{comultiplication}} \bigvee_\alpha X^{(n)} \to \bigvee_\alpha e_\alpha(X^{(n)}),$$

is a homotopy equivalence. Thus we have

> *decompositions of the identity in* $\mathbb{Z}_{(p)}(S_n)$ $\Rightarrow$ *functorial decompositions of* $X^{(n)}$ *for* $p$*-local suspensions* $X$.

Let $\bar{H}_*(X)$ be the mod $p$ homology of $X$. By the Künneth Theorem, we have $\bar{H}_*(X^{(n)}) = \bar{H}_*(X)^{\otimes n}$. Let $V$ denote $\bar{H}_*(X)$. Then $V$ is a vector space over $\mathbb{Z}/p$. By applying the homology functor $\bar{H}_*$, we have

> *functorial decompositions of* $X^{(n)}$ *for* $p$*-local suspensions* $X$ $\Rightarrow$ *functorial decompositions of the tensor product* $V^{\otimes n}$ *for vector spaces* $V$ *over* $\mathbb{Z}/p$.

Algebraic arguments shows that

> *functorial decompositions of* $V^{\otimes n}$ $\Leftrightarrow$ *decompositions of the identity in* $\mathbb{Z}/p(S_n)$.

Modular representation theory of symmetric groups shows that

> *decompositions of the identity in* $\mathbb{Z}/p(S_n)$ $\Leftrightarrow$ *decompositions of the identity in* $\mathbb{Z}_{(p)}(S_n)$.

Thus we answer the first question as follows:

> *functorial decompositions of* $X^{(n)}$ *for* $p$*-local suspensions* $X$ $\Leftrightarrow$ *functorial decompositions of* $n$*-fold tensor products over* $\mathbb{Z}/p$ $\Leftrightarrow$ *modular representation theory of* $S_n$.

Now let's consider the second question, that is how to know the homology of functorial factors of $X^{(n)}$ for a given $X$. (Again we assume that $X$ is a $p$-local suspension.) Let $V = \bar{H}_*(X)$ be a given vector space over $\mathbb{Z}/p$. Let $W$ be a general vector space over $\mathbb{Z}/p$. By the statement above, this question is reduced to an algebraic question which is given as follows:

> *Let $W^{\otimes n} \cong A(W) \oplus B(W) \oplus \cdots$ be a functorial decomposition. What are $A(V)$, $B(V)$, $\cdots$ for a given vector space $V$?*

Now we do observations in algebra. Let $\mathbf{k}$ be the ground field. (In our case $\mathbf{k} = \mathbb{Z}/p$.) Modular representation theory of symmetric groups shows that

> *decompositions of the identity in $\mathbf{k}(S_n)$ only depends on the characteristic of $\mathbf{k}$.*

Thus we have

> *Let $\mathbf{k}$ be any extension field of $\mathbb{Z}/p$, that is $\mathbf{k}$ is of characteristic $p$. Then $A^{\mathbf{k}}(V \otimes_{\mathbb{Z}/p} \mathbf{k}) \cong A(V) \otimes_{\mathbb{Z}/p} \mathbf{k}$, where $A^{\mathbf{k}}$ is the corresponding functor of $A$ by changing the ground field $\mathbb{Z}/p$ to be $\mathbf{k}$.*

In other words, we can change the background field $\mathbb{Z}/p$ to be any field $\mathbf{k}$ of characteristic $p$. So we assume that $V$ is a given vector space over $\mathbf{k}$ and we intend to know $A(V) = A^{\mathbf{k}}(V)$.

Observe that $A$ is a functor on vector spaces. So any endomorphism $\phi\colon V \to V$ induces an endomorphism $A(\phi)\colon A(V) \to A(V)$. In particular, $A(V)$ is a module over the general linear group $GL_m(\mathbf{k})$, where $m = \dim V$. Thus we have

*The evaluation of functorial decompositions of the n-fold tensor product on* $V^{\otimes n}$ $\Rightarrow$ *decompositions of* $V^{\otimes n}$ *over* $GL_m(\mathbf{k})$.

From the representation theory, it turns out that the inverse of this statement is also true if the order of the ground field $\mathbf{k}$ is sufficiently large. More precisely

*Let* $\mathbf{k}$ *be a field of characteristic* $p$ *and of sufficiently large order. Suppose that* $V^{\otimes n} \cong M_1 \oplus M_2 \oplus \cdots$ *be a decomposition of* $V^{\otimes n}$ *over* $GL_m(\mathbf{k})$. *Then there is a functorial decomposition* $W^{\otimes n} \cong A(W) \oplus B(W) \oplus \cdots$ *such that the evaluation* $A(V) = M_1,\ B(V) = M_2,\ \cdots$ .

In other words, we can find all $A(V)$ by taking a large field $\mathbf{k}$ and doing decompositions of $V^{\otimes n}$ over $GL_m(\mathbf{k})$. Let's take a particular example to see how these machines work well.

Assume that $p = 2$ and $\dim V = 2$. (In this case, $X$ is a $p$-local two-cell complex. For instance, $X$ is a suspension of $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$, $\mathbb{K}P^2$. In general, $X$ can be the homotopy cofiber of **ANY** elements in the stable homotopy groups of spheres.) Let $\{u, v\}$ be a basis for $V$. In this case, we can actually give a complete decomposition of $V^{\otimes n}$ over $GL_2(\mathbf{k})$.

It is easy to show that $V$ and $V^{\otimes 2}$ are indecomposable over $GL_2(\mathbf{k})$. The first decomposable case is:

$$V^{\otimes 3} \cong L_2(V) \otimes V \oplus L_2(V) \otimes V \oplus P_2(V) \otimes V = (2L_2(V) \oplus P_2(V)) \otimes V,$$

as modules over $GL_2(\mathbf{k})$, where $L_n(W) = \{[[a_1, a_2], \cdots, a_n] \in W^{\otimes n}\}$ is the set of $n$-fold Lie elements and $P_{2^s}(W) = \{a^{2^s} | a \in W\}$.

(Note: 1) $\dim L_2(V) = 1$. 2) $\dim P_{2^s}(V) = 2$ with a basis $\{u^{2^s}, v^{2^s}\}$. 3) For any module $W$, there is a functorial decomposition $W^{\otimes 3} \cong L_3(W) \oplus L_3(W) \oplus M_3(W)$. But $L_3(W) \cong L_2(W) \otimes W$ if and only if $\dim W \leq 2$. In other words, the general functorial decomposition of $W^{\otimes 3}$ can not be written down as the form above, that is this is the special property for the case $\dim V = 2$.)

Since $\dim P_{2^s}(V) = 2$, we have

$P_{2^s}(V)^{\otimes 3} \cong (2L_2(P_{2^s}(V)) \oplus P_2(P_{2^s}(V))) \otimes P_{2^s}(V) = (2L_2(V)^{\otimes n} \oplus P_{2^{s+1}}(V))) \otimes P_{2^s}(V)$

as modules over $GL_2(\mathbf{k})$. It turns out that this gives an algorithm to completely determine decompositions of $V^{\otimes n}$ over the Steenrod algebra. Let's look at one more example ($n = 7$). Write $MN$ for $M \otimes N$, $M + N$ for $M \oplus N$, $L_2$ for $L_2(V)$ and $P_{2^s}$ for $P_{2^s}(V)$.

$V^7 = V^3V^3V = ((2L_2 + P_2)V)^2V = (2L_2 + P_2)^2V^3 = (2L_2 + P_2)^3V$

$= 8L_2^3V + 12L_2^2P_2V + 6L_2P_2^2V + P_2^3V$

$= 8L_2^3V + 12L_2^2P_2V + 6L_2P_2^2V + (2L_2^2 + P_4)P_2V$

$= 8L_2^3V + 14L_2^2P_2V + 6L_2P_2^2V + P_4P_2V.$

This show that $X^{(7)}$ has four factors (multiplicities are 8, 14, 6 and 1). By using homotopy theory, when $X = P^n(2) = \Sigma^{n-2}\mathbb{R}P^2$ with $n \geq 3$, we can actually show that

$$(P^n(2))^{(7)} \simeq \mathbf{H}P^2 \wedge \mathbf{C}P^2 \wedge P^{7n-12}(2) \vee \bigvee^{6} (\mathbf{C}P^2)^{(2)} \wedge P^{7n-9}(2) \vee \bigvee^{14} \mathbf{C}P^2 \wedge P^{7n-6}(2) \vee \bigvee^{8} P^{7n-3}(2).$$

(Note: This formula gives an interesting relation among different projective planes.)

Let $m$ be a non-negative integer. Let $\epsilon(m)$ be defined by $\epsilon(m) = 2$ if $m > 0$ and even, $\epsilon(m) = 1$ if $m$ is odd, and $\epsilon(0) = 0$. A decomposition for general $V^{\otimes n}$ is as follows.

*There is a decomposition over $Gl_2(\mathbf{k})$*

$$V^{\otimes n} \cong \bigoplus_{\substack{0 \le i_l \le \frac{i_{l-1} - \epsilon(i_{l-1})}{2} \\ i_{l-1} > \epsilon(i_{l-1}) \\ 1 \le l \le s \\ 0 \le i_s \le \epsilon(i_s)}} k_I L_2^{a_I} P_{2^s}^{\epsilon(i_s)} \cdot P_{2^{s-1}}^{\epsilon(i_{s-1})} \cdots P_2^{\epsilon(i_1)} P_1^{\epsilon(n)},$$

*where*

$$a_I = \frac{1}{2}(n - 2^s \epsilon(i_s) - 2^{s-1} \epsilon_(i_{s-1}) - \cdots - 2\epsilon(i_1) - \epsilon(n))$$

*and*

$$k_I = \binom{\frac{n-\epsilon(n)}{2}}{i_1} \binom{\frac{i_1 - \epsilon(i_1)}{2}}{i_2} \cdots \binom{\frac{i_{s-1} - \epsilon(i_{s-1})}{2}}{i_s} 2^{(n - (\epsilon(n) + i_1 + \epsilon(i_1) + \cdots + i_{s-1} + \epsilon(i_{s-1}) + i_s))/2}.$$

(This general formula, which looks complicated, can be elementarily worked out by induction and using $P_{2^s}^3 = (2L_2^s + P_{2^{s+1}})P_{2^s}$.) Using arguments in homotopy theory, we have the following surprised result.

**Theorem 0.1.** *The decomposition above is a complete decomposition over the Steenrod algebra if $X$ has a non-trivial Steenrod operation, that is, each factor is indecomposable over the Steenrod algebra. In particular, this formula gives a specific information about the homology of each homotopy indecomposable factors of $X^{(n)}$ (as an* **individual space***) when $X$ is a suspension of a projective plane.*

**Note:** In general, if $A(X)$ is a functorial indecomposable homotopy retract of $X^{(n)}$, it is possible that $A(X)$ has a further decomposition for a given space $X$. (e.g. $X$ is a wedge of spheres.) This theorem shows that when $X$ is a suspension of a projective plane, functorial decompositions of $X^{(n)}$ $\Leftrightarrow$ individual space decompositions of $X^{(n)}$. Thus modular representation theory of the symmetric groups gives not only functorial decompositions for $X^{(n)}$ but also individual space decompositions in the cases above. In these cases, the homology of each indecomposable factor is just a suspension of the following module:

$$P_{2^k}P_{2^{k-1}}\cdots P_1 P_{2^{a_1}}P_{2^{a_2}}\cdots P_{2^{a_s}}$$

for some sequence $0\leq a_1<a_2<\cdots<a_s\leq k$ which is a submodule of $\Lambda_{2^{k+1}-1}(V)\otimes\Lambda_{2^{k+1}-1}(V)$. (Note: It looks a good question whether these are all geometrically realizable submodules of $\Lambda_{2^{k+1}-1}(V)^{\otimes 2}$. Geometrical realization of modules over the Steenrod algebra is one of very interesting topics in homotopy theory.) We can explain more about this module over the Steenrod algebra:

Let $M_k(X)=P_{2^k}P_{2^{k-1}}\cdots P_1$. If $X$ is a suspension of $\mathbb{R}P^2$, then $M_k(X)$ is the exterior algebra generated by $Sq^1, Sq^2, Sq^4, \cdots, Sq^{2^k}$. If $X$ is a suspension of $\mathbb{K}P^2$, then $M_k(X)$ is the exterior algebra generated by $Sq^8, Sq^{16}, \cdots, Sq^{2^{k+3}}$.

A geometric realization of $M_k(X)$ can be taken the smallest retract of $X^{(2^{k+1}-1)}$ which contains the bottom cell. (Note: it is possible that there are different spaces realizing $M_k(X)$. For instance, $X$ is a suspension of $\mathbb{R}P^2$ and $k = 15$, a suspension of $\mathbb{K}P^2 \wedge \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$ is a different realizing space.) The canonical inclusion $\Sigma^? X^{(2^k-1)} \to X^{(2^{k+1}-1)}$ induces a spectrum which is a geometrical realization of the abelianizer of the Steenrod algebra if $X$ is a suspension of $\mathbb{R}P^2$. This spectrum is different from $\mathbb{R}P^\infty$. When $X$ is a suspension of $\mathbb{K}P^2$, we get a spectrum which cohomology is the exterior algebra generated by $Sq^8, Sq^{16}, \cdots$ and so one-cell in each dimension of $8n$. (Note: all factors in $X^{(n)}$ are self-dual and so possibly we can construct spectrum which have cells from $-\infty$ to $+\infty$ using these factors.) It looks that we can construct 'new' spectra in this way, but the question to me is whether we can get any special properties for these spectra.

**Note:** In stable homotopy category, 2-local two-cell complexes $X \leftrightarrow$ 2-primary elements in the stable homotopy group $\pi_*^s(S^0)$ by the relation that $X$ is taken as the homotopy cofiber of a map $\alpha\colon S^k \to S^0$. The decompositions above give that for each $\alpha \in \pi_*^s(S^0)$ there is a family of family of spaces with specific information in homology.

Now we give some remarks about this topic. One may like to see whether the decompositions for two-cell complexes can be extended to general case. This is tightly related to modular representation theory of the symmetric group. Since modular representation theory of symmetric groups is still quite unknown, functorial decompositions of self smashes are unknown in general. On the other hand, it looks a good question:

*Can we get any 'new' information about modular representation theory of the symmetric groups by studying functorial decompositions of self smashes?*

**History Remarks:** These decomposition formulas are given in one of my joint papers with Paul Selick. (It is still a preprint.) In 1980s, Mike Hopkins, Doug Ravenel, Jeff Smith and others have been to study some particular retracts of self smashes using representation theory. These retracts play a key role in Hopkins' nilpotence theorem. Fred Cohen and I have been to apply the formula $P^n(2)^{(3)} \simeq P^{3n-1}(2) \vee P^{3n-1}(2) \vee \Sigma^{3n-6}\mathbb{C}P^2 \wedge \mathbb{R}P^2$ to show that there are infinitely many $\mathbb{Z}/8$-summands in $\pi_*(P^n(2))$. (Barratt conjecture states that $8\pi_*(P^n(2)) = 0$. So if the Barratt conjecture is true (it is still quite open), 8 is the best possible exponent for $\pi_*(P^n(2))$.) The specific geometrical decomposition of $P^n(2)^{(7)}$ was given in my thesis using classical homotopy theory. When I met Jeff Smith in 1993 or so, he said that he is interested in the smallest factor of $P^n(2)^{(k)}$ that contains the bottom cell. (Somehow now we can tell all what he wanted.) There are several people have been to study splittings of $P^n(2)^{(k)}$ using representation theory. (So our methods or ideas are not new, but the result is new.) The multiplicity $k_I$ in Theorem 0.1 is the multiplicity of primitive projective $\mathbb{Z}/2(S_n)$-modules corresponding to two-row Young diagrams. This was known in representation theory. Furthermore there is a similar multiplicity formula in odd prime cases. This suggested that it should have a specific decomposition formula for $X^{(n)}$ when $X$ is a $p$-local two-cell suspension for $p > 2$.

Another interesting question in homotopy theory is how to decompose a loop space into a product of other spaces. A particular case is how to decompose $\Omega\Sigma X$. It turns out this question is related to modular representation theory of symmetric groups as well.

Let $\bar{V}$ be an $n$-dimensional $\mathbf{k}$-module generated by letters $x_1, \cdots, x_n$. Let $\mathrm{Lie}(n)$ be the submodule of $\bar{V}^{\otimes n}$ spanned by $\{[[x_{\sigma(1)}, x_{\sigma(2)}], \cdots, x_{\sigma(n)}] \in \bar{V}^{\otimes n} | \sigma \in S_n\}$. Roughly speaking, $\mathrm{Lie}(n)$ is given by the set of general $n$-fold commutators. The symmetric group $S_n$ acts on $\mathrm{Lie}(n)$ by permuting letters. So $\mathrm{Lie}(n)$ is a module over $S_n$. We have

*functorial decompositions of* $\Omega\Sigma X \Leftrightarrow$ *modular representation theory of* $\mathrm{Lie}(n)$

$\Leftrightarrow$ *functorial coalgebra decompositions of the tensor algebra* $T(V)$.

It is known in homotopy theory that there is a tight relation between $\Omega\Sigma X$ and self-smashes. For instance,

$$[\Omega\Sigma X, \Omega\Sigma X] \cong [\Sigma\Omega\Sigma X, \Sigma X] \cong [\Sigma \bigvee_{n=1}^{\infty} X^{(n)}, \Sigma X],$$

where the last isomorphism is NOT a group isomorphism.

**Question:** *Let* $X$ *be a* 2*-local two-cell suspension with a nontrivial Steenrod operation. Is it true that*

*decompositions of* $\Omega\Sigma X \Leftrightarrow$ *coalgbera decompositions of* $H_*(\Omega\Sigma X) = T(\bar{H}_*(X))$

*over the Steenrod algebra?*

*If so, can we work out a specific complete decomposition.*

For general $X$, we have the following question.

**Question:** How to determine the maximal (largest) projective $S_n$-submodule, denoted by $\mathrm{Lie}^{\max}(n)$, of $\mathrm{Lie}(n)$?

$\mathrm{Lie}^{\max}(n)$ is unique up to isomorphism. $\mathrm{Lie}(n)$ itself is projective if and only if $n \not\equiv 0 \bmod p$. In this case, $\mathrm{Lie}^{\max}(n) = \mathrm{Lie}(n)$. The most interesting case to us is $\mathrm{Lie}^{\max}(p^r)$. So far we only know $\mathrm{Lie}^{\max}(p^r)$ for $r \leq 3$ when $p = 2$ and $r = 1$ when $p > 2$. In other words $\mathrm{Lie}^{\max}(p^2)$ is still unknown for $p > 2$. For $p = 2$, it took a lot of calculations to determine $\mathrm{Lie}^{\max}(8)$.

(Note: Although $\mathrm{Lie}^{\max}(n)$ is not completely known, we can get a lot of information on it using homotopy theory.)

Now we end this talk with following picture

- *functorial decompositions of* $X^{(n)}$ $\Leftrightarrow$ *representation theory of symmetric groups.*
- *restricted on individual spaces* $\Leftrightarrow$ *representation theory of* $V^{\otimes n}$ *over the general linear group.*
- *functorial decompositions of loop suspensions* $\Leftrightarrow$ *representation theory of* $\mathrm{Lie}(n)$.
- *restricted to individual spaces* $\overset{?}{\Leftrightarrow}$ *representation theory of* $L_n(V)$ *over the general linear group.*