

COMBINATORIAL DESCRIPTIONS OF HOMOTOPY GROUPS

JIE WU

0.1. Simplicial Groups. Let \mathcal{O} be the category of finite ordered sets and ordered functions (that is $f(x) \leq f(y)$ if $x \leq y$). A *simplicial set* X means a cofunctor from \mathcal{O} to the category of sets. This is equivalent to the usual definition of simplicial sets in the following sets:

Let X be a cofunctor from \mathcal{O} to sets. Let $X_n = X(\{0, 1, \dots, n\})$. The face function $d_j: X_n \rightarrow X_{n-1}$ is induced by the inclusion $d^j: \{0, \dots, n-1\} \rightarrow \{0, \dots, n\}$ with $d^j(i) = i$ for $i < j$, $d^j(i) = i + 1$ for $i \geq j$ for $0 \leq j \leq n$. The degeneracy function $s_j: X_n \rightarrow X_{n+1}$ is induced by $s^j: \{0, \dots, n+1\} \rightarrow \{0, \dots, n\}$ with $s^j(i) = i$ for $i \leq j$ and $s^j(i) = i - 1$ for $i > j$ for $0 \leq j \leq n$. The simplicial identities holds for d_i 's and s_j 's. Conversely, any simplicial set defines uniquely a cofunctor from \mathcal{O} to sets.

It is well-known that the category of simplicial sets is equivalent to the category of *CW*-complexes. So sometimes a space means a simplicial set in homotopy theory. **Note.** A difference between simplicial set and simplicial complex is: a simplicial set means a cofunctor from \mathcal{O} to sets and a simplicial complex means a cofunctor from the category of finite ordered sets and strictly ordered functions (that is $f(x) < f(y)$ if $x < y$) to the category of sets. In other words, there are no s_j 's for simplicial complexes.

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In general, let \mathcal{C} be any category. A simplicial object over \mathcal{C} means a cofunctor from \mathcal{O} to \mathcal{C} . In particular, if \mathcal{C} is the category of groups, then a simplicial object over \mathcal{C} is called a *simplicial group*. In other words, a simplicial group G means a sequence of groups $\{G_n\}_{n \geq 0}$ with homomorphisms $d_i: G_n \rightarrow G_{n-1}$ and $s_i: G_n \rightarrow G_{n+1}$, for $0 \leq i \leq n$, that satisfy the simplicial identities.

Roughly speaking simplicial groups are topological groups (Note: any loop space is (weak) homotopy equivalent to a topological group and any topological group is a loop space.)

Since simplicial sets (groups) are just cofunctors, any ‘functorial statements’ on sets (groups) canonically extend to statements on spaces (loop spaces). Let’s take an example to see how this works.

Example. By the van-Kampen Theorem, we have $B(G_1 * G_2) \simeq BG_1 \vee BG_2$, where $G_1 * G_2$ is the free product of G_1 and G_2 . Since this is ‘functorial’, one has $B(G_1 * G_2) \simeq BG_1 \vee BG_2$ for any simplicial groups G_1 and G_2 , that is $G_1 * G_2 \simeq \Omega(BG_1 \vee BG_2)$ for any simplicial groups G_1 and G_2 .

Consider the canonical quotient homomorphism $q: G_1 * G_2 \rightarrow G_1 \times G_2$. It is easy to show that the kernel of q is a free group (freely) generated by $[x, y]$ for $x \in G_1$ and $y \in G_2$, that is there is a natural isomorphism

$$F(G_1 \wedge G_2) \rightarrow \text{Ker}(q),$$

where $F(X)$ is the (reduced) free group generated by a pointed set X , that is the base point $*$ = 1 in $F(X)$. Thus we have short exact sequence of simplicial groups

$$F(G_1 \wedge G_2) \hookrightarrow G_1 * G_2 \twoheadrightarrow G_1 \times G_2.$$

This is fibration because it is a group extension. Since there is cross-section (simplicial map) $G_1 \times G_2 \rightarrow G_1 * G_2$, there is a decomposition as spaces (simplicial sets):

$$G_1 * G_2 \cong F(G_1 \wedge G_2) \times G_1 \times G_2.$$

Let $X = BG_1$ and $Y = BG_2$. Then this splitting is:

$$\Omega(X \vee Y) \simeq \Omega\Sigma(\Omega X \wedge \Omega Y) \times \Omega X \times \Omega Y$$

for any path-connected spaces X and Y . This formula is called the Hilton-Milnor Theorem or the Hilton-Milnor type theorem. When X and Y are suspensions, this statement is the well-known Hilton-Milnor Theorem. B. Gray and K. A. Hardie generalized the Hilton-Milnor Theorem as above by using geometric arguments, respectively in the early of 1970s. Berstein (published in Trans AMS 1965 and later A. Clark and L. Smith (Pacific J. Math.) in 1968, E. Katz (Proc. AMS) in 1976) studied the homology of the free product of simplicial groups. By using simplicial groups, we can give a very short (and different) proof of the Hilton-Milnor theorem as above. (The homology of the free product can be easily determined by looking this geometric decomposition.)

Note. Let G_0 be a simplicial subgroup of G_1 and G_2 and let $G_1 *_{G_0} G_2$ be the free product over G_0 . Then $G_1 *_{G_0} G_2 \simeq \Omega(BG_1 \cup_{BG_0} BG_2)$.

Question. Can we give a generalization of the Hilton-Milnor theorem by considering free products with amalgamations?

0.2. Combinatorial Description of $\pi_*(S^3)$. Let X be a simplicial group. The homotopy group $\pi_*(X)$ is defined by $\pi_*(X) = \pi_*(|X|)$, that is the homotopy groups of the geometric realization of X . For a simplicial group G , the homotopy groups $\pi_*(G)$ can be determined as follows:

The Moore chain complex NG is defined by

$$N_n(G) = \bigcap_{j>0} \text{Ker}(d_j: G_n \rightarrow G_{n-1}).$$

Then NG with $d_0|_{NG}: N_n(G) \rightarrow N_{n-1}(G)$ is a (non-commutative in general) chain complex, that is $d_0|_{N_n(G)} \circ d_0|_{N_{n+1}(G)} = 1$ is a trivial homomorphism.

Moore's theorem is:

$$\pi_n(G) = H_n(NG, d_0),$$

more precisely

$$\pi_n(G) = \bigcap_{j \geq 0} \text{Ker}(d_j: G_n \rightarrow G_{n-1}) / d_0 \left(\bigcap_{j > 0} \text{Ker}(d_j: G_{n+1} \rightarrow G_n) \right).$$

For a simplicial group G , we write $\mathcal{Z}_n(G)$, called n -cycles, for $\bigcap_{j \geq 0} \text{Ker}(d_j: G_n \rightarrow G_{n-1})$ and $\mathcal{B}_n(G)$, called n -boundaries, for $d_0(N_{n+1}(G))$. Thus

$$\pi_n(G) = \frac{n - \text{cycles}}{n - \text{boundaries}}.$$

Note. In theory, to study the homotopy groups of a given space X , we need to: 1) find a simplicial groups model for (loop spaces) of X , 2) determine n -cycles and 3) determine n -boundaries.

There are some simplicial group models. For instance,

- (1). Kan's G -construction is a loop functor;
- (2). Milnor's construction is a loop suspension functor. Milnor's $F(K)$ -construction is as follows: Let K be a pointed simplicial set. Then $F(K)$ is the free group generated by K modulo the single relation $* = 1$. $F(K) \simeq \Omega \Sigma K$.
- (3). Carlsson's construction: Let G be a simplicial group and let X be a pointed simplicial set. Let $G \otimes X$ be the (reduced) free product of G with indices over

X . Then $G \otimes X \simeq \Omega(BG \wedge X)$. For instance, $\mathbb{Z}_{(p)} \otimes X$ is the p -localization of $\Omega\Sigma X$.

(4). others like simplicial group models for $\Omega^\infty\Sigma^\infty$, $\Omega^n\Sigma^n$.

Let G be a simplicial group. Then it is easy to show that $\mathcal{B}_n(G)$ is a normal subgroup of G_n . Furthermore we have a general theorem:

Suppose that $\pi_0(G)$ acts trivially on $\pi_n(G)$. Then $\pi_n(G) = \mathcal{Z}_n/\mathcal{B}_n$ is contained in the center of $G_n/\mathcal{B}_n(G)$. If G_{n-1} is centerless, then $\pi_n(G)$ is equal to the center of $G_n/\mathcal{B}_n(G)$.

This one was first given in my thesis.

Example. $\pi_*(\Sigma K(\pi, 1))$. By the Carlsson's construction $F^\pi(S^1) \simeq \Omega(S^1 \wedge B\pi) = \Omega(\Sigma K(\pi, 1))$. So we just need to consider $\pi_*(F^\pi(S^1))$. By the definition of the 1-sphere, $S^1 = \Delta[1]/\partial\Delta[1]$, more precisely,

$S_0^1 = \{*\}$, $S_1^1 = \{*, \sigma\}$, $S_2^1 = \{*, s_0\sigma, s_1\sigma\}$, $S_3^1 = \{*, s_2s_1\sigma, s_2s_0\sigma, s_1s_0\sigma\}$, in general

$$S_{n+1}^1 = \{*, x_0, \dots, x_n\},$$

where $x_i = s_n \cdots \hat{s}_i \cdots s_0\sigma$.

It follows that $F^\pi(S^1)_n$ is the n -fold free product of π , that is $F^\pi(S^1)_n = \pi_{x_0} * \cdots * \pi_{x_{n-1}} = \pi_0 * \cdots * \pi_{n-1}$, where $\pi_i = \pi_{x_i} = \pi$.

Let's compute 2-cycles \mathcal{Z}_2 . By definition,

$$\mathcal{Z}_2 = \text{Ker}(d_0) \cap \text{Ker}(d_1) \cap \text{Ker}(d_2).$$

By the simplicial identities, $d_0s_0\sigma = \sigma$, $d_0s_1\sigma = *$. Thus $d_0: F^\pi(S^1)_2 = \pi_0 * \pi_1 \rightarrow F_1^\pi = \pi$ is the projection, that is $d_0|_{\pi_0}$ is the identity map and $d_0|_{\pi_1}$ is the trivial map.

Similarly, $d_2 s_0 \sigma = 1$ and $d_2 s_1 \sigma = \sigma$. It follows that $d_2: \pi_0 * \pi_1 \rightarrow \pi$ is given by $d_2|_{\pi_0}$ is trivial and $d_2|_{\pi_1}$ is the identity map. Thus

$\text{Ker } d_0 \cap \text{Ker } d_2$ is the kernel of the canonical quotient $\pi * \pi \rightarrow \pi \times \pi$ and so $\text{Ker } d_0 \cap \text{Ker } d_1$ is isomorphic to $F(\pi_0 \wedge \pi_1)$, the subgroup of $\pi_0 * \pi_1$ generated by the commutators $[a, b]$ for $a \in \pi_0$ and $b \in \pi_1$.

Since $d_1 s_0 \sigma = \sigma$ and $d_1 s_1 \sigma = \sigma$, $d_1: \pi_0 * \pi_1 \rightarrow \pi$ is induced by $d_1|_{\pi_0} = d_1|_{\pi_1} = \text{id}$. Thus there is a short exact sequence

$$\mathcal{Z}_2 \hookrightarrow F(\pi \wedge \pi) \xrightarrow{q} \Gamma_2 \pi,$$

where $q(a \wedge b) = [a, b]$ in π for $a \in \pi_0 = \pi$ and $b \in \pi_1 = \pi$.

We can determine n -boundaries \mathcal{B}_n as follows:

Let $\{g^{(\alpha)} | \alpha \in J\}$ be a set of generators for π and so π_j is generated by $\{g_j^{(\alpha)} | \alpha \in J\}$. Then \mathcal{B}_{n+1} is the normal subgroup of $F_{n+1}^\pi = \pi_0 * \cdots * \pi_n$ generated by

$$\beta^t(y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \dots, y_{i_t}^{(\alpha_t)\epsilon_t}),$$

where

- 1) $\epsilon_j = \pm 1$;
- 2) $-1 \leq i_s \leq n$;
- 3) all integers in $\{-1, 0, 1, 2, \dots, n\}$ appear as at least one of the integers i_s ;
- 4) $y_{-1}^{(\alpha)} = g_0^{(\alpha)-1}$, $y_j = g_j^{(\alpha)} g_{j+1}^{(\alpha)-1}$ for $1 \leq j \leq n-1$ and $y_n^{(\alpha)} = g_n^{(\alpha)}$;
- 5) for each $t \geq n+2$, β^t runs over all of the bracket arrangements of weight t .

For instance, \mathcal{B}_2 is generated by elements like $[[a_0, b_0 b_1^{-1}], c_1]$ for $a, b, c \in \pi$ and \dots .

By our general theorem, this gives a combinatorial description of $\pi_n(F^\pi(S^1))$ as the center of $F^\pi(S^1)_n/\mathcal{B}_n$ for $n \neq 2$. For $\pi_2(F^\pi(S^1)) = \pi_3(\Sigma K(\pi, 1))$ is just $\mathcal{Z}_2/\mathcal{B}_2$.

Question. Determine $\pi_2(F^\pi(S^1)) = \pi_3(\Sigma K(\pi, 1))$.

Note: $\pi_3(\Sigma K(\pi, 1))$ was determined by Brown-Loday using other methods so-called tensor products of non-abelian groups. It would be interesting if we can use simplicial groups to get Brown-Loday's theorem. (Because we also have a general description of the higher homotopy groups. This may leads to give a 'generalization' of their 'tensor products' of non-commutative groups to give a possible determination of higher homotopy groups.)

Note. By the Kan-Thurston, for any path-connected space X , there is a group π and a map $f: K(\pi, 1) \rightarrow X$ such that f is a homology equivalence. Thus $\Sigma f: \Sigma K(\pi, 1) \rightarrow \Sigma X$ is a homotopy equivalence. In some sense, the determination of $\pi_*(\Sigma K(\pi, 1))$ is just the determination of the homotopy groups of suspensions. But the group π depends on X and is complicated in general. For instance, if X is simply connected, say S^n with $n > 1$, then π has to be a perfect group. Also it does not seem that π is uniquely determined by X . On the other hand, assume that π and π' are two different groups with homology equivalences $f: K(\pi, 1) \rightarrow X$ and $f': K(\pi', 1) \rightarrow X$. Then $\Sigma K(\pi, 1) \simeq \Sigma K(\pi', 1)$. In particular, $\pi_*(F^\pi(S^1)) \cong \pi_*(F^{\pi'}(S^1))$. Our combinatorial description of the homotopy groups depends on the choice of the group π , but the homotopy groups are independent on the choice of π . This suggests that there might be certain relations between our combinatorial descriptions of the homotopy groups and the homology of π . (But to get the same homotopy groups, we need more than the homology of π , i.e., we need homology equivalence $K(\pi, 1) \longrightarrow X \longleftarrow K(\pi', 1)$.)

(Note: By looking at the construction, $F^\pi(S^1)_n$ is the n -fold self coproduct of π and $B\pi_n = K(\pi, 1)_n$ is the n -fold product of π . There is a canonical quotient from $F^\pi(S^1)_n \rightarrow B\pi_n$. But this map is NOT a simplicial map in general. On the other hand, there is a canonical simplicial map $: B\pi = K(\pi, 1) \rightarrow F^\pi(S^1) = \Omega\Sigma K(\pi, 1)$.)

By taking $\pi = \mathbb{Z}$, we have the following theorem: (Note: $F^{\mathbb{Z}}(S^1) = F(S^1) = \Omega S^2$ and so $F(S^1)_n$ is the free group generated by x_0, \dots, x_{n-1} or y_0, \dots, y_{n-1} , where $y_i = x_{i-1}x_i^{-1}$.)

$\pi_{n+2}(S^2) = \pi_{n+1}(F(S^1))$ is the center of the group W_{n+2} generated by y_0, \dots, y_n modulo the relations: $\beta^t(z_1^{\epsilon_1}, z_2^{\epsilon_2}, \dots, z_t^{\epsilon_t})$, where

- 1) $\epsilon_j = \pm 1$;
- 2) $z_s \in \{y_{-1}, y_0, y_1, \dots, y_n\}$ with $y_{-1} = (y_0 y_1 y_2 \cdots y_n)^{-1}$;
- 3) all elements in $\{y_{-1}, y_0, y_1, \dots, y_n\}$ appear as at least one of the z_j ;
- 4) for each $t \geq n + 2$, β^t runs over all of the commutator bracket arrangements of weight t .

Note. In $F(S^1)_{n+1}$ is generated by $x_i = s_n \cdots \hat{s}_i \cdots s_0 \sigma$. By letting $y_j = x_j x_{j+1}^{-1}$, the face homomorphism $d_j: F(S^1)_{n+1} = F(y_0, \dots, y_n) \rightarrow F(S^1)_n = F(y_0, \dots, y_{n-1})$ is given as follows:

For $j > 0$, d_j is just a coordinate projection, that is $d_j(y_i) = y_i$ for $i < j - 1$, $d_j(y_{j-1}) = 1$ and $d_j(y_i) = y_{i-1}$ for $i \geq j$. d_0 is given by: $d_0(y_0) = y_{-1} = (y_0 \cdots y_{n-1})^{-1}$ and $d_0(y_i) = y_{i-1}$ for $i > 0$.

We determined the generators for the Moore chain complex $NF(S^1) = \bigcap_{j>0} \text{Ker } d_j$ that is the intersection of the projections. By using this, then we have the generators for the boundaries $\mathcal{B}(F(S^1)) = d_0(NF(S^1))$.

0.3. Braid Group Actions. Let $F(S^1)$ be Milnor's construction of the simplicial 1-sphere S^1 . Then $F(S^1)_{n+1} = F(y_0, \dots, y_n)$ is a free group generated by letters y_0, \dots, y_n with the following simplicial structure

$$d_j y_k = \begin{cases} y_{k-1} & \text{for } j \leq k, \\ 1 & \text{for } j = k + 1, \\ y_k & \text{for } j > k + 1, \end{cases}$$

and

$$s_j y_k = \begin{cases} y_{k+1} & \text{for } j \leq k, \\ y_k y_{k+1} & \text{for } j = k + 1, \\ y_k & \text{for } j > k + 1, \end{cases}$$

for $0 \leq j \leq n + 1$, where $y_{-1} = (y_0 \cdots y_{n-1})^{-1}$ in $F(S^1)_n$. Let the braid group B_{n+1} act on $F(S^1)_{n+1} = F(y_0, \dots, y_n)$ in the usual way, that is,

$$\sigma_k(y_j) = \begin{cases} y_{k+1} & \text{if } j = k \\ y_{k+1}^{-1} y_k y_{k+1} & \text{if } j = k + 1 \\ y_j & \text{otherwise} \end{cases}$$

for $0 \leq k \leq n - 1$. Let σ_{-1} be an automorphism of $F(y_0, \dots, y_n)$ defined by

$$\sigma_{-1}(y_0) = y_0^{-1} y_{-1} y_0 = y_0^{-1} y_n^{-1} \cdots y_1^{-1} \quad \sigma(y_j) = y_j \quad \text{for } j > 0.$$

The subgroup of the automorphism group of $F(y_0, \dots, y_n)$ generated by σ_j for $-1 \leq j \leq n - 1$ is the braid group B_{n+2} . By direct calculation, we have

Proposition 0.1. *The following identities hold for the braid groups action on $F(S^1)$:*

$$(1) \quad d_j \sigma_k = \begin{cases} \sigma_{k-1} d_j & j \leq k \\ d_{k+2} & j = k+1 \\ d_{k+1} & j = k+2 \\ \sigma_k d_j & j > k+2 \end{cases}$$

$$(2) \quad s_j \sigma_k = \begin{cases} \sigma_{k+1} s_j & j \leq k \\ \sigma_{k+1} \circ \sigma_k \circ s_{k+2} & j = k+1 \\ \sigma_k \circ \sigma_{k+1} \circ s_{k+1} & j = k+2 \\ \sigma_k s_j & j > k+2 \end{cases}$$

Definition 0.2. A simplicial group G is called *braided* if there is a braid group B_{n+1} -action on G_n for each n such that the identities 1 and 2 are satisfied, where B_{n+1} is considered as the braid group generated by $\sigma_{-1}, \sigma_0, \dots, \sigma_{n-2}$.

One can show that:

If G is a braided simplicial group, then the loop space ΩG and the Postnikov systems of G are braided. In particular, the loop spaces and the Postnikov systems of $F(S^1)$ are braided.

Let K_n be the pure braided group, that is, K_n is that kernel of the canonical epimorphism $B_n \rightarrow \Sigma_n$. Let B_{n+1} be the subgroup of B_{n+2} generated by σ_j for $j \geq 0$. Recall that

$$\pi_{n+1}(F(S^1)) = \mathcal{Z}(F(S^1))_{n+1} / \mathcal{B}(F(S^1))_{n+1}.$$

Consider the actions of two braided groups B_{n+2} and B_{n+1} on $F(S^1)_{n+1}/\mathcal{B}(F(S^1))_{n+1}$.

We have

Theorem 0.3. *If $n \geq 2$, then in $F(S^1)_{n+1}/\mathcal{B}(F(S^1))_{n+1}$,*

1) *the fixed set of the pure braided group K_{n+1} -action is*

$$\mathbb{Z} \oplus \pi_{n+1}(F(S^1));$$

2) *the fixed set of K_{n+2} -action is*

$$\pi_{n+1}(F(S^1)).$$

0.4. A Representation of $F(S^1)$. In this subsection, the ground ring R is \mathbb{Z} or \mathbb{Z}/p . Let X be a pointed set. Let $A(X)$ be the algebra of non-commutative formal power series with variables in any finite subset of X over R modulo the single relation that $* = 1$, where $*$ is the base point of X . Let X be a pointed simplicial set. The simplicial algebra $A(X)$ is defined by applying the functor A to X . Let $X = S^1$ be the simplicial circle. Then

1) There is a choice of generators in $A(S^1)_{n+1}$ such that

$$A(S^1)_{n+1} = A(x_0, x_1, \dots, x_n)$$

is the associated algebra of the non-commutative formal power series in variables x_0, x_1, \dots, x_n over R .

2) The simplicial structure on $A(S^1)$ is given by

$$d_j x_k = \begin{cases} x_{k-1} & \text{for } j \leq k, \\ 0 & \text{for } j = k + 1, \\ x_k & \text{for } j > k + 1, \end{cases}$$

and

$$s_j x_k = \begin{cases} x_{k+1} & \text{for } j \leq k, \\ x_k + x_{k+1} + x_k x_{k+1} & \text{for } j = k + 1, \\ x_k & \text{for } j > k + 1, \end{cases}$$

for $0 \leq j \leq n + 1$, where

$$x_{-1} = (1 + x_{n-1})^{-1}(1 + x_{n-2})^{-1} \cdots (1 + x_0)^{-1} - 1$$

in $A(S^1)_n$.

Let the Braided group B_{n+1} act on $A(S^1)_{n+1}$ as follows.

For each $0 \leq k \leq n - 1$, $\sigma_k: A(S^1)_{n+1} \rightarrow A(S^1)_{n+1}$ is an automorphism of algebras with

$$\sigma_k(x_j) = \begin{cases} x_{k+1} & \text{if } j = k \\ (1 + x_{k+1})^{-1}(1 + x_k)(1 + x_{k+1}) - 1 & \text{if } j = k + 1 \\ x_j & \text{otherwise} \end{cases}$$

for $0 \leq k \leq n - 1$.

Let σ_{-1} be an automorphism of $A(S^1)_{n+1}$ defined by

$$\sigma_{-1}(x_0) = (1 + x_0)^{-1}(1 + x_{-1})(1 + x_0) - 1 = (1 + x_0)^{-1}(1 + x_n)^{-1} \cdots (1 + x_1)^{-1} - 1$$

and $\sigma(x_j) = x_j$ for $j > 0$. The subgroup of the automorphism group of $A(S^1)_{n+1}$ generated by σ_j for $-1 \leq j \leq n - 1$ is the braid group B_{n+2} . Let

$$e: F(S^1) \rightarrow A(S^1)$$

be the canonical representation, that is

$$e(y_j) = 1 + x_j$$

for each j .

Proposition 0.4. *The simplicial algebra $A(S^1)$ is a braided simplicial algebra, that is, the braided action satisfies the identities 1 and 2 in Proposition 0.1. Furthermore, the representation $e: F(S^1) \rightarrow A(S^1)$ is a braided representation, that is e commutes with the braid group action and the simplicial structure.*

The proof is straight forward.

Since the representation $e: F(S^1) \rightarrow A(S^1)$ is faithful, we have

Proposition 0.5. *Let $w \in F(S^1)_{n+1}$. Then*

- 1) $w \in \mathcal{Z}(F(S^1))_{n+1}$ if and only if $e(w) - 1 \in \mathcal{Z}(A(S^1))_{n+1}$;
- 2) $w \in NF(S^1)_{n+1}$ if and only if $e(w) - 1 \in NA(S^1)_{n+1}$.

Let $w = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_t}^{n_t}$ be a monomial in $A(x_0, \cdots, x_n)$. We call w is *non-degenerate* if the set

$$\{i_1, \cdots, i_t\} = \{0, \cdots, n\},$$

that is each letter x_j with $0 \leq j \leq n$ appears in w at least once.

Theorem 0.6. *Let f be a series in $A(S^1)_{n+1}$. Then $f \in NA(S^1)_{n+1}$ if and only if f is a formal series of non-degenerate monomials.*