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HOMOTOPY GROUPS AND BRAIDS

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In this talk, we will give some relations between the homotopy groups of the 2-sphere and the braid groups. We should point out that this is one of research projects in the National University of Singapore. We (Jon Berrick, Kai Meng Tan, Yan Loi Wong and I) have a weekly informal seminar to discuss braids, symmetric groups and homotopy theory. Jon Berrick lets me to say hello to everybody here.

The word group $W_n$ is defined as follows:

$\textit{generators: } y_1, y_2, \cdots, y_n$;

$\textit{relations: } (1). \text{ the product element } y_1 y_2 \cdots y_n \text{ and } (2). \text{ all iterated commutators}$

$$[y_{i_1}^{\pm 1}, \cdots, y_{i_t}^{\pm 1}]$$

with $\{i_1, \cdots, i_t\} = \{1, \cdots, n\}$, that is each generator occurs at least once.

Let $F(S^1)$ be the Milnor’s construction of the circle $S^1$. Then $W_n = F(S^1)_{n-1}$/boundaries. An important property of the group $W_n$ is

$\textbf{Theorem.}$ The center $Z(W_n)$ is isomorphic to $\pi_n(S^2)$.

0.1. $\textbf{The geometry on } W_n \textbf{ and the braid group action.}$ Let $\hat{F}_n$ be the group defined by

$\textit{generators: } y_1, \cdots, y_n \text{ with the single relation: } y_1 y_2 \cdots y_n$.

Clearly $\hat{F}_n \cong F_{n-1}$ the free group of rank $n - 1$. Let $\check{W}_n$ be the group defined by

\[\hat{F}_n \cong F_{n-1}\]

\[\check{W}_n\]
generators: \( y_1, y_2, \ldots, y_n \);
all iterated commutators
\[
[y_{i_1}^{\pm 1}, \ldots, y_{i_t}^{\pm 1}]
\]
with \( \{i_1, \ldots, i_t \} = \{1, \ldots, n\} \).

As a group \( \tilde{W}_n = F(S^1)_n \)/Moore chain complex. The group \( W_n \) is the quotient of \( \tilde{W}_n \) by the single relation \( y_1 y_2 \cdots y_n = 1 \). We also keep in mind that there is a short exact sequence
\[
\text{Ker}(d_0)/\text{cycles} \hookrightarrow \tilde{W}_n \twoheadrightarrow W_n.
\]
We are going to construct a space \( X_n \) such that \( \pi_1(X_n) = W_n \). First we construct a space \( \tilde{X}_n \) such that \( \pi_1(\tilde{X}_n) = \tilde{W}_n \). Let \( Q_n = \{q_1, \ldots, q_n\} \) be the set of distinguished points in \( D^2 \setminus S^1 \) and let \( Q_{ni} = Q_n \setminus \{q_i\} \). The inclusions \( D^2 \setminus Q_n \hookrightarrow D^2 \setminus Q_{ni} \) induces the map
\[
\phi_n : D^2 \setminus Q_n \longrightarrow \prod_{i=1}^n D^2 \setminus Q_{ni} \quad x \rightarrow (x, \ldots, x).
\]
Then
\[
\tilde{W}_n \cong \text{Im}(\phi_n) : \pi_1(D^2 \setminus Q_n) \to \pi_1(\prod_{i=1}^n D^2 \setminus Q_{ni})
\]
Thus there exists a (unique) space \( \tilde{X}_n \) such that
\[
\begin{array}{c}
\begin{array}{c}
D^2 \setminus Q_n \\
\downarrow \\
D^2 \setminus Q_n
\end{array} \hookrightarrow \tilde{X}_n \\
\text{covering} \\
\begin{array}{c}
\begin{array}{c}
\prod_{i=1}^n D^2 \setminus Q_{ni}
\end{array}
\end{array}
\end{array}
\]
with \( \pi_1(\tilde{X}_n) = \tilde{W}_n \). Then \( \tilde{X}_n \simeq K(\tilde{W}_n, 1) \). Let \( X_n \) be defined by the push out

\[
\begin{array}{c}
\xymatrix{
D^2 \setminus Q_n \ar[r] \ar[d] & \tilde{X}_n \ar[d] \\
S^2 \setminus Q_n \ar[r] & X_n = \tilde{X}_n \bigcup_{D^2 \setminus Q_n} S^2 \setminus Q_n.
}
\end{array}
\]

In sense, the space \( X_n \) is a ‘blow-up’ of \( S^2 \setminus Q_n \) along the map \( \phi_n \). By the Seifert-van Kampen theorem, \( \pi_1(X_n) = W_n \). (Note. \( X_n \) is not \( K(W_n, 1) \) because \( X_n \) is an \( n \)-dimensional \( CW \)-complex but \( W_n \) does contain finite subgroups in general.)

Recall (one of the definitions of the braid group)

\[
The braid group \( B_n \), as a subgroup of \( \text{Aut}(F_n) \), is the group of automorphisms of \( \pi_1(D^2 \setminus Q_n) \) which are induced by autohomeomorphisms of \( D^2 \setminus Q_n \) that keep the boundary of \( D^2 \) fixed pointwise.
\]

Let \( \beta \in B_n \) and let \( h \) be an autohomeomorphism of \( D^2 \setminus Q_n \) representing \( \beta \). Then \( h \) has a unique extension \( \hat{h} \) to \( D^2 \). The map \( \hat{h} : D^2 \to D^2 \) is isotopic to the identity map. Let \( h_t \) be an isotopy between \( \hat{h} \) and the identity. Then the strings defined by \( h_t(q_i) \) are exactly the geometric braid corresponding to \( \beta \).

The restriction of \( \hat{h} \) to \( D^2 \setminus Q_{ni} \) gives a map \( h_i : D^2 \setminus Q_{ni} \to Q_{n\mu(\beta)(i)} \), where \( \mu : B_n \to \Sigma_n \) is the quotient map. (Note. As a geometric braid, \( h_i \) is a braid with \( n-1 \) components obtained from \( \beta \) by deleting the \( i \)-th string.) The autohomeomorphism
(h_1, \ldots, h_n) of \prod_{i=1}^n D^2 \setminus Q_{n_i} induces an autohomeomorphism \bar{h} of \bar{X}_n such that

\[
\begin{array}{c}
D^2 \setminus Q_n \xrightarrow{h} \bar{X}_n \\
\downarrow \quad \quad \quad \downarrow \\
D^2 \setminus Q_n \xrightarrow{\bar{h}} \bar{X}_n.
\end{array}
\]

Let \bar{h} be the canonical extension of h to S^2 \setminus Q_n. Then the map \bar{h} \cup \bar{h} is an autohomeomorphism of X_n. This gives the braid group action on W_n = \pi_1(X_n) and so the push-out

\[
\begin{array}{c}
F_n \rightarrow W_n \\
\downarrow \\
\hat{F}_n \rightarrow W_n
\end{array}
\]

is a diagram over the braid group B_n.

**Theorem 0.1.** Let B_n act on W_n induced by its action on F_n. Then

1) The fixed set of the pure braid group action on W_n is the center \pi_n(S^2) of W_n.

2) The fixed set of B_n-action on W_n is \{\alpha \in \pi_n(S^2) | 2\alpha = 0\}.

**Note.** We proved this result by considering the braid groups action on the simplicial group F(S^1) instead of geometrical description above. But this description gives another interesting relation between the homotopy and braids. (We will explain this in details later.)

**Note.** There is a geometrical description of the group \hat{W}_n = W_n/center = F(S^1)_{n-1}/cycles given as follows.
Consider the diagonal map

$$\psi_n: S^2 \setminus Q_n \hookrightarrow \prod_{i=1}^{n} S^2 \setminus Q_{ni}, \quad x \to (x, \cdots, x).$$

There is a (unique) space $\tilde{X}_n$ such that

\begin{center}
\begin{xy}
  0;0;0*+!R{\psi_n};0;0*+!L{S^2 \setminus Q_n};0;0*+!C{\tilde{X}_n};
  0;0;0*+!R{\prod_{i=1}^{n} S^2 \setminus Q_{ni}};
  0;0;0*+!C{\text{covering}};
\end{xy}
\end{center}

with $\pi_1(\tilde{X}_n) = \bar{W}_n$. The space $\tilde{X}$ is $K(\bar{W}_n, 1)$ and so the fundamental group of the homotopy fibre of $X_n \to \tilde{X}_n$ is $\pi_n(S^2)$. In group theory, the cycles $\mathcal{Z}(F(S^1))_{n-1}$ is the intersection of the coordinate projections $d_i: \hat{F}_n \to \hat{F}_{n-1}$ for $1 \leq i \leq n$, where $d_i$ sends $\hat{y}_i$ to 1 and sends the other generators for $\hat{F}_n$ to the generators for $\hat{F}_{n-1}$.

0.2. The Kernel of $B_n \to \text{Aut}(W_n)$. The $B_n$-action on $W_n$ induces a homomorphism $\theta: B_n \to \text{Aut}(W_n)$. It turns out that $\theta$ has a nontrivial kernel. Below we want to say what is the kernel of $\theta$.

Let $C_n$ be the subset of $B_n$ consisting of all geometric braids $\beta$ with the property that $\beta$ reduces to a trivial braid after removing ANY one of the strings. Then $C_n$ is a normal subgroup of $B_n$.

The links obtained from the braids in $C_n$ are Borromean Rings. (That is, a link $L$ with $n$ components such that any $n - 1$ components of $L$ are unlinked.)

**Theorem 0.2.** Suppose that $n \geq 4$. Then the kernel of $B_n \to \text{Aut}(W_n)$ is the normal subgroup of $B_n$ generated by the center of $B_n$ and $C_n$. Thus

$$\text{Ker}(B_n \to \text{Aut}(W_n)) \cong \mathbb{Z} \times C_n.$$
Sketch of Proof. 1) \( \text{Ker}(B_n \to \text{Aut}(\hat{F}_n)) = Z(B_n) \) for \( n \geq 3 \).

2) \( \text{Ker}(B_n \to \text{Aut}(\hat{W}_n)) = C_n \) for any \( n \).

3) \( \text{Ker}(B_n \to \text{Aut}(\hat{W}_n)) = Z(B_n) \times C_n \) for \( n \geq 4 \).

Now we give another relation between the homotopy and the braids. Recall that the pure braid group \( K_n \) is isomorphic to \( \pi_1(F(\mathbb{R}^2, n)) \). Let \( R_n \) be the kernel of

\[ K_n = \pi_1(F(\mathbb{R}^2, n)) \to \pi_1(F(S^2, n)) \]

induced by the inclusion of configuration spaces \( F(\mathbb{R}^2, n) \to F(S^2, n) \) on the fundamental groups. Then we have

**Theorem 0.3.** There is an isomorphism of groups \( R_n \cap C_n \cong Z(F(S^1))_{n-1} \) for \( n \geq 4 \).

**Sketch of Proof.** The \( K_n \)-action on \( F_n \) induces an action on \( \hat{F}_n \). Then, by a result in Birman’s book, \( R_n \) is the inner automorphism group of \( \hat{F}_n \). Thus for any \( \phi \in R_n \) there is a unique word \( w \in \hat{F}_n \) such that \( \phi(x) = w^{-1}xw = \chi_w(x) \). Then check that \( w \) is a cycle if and only if \( \phi \in R_n \cap C_n \).

**Note.** We are trying to see what’s the relations between the braids and the boundaries of \( F(S^1) \). The relation above tells that it is possible that the homotopy groups are certain linking invariants (or certain derived functor on certain links).

**Note.** For \( n \geq 4 \), \( R_n \cap C_n \) is the same as the intersection of \( C_n \) with the kernel of \( B_n \) to the mapping class group on the \( n \)-punctured sphere. In Briman’s book, he asked to determine free generators for the group \( R_n \cap C_n \). (Question 23 listed in the end of the book.) As we see above, this sounds a very difficult questions and it essentially talks about the homotopy groups of the sphere.