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The LS-category of Certain 3-cell Complexes

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★**Harper's Question:** Find 2-local non-suspension co-H-spaces $X = P^3(2) \cup_f e^n$.

Note. The LS-category of $P^3(2) \cup_f e^n$ is either 1 or 2.

• **Observation 1:** There does not exist 2-local non-suspension two-cell co-H-space.

Proof. Let $X = S^n \cup_f e^{n+k+1}$ be a co-H-space. Then the attaching map $f\colon S^{n+k} \to S^n$ is a co-H-map. Let $f'\colon S^{n+k-1} \to \Omega S^n$ be the adjoint map of f. Then the composite

$$S^{n+k-1} \xrightarrow{f'} \Omega S^n \xrightarrow{H} \Omega S^{2n-1}$$

is null homotopic, where H is the Hopf invariant. Thus f' lifts to the homotopy fibre S^{n-1} of H by the EHP-sequence and so there is a map $g\colon S^{n+k-1} \to S^{n-1}$ such that $f \simeq \Sigma g$. It follows that $X \simeq \Sigma(S^{n-1} \cup_g e^{n+k})$. □

- To get 2-local non-suspension co-H-spaces, one needs at least 3-cells.

★Observation 2.

- $X = (\Sigma Y) \cup_f e^{n+1}$ with $n > \dim Y$ is a suspension if and only if the attaching $f\colon S^n \to \Sigma Y$ is a suspension.
- $X = (\Sigma Y) \cup_f e^{n+1}$ with $n > \dim Y \geq 2$ is a co-H-space if and only if the attaching map $f\colon S^n \to \Sigma Y$ is a co-H-map (with respect to a comultiplication on Y induced by X).

★Observation 3.

- The suspension $E_*\colon \pi_n(\mathbb{R}\mathrm{P}^2) \to \pi_{n+1}(P^3(2))$ is zero for $n \geq 4$.
- NO suspension $X = P^3(2) \cup_f e^n$ for $n \geq 5$ unless $f = 0$.
- Harper's question: Find co-H-maps $f\colon S^n \to P^3(2)$.
- Note. There are two comultiplications on $P^3(2)$. If f is a co-H-map with respect to one of them, then so is with respect to another.

★ **Hopf Invariants.** Let X and Y be path-connected spaces. Recall that there is a fibre sequence

$$\Sigma\Omega X \wedge \Omega Y \xrightarrow{\phi} X \vee Y \xrightarrow{q} X \times Y$$

and the adjoint $\phi'\colon \Omega X \wedge \Omega Y \to \Omega(X \vee Y)$ is the Samelson product $[i_1, i_2]$, where $i_1\colon \Omega X \to \Omega(X \vee Y)$ and $i_2\colon \Omega Y \to \Omega(X \vee Y)$ are the canonical inclusions. Let θ_X and $\theta_Y\colon X \vee Y \to X \vee Y$ be the maps defined by the composites

$$X \vee Y \xrightarrow{\text{pinch}} X \hookrightarrow X \vee Y \quad \text{and} \quad X \vee Y \xrightarrow{\text{pinch}} Y \hookrightarrow X \vee Y,$$

respectively. Let $\tilde{H}\colon \Omega(X \vee Y) \to \Omega(X \vee Y)$ be a map such that the homotopy class $[\tilde{H}] = [\mathrm{id}][\theta_Y]^{-1}[\theta_X]^{-1}$ in the group $[\Omega(X \vee Y), \Omega(X \vee Y)]$. Then

- The composite $q \circ \tilde{H}\colon \Omega(X \vee Y) \to \Omega X \times \Omega Y$ is null homotopic and so the map $\tilde{H}$ lifts to the fibre $\Omega\Sigma(\Omega X \wedge \Omega Y)$ (uniquely) up to homotopy, denoted by H.
- $$\Omega\Sigma(\Omega X \wedge \Omega Y) \xrightarrow{\Omega\phi} \Omega(X \vee Y) \xrightarrow{H} \Omega\Sigma(\Omega X \wedge \Omega Y)$$

is homotopic to the identity map with a fibre sequence

- $$\Omega X \times \Omega Y \xrightarrow{i_1 \cdot i_2} \Omega(X \vee Y) \xrightarrow{H} \Omega\Sigma(\Omega X \wedge \Omega Y).$$

This defines a particular choice of the Hopf map $H\colon \Omega(X\vee Y)\to \Omega\Sigma(\Omega X\wedge\Omega Y)$.

★ *Let Y and Z be a path connected co-H-spaces and let $f\colon \Sigma Y\to Z$ be any map. Then f is a co-H-map if and only if the composite* $Y\xrightarrow{f'}\Omega Z\xrightarrow{\Omega\mu'}\Omega(Z\vee Z)\xrightarrow{H}\Omega\Sigma(\Omega Z\wedge\Omega Z)$ *is null homotopic, where f' is the adjoint map of f.*

- Let Z be a co-H-space. The composite $\Omega Z\xrightarrow{\Omega\mu'}\Omega(Z\vee Z)\xrightarrow{H}\Omega\Sigma(\Omega Z\wedge\Omega Z)$ is called a *Hopf map* for the co-H-space Z and we abbreviate H for this map. Note that the Hopf map $H\colon \Omega Z\to\Omega\Sigma(\Omega Z\wedge\Omega Z)$ depends on the choice of comultiplications on Z.

- Let $F_H(Z)$ be the homotopy fibre of the Hopf map $H\colon \Omega Z\to \Omega\Sigma(\Omega Z\wedge\Omega Z)$ with induced map $\lambda=\lambda_Z\colon F_H(Z)\to\Omega Z$. This gives a homotopy functor F_H from co-H-spaces to spaces.

- Let $\mathcal{P}_n(Z)$ be the subset of $\pi_n(Z)$ consisting of homotopy classes represented by co-H-maps. Then
- $\mathcal{P}_n(Z)=\mathrm{Im}(\pi_{n-1}(F_H(Z))\to\pi_{n-1}(\Omega Z))$.

The Results. Now we consider our case where $Z = \Sigma\mathbb{R}\mathrm{P}^2$. We write $P^n(2)$ for $\Sigma^{n-2}\mathbb{R}\mathrm{P}^2$. In our notation, $P^3(2) = \Sigma\mathbb{R}\mathrm{P}^2$. Let $\mathbb{R}\mathrm{P}^b_a = \mathbb{R}\mathrm{P}^b/\mathbb{R}\mathrm{P}^{a-1}$ and let $X\langle n\rangle$ be the n-connected cover of a space X. Then the homotopy fibre of the inclusion $P^3(2) \hookrightarrow BSO(3)$ is $\Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)$ and so there is a fibre sequence

$$SO(3) \longrightarrow \Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2) \longrightarrow P^3(2).$$

It follows that there is a fibre sequence

$$\Omega(P^3(2)\langle 2\rangle) \xrightarrow{\partial} S^3 \longrightarrow \Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2) \longrightarrow P^3(2)\langle 2\rangle,$$

where the map $S^3 \to \Sigma\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)$ is of degree 4 into the bottom cell of target space. This fibre sequence induces a splitting of $\Omega^3 P^3(2)$. In particular,

$$\pi_*(\Sigma\mathbb{R}\mathrm{P}^2) = \pi_*(\mathbb{R}\mathrm{P}^4_2 \vee P^6(2)) \oplus \pi_{*-1}(S^3)$$

for $* \geq 5$. Let $S^3\{2\}$ be the homotopy fibre of degree 2 map from S^3 to S^3.

Our main result is as follows.

★**Theorem.** *Let* $\partial\colon \Omega(P^3(2)\langle 2\rangle) \to S^3$ *be defined above.*

1. *The composite* $F_H(P^3(2))\langle 1\rangle \longrightarrow \Omega(P^3(2)\langle 2\rangle) \xrightarrow{\partial} S^3$ *lifts to* $S^3\{2\}$*.*

2. *Let* $\theta\colon F_H(P^3(2))\langle 1\rangle \to S^3\{2\}$ *be a resulting lifting. Then* θ *has a cross-section and so* $S^3\{2\}$ *is a retract of the universal cover of* $F_H(P^3(2))$*.*

- Since the space $S^3\{2\}$ is indecomposable, we determine the "*smallest retract*" of $F_H(P^3(2))\langle 1\rangle$ which contains the bottom cell.

- From the commutative diagram

$$\begin{tikzcd}
\pi_*(F_H(P^3(2))) \arrow[r] \arrow[d, "\theta_*"] & \pi_*(\Omega P^3(2)) \arrow[d, "\partial_*"] \\
\pi_*(S^3\{2\}) \arrow[r] & \pi_*(S^3),
\end{tikzcd}$$

we obtain that

- $\mathrm{Im}(\partial_*\colon \mathcal{P}_{*+1}(P^3(2)) \to \pi_*(S^3)) = \{\alpha \in \pi_*(S^3) \mid 2\alpha = 0\}$.

★ Our Answer to Harper's Question:

- For each $\alpha \in \pi_n(S^3)$ with $2\alpha = 0$, there is a co-H-map $f\colon S^{n+1} \to P^3(2)$ such that $\partial_*([f]) = \alpha$ and so there is a corresponding co-H-space $X = \Sigma\mathbb{R}\mathrm{P}^2 \cup_f e^{n+2}$, which is not a suspension when $n \geq 3$. The first example is $X = \Sigma\mathbb{R}\mathrm{P}^2 \cup_f e^6$, where f corresponds to $\eta \in \pi_4(S^3) = \mathbb{Z}/2$.

★ **Question:** Is $\partial_*\colon \mathcal{P}_{*+1}(P^3(2)) \to \{\alpha \in \pi_*(S^3) \mid 2\alpha = 0\}$ an isomorphism for $* \geq 4$?

★ The Map $\theta\colon F_H(P^3(2))\langle 1\rangle \to S^3\{2\}$

Observe that $P^3(2)\xrightarrow{[2]} P^3(2)\hookrightarrow BSO(3)$ is null homotopic, where $[k]\colon Z\to Z$ is a map of degree k for a co-H-space Z. It follows that the composite

$$\Omega P^3(2)\xrightarrow{\Omega[2]}\Omega P^3(2)\xrightarrow{\partial} SO(3)$$

is null homotopic. Thus

$$2\circ\partial\circ\lambda\simeq\partial\circ(2\circ\lambda)\simeq\partial\circ(\Omega[2]\circ\lambda)\simeq *\colon F_H(P^3(2)))\to SO(3),$$

where we use the fact that

$k\circ\lambda\simeq\Omega[k]\circ\lambda\colon F_H(Z)\colon F_H(Z)\to\Omega Z.$

It follows that there is a homotopy commutative diagram

$$\begin{tikzcd}
F_H(P^3(2)) \arrow[r] \arrow[d, "\theta"] & \Omega P^3(2) \arrow[d, "\partial"] \\
SO(3)\{2\} \arrow[r] & SO(3)
\end{tikzcd}$$

and so, by taking universal covering, we obtain a map $\theta\colon F_H(P^3(2))\langle 1\rangle\to S^3\{2\}$.

★ The Map $S^3\{2\} \to F_H(P^3(2))$

Now we give the sketch of ideas how to construct a cross-section map for $\theta\colon F_H(P^3(2))\langle 1\rangle \to S^3\{2\}$. Let $q\colon S^2 \to \mathbb{R}\mathrm{P}^2$ be the quotient map.

★ *Let $f\colon \Sigma Y \to P^3(2)$ be any co-H-map. Then the composite*

$$Y\times J(S^2) \xrightarrow{f'\times\Omega\Sigma q} \Omega P^3(2)\times\Omega P^3(2) \xrightarrow{\mu} \Omega P^3(2) \xrightarrow{H} \Omega\Sigma((\Omega P^3(2))^{(2)})$$

is null homotopic.

The proof of this lemma requires the combinatorial calculation of the Hopf invariant H.

Let $\phi\colon P^4(2) \to P^3(2)$ be the map in the cofibre sequence $\mathbb{R}\mathrm{P}^2 \hookrightarrow \mathbb{R}\mathrm{P}^4 \longrightarrow P^4(2) \xrightarrow{\phi} P^3(2)$.

★ *Let $\phi\colon P^4(2) \to P^3(2)$ be the map defined above. Then*

1) *ϕ restricted to S^3 is homotopic to $\Sigma q\colon S^3 \to P^3(2)$ and*

2) *ϕ is a co-H-map*

The Construction of the Map $S^3\{2\} \to F_H(P^3(2))$:

Consider the homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
\Omega S^3 \arrow[r] \arrow[d, "\Omega g"] & S^3\{2\} \arrow[r] \arrow[d, "\bar{g}"] & S^3 \arrow[r, "{[2]}"] \arrow[d] & S^3 \arrow[d, hook, "g"] \\
\Omega P^4(2) \arrow[r, equal] & \Omega P^4(2) \arrow[r] & * \arrow[r] & P^4(2).
\end{tikzcd}$$

Since the fibre sequence $\Omega S^3 \longrightarrow S^3\{2\} \longrightarrow S^3$ is principal, there is a right $J(S^2)$-action $\mu\colon S^3\{2\} \times J(S^2) \longrightarrow S^3\{2\}$ with a homotopy commutative diagram

$$\begin{tikzcd}
S^3\{2\} \times J(S^2) \arrow[r, "\mu"] \arrow[d, "\bar{g} \times \Omega g"'] & S^3\{2\} \arrow[d, "\bar{g}"] \\
J(P^3(2)) \times J(P^3(2)) \arrow[r, "\mu"] & J(P^3(2)).
\end{tikzcd}$$

Let $\tilde{s}\colon S^3\{2\} \to J(\mathbb{R}\mathrm{P}^2)$ be the composite

$$S^3\{2\} \xrightarrow{\bar{g}} J(P^3(2)) \xrightarrow{\Omega\phi} J(\mathbb{R}\mathrm{P}^2).$$

It follows that there is a homotopy commutative diagram

$$\begin{tikzcd}
S^3\{2\} \times J(S^2) \arrow[r, "\mu"] \arrow[d, "\tilde{s} \times J(q)"'] & S^3\{2\} \arrow[d, "\tilde{s}"] \\
J(\mathbb{R}\mathrm{P}^2) \times J(\mathbb{R}P^2) \arrow[r, "\mu"] & J(P^3(2)).
\end{tikzcd}$$

The composite

$$P^3(2) \times J(S^2) \xrightarrow{\mu} S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{R}\mathrm{P}^2) \xrightarrow{H} \Omega\Sigma(J(\mathbb{R}\mathrm{P}^2))^{(2)}$$

is null homotopic by the two lemmas above. By the suspension splitting of $S^3\{2\}$, the map

$$\mu^*\colon [S^3\{2\}, \Omega W] \longrightarrow [P^3(2) \times J(S^2), \Omega W]$$

is a monomorphism for any W. Thus the composite

$$S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{R}\mathrm{P}^2) \xrightarrow{H} \Omega\Sigma(J(\mathbb{R}\mathrm{P}^2))^{(2)}$$

is null homotopic and so the map $\tilde{s}$ lifts to $F_H(P^3(2))$. Let $\bar{s}\colon S^3\{2\} \to F_H(P^3(2))$ be a lifting of $\tilde{s}$. Since $S^3\{2\}$ is simply connected, the map $\bar{s}$ lifts to the universal cover $F_H(P^3(2))\langle 1\rangle$ and let $s\colon S^3\{2\} \to F_H(P^3(2))\langle 1\rangle$ be a lifting of $\bar{s}$. Then composite

$$S^3\{2\} \xrightarrow{s} F_H(P^3(2))\langle 1\rangle \xrightarrow{\theta} S^3\{2\}$$

is a homotopy equivalence because it induces an isomorphism on H_3 of the atomic space $S^3\{2\}$.

Note. For the first non-suspension co-H-space $X = \Sigma\mathbb{R}\mathrm{P}^2 \cup_f e^6$, the attaching map f is given by the composite

$$S^5 \xrightarrow{\bar{\eta}} P^4(2) \xrightarrow{\phi} P^3(2),$$

where $\bar{\eta}\colon S^5 \to P^4(2)$ is a lifting of $S^5 \to S^4$. The map $\bar{\eta}$ is a suspension, but $\phi \circ \bar{\eta}$ is *not*.

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