The $LS$-category of Certain 3-cell Complexes

Jie Wu

Department of Mathematics
National University of Singapore
Singapore 117543
Republic of Singapore
e-mail: matwuj@nus.edu.sg
Web: www.math.nus.edu.sg/~matwujie
**Harper’s Question:** Find 2-local non-suspension co-$H$-spaces $X = P^3(2) \cup_f e^n$.

**Note.** The $LS$-category of $P^3(2) \cup_f e^n$ is either 1 or 2.

- **Observation 1:** There does not exist 2-local non-suspension two-cell co-$H$-space.

**Proof.** Let $X = S^n \cup_f e^{n+k+1}$ be a co-$H$-space. Then the attaching map $f: S^{n+k} \to S^n$ is a co-$H$-map. Let $f': S^{n+k-1} \to \Omega S^n$ be the adjoint map of $f$. Then the composite

$$S^{n+k-1} \xrightarrow{f'} \Omega S^n \xrightarrow{H} \Omega S^{2n-1}$$

is null homotopic, where $H$ is the Hopf invariant. Thus $f'$ lifts to the homotopy fibre $S^{n-1}$ of $H$ by the $EHP$-sequence and so there is a map $g: S^{n+k-1} \to S^{n-1}$ such that $f \simeq \Sigma g$. It follows that $X \simeq \Sigma(S^{n-1} \cup_g e^{n+k})$. □
• To get 2-local non-suspension co-$H$-spaces, one needs at least 3-cells.

★Observation 2.
• $X = (\Sigma Y) \cup_f e^{n+1}$ with $n > \dim Y$ is a suspension if and only if the attaching $f: S^n \to \Sigma Y$ is a suspension.
• $X = (\Sigma Y) \cup_f e^{n+1}$ with $n > \dim Y \geq 2$ is a co-$H$-space if and only if the attaching map $f: S^n \to \Sigma Y$ is a co-$H$-map (with respect to a comultiplication on $Y$ induced by $X$).

★Observation 3.
• The suspension $E_*: \pi_n(\mathbb{R}P^2) \to \pi_{n+1}(P^3(2))$ is zero for $n \geq 4$.
• NO suspension $X = P^3(2) \cup_f e^n$ for $n \geq 5$ unless $f = 0$.
• Harper’s question: Find co-$H$-maps $f: S^n \to P^3(2)$.
• Note. There are two comultiplications on $P^3(2)$. If $f$ is a co-$H$-map with respect to one of them, then so is with respect to another.
Hopf Invariants. Let $X$ and $Y$ be path-connected spaces. Recall that there is a fibre sequence

$$\Sigma \Omega X \land \Omega Y \xrightarrow{\phi} X \lor Y \xrightarrow{q} X \times Y$$

and the adjoint $\phi': \Omega X \land \Omega Y \to \Omega(X \lor Y)$ is the Samelson product $[i_1, i_2]$, where $i_1: \Omega X \to \Omega(X \lor Y)$ and $i_2: \Omega Y \to \Omega(X \lor Y)$ are the canonical inclusions. Let $\theta_X$ and $\theta_Y$: $X \lor Y \to X \lor Y$ be the maps defined by the composites $X \lor Y \xrightarrow{\text{pinch}} X \xleftarrow{} X \lor Y$ and $X \lor Y \xrightarrow{\text{pinch}} Y \xleftarrow{} X \lor Y$, respectively. Let $\tilde{H}: \Omega(X \lor Y) \to \Omega(X \lor Y)$ be a map such that the homotopy class $[\tilde{H}] = [\text{id}][\theta_Y]^{-1}[\theta_X]^{-1}$ in the group $[\Omega(X \lor Y), \Omega(X \lor Y)]$. Then

- The composite $q \circ \tilde{H}: \Omega(X \lor Y) \to \Omega X \times \Omega Y$ is null homotopic and so the map $\tilde{H}$ lifts to the fibre $\Omega\Sigma(\Omega X \land \Omega Y)$ (uniquely) up to homotopy, denoted by $H$.

- $\Omega\Sigma(\Omega X \land \Omega Y) \xrightarrow{\Omega \phi} \Omega(X \lor Y) \xrightarrow{H} \Omega\Sigma(\Omega X \land \Omega Y)$ is homotopic to the identity map with a fibre sequence

- $\Omega X \times \Omega Y \xrightarrow{i_1 \cdot i_2} \Omega(X \lor Y) \xrightarrow{H} \Omega\Sigma(\Omega X \land \Omega Y)$. 
This defines a particular choice of the Hopf map \( H : \Omega(X \vee Y) \to \Omega \Sigma(\Omega X \wedge \Omega Y) \).

\[ \star \text{Let } Y \text{ and } Z \text{ be a path connected co-}H\text{-spaces and let } f : \Sigma Y \to Z \text{ be any map. Then } f \text{ is a co-}H\text{-map if and only if the composite } \]
\[ Y \xrightarrow{f'} \Omega Z \xrightarrow{\Omega \mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega \Sigma(\Omega Z \wedge \Omega Z) \]
\[ \text{is null homotopic, where } f' \text{ is the adjoint map of } f. \]

\[ \bullet \text{Let } Z \text{ be a co-}H\text{-space. The composite } \Omega Z \xrightarrow{\Omega \mu'} \Omega(Z \vee Z) \xrightarrow{H} \Omega \Sigma(\Omega Z \wedge \Omega Z) \text{ is called a Hopf map for the co-}H\text{-space } Z \text{ and we abbreviate } H \text{ for this map. Note that the Hopf map } H : \Omega Z \to \Omega \Sigma(\Omega Z \wedge \Omega Z) \text{ depends on the choice of comultiplications on } Z. \]

\[ \bullet \text{Let } F_H(Z) \text{ be the homotopy fibre of the Hopf map } H : \Omega Z \to \Omega \Sigma(\Omega Z \wedge \Omega Z) \text{ with induced map } \lambda = \lambda_Z : F_H(Z) \to \Omega Z. \text{ This gives a homotopy functor } F_H \text{ from co-}H\text{-spaces to spaces.} \]

\[ \bullet \text{Let } P_n(Z) \text{ be the subset of } \pi_n(Z) \text{ consisting of homotopy classes represented by co-}H\text{-maps. Then } \]
\[ P_n(Z) = \text{Im}(\pi_{n-1}(F_H(Z)) \to \pi_{n-1}(\Omega Z)). \]
The Results. Now we consider our case where $Z = \Sigma \mathbb{R}P^2$. We write $P^n(2)$ for $\Sigma^{n-2}\mathbb{R}P^2$. In our notation, $P^3(2) = \Sigma \mathbb{R}P^2$.

Let $\mathbb{R}P^b_a = \mathbb{R}P^b / \mathbb{R}P^{a-1}$ and let $X\langle n \rangle$ be the $n$-connected cover of a space $X$. Then the homotopy fibre of the inclusion $P^3(2) \hookrightarrow BSO(3)$ is $\Sigma \mathbb{R}P^4_2 \vee P^6(2)$ and so there is a fibre sequence

$$SO(3) \longrightarrow \Sigma \mathbb{R}P^4_2 \vee P^6(2) \longrightarrow P^3(2).$$

It follows that there is a fibre sequence

$$\Omega(P^3(2)\langle 2 \rangle) \longrightarrow S^3 \longrightarrow \Sigma \mathbb{R}P^4_2 \vee P^6(2) \longrightarrow P^3(2)\langle 2 \rangle,$$

where the map $S^3 \to \Sigma \mathbb{R}P^4_2 \vee P^6(2)$ is of degree 4 into the bottom cell of target space. This fibre sequence induces a splitting of $\Omega^3 P^3(2)$. In particular,

$$\pi_* (\Sigma \mathbb{R}P^2) = \pi_*(\mathbb{R}P^4_2 \vee P^6(2)) \oplus \pi_{*-1}(S^3)$$

for $* \geq 5$. Let $S^3\{2\}$ be the homotopy fibre of degree 2 map from $S^3$ to $S^3$. 


Our main result is as follows.

**Theorem.** Let $\partial : \Omega(P^3(2)) \to S^3$ be defined above.

1. The composite $F_H(P^3(2)) \langle 1 \rangle \to \Omega(P^3(2)) \langle 2 \rangle \to S^3$ lifts to $S^3\{2\}$.

2. Let $\theta : F_H(P^3(2)) \langle 1 \rangle \to S^3\{2\}$ be a resulting lifting. Then $\theta$ has a cross-section and so $S^3\{2\}$ is a retract of the universal cover of $F_H(P^3(2))$.

- Since the space $S^3\{2\}$ is indecomposable, we determine the “smallest retract” of $F_H(P^3(2))\langle 1 \rangle$ which contains the bottom cell.

- From the commutative diagram

\[
\begin{array}{ccc}
\pi_*(F_H(P^3(2))) & \longrightarrow & \pi_*(\Omega P^3(2)) \\
\downarrow \theta_* & & \downarrow \partial_* \\
\pi_*(S^3\{2\}) & \longrightarrow & \pi_*(S^3),
\end{array}
\]

we obtain that

- $\text{Im}(\partial_* : \mathcal{P}_{*+1}(P^3(2)) \to \pi_*(S^3)) = \{ \alpha \in \pi_*(S^3) | 2\alpha = 0 \}$. 

Our Answer to Harper’s Question:

• For each $\alpha \in \pi_n(S^3)$ with $2\alpha = 0$, there is a co-$H$-map $f : S^{n+1} \to P^3(2)$ such that $\partial_*([f]) = \alpha$ and so there is a corresponding co-$H$-space $X = \Sigma \mathbb{RP}^2 \cup_f e^{n+2}$, which is not a suspension when $n \geq 3$. The first example is $X = \Sigma \mathbb{RP}^2 \cup_f e^6$, where $f$ corresponds to $\eta \in \pi_4(S^3) = \mathbb{Z}/2$.

★ Question: Is $\partial_* : P_{*+1}(P^3(2)) \to \{ \alpha \in \pi_*(S^3) | 2\alpha = 0 \}$ an isomorphism for $* \geq 4$?
The Map $\theta : F_H(P^3(2))(1) \to S^3\{2\}$

Observe that $P^3(2) \xrightarrow{[2]} P^3(2) \hookrightarrow BSO(3)$ is null homotopic, where $[k] : Z \to Z$ is a map of degree $k$ for a co-$H$-space $Z$. It follows that the composite

$$\Omega P^3(2) \xrightarrow{[2]} \Omega P^3(2) \xrightarrow{\partial} SO(3)$$

is null homotopic. Thus

$$2 \circ \partial \circ \lambda \simeq \partial \circ (2 \circ \lambda) \simeq \partial \circ (\Omega[2] \circ \lambda) \simeq * : F_H(P^3(2))) \to SO(3),$$

where we use the fact that

$$k \circ \lambda \simeq \Omega[k] \circ \lambda : F_H(Z) : F_H(Z) \to \Omega Z.$$

It follows that there is a homotopy commutative diagram

$$\begin{array}{ccc}
F_H(P^3(2)) & \longrightarrow & \Omega P^3(2) \\
\downarrow \theta & & \downarrow \partial \\
SO(3)\{2\} & \longrightarrow & SO(3)
\end{array}$$

and so, by taking universal covering, we obtain a map

$$\theta : F_H(P^3(2))(1) \to S^3\{2\}.$$
The Map $S^3\{2\} \to F_H(P^3(2))$

Now we give the sketch of ideas how to construct a cross-section map for $\theta: F_H(P^3(2))(1) \to S^3\{2\}$. Let $q: S^2 \to \mathbb{R}P^2$ be the quotient map.

Let $f: \Sigma Y \to P^3(2)$ be any co-$H$-map. Then the composite

$$Y \times J(S^2) \xrightarrow{f \times \Omega \Sigma q} \Omega P^3(2) \times \Omega P^3(2) \xrightarrow{\mu} \Omega P^3(2) \xrightarrow{H} \Omega \Sigma((\Omega P^3(2))^{(2)})$$

is null homotopic.

The proof of this lemma requires the combinatorial calculation of the Hopf invariant $H$.

Let $\phi: P^4(2) \to P^3(2)$ be the map in the cofibre sequence

$$\mathbb{R}P^2 \xrightarrow{c} \mathbb{R}P^4 \xrightarrow{\phi} P^4(2) \xrightarrow{\phi} P^3(2).$$

Let $\phi: P^4(2) \to P^3(2)$ be the map defined above. Then

1) $\phi$ restricted to $S^3$ is homotopic to $\Sigma q: S^3 \to P^3(2)$

and

2) $\phi$ is a co-$H$-map
The Construction of the Map $S^3\{2\} \to F_H(P^3(2))$:

Consider the homotopy commutative diagram of fibre sequences

$$
\begin{array}{cccccc}
\Omega S^3 & \longrightarrow & S^3\{2\} & \longrightarrow & S^3 & \longrightarrow S^3 \\
\downarrow \Omega g & & \downarrow \bar{g} & \downarrow g \\
\Omega P^4(2) & \longrightarrow & \Omega P^4(2) & \longrightarrow & \ast & \longrightarrow P^4(2).
\end{array}
$$

Since the fibre sequence $\Omega S^3 \longrightarrow S^3\{2\} \longrightarrow S^3$ is principal, there is a right $J(S^2)$-action $\mu : S^3\{2\} \times J(S^2) \longrightarrow S^3\{2\}$ with a homotopy commutative diagram

$$
\begin{array}{cccccc}
S^3\{2\} \times J(S^2) & \longrightarrow & \ast & \longrightarrow & S^3\{2\} \\
\downarrow \bar{g} \times \Omega g & & \downarrow \bar{g} \\
J(P^3(2)) \times J(P^3(2)) & \longrightarrow & J(P^3(2)).
\end{array}
$$
Let $\tilde{s}: S^3\{2\} \to J(\mathbb{R}P^2)$ be the composite

$$S^3\{2\} \xrightarrow{\tilde{g}} J(P^3(2)) \xrightarrow{\Omega \phi} J(\mathbb{R}P^2).$$

It follows that there is a homotopy commutative diagram

$$
\begin{array}{ccc}
S^3\{2\} \times J(S^2) & \xrightarrow{\mu} & S^3\{2\} \\
\tilde{s} \times J(q) & & \tilde{s} \\
J(\mathbb{R}P^2) \times J(\mathbb{R}P^2) & \xrightarrow{\mu} & J(P^3(2)).
\end{array}
$$

The composite

$$P^3(2) \times J(S^2) \xrightarrow{\mu} S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{R}P^2) \xrightarrow{H} \Omega \Sigma(J(\mathbb{R}P^2))^{(2)}$$

is null homotopic by the two lemmas above. By the suspension splitting of $S^3\{2\}$, the map

$$\mu^*: [S^3\{2\}, \Omega W] \to [P^3(2) \times J(S^2), \Omega W]$$

is a monomorphism for any $W$. Thus the composite

$$S^3\{2\} \xrightarrow{\tilde{s}} J(\mathbb{R}P^2) \xrightarrow{H} \Omega \Sigma(J(\mathbb{R}P^2))^{(2)}$$
is null homotopic and so the map $\tilde{s}$ lifts to $F_H(P^3(2))$. Let $\bar{s}: S^3\{2\} \to F_H(P^3(2))$ be a lifting of $\tilde{s}$. Since $S^3\{2\}$ is simply connected, the map $\bar{s}$ lifts to the universal cover $F_H(P^3(2))\langle 1 \rangle$ and let $s: S^3\{2\} \to F_H(P^3(2))\langle 1 \rangle$ be a lifting of $\bar{s}$. Then composite

$$S^3\{2\} \xrightarrow{s} F_H(P^3(2))\langle 1 \rangle \xrightarrow{\theta} S^3\{2\}$$

is a homotopy equivalence because it induces an isomorphism on $H_3$ of the atomic space $S^3\{2\}$.

**Note.** For the first non-suspension co-$H$-space $X = \Sigma \mathbb{RP}^2 \cup_f e^6$, the attaching map $f$ is given by the composite

$$S^5 \xrightarrow{\bar{\eta}} P^4(2) \xrightarrow{\phi} P^3(2),$$

where $\bar{\eta}: S^5 \to P^4(2)$ is a lifting of $S^5 \to S^4$. The map $\bar{\eta}$ is a suspension, but $\phi \circ \bar{\eta}$ is not.