Joint with Fred Cohen, Jon Berrick, Yan Loi Wong

We will give some relations between the homotopy groups of the 2-sphere and the braid groups.

We should point out that this is one of research projects in the National University of Singapore. We (Jon Berrick, Kai Meng Tan, Yan Loi Wong and I) have a weekly informal seminar to discuss braids, symmetric groups and homotopy theory.

In addition to the connections between the braids and the homotopy which we are going to talk today, there is another interesting connection between the modular representation theory of the symmetric groups and some ‘natural’ problems in the homotopy theory.

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Configurations and Braids

• Let $M$ be any (path-connected) space. The ordered configuration space $F(M, n)$ is defined by
  \[ F(M, n) = \{(x_1, x_2, \cdots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\} \]

• The symmetric group $\Sigma_n$ acts on $F(M, n)$ by permuting coordinates. The unordered configuration space $B(M, n)$ is defined by $F(M, n)/\Sigma_n$ and so there is a covering $p: F(M, n) \longrightarrow B(M, n)$ with fibre $\Sigma_n$.

• The braid group $B_n(M)$ of $n$ strings over $M$ is defined by
  \[ B_n(M) = \pi_1(B(M, n)) \]

Choose a base point $(q_1, q_2, \cdots, q_n)$ for $F(M, n)$. Let $\omega: S^1 \to B(M, n)$ be a loop. Then there is a lifting path $\lambda: [0, 1] \to F(M, n)$ such that

• $\lambda(0) = (q_1, q_2, \cdots, q_n)$, $\lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$ and $p(\lambda) = \omega$. Thus

• $\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$ with $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$. We obtain $n$ strings $\lambda_i(t)$ in the cylinder $M \times I$ starting at $q_i$ and ending with $q_{\sigma(i)}$ for some $\sigma$. The multiplication is given by the composition of strings.

• The pure braid group $P_n(M)$ is defined by $P_n(M) = \pi_1(F(M, n))$.

The pure braids are $n$ strings $\lambda_i(t)$ in $M \times I$ starting at $q_i$ and ending with $q_i$. 
Brunnian Braids

Consider the coordinate projections
\[ d_i : F(M, n+1) \to F(M, n) \quad (x_0, x_1, \cdots, x_n) \mapsto (x_0, x_1, \cdots, \hat{x}_i, \cdots, x_n). \]

The continuous map \( d_i \) induces, by taking the fundamental group,

\( \bullet \) a group homomorphism \( d_i = d_i^* : P_{n+1}(M) \to P_n(M) \) and

\( \bullet \) a function \( d_i : B_{n+1}(M) \to B_n(M) \) given by
\[ (\lambda_0(t), \cdots, \lambda_n(t)) \mapsto (\lambda_0(t), \cdots, \hat{\lambda_i}(t), \cdots, \lambda_n(t)), \]
that is, deleting the \((i+1)\)-th string for \( 0 \leq i \leq n \).

\( \bullet \) The sequence of groups \( \{ P_{n+1}(M) \}_{n \geq 0} \) with \( d_i \) forms so-called \( \Delta \)-group which we will describe.

\( \bullet \) \( \{ B_{n+1}(M) \}_{n \geq 0} \) with \( d_i \) forms so-called crossed \( \Delta \)-group.

A braid \( \beta \in B_{n+1}(M) \) is called Brunnian if \( d_i(\beta) = 1 \) for all \( 0 \leq i \leq n \).

In other words, the group of Brunnian braids \( Br_{n+1}(M) \) is given by

\( \bullet \) \( Br_{n+1}(M) : = \bigcap_{i=0}^{n} \text{Ker}(d_i). \)

The classical Borromean Ring is a link by closing up a Brunnian braid of 3 strings over \( D^2 \).
Let $X$ be a space. Let’s review the definition of $H_*(X)$. First we consider a sequence of sets $S_*(X) = \{S_n(X)\}_{n \geq 0}$, where $S_n(X)$ is the set of continuous maps from the $n$-simplex $\Delta[n]$ to $X$. The inclusion of the $(i+1)$-th face $d^i: \Delta[n-1] \to \Delta[n]$, $0 \leq i \leq n$, induces a function

$$d_i: S_n(X) = \text{Map}(\Delta[n], X) \to S_{n-1}(X) = \text{Map}(\Delta[n-1], X).$$

Then we have the differential $\partial = \sum_{i=0}^{n}(-1)^i d_i: \mathbb{Z}(S_n(X)) \to \mathbb{Z}(S_{n-1}(X))$ and $H_*(X) = H(\mathbb{Z}(S_n(X)); \partial)$. 

- One may ask whether there is a similar combinatorial definition of the homotopy groups $\pi_*(X)$, where, by definition, $\pi_n(X)$ is the set of the homotopy classes of pointed maps from the sphere $S^n$ to $X$. The answer is “Yes” and it has been much studied since 1950s. People found that the singular simplicial set $S_*(X)$ actually control the homotopy type of the space $X$ in some sense, where one has to add degeneracies $s_i: S_n(X) \to S_{n+1}(X)$, $0 \leq i \leq n$, which are induced by maps $s^i: \Delta[n+1] \to \Delta[n]$. For instance, there are two functions $s^0, s^1: \Delta[1] = [0, 1] \to \Delta[0]$ given by $s^0(t) = 1$ and $s^1(t) = 0$. 
The abstract version of $S_*(X)$ is simplicial set. A simplicial set $X$ means a sequence of sets $X = \{X_n\}_{n \geq 0}$ with faces $d_j: X_n \to X_{n-1}$ and degeneracies $s_j: X_n \to X_{n+1}$ for $0 \leq j \leq n$ such that “simplicial identities” hold. The difference between simplicial sets and simplicial complexes is that: one needs “degeneracies” for simplicial sets. We also call an abstract simplicial complex a $\Delta$-set.

- A simplicial set is in one-to-one correspondence with a cofunctor from finite ordered sets $O$ to sets. Here the morphisms in $O$ are function $f$ such that $f(x) \leq f(y)$ whenever $x \leq y$. Objects in $O$ are given by $n = \{0, 1, \ldots, n\}$. The coface $d^i: n - 1 \to n$ is given by the ordered embedding such that $i$ does not lie the image. The codegeneracy $s^i: n + 1 \to n$ is given by $s^i(i) = s^i(i + 1) = i$ and maps others in order.
- A $\Delta$-set is in one-to-one correspondence with a cofunctor from finite strictly ordered sets to sets.

A simplicial group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with face homomorphisms and degeneracy homomorphisms. A $\Delta$-group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with only face homomorphisms.
• The geometric realization $|G|$ of a simplicial group $G$ is a topological group. Let $G = \{G_n\}_{n \geq 0}$ be a simplicial group.

• The Moore complex: $N_n G = \bigcap_{j=1}^{n} \text{Ker}(d_j : G_n \to G_{n-1})$;

• The Moore cycles: $Z_n G = \bigcap_{j=0}^{n} \text{Ker}(d_j : G_n \to G_{n-1})$;

• The Moore boundaries: $B_n G = d_0(N_{n+1} G)$.

• The sequence of groups $NG = \{N_n G\}$ with $d_0$ is a (non-commutative in general) chain complex. The classical theorem due to John Moore is

★ Theorem. For any simplicial group $G$, the homotopy group $\pi_n(|G|) \cong Z_n(G)/B_n(G)$. In other words, the homotopy groups of the space $|G|$ can be 'computed' combinatorially.

• If $G$ is a $\Delta$-group, then one still has $NG$, $Z(G)$ and $B(G)$ defined in the same way and then $\pi_n(G)$ is defined to be the coset of $Z(G)$ by $B(G)$. 
The Homotopy Groups of Spheres

The central problem in homotopy theory is to study the homotopy groups \( \pi_* (S^n) \). Some classical results are as follows.

\[ \star \quad \pi_1 (S^1) = \mathbb{Z} \text{ and } \pi_r (S^1) = 0 \text{ for } r \geq 2. \]

\[ \star \quad \pi_r (S^n) \text{ for } r < n \text{ and } \pi_n (S^n) = \mathbb{Z}. \]

\[ \star \quad \text{[Hopf]} \quad \pi_r (S^2) = \pi_r (S^3) \text{ for } r \geq 3, \quad \pi_r (S^4) = \pi_{r-1} (S^3) \oplus \pi_r (S^7), \quad \pi_r (S^8) = \pi_{r-1} (S^7) \oplus \pi_r (S^{15}). \]

\[ \star \quad \text{[Serre]} \quad \pi_r (S^{2n+1}) \text{ is finite for } r > 2n + 1 \text{ and so is } \pi_r (S^{2n}) \text{ for } r > 2n \text{ and } r \neq 4n - 1. \quad \pi_{4n-1} (S^{2n}) = \mathbb{Z} \oplus \text{finite}. \]

\[ \star \quad \text{[James]} \quad 4 \cdot (2 - \text{torsion of } \pi_* (S^3)) = 0. \]

\[ \star \quad \text{[Selick]} \quad p \cdot (p - \text{torsion of } \pi_* (S^3)) = 0 \text{ for } p > 2. \]

\[ \star \quad \text{[Cohen-Moore-Neisendorfer]} \quad p^n \cdot (p - \text{torsion of } \pi_* (S^{2n+1})) = 0 \text{ for } p > 2. \]

\[ \star \quad \pi_r (S^3) \text{ has been computed up to } r \leq 64 \text{ or so by various people.} \]

But so far NO good methods can compute general homotopy groups of \( S^n \) for \( n \geq 2 \).
By using simplicial groups, we have the following combinatorial descriptions of $\pi_*(S^2)$.

$\star$ [my thesis] $\pi_n(S^2)$ is isomorphic to the center of the word group $G_n$ defined by

- generators: $x_1, x_2, \ldots, x_n$.
- relation 1: the product $x_1 x_2 \cdots x_n = 1$ and
- relation 2: all iterated commutators
  
  $[x_{i_1}^{\pm 1}, x_{i_2}^{\pm 1}, \ldots, x_{i_t}^{\pm 1}] = 1$

  if each generator $x_j$ occurs in the sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ at least once.

- For instance, $n = 2$, these commutators are $[x_1, x_2]$, $[[x_1, x_2], x_1]$, $[[[x_1, x_2^{-1}], x_2], [x_1, x_2]]$ and etc.

- The Artin representation of the braid group $B_n$ on the free group $F(x_1, \ldots, x_n)$ induces a $B_n$-action on the group $G_n$.

$\star$ [-] The center of $G_n$ is equal to the fixed set of the pure braid group $P_n$ action on $G_n$.

Thus $\pi_{n+1}(S^2)$ is isomorphic to the fixed set of the pure braid group action on $G_n$. 

Our Main Results

Consider the $\Delta$-group $\mathcal{P}(S^2) = \{P_{n+1}(S^2)\}$. We have the following interesting result.

★ Theorem. The homotopy group $\pi_n(S^3)$ isomorphic to

$$
\pi_n(\mathcal{P}(S^2)) = \mathcal{Z}_n(\mathcal{P}(S^2))/\mathcal{B}_n(\mathcal{P}(S^2))
$$

for $n \geq 4$.

- By definition,

$$
\mathcal{Z}_n = \bigcap_{j=0}^{n+1} \ker(d_j: P_{n+1}(S^2) \to P_n(S^2)), \text{(The Brunnian braids over } S^2).)
$$

$$
\mathcal{B}_n = d_0(\bigcap_{j=1}^{n+1} \ker(d_j: P_{n+2}(S^2) \to P_{n+1}(S^2)). \text{(Certain subgroup of Brunnian braids over } S^2).)
$$

- Roughly speaking, consider the sequence of pure braids over $S^2$ and define the faces to be the homomorphism induced by deleting strings. Then the “derived” groups obtained from these groups are the homotopy groups of the 3-sphere.

- $S^2 = \mathbb{C}P^1$ and $P_n(\mathbb{C}P^1)$ is the fundamental group of the variety $F(\mathbb{C}P^1, n)$. The mapping class group on $S^2$ is the quotient of $B_n(S^2)$ by its center $\mathbb{Z}/2$ for $n \geq 3$. 

Our next result is as follows. The canonical embedding $D^2 \subseteq S^2$ induces a group homomorphism $\text{Br}_n(D^2) \longrightarrow \text{Br}_n(S^2)$.

**Theorem.** There is an exact sequence

$$1 \longrightarrow \text{Br}_{n+2}(S^2) \xrightarrow{\partial} \text{Br}_{n+1}(D^2) \longrightarrow \text{Br}_{n+1}(S^2) \longrightarrow \pi_n(S^3) \longrightarrow 1$$

for $n \geq 4$.

- We have not checked the map $\partial$ yet. I guess that this map might be obtained by making one of the strings over $S^2$ as $\infty \times I$ and pushing others to be strings over $D^2$.

- For instance, $\text{Br}_5(S^2)$ modulo $\text{Br}_5(D^2)$ is $\pi_4(S^3) = \mathbb{Z}/2$. The other low homotopy groups of $S^3$ are as follows:

  \[
  \pi_5(S^3) = \mathbb{Z}/2, \pi_6(S^3) = \mathbb{Z}/12, \pi_7(S^3) = \mathbb{Z}/2, \pi_8(S^3) = \mathbb{Z}/2, \pi_9(S^3) = \mathbb{Z}/3, \pi_{10}(S^3) = \mathbb{Z}/15, \text{ and etc.}
  \]

Thus, up to certain range, $\text{Br}_{n+1}(S^2)$ modulo $\text{Br}_{n+1}(D^2)$ are known by non-trivial calculations of $\pi_*(S^3)$. 
Discussions and Questions

• By Serre’s theorem, $\text{Br}_{n+1}(S^2)$ modulo $\text{Br}_{n+1}(D^2)$ must be finite abelian for $n \geq 4$. We checked that $\text{Br}_4(S^2)$ modulo $\text{Br}_4(D^2)$ is non-abelian and infinite. (Why the Brunnian braids of 4 strings are different from other cases? Any geometric explanation?)

• There are families of elements in $\pi_*(S^3)$ which are known as periodic elements obtained by using $K$-theory. Our exact sequence suggests that there are families of Brunnian braids over $S^2$ which represents these elements. Is it possible to describe these braids?

• Let $p$ be an odd prime. Selick’s theorem says that the $p$-torsion component of $\pi_*(S^3)$ is a direct sum of $\mathbb{Z}/p$, which is just a vector space over $\mathbb{Z}/p$. This means that the composite $\text{Br}_{n+1}(S^2) \longrightarrow \pi_n(S^3) \xrightarrow{\text{proj}} p$ – torsion of $\pi_n(S^3)$ factors through the group algebra $\mathbb{Z}/p(\text{Br}_{n+1}(S^2))$. Namely, the $p$-torsion of $\pi_n(S^3)$ is a quotient of the group algebra $\mathbb{Z}/p(\text{Br}_{n+1}(S^2))$. The 2-torsion of $\pi_n(S^3)$ is a quotient of $\mathbb{Z}/4(\text{Br}_{n+1}(S^2))$.

• $\mathbb{Q}(\pi_n(S^3))$ is a finite dimensional quotient algebra of $\mathbb{Q}(\text{Br}_{n+1}(S^2))$ for $n \geq 4$. Does the representation theory of braid groups/mapping class groups tell anything about the homotopy?

• Given a braid on $D^2$, we can obtain a link in $\mathbb{R}^3$ by closing up it. Now given a braid on $S^2$, it seems not clear to me how to get a link in $\mathbb{R}^3$. Since a braid on $S^2$ are strings in $S^2 \times I$, it looks that we at least have a link in the 3-manifold $S^2 \times S^1$. Do any link invariants tell anything about the homotopy?
Some References to this talk.


3) Fred Cohen and Jie Wu, *A representation of homotopy groups into braids*, preprint (not finished yet).

4) Jon Berrick, Fred Cohen, Yanloi Wong and Jie Wu, *Braids, configurations and the homotopy groups*, preprint (not finished yet).


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