Joint with Fred Cohen (University of Rochester), Jon Berrick (National University of Singapore), Yan Loi Wong (National University of Singapore)

I would like to thank Beijing University for giving me a chance to talk this topic.

We will give some relations between the homotopy groups of the 2-sphere and the braid groups. The key point of this talk is to describe the ideas how to establish these relations.

We should point out that this is one of research projects in the National University of Singapore. We (Jon Berrick, Kai Meng Tan, Yan Loi Wong and I) had a weekly informal seminar to discuss braids, symmetric groups and homotopy theory. Fred Cohen and Vershinin will visit NUS in July-August for joint-study on this project.

In addition to the connections between the braids and the homotopy which we are going to talk today, there is another interesting connection between the modular representation theory of the symmetric groups and some ‘natural’ problems in the homotopy theory.

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Configurations and Braids

• Let $M$ be any (path-connected) space. The ordered configuration space $F(M, n)$ is defined by

$$F(M, n) = \{(x_1, x_2, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}.$$  

• It is still an open problem how to determine the homology of $F(M, n)$ for a general manifold $M$. (Even rational homology.)

• The symmetric group $\Sigma_n$ acts on $F(M, n)$ by permuting coordinates. The ordered configuration space $B(M, n)$ is defined by $F(M, n)/\Sigma_n$ and so there is a covering $p: F(M, n) \longrightarrow B(M, n)$ with fibre $\Sigma_n$.

• The braid group $B_n(M)$ of $n$ strings over $M$ is defined by

$$B_n(M) = \pi_1(B(M, n)).$$

Choose a base point $(q_1, q_2, \cdots, q_n)$ for $F(M, n)$. Let $\omega: S^1 \rightarrow B(M, n)$ be a loop. Then there is a lifting path $\lambda: [0, 1] \rightarrow F(M, n)$ such that

• $\lambda(0) = (q_1, q_2, \cdots, q_n)$, $\lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$ and $p(\lambda) = \omega$. Thus

• $\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$ with $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$. We obtain $n$ strings $\lambda_i(t)$ in the cylinder $M \times I$ starting at $q_i$ and ending with $q_{\sigma(i)}$ for some $\sigma$. The multiplication is given by the composition of strings.

• The pure braid group $P_n(M)$ is defined by $P_n(M) = \pi_1(F(M, n))$.

The pure braids are $n$ strings $\lambda_i(t)$ in $M \times I$ starting at $q_i$ and ending with $q_i$. 


• When \( M \) is the unit disk \( D^2 \), \( B_n = B_n(D^2) \) is the classical Artin braid group. **Any link** can be obtained by closing up an (Artin) braid.
• In addition to low dimensional topology, the braids and configurations are used a lot in many other areas such as algebraic geometry, number theory, mathematical physics and etc.

**Brunnian Braids**

Consider the coordinate projections
\[
d_i : F(M, n + 1) \to F(M, n) \quad (x_0, x_1, \cdots, x_n) \mapsto (x_0, x_1, \cdots, \hat{x}_i, \cdots, x_n).
\]
The continuous map \( d_i \) induces, by taking the fundamental group,
• a group homomorphism \( d_i = d_i^* : P_{n+1}(M) \to P_n(M) \) and
• a function \( d_i : B_{n+1}(M) \to B_n(M) \) given by
\[
(\lambda_0(t), \cdots, \lambda_n(t)) \mapsto (\lambda_0(t), \cdots, \hat{\lambda}_i(t), \cdots, \lambda_n(t)),
\]
that is, deleting the \((i + 1)\)-th string for \( 0 \leq i \leq n \).
• The sequence of groups \( \{P_{n+1}(M)\}_{n \geq 0} \) with \( d_i \) forms so-called \( \Delta \)-group which we will describe.
• \( \{B_{n+1}(M)\}_{n \geq 0} \) with \( d_i \) forms so-called **crossed** \( \Delta \)-group.

A braid \( \beta \in B_{n+1}(M) \) is called **Brunnian** if \( d_i(\beta) = 1 \) for all \( 0 \leq i \leq n \). In other words, the group of Brunnian braids \( Br_{n+1}(M) \) is given by
• \( Br_{n+1}(M) : = \bigcap_{i=0}^{n} \text{Ker}(d_i) \).

The classical **Borromean Ring** is a link by closing up a Brunnian braid of 3 strings over \( D^2 \).
• I have been confused about Borromean and Brunnian. Murray Gerstenhaber said that “Borromean” is a town in Italy and “Brunnian” is a mathematician.
Simplicial Groups and $\Delta$-Groups

Let $X$ be a space. Let’s review the definition of $H_*(X)$. First we consider a sequence of sets $S_*(X) = \{S_n(X)\}_{n \geq 0}$, where $S_n(X)$ is the set of continuous maps from the $n$-simplex $\Delta[n]$ to $X$. The inclusion of the $(i + 1)$-th face $d^i: \Delta[n - 1] \rightarrow \Delta[n]$, $0 \leq i \leq n$, induces a function

$$d_i: S_n(X) = \text{Map}(\Delta[n], X) \rightarrow S_{n-1}(X) = \text{Map}(\Delta[n - 1], X).$$

Then we have the differential $\partial = \sum_{i=0}^{n} (-1)^i d_i: Z(S_n(X)) \rightarrow Z(S_{n-1}(X))$ and $H_*(X) = H(Z(S_n(X)); \partial)$.

- One may ask whether there is a similar combinatorial definition of the homotopy groups $\pi_*(X)$, where, by definition, $\pi_n(X)$ is the set of the homotopy classes of pointed map from the sphere $S^n$ to $X$. The answer is “Yes” and it has been much studied since 1950s. People found that the singular simplicial set $S_*(X)$ actually control the homotopy type of the space $X$ in some sense, where one has to add degeneracies $s_i: S_n(X) \rightarrow S_{n+1}(X)$, $0 \leq i \leq n$, which are induced by maps $s^i: \Delta[n + 1] \rightarrow \Delta[n]$. For instance, there are two functions $s^0, s^1: \Delta[1] = [0, 1] \rightarrow \Delta[0]$ given by $s^0(t) = 1$ and $s^1(t) = 0$.

- The abstract version of $S_*(X)$ is simplicial set. A simplicial set $X$ means a sequence of sets $X = \{X_n\}_{n \geq 0}$ with faces $d_j: X_n \rightarrow X_{n-1}$ and degeneracies $s_j: X_n \rightarrow X_{n+1}$ for $0 \leq j \leq n$ such that “simplicial identities” hold. The difference between simplicial sets and simplicial complexes is that: one needs “degeneracies” for simplicial sets. We also call an abstract simplicial complex a $\Delta$-set.
A simplicial set is in one-to-one correspondence with a cofunctor from finite ordered sets $\mathcal{O}$ to sets. Here the morphisms in $\mathcal{O}$ are function $f$ such that $f(x) \leq f(y)$ whenever $x \leq y$. Objects in $\mathcal{O}$ are given by $n = \{0, 1, \ldots, n\}$. The coface $d^i: n - 1 \to n$ is given by the ordered embedding such that $i$ does not lie the image. The codegeneracy $s^i: n + 1 \to n$ is given by $s^i(i) = s^i(i + 1) = i$ and maps others in order.

A $\Delta$-set is in one-to-one correspondence with a cofunctor from finite strictly ordered sets to sets.

A simplicial group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with face homomorphisms and degeneracy homomorphisms. A $\Delta$-group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with only face homomorphisms.

The geometric realization $|G|$ of a simplicial group $G$ is a topological group. Let $G = \{G_n\}_{n \geq 0}$ be a simplicial group.

- The Moore complex: $N_nG = \cap_{j=1}^n \text{Ker}(d_j: G_n \to G_{n-1})$;
- The Moore cycles: $Z_nG = \cap_{j=0}^n \text{Ker}(d_j: G_n \to G_{n-1})$;
- The Moore boundaries: $B_nG = d_0(N_{n+1}G)$.

The sequence of groups $NG = \{N_nG\}$ with $d_0$ is a (non-commutative in general) chain complex. The classical theorem due to John Moore is

$\star$ **Theorem.** For any simplicial group $G$, the homotopy group $\pi_n(|G|) \cong Z_n(G)/B_n(G)$. 
• In other words, suppose that we know the homotopy type of $|G|$, the homotopy groups of the space $|G|$ can be ‘computed’ combinatorially.
• It is a good question how to determine the homotopy type of $|G|$. J. Milnor, D. Kan and G. Carlsson give some constructions for answering this question. I also have a paper on this topic.
• If $G$ is a $\Delta$-group, then one still has $NG$, $Z(G)$ and $B(G)$ defined in the same way and then $\pi_n(G)$ is defined to be the coset of $Z(G)$ by $B(G)$.
• So far $\Delta$-groups have been less studied. We are still playing with (good) examples.
The Homotopy Groups of Spheres

• By definition, let $X$ be a pointed topological space. Then the homotopy group $\pi_n(X) := [S^n, X]$, is the set of the (pointed) homotopy classes of (pointed) continuous maps from the n-sphere $S^n$ to $X$.

• $\pi_0(X)$ is the set of path-connected components of $X$, which is not a group in general. $\pi_1(X)$ is a group, but non-commutative in general. $\pi_n(X)$ is an abelian group for $n \geq 2$.

The central problem in homotopy theory is to study the homotopy groups $\pi_*(S^n)$. Some classical results are as follows.

★ $\pi_1(S^1) = \mathbb{Z}$ and $\pi_r(S^1) = 0$ for $r \geq 2$.

★ $\pi_r(S^n) = 0$ for $r < n$ and $\pi_n(S^n) = \mathbb{Z}$.

★ [Hopf] $\pi_r(S^2) = \pi_r(S^3)$ for $r \geq 3$, $\pi_r(S^4) = \pi_{r-1}(S^3) \oplus \pi_r(S^7)$, $\pi_r(S^8) = \pi_{r-1}(S^7) \oplus \pi_r(S^{15})$.

★ [Serre] $\pi_r(S^{2n+1})$ is finite for $r > 2n+1$ and so is $\pi_r(S^{2n})$ for $r > 2n$ and $r \neq 4n - 1$. $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus$ finite.

★ [James] $4 \cdot (2 - \text{torsion of } \pi_*(S^3)) = 0$.

★ [Selick] $p \cdot (p - \text{torsion of } \pi_*(S^3)) = 0$ for $p > 2$.

★ [Cohen-Moore-Neisendorfer] $p^n \cdot (p - \text{torsion of } \pi_*(S^{2n+1})) = 0$ for $p > 2$.

★ $\pi_r(S^3)$ has been computed up to $r \leq 64$ or so by various people. But so far NO good methods can compute general homotopy groups of $S^n$ for $n \geq 2$. Our theorems (below) give some global structures.

• Čech defined the higher homotopy groups, but abandoned them because they are abelian. (1930s)
• It was originally conjectured that the homotopy groups of spheres are isomorphic to their homology groups. Then Heinz Hopf invented the Hopf map.

• By using simplicial groups, we have the following combinatorial descriptions of $\pi_*(S^2)$.

  ★[From my thesis, appeared in Math. Proc. Camb. Philos. Soc.] $\pi_n(S^2)$ is isomorphic to the center of the word group $G_n$ defined by

  • generators: $x_1, x_2, \ldots, x_n$.
  • relation 1: the product $x_1 x_2 \cdots x_n = 1$ and
  • relation 2: all iterated commutators

$$[x_{i_1}^{\pm 1}, x_{i_2}^{\pm 1}, \ldots, x_{i_t}^{\pm 1}] = 1$$

if each generator $x_j$ occurs in the sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ at least once.

• For instance, $n = 2$, these commutators are $[x_1, x_2]$, $[[x_1, x_2], x_1]$, $[[[x_1, x_2^{-1}], x_2], [x_1, x_2]]$ and etc.

• The Artin representation of the braid group $B_n$ on the free group $F(x_1, \cdots, x_n)$ induces a $B_n$-action on the group $G_n$.

  ★[To appear in Proc. LMS] The center of $G_n$ is equal to the fixed set of the pure braid group $P_n$ action on $G_n$.

Thus $\pi_{n+1}(S^2)$ is isomorphic to the fixed set of the pure braid group action on $G_n$. 
Our Main Results

Consider the $\Delta$-group $\mathcal{P}(S^2) = \{P_{n+1}(S^2)\}$. We have the following interesting result.

$\star$ **Theorem.** The homotopy group $\pi_n(S^3)$ isomorphic to

$$
\pi_n(\mathcal{P}(S^2)) = \mathcal{Z}_n(\mathcal{P}(S^2))/\mathcal{B}_n(\mathcal{P}(S^2))
$$

for $n \geq 4$.

- By definition,

  $$
  \mathcal{Z}_n = \bigcap_{j=0}^{n+1} \ker(d_j: P_{n+1}(S^2) \to P_n(S^2)), \text{ (The Brunnian braids over } S^2)\text{.)}
  $$

  $$
  \mathcal{B}_n = d_0(\bigcap_{j=1}^{n+1} \ker(d_j: P_{n+2}(S^2) \to P_{n+1}(S^2))). \text{ (Certain subgroup of Brunnian braids over } S^2)\text{.)}
  $$

- Roughly speaking, consider the sequence of pure braids over $S^2$ and define the faces to be the homomorphism induced by deleting strings. Then the “derived” groups obtained from these groups are the homotopy groups of the 3-sphere.

- $S^2 = \mathbb{CP}^1$ and $P_n(\mathbb{CP}^1)$ is the fundamental group of the variety $F(\mathbb{CP}^1, n)$. The mapping class group on $S^2$ is the quotient of $B_n(S^2)$ by its center $\mathbb{Z}/2$ for $n \geq 3$. 
Our next result is as follows. The canonical embedding $D^2 \subseteq S^2$ induces a group homomorphism $\text{Br}_n(D^2) \longrightarrow \text{Br}_n(S^2)$.

\textbf{★ Theorem.} There is an exact sequence
\[ 1 \longrightarrow \text{Br}_{n+2}(S^2) \overset{\partial}{\longrightarrow} \text{Br}_{n+1}(D^2) \longrightarrow \text{Br}_{n+1}(S^2) \longrightarrow \pi_n(S^3) \longrightarrow 1 \]
for $n \geq 4$.

• For instance, $\text{Br}_5(S^2)$ modulo $\text{Br}_5(D^2)$ is $\pi_4(S^3) = \mathbb{Z}/2$. The other low homotopy groups of $S^3$ are as follows:

\[ \pi_5(S^3) = \mathbb{Z}/2, \pi_6(S^3) = \mathbb{Z}/12, \pi_7(S^3) = \mathbb{Z}/2, \pi_8(S^3) = \mathbb{Z}/2, \pi_9(S^3) = \mathbb{Z}/3, \pi_{10}(S^3) = \mathbb{Z}/15, \text{and etc.} \]

Thus, up to certain range, $\text{Br}_{n+1}(S^2)$ modulo $\text{Br}_{n+1}(D^2)$ are known by non-trivial calculations of $\pi_*(S^3)$.

• Question 23 in the end of Birman’s red book, J. Birman, \textit{Braids, Links and Mapping Class Groups}, Ann. of Math. Studies, vol. 82, Princeton Univ. Press, Princeton, NJ, 1975, essentially she asked to find the free generators of $\text{Br}_n(S^2)$. If her old question were answered, then, together with some of my works, one has the combinatorial determination of the homotopy groups $\pi_n(S^2)$ by listing generators and relations. Actually, for the purpose of determining generators and relations for $\pi_n(S^2)$, we only need a weak version of Birman’s question.

★ Determine a set of generators for $\text{Br}_n(S^2)$ for $n \geq 5$. It would be very interesting if one can describe the generators for $\text{Br}_n(S^2)$ as certain invariants, say certain link invariants or anything else. One of the ideas might be to construct links in $S^2 \times S^1$ by closing up Brunnian braids in $\text{Br}_n(S^2)$ and then consider certain invariants.
Let $B_n = B_n(D^2)$ be the classical braid groups and let $P_n = P_n(D^2)$.

There is also a striking presentation of the homotopy groups of the sphere solely in terms of Brunnian braids over the disk. First we describe an operation $\tilde{\partial} : B_{n+1} \to B_n$ as follows. Let $\delta : F(\mathbb{C}, n + 1) \to F(\mathbb{C}, n)$ be the map defined by

$$\delta(z_0, z_1, \ldots, z_n) = \left( \frac{1}{z_1 - z_0}, \frac{1}{z_2 - z_0}, \ldots, \frac{1}{z_n - z_0} \right),$$

corresponding geometrically to the reflection map in $\mathbb{C}$ about the unit circle centered at $z_0$. We can show that on fundamental groupoids $\delta$ induces a function $\tilde{\partial} : B_{n+1} \to B_n$ that restricts to a group homomorphism from $P_{n+1}$ to $P_n$ and from $\text{Br}_{n+1}(D^2)$ to $\text{Br}_n(D^2)$. From the braid relations, there is a canonical involution homomorphism $\chi : B_n \to B_n$ that sends each standard generator to its inverse. Likewise it restricts to a group homomorphism from $P_n$ to $P_n$ and from $\text{Br}_n(D^2)$ to $\text{Br}_n(D^2)$.

Composing $\chi$ with $\tilde{\partial}$ gives a homomorphism $\partial$ on $\text{Br}_{n+1}(D^2)$ that maps into $\text{Br}_n(D^2)$ and has the further property that $\partial \circ \partial$ is trivial. We therefore obtain a ‘chain complex’ of nonabelian groups

$$(\text{Br}(D^2), \partial) : \cdots \to \text{Br}_{n+1}(D^2) \xrightarrow{\partial} \text{Br}_n(D^2) \xrightarrow{\partial} \text{Br}_{n-1}(D^2) \to \cdots.$$ 

The homology of this chain complex is a very pleasant surprise ...

★ Theorem. For all $n$ there is an isomorphism of groups

$$H_n(\text{Br}(D^2)) \cong \pi_n(S^2).$$
Let $\Gamma = \{\Gamma_n\}_{n \geq 0}$ be the sequence of groups defined by $\Gamma_0 = 1$ and, for $n \geq 1$, $\Gamma_n = P_n$ with the faces $d_0 = \partial$, and, for $1 \leq i \leq n$, $d_i$ given by deleting the $i$th string. Then $\Gamma$ is a $\Delta$-group.

\textbf{★ Theorem.} $\pi_*(\Gamma) = \pi_*(S^2)$.

- Note that $S^2$ is NOT an $H$-space. There is NO simplicial group model for $S^2$. (The geometric realization of a simplicial group is always a loop space.) This result says that there is a $\Delta$-group model for $S^2$, and these groups are just given by \textbf{Artin pure braid groups}!
- This result suggests that, if we like to study homotopy groups combinatorially, it might not be a bad idea to expand our category of simplicial groups to $\Delta$-groups. Rationally we know that basically DGA is a model for spaces. In $p$-local cases, it might not be a bad idea to systematically study $\Delta$-groups.

\textbf{Some References to this talk.}


3) Fred Cohen and Jie Wu, \textit{A representation of homotopy groups into braids}, preprint.

4) Jon Berrick, Fred Cohen, Yanloi Wong and Jie Wu, \textit{Braids, configurations and the homotopy groups}, preprint.