0.1. Why Do We Want to Decompose a Space? Basically the goal of mathematics is to classify certain objects. For instance, we are able to classify 2-dimensional manifolds. Then we are trying to classify 3-manifolds while Poincaré conjecture sounds difficult to be solved. In homotopy theory, a general question is how to classify spaces (up to homotopy). The general idea for classifying spaces is: (1). Decompose spaces (up to homotopy) into “smaller spaces”; and (2). Study “indecomposable factors” (atomic spaces).

A decomposition of a space $X$ means $X \simeq Y \vee Z$ or $X \simeq Y \times Z$. Usually we want to decompose a co-$H$-space as a wedge of smaller spaces and an $H$-space as a product of smaller spaces.

**Exercise.** A path-connected co-$H$-space does not have a proper product decomposition. A path-connected $H$-space does not have a proper wedge decomposition. (A proper decomposition here means that each factor has non-trivial mod $p$ homology.)

**Sketch of Proof.** I only prove that a co-$H$-space does not have a proper product decomposition. For the second statement, you may use the fact that the homology of an $H$-space is a Hopf algebra. We write $H_*(X)$ for $H_*(X; \mathbb{Z}/p)$. Let $X$ be a co-$H$-space. Then there is a comultiplication $\mu': X \to X \vee X$, that is, there is homotopy commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\mu'} & X \vee X \\
\downarrow \cong & & \downarrow \\
X & \xrightarrow{\Delta} & X \times X \\
\end{array}
$$

It follows that the reduced diagonal $\tilde{\Delta}: X \xrightarrow{\Delta} X \times X \xrightarrow{\text{pinch}} X \wedge X = X \times X/(X \vee X)$ is null homotopic. Let $\psi = \Delta_*: H_*(X) \to H_*(X \times X) = H_*(X) \otimes H_*(X)$ be the diagonal
(or comultiplication). For $x \in \tilde{H}_s(X)$, the reduced diagonal $\bar{\psi}(x) = \psi(x) - x \otimes 1 - 1 \otimes x$, that is, $\bar{\psi} = \bar{\Delta}_s: \tilde{H}_s(X) \rightarrow \tilde{H}_s(X \wedge X) = \tilde{H}_s(X) \otimes \tilde{H}_s(X) \subseteq H_s(X) \otimes H_s(X)$. Since $\bar{\Delta}$ is null homotopic, $\bar{\psi} = \bar{\Delta}_s = 0$. In other words, $\psi(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \tilde{H}_s(X)$.

Suppose that $X \simeq Y \times Z$ is a proper decomposition. Then each $Y$ and $Z$ is a retract of $X$, for instance, $Y \leftarrow Y \times Z \simeq X \times Y \times Z \rightarrow \text{proj.} Y$ is the identity. It follows that each $Y$ and $Z$ is a co-$H$-space. Let $a \neq 0 \in \tilde{H}_s(Y)$ and $b \neq 0 \in \tilde{H}_s(Z)$. Then $a \otimes b \in H_s(Y) \otimes H_s(Z) = H_s(X)$. From the above,

$$\psi(a) = 1 \otimes a + a \otimes 1 \quad \psi(b) = 1 \otimes b + b \otimes 1$$

$$\implies \psi(a \otimes b) = T_{2,3}(1 \otimes a + a \otimes 1) \otimes (1 \otimes b + b \otimes 1)$$

$$= (1 \otimes 1) \otimes (a \otimes b) + (a \otimes 1) \otimes (1 \otimes b) + (-1)^{|a||b|} (1 \otimes b) \otimes (a \otimes 1) + (a \otimes b) \otimes (1 \otimes 1)$$

$$\implies \bar{\psi}(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b) + (-1)^{|a||b|} (1 \otimes b) \otimes (a \otimes 1) \neq 0.$$  

This contradicts to that $\bar{\psi} = 0$. \hfill \Box

**Note.** From the proof, you can see that we only need the condition that $\bar{\Delta}_s: \tilde{H}_s(X) \rightarrow \tilde{H}_s(X) \otimes \tilde{H}_s(X)$ is zero. (So you can give a little generalization of the statement.)

Also you can think similar question: By assuming that $k$-fold reduced diagonal $\bar{\Delta}_k: X \rightarrow X^{(k)} = \bigwedge_{i=1}^k X$ induces a zero map in the reduced homology, what can you say about possible product decompositions.

0.2. **Basic Ideas for Decompositions.** I assume that all spaces (below) are simply-connected $CW$-complexes. In this case, we can use Whitehead theorem that $X \simeq Y$ if and only if there exists $f: X \rightarrow Y$ such that $f_*: H_*(X) \xrightarrow{\cong} H_*(Y)$. (So we can play a lot of homology.) Usually integral homology is more difficult to be computed than rational and mod $p$ homology. This is one of reasons that we consider $p$-local spaces rather than integral spaces. For $p$-local simply connected spaces of finite type, $\bullet$ $X \simeq Y$ if and only if $\exists f: X \rightarrow Y$ with $f_*: H_*(X; \mathbb{Z}_{(p)}) \xrightarrow{\cong} H_*(Y; \mathbb{Z}_{(p)})$, where $\mathbb{Z}_{(p)}$ is the ring of $p$-local integers; if and only if $\exists f: X \rightarrow Y$ such that

$$f: H_*(X; \mathbb{Z}/p) \xrightarrow{\cong} H_*(Y; \mathbb{Z}/p) \quad \text{and} \quad f: H_*(X; \mathbb{Q}) \xrightarrow{\cong} H_*(Y; \mathbb{Q}).$$
For simply connected $p$-completed spaces, $X \simeq Y$ if and only if $\exists f : X \to Y$ with $f_* : H_*(X; \mathbb{Z}/p) \xrightarrow{\cong} H_*(Y; \mathbb{Z}/p)$.

Assume that $X \simeq Y \vee Z$. Let $f_Y : X \to X$ be the composite $X \longrightarrow Y \vee Z \xrightarrow{\text{proj}} Y \leftarrow Y \vee Z \longrightarrow X$. Then $f_Y$ is a homotopy idempotent, that is, $f_Y \circ f_Y \simeq f_Y : X \to X$. In other words, if we know a decomposition, we obtain homotopy idempotents. (Similar situation happens for product decompositions.)

Conversely, suppose that there is a self map $f : X \to X$ (unnecessary homotopy idempotent. The point is that usually it is difficult to see whether one map is homotopic to another.). We consider the homotopy colimit of the sequence

$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots \xrightarrow{\text{hocolim}_f X}$$

with $\tilde{f} : X \to \text{hocolim}_f X$ and $H_*(\text{hocolim}_f X) = \text{colim}_f H_*(X)$ (the stable fixed points of $f_*$). (Note. Under certain conditions (for instance, the homotopy group $\pi_n(X)$ are finite for all $n$), $\text{hocolim}_f X$ is a retract of $X$.) This gives some basic ideas for decomposing spaces, namely, if we know the explicit information on $f_* : H_*(X) \to H_*(X)$, then we obtain a map $\tilde{f} : X \to \text{hocolim}_f X$ with workable homology information. Suppose that we have two maps $g, f : X \to X$ with “enough” good information in homology. Then we can try to show the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{\tilde{f} \times \tilde{g}} \text{hocolim}_f X \times \text{hocolim}_g X$$

is a homotopy equivalence by looking at the homology, or if $X$ is a co-$H$-space, we can try the composite

$$X \xrightarrow{\mu'} X \vee X \xrightarrow{\tilde{f} \vee \tilde{g}} \text{hocolim}_f X \vee \text{hocolim}_g X.$$

**Example.** Let $X$ be simply-connected $p$-local co-$H$-$CW$-complex. Suppose that there is a self map $f : X \to X$ such that $f_* \circ f_* = f_* : H_*(X) \to H_*(X)$, where $H_*(X) = H_*(X; \mathbb{Z}(p))$. Then $X \simeq Y \vee Z$ with $H_*(Y) = \text{Im}(f_* : H_*(X) \to H_*(X))$. 

\[ \]
Proof. For path-connected co-H-space, the homotopy inverse $-1$ is defined. Let $g = \text{id}_X - f$ be the composite

$$X \xrightarrow{\mu'} X \vee X \xrightarrow{\text{id}_X \vee f \cdot (-1)} X \vee X \xrightarrow{\text{fold}} X.$$  

In homology, $g_\ast(x) = x - f_\ast(x)$ for $x \in \tilde{H}_\ast(X)$. Let $Y = \hocolim_f X$ and let $Z = \hocolim_g X$. Since $f_\ast$ is an idempotent, so is $g_\ast$ and hence $H_\ast(Y) = \text{Im}(f_\ast)$ and $H_\ast(Z) = \text{Im}(g_\ast)$. Consider the composite

$$\phi: X \xrightarrow{\mu'} X \vee X \xrightarrow{\tilde{f} \vee \tilde{g}} Y \vee Z$$

In homology

$$\phi_\ast: \tilde{H}_\ast(X) \to \tilde{H}_\ast(Y \vee Z) = \tilde{H}_\ast(Y) \oplus \tilde{H}_\ast(Z), \quad x \mapsto (f_\ast(x), g_\ast(x)) = (f_\ast(x), x - f_\ast(x))$$

is an isomorphism. Thus $X \simeq Y \vee Z$. \hfill $\square$

Exercise. Let $X$ be simply-connected $p$-local $H$-CW-complex. Suppose that there is a self map $f: X \to X$ such that $f_\ast \circ f_\ast = f_\ast: H_\ast(X) \to H_\ast(X)$, where $H_\ast(X) = H_\ast(X; \mathbb{F})$ for $\mathbb{F} = \mathbb{Z}/p, \mathbb{Q}$. Then $X \simeq Y \times Z$ with $H_\ast(Y) = \text{Im}(f_\ast: H_\ast(X) \to H_\ast(X))$.

0.3. Examples of Product Decompositions. Now we some classical examples.

Example 1. If $p > 2$, then $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$.

Idea of Proof. My proof may be different from the usual text books. I just display some “canonical self maps” and leave the computational work to you.

Recall that $H_\ast(\Omega S^{2n}) = H_\ast(\Omega S^{2n}; \mathbb{Z})$ is the tensor algebra $T(x)$ generated by $x$ with dimension $|x| = 2n - 1$. A basis for $H_\ast(\Omega S^{2n})$ is given by $1, x, x^2, x^3, \ldots$.

Observation 1. Let $[k]: S^{2n} \to S^{2n}$ be a map of degree $k$. Then we obtain a family of maps $\Omega[k]: \Omega S^{2n} \to \Omega S^{2n}$. Since $\Omega[k]$ is a loop map, $\Omega[k]_\ast: H_\ast(\Omega S^{2n}) \to H_\ast(\Omega S^{2n})$ is an algebraic map. Because $\Omega[k]_\ast(x) = kx$, we obtain

$$\Omega[k]_\ast(x^s) = (kx)^s = k^sx^s.$$
Observation 2. Since $\Omega S^{2n}$ is an $H$-space, there are power maps $l: \Omega S^{2n} \to \Omega S^{2n}$, $y \mapsto y^l$. For instance, $2: \Omega S^{2n} \to \Omega S^{2n}$ is the composite

$$
\Omega S^{2n} \xrightarrow{\Delta} \Omega S^{2n} \times \Omega S^{2n} \xrightarrow{\mu} \Omega S^{2n}.
$$

We can compute

$$
2_* = \mu_* \circ \Delta_* : H_*(\Omega S^{2n}) = T(x) \to H_*(\Omega S^{2n}) = T(x),
$$

where $\psi = \Delta_*$ is the comultiplication. Since $|x|$ is odd, $x^2$ is primitive, that is, $\psi(x^2) = 1 \otimes x^2 + x^2 \otimes 1$. It follows that

$$
2_*(x^2) = 1 \cdot x^2 + x^2 \cdot 1 = 2x^2.
$$

In general,

$$
\psi(x^{2s}) = (1 \otimes x^2 + x^2 \otimes 1)^s = \sum_{i+j=s} \left( \begin{array}{c} s \\ i \end{array} \right) x^{2i} \otimes x^{2j} \implies 2_*(x^{2s}) = \left( \sum_{i+j=s} \right) x^{2s} = 2^s x^{2s}
$$

Similarly

$$
\psi(x^{2s+1}) = \psi(x \cdot x^{2s}) = \psi(x) \cdot \psi(x^{2s}) = (x \otimes 1 + 1 \otimes x) \cdot \sum_{i+j=s} \left( \begin{array}{c} s \\ i \end{array} \right) x^{2i} \otimes x^{2j}
$$

$$
= \sum_{i+j=s} \left( \begin{array}{c} s \\ i \end{array} \right) x^{2i+1} \otimes x^{2j} + \sum_{i+j=s} \left( \begin{array}{c} s \\ i \end{array} \right) x^{2i} \otimes x^{2j+1}
$$

$$
\implies 2_*(x^{2s+1}) = 2 \left( \sum_{i+j=s} \right) x^{2s} = 2^{s+1} x^{2s}
$$

(Note. $\Omega[2]_*(x^{2s}) = 2^{2s} x^{2s} \neq 2_*(x^{2s})$. In particular, $\Omega[2] \not\cong 2$.)

You can compute $l_*: H_*(\Omega S^{2n}) \to H_*(\Omega S^{2n})$ for other $l$. For $l = -1$, we have $(-1)_*(x) = -x$, $(-1)_*(x^2) = -x^2$, $(-1)_*(x^{2s}) = (-1)^s x^{2s}$ and $(-1)_*(x^{2s+1}) = (-1)^{s+1} x^{2s+1}$. (Note. $(-1)_*$ is a graded anti-homomorphism, that is, $(-1)_*(ab) = (-1)^{|a||b|} (-1)_*(a) \cdot (-1)_*(b)$.)

Construction of Self Maps. First let try $f: \Omega S^{2n} \to \Omega S^{2n}$ with hocolim$_f \Omega S^{2n} \simeq S^{2n-1}$. The map $f$ should satisfy:

$$
f_*(x) = x \quad f_*(x^2) = 0
$$
(Prove that if \( f \) satisfies these conditions, then \( \text{hocolim}_f \Omega S^{2n} \simeq S^{2n-1} \).)

Consider \( g = \Omega[k] * l \). Then \( g_*(x) = (k + l)x \). Since \( x^2 \) is primitive,

\[
g_*(x^2) = \Omega[k]_*(x^2) + l_*(x^2) = (k^2 + l)x^2.
\]

Suppose that \( l = -k^2 \). Then \( g_*(x^2) = 0 \) and \( g_*(x) = (k - k^2)x = k(1 - k)x \). Let \( k = 2 \), that is, \( g = \Omega[2] * (-4) \). Then \( g_*(x) = 2x \) and \( g_*(x^2) = 0 \).

**What do we need from localization?** For a \( p \)-local \( H \)-space \( X \), the \( q \)-th power \( q: X \to X \) is a homotopy equivalence if \( q \not\equiv 0 \mod p \). In other words, the \( q \)-th root \( q^{-1}: X \to X \) is defined. In particular, for \( p \)-local \( H \)-spaces with \( p > 2 \), \( \frac{1}{2}: X \to X \) is defined.

For \( p \)-localization of \( \Omega S^{2n} \) with \( p \neq 2 \), let \( f \) be defined to be the composite

\[
\Omega S^{2n} \xrightarrow{\Omega[2] * (-4)} \Omega S^{2n} \xrightarrow{\frac{1}{2}} \Omega S^{2n}.
\]

Then, \( f_* \) satisfies two conditions above in \( H_*(\Omega S^{2n}; \mathbb{Z}(p)) \).

(Note. This result only holds after \( p \)-localization with \( p \neq 2 \) unless \( 2n = 2, 4, 8 \), that is, integrally, \( \Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1} \) if and only if \( 2n = 2, 4, 8 \). This is the famous Hopf invariant one problem, and it was solved by Adams in 1950s.) \( \square \)

**Generalization.** In this example, the key point is to play with the self maps \( \Omega[k] \) and \( l \) of \( \Omega S^{2n} \). Let \( X \) be any pointed space. Consider the space \( \Omega \Sigma X \), where \( \Sigma X = S^1 \wedge X \).

The degree map \([k]: \Sigma X \to \Sigma X\) is defined to be \([k] \wedge \text{id}_X: S^1 \wedge X \to S^1 \wedge X\). In homology, \([k]_*(x) = kx\) for \( x \in \bar{H}_*(\Sigma X)\). Thus the maps \( \Omega[k]: \Omega \Sigma X \to \Omega \Sigma X \) are defined. Similarly, we have power maps \( l: \Omega \Sigma X \to \Omega \Sigma X \). By studying these canonical maps together with other functorial self maps of \( \Omega \Sigma X \), we obtain a lot of information of functorial decompositions of the spaces \( \Omega \Sigma X \), See References (1), (4) and (5). In Reference (1), we solved the Cohen conjecture.

**Example 2.** After any \( p \)-localization, \( \Omega S^{2n+1} \) is indecomposable for \( n > 0 \).
Proof. In this example, I am trying to explain some ideas how to prove that a space is indecomposable by playing with cohomology ring. I write $H_*(X)$ for $H_*(X; \mathbb{Z}(p))$ in this example.

Suppose that $\Omega S^{2n+1} \simeq Y \times Z$. We show that either $Y \simeq \ast$ or $Z \simeq \ast$. Equivalently, either $Y$ or $Z \simeq \Omega S^{2n+1}$. Since $H_{2n}(\Omega S^{2n+1}) = \mathbb{Z}(p)$, we may assume that $H_{2n}(Y) = \mathbb{Z}(p)$. Let $f$ be the composite $\Omega S^{2n+1} \xrightarrow{\text{proj}} Y \xhookrightarrow{} \Omega S^{2n+1}$. It suffices to show that $f$ is a homotopy equivalence by showing that $f^*: H^*(\Omega S^{2n+1}) \xrightarrow{\cong} H^*(\Omega S^{2n+1})$.

Recall that $H^*(\Omega S^{2n+1})$ is a commutative divided algebra, that is, $H^*(\Omega S^{2n+1})$ has a basis $\gamma_i$ for $i \geq 0$ with $|\gamma_0| = 1$, $|\gamma_i| = 2n$, $\gamma_i \cdot \gamma_j = \binom{i+j}{i} \gamma_{i+j}$. In particular, $\gamma_1^k = k! \gamma_k$.

Now since $f^*: H^{2n}(\Omega S^{2n+1}) \to H^{2n}(\Omega S^{2n+1})$ is an isomorphism, $f_*(\gamma_1) = \alpha \gamma_1$ for some unit element $\alpha \in \mathbb{Z}(p)$. Since $f^*$ is a morphism of algebras,

$$k! f^*(\gamma_k) = f^*(\gamma_1^k) = f^*(\gamma_1)^k = \alpha^k \gamma_1^k = k! \alpha^k \gamma_k.$$

Thus $f^*(\gamma_k) = \alpha^k \gamma_k$ and so $f^*$ is an isomorphism in cohomology. \hfill $\square$

0.4. Examples of Wedge Decompositions. Now we propose a question how to decompose $n$-fold self smash product of the projective plane $\mathbb{R}P^2$. This question was answered in Reference (2), where modular representation theory of symmetric groups is used much. (Our answer is actually much more general. But $\mathbb{R}P^2$ is a typical example.) In this section, we discuss the first three cases. We play with Steenrod operations and, in the third case, we play a little with symmetric group action. The homology $H_*(X)$ is mod 2 homology. Let $P^n(2) = \Sigma^{n-2} \mathbb{R}P^2$ for $n \geq 2$, called mod 2 Moore space, that is, $H_{n-1}(P^n(2); \mathbb{Z}) = \mathbb{Z}/2$, and other homology groups (except $(n-1)$-th and 0-th) are zero. The mod 2 homology $\tilde{H}_*(P^n(2))$ has a basis $u$, $v$ with $|u| = n - 1$, $|v| = n$ and $Sq^1 v = u$. (Note. $P^n(2)$ is the homotopy cofibre of the degree map $[2]: S^{n-1} \to S^{n-1}$.)
Basically we only need few properties of Steenrod operations: (1). $f_* \circ Sq^n = Sq^n \circ f_*$ for any map $f: X \to Y$; and (2). (The Cartan Formula)

$$Sq^n(a \otimes b) = \sum_{i+j=n} Sq^i a \otimes Sq^j b$$

for $a \otimes b \in H_*(X) \otimes H_*(Y) = H_*(X \times Y)$, or $a \otimes b \in \bar{H}_*(X) \otimes \bar{H}_*(Y) = \bar{H}_*(X \wedge Y)$, where $Sq^0 x = x$.

**Example 1.** $P^n(2)$ is indecomposable for all $n \geq 2$.

*Proof.* Let $f: P^n(2) \to P^n(2)$ be any map such that $f_*: H_{n-1}(P^n(2)) \xrightarrow{\cong} H_{n-1}(P^n(2))$, that is $f_*(u) = u$. Then

$$Sq^1 f_*(v) = f_*(Sq^1 v) = f_*(u) = u \neq 0.$$ 

Thus $f_*(v) \neq 0$ and so $f_*(v) = v$. \hfill \Box

**Example 2.** $P^n(2) \wedge P^n(2)$ is indecomposable for all $n \geq 2$.

*Proof.* Let $f: P^n(2) \wedge P^n(2) \to P^n(2) \wedge P^n(2)$ be any map such that $f_*: H_{2(n-1)}(P^n(2) \wedge P^n(2)) \xrightarrow{\cong} H_{2(n-1)}(P^n(2) \wedge P^n(2))$. Now $\bar{H}_*(P^n(2) \wedge P^n(2)) = \bar{H}_*(P^n(2)) \otimes \bar{H}_*(P^n(2))$ has a basis $\{u \otimes u, u \otimes v, v \otimes u, v \otimes v\}$. We write $ab$ for $a \otimes b$. The Steenrod operations are given by

1) $Sq^2 v^2 = Sq^1 v Sq^1 v = u^2$ by the Cartan formula because $Sq^2 v = 0$. 
2) $Sq^1 v^2 = Sq^1 v \cdot v + v \cdot Sq^1 v = uv + vu$.
3) $Sq^1 uv = Sq^1 uv = u^2$.

Since $f_*(u^2) = u^2$ and $Sq^2 v^2 = u^2$, we have $f_*(v^2) = v^2$. It follows that

$$f_*(uv + vu) = f_*(Sq^1 v^2) = Sq^1 f_*(v^2) = Sq^1 v^2 = uv + vu.$$ 

Now because $Sq^1 f_*(uv) = f_*(Sq^1 uv) = f_*(u^2) = u^2$, we have $f_*(uv) = uv$ or $vu$. In both cases, we have $f_*: \bar{H}_*(P^n(2) \wedge P^n(2)) \xrightarrow{\cong} \bar{H}_*(P^n(2) \wedge P^n(2))$. \hfill \Box
The first nontrivial decomposition comes from $(P^n(2))^{(3)}$, the 3-fold self smash. Note $\dim \bar{H}^\ast((P^n(2))^{(3)}) = 2^3 = 8$ with a basis

$$v^3$$
$$Sq^1_v^3 \alpha \beta$$
$$Sq^2_v^3 Sq^1_v \alpha Sq^1_v \beta$$
$$Sq^3_v^3 = u^3,$$

where $\beta = [[u, v], v]$ and $[a, b] = ab - ba$. Write $V = \bar{H}_\ast(P^n(2))$. As a module over the Steenrod algebra, $V^{\otimes 3} = \bar{H}_\ast((P^n(2))^{(3)})$ admits a decomposition

$$V^{\otimes 3} \cong M \oplus L \oplus L,$$

with $\dim M = 4$ and $\dim L = 2$. This tells that there is a chance that

$$(P^n(2))^{(3)} \simeq Y \vee Z \vee Z$$

with $\bar{H}_\ast(Y) = M$ and $\bar{H}_\ast(Z) = L$. For simplicity, we only show how to construct the spaces $Y$, and we assume that $n \geq 3$. (You can see References (2) or (8) for details.)

Let the symmetric group $\Sigma_n$ act on $X^{(n)}$ by permuting coordinates. Let $\sigma = (1, 2, 3)$. Since $P^n(2)$ is a co-$H$-space for $n \geq 3$, let

$$f = \text{id} + \sigma + \sigma^2 : (P^n(2))^{(3)} \rightarrow (P^n(2))^{(3)}.$$  

Then

$$f_\ast(v^3) = 3v^3 = v^3 \quad f_\ast(uw^2) = f_\ast(vuv) = f_\ast(v^2u) = uv^2 + uv + v^2u = Sq^1_v^3$$

$$f_\ast(u^2v) = f_\ast(uvu) = f_\ast(vu^2) = u^2v + uv + vu^2 = Sq^2_v^3 \quad f_\ast(u^3) = 3u^3 = u^3.$$

It follows that $f_\ast \circ f_\ast = f_\ast$. Let $Y = \text{hocolim}_f(P^n(2))^{(3)}$. Then $\bar{H}_\ast(Y) = M$.

0.5. **Functorial Decompositions of Self Smash Products and the Modular Representation Theory of Symmetric Groups.** Now we consider a general question. I assume that any spaces are path-connected $p$-local suspensions $X$ in this section. We want to see how to produce decompositions

$$X^{(n)} \simeq A(X) \vee B(X) \vee \cdots,$$
where $A$ and $B$ are functors on $X$. The key idea is to apply the modular representation theory of symmetric groups.

First let $\Sigma_n$ act on $X^{(n)}$ by permuting coordinates. Thus, for each $\sigma \in \Sigma_n$, we obtain a map $\sigma: X^{(n)} \to X^{(n)}$. Note that

$$\sigma_*: \tilde{H}_*(X^{(n)}) = \tilde{H}_*(X)^{\otimes n} \to \tilde{H}_*(X^{(n)}) = \tilde{H}_*(X)^{\otimes n}$$

is a graded permutation of coordinates, where the homology is taken over the fields $\mathbb{Z}/p$ or $\mathbb{Q}$.

Next for $\sigma + \tau \in \mathbb{Z}(\Sigma_n)$, because $X^{(n)}$ is a co-$H$-space, we can define a map

$$\sigma + \tau: X^{(n)} \to X^{(n)}$$

such that, in homology, $(\sigma + \tau)_* = \sigma_* + \tau_*$. In general, for any element $\alpha \in \mathbb{Z}(\Sigma_n)$, we can define a map $\alpha: X^{(n)} \to X^{(n)}$ with the desired homological properties.

Since we assume that $X$ is a $p$-local co-$H$-spaces, the degree maps $[q]: X \to X$ is a homotopy equivalence (homotopy invertible) and so, for any $\alpha \in \mathbb{Z}(p)\mathbb{Z}(\Sigma_n)$, we can define a map $\alpha: X^{(n)} \to X^{(n)}$ with the desired homological properties.

Now suppose that

$$1 = \sum_{i=1}^{q} e_i$$

is an orthogonal decomposition of the identity in $\mathbb{Z}(p)(S_n)$ in terms of primitive idempotents. Then we have the homotopy decomposition given by

$$X^{(n)} \xrightarrow{\text{comulti.}} \bigvee_{i=1}^{q} X^{(n)} \xrightarrow{\bigvee_{i=1}^{q} e_i} \bigvee_{i=1}^{q} \text{hocolim}_{e_i} X^{(n)}$$

with

$$\tilde{H}_*(\text{hocolim}_{e_i} X^{(n)}) = \text{Im}(e_{i*}: (\tilde{H}_*(X))^{\otimes n} \to (\tilde{H}_*(X))^{\otimes n})$$.

By using modular representation theory (the primitive idempotents correspond to Young diagrams), we produce a functorial decompositions of self smash product of two-cell suspensions. (The homology of each factor can be explicitly determined.) Then we take $X = \mathbb{R}P^2$. (Note. $\mathbb{R}P^2$ is NOT a co-$H$-space, but there are some techniques for solving this trouble in Reference (2).) By using Steenrod operations,
we show that each factor is indecomposable. In other words, we give a complete homotopy decomposition of self smash spaces of $\mathbb{R}P^2$, and this decomposition just comes from **functorial decompositions** of self smash spaces of general spaces!!

**Note.** Modular representation theory of the symmetric groups is much more complicated than rational representation theory. The decomposition of the identity is the fundamental problem in the modular representation theory, and it still remains open (similar to the homotopy groups of spheres). Because of this reason, we are unable to determine a complete functorial decomposition of self smash spaces of three-or-more-cell suspensions.

I also should point out that the problem on “functorial decompositions of self smash spaces of suspensions” is equivalent to the fundamental problem in the modular representation theory of the symmetric groups. This sounds an interesting connection between topology and representation theory.
Some References to this talk.

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