BRAIDS AND HOMOTOPY GROUPS

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The $\Delta$ and Simplicial-Structure on Configurations

• Let $M$ be any space. The ordered configuration space $F(M, n + 1)$ by definition:
  \[ F(M, n + 1) = \{(x_0, x_1, x_2, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j \}. \]

• Consider the sequence of spaces \(\{F(M, n + 1)\}_{n \geq 0}\) with coordinate projections: \(d_i: F(M, n + 1) \to F(M, n)\)
  \[(x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).\]

• Let $A$ be any pointed space. Assume that $M$ has a good base-point (roughly speaking there is an embedding of $\mathbb{R}^+ = [0, \infty)$ into $M$), for instance, $M$ has a whisker. Then
  \[
  \star \text{ the sequence of sets } \{[A, F(M, n+1)]\}_{n \geq 0} \text{ forms a } \Delta\text{-set with faces } d_i \text{ induced by the above maps, namely } d_j d_i = d_i d_{j+1} \text{ for } i \leq j.
  \]

• Note. The only point is that coordinate projections do not preserve base-points, but this base-point trouble can be overcome by assuming that $M$ has a good base-point.

  \[
  \star \text{ If } A \text{ is a cogroup, then } \{[A, F(M, n+1)]\}_{n \geq 0} \text{ is a } \Delta\text{-group. In particular, the sequence of fundamental groups } \{\pi_1(F(M, n+1))\}_{n \geq 0} \text{ is a } \Delta\text{-group.}
  \]

• Note. For unordered configuration space $B(M, n) = F(M, n) / \Sigma_n$, the sequence of groups \(\{\pi_1(B(M, n+1))\}_{n \geq 0}\) is a crossed $\Delta$-group, roughly speaking, faces are only functions (not group homomorphisms) satisfying certain crossed conditions.
Under certain conditions, \( \{[A, F(M, n + 1)]\}_{n \geq 0} \) can be a simplicial set (simplicial group if \( A \) is a cogroup). Our idea is to construct degeneracy \( s_i : F(M, n + 1) \to F(M, n + 2) \) given something like:

\[
(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x'_i, x_{i+1}, \ldots, x_n),
\]

where, roughly speaking, \( x'_i \) is a point very close to \( x_i \) but different from \( \{x_0, \ldots, x_n\} \), and the function \( x'_i(x_0, \ldots, x_n) \) should be continuous.

We consider the case that \( M \) is a metric space with a so-called steady flow. In the case that \( M \) is a differentiable manifold, our condition is equivalent to that \( M \) has a (continuous) nonvanishing vector field (or equivalently \( M \) has zero Euler characteristic).

The most interesting examples in our work are:

- \( \{\pi_1(F(S^2, n + 1))\}_{n \geq 0} \) is a \( \Delta \)-group (but not simplicial group).
- \( \{\pi_1(F(D^2, n + 1))\}_{n \geq 0} \) is a simplicial group.

I will explain the relations between these examples and the general homotopy groups of the sphere.

**Note.** One could compare these ideas with Cohen groups, where roughly speaking the Cohen groups are obtained from the equalizers of the faces of the \( \Delta \)-group \( \{[X^{n+1}, \Omega Y]\}_{n \geq 0} \) and where the \( \Delta \)-structure is obtained by considering coordinate inclusions \( X^n \to X^{n+1} \). If one also consider the coordinate projections of \( X \)'s, then one gets \( \Delta \) and \( \text{co-}\Delta \) on \( \{[X^{n+1}, \Omega Y]\}_{n \geq 0} \) with relations between faces and cofaces. From this, Hopf invariants can be obtained combinatorially by working out formulae on cofaces.
Braids

Consider the covering \( p: F(M, n) \longrightarrow B(M, n) = F(M, n)/\Sigma_n \) with fibre \( \Sigma_n \).

- The **braid group** \( B_n(M) \) of \( n \) strings over \( M \) is defined by
  \[
  B_n(M) = \pi_1(B(M, n)).
  \]

The intuitive description is as follows. Choose a base point \((q_1, q_2, \cdots, q_n)\) for \( F(M, n) \). Let \( \omega: S^1 \to B(M, n) \) be a loop. Then there is a lifting path \( \lambda: [0, 1] \to F(M, n) \) such that
  - \( \lambda(0) = (q_1, q_2, \cdots, q_n), \lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)}) \) for some \( \sigma \in \Sigma_n \) and \( p(\lambda) = \omega \). Thus
  - \( \lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)) \) with \( \lambda_i(t) \neq \lambda_j(t) \) for \( i \neq j \) and \( 0 \leq t \leq 1 \). We obtain \( n \) strings \( \lambda_i(t) \) in the cylinder \( M \times I \) starting at \( q_i \) and ending with \( q_{\sigma(i)} \) for some \( \sigma \). The multiplication is given by the composition of strings.
  - The pure braid group \( P_n(M) \) is defined by \( P_n(M) = \pi_1(F(M, n)) \).

The pure braids are \( n \) strings \( \lambda_i(t) \) in \( M \times I \) starting at \( q_i \) and ending with \( q_i \).

- When \( M \) is the unit disk \( D^2 \), \( B_n = B_n(D^2) \) is the classical Artin braid group. **Any link** can be obtained by closing up an (Artin) braid.
Brunnian Braids

Consider the coordinate projections
\[ d_i : F(M, n+1) \to F(M, n) \quad (x_0, x_1, \ldots, x_n) \mapsto (x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n). \]

The map \( d_i \) induces, by taking the fundamental group,

- a group homomorphism \( d_i = d_{i*} : P_{n+1}(M) \to P_n(M) \) and
- a function \( d_i : B_{n+1}(M) \to B_n(M) \) given by

\[(\lambda_0(t), \ldots, \lambda_n(t)) \mapsto (\lambda_0(t), \ldots, \hat{\lambda}_i(t), \ldots, \lambda_n(t)),\]

that is, deleting the \((i+1)\)-th string for \(0 \leq i \leq n\).

A braid \( \beta \in B_{n+1}(M) \) is called Brunnian if \( d_i(\beta) = 1 \) for all \(0 \leq i \leq n\).

In other words, the group of Brunnian braids \( \text{Br}_{n+1}(M) \) is given by

- \( \text{Br}_{n+1}(M) : = \bigcap_{i=0}^{n} \ker(d_i : B_{n+1}(M) \to B_n(M)). \)

The classical Borromean Rings is a link by closing up a Brunnian braid of 3 strings over \( D^2 \).
Moore Cycles and Moore Chains of \(\Delta\)-groups

We pursue John Moore’s notion for simplicial groups. Let \(G = \{G_n\}_{n \geq 0}\) be a \(\Delta\)-group (not necessarily simplicial group).

- **Moore chains:**
  \[ N_n G = \bigcap_{i > 0} \ker(d_i : G_n \to G_{n-1}) \]
  This gives a chain complex of (non-commutative in general) groups:
  \[ \cdots \to N_{n+1} G \xrightarrow{d_0} N_n G \xrightarrow{d_0} N_{n-1} G \to \cdots, \]
  that is, \(d_0 \circ d_0\) is the trivial homomorphism in \(N G = \{N_n G\}_{n \geq 0}\).

- **Moore cycles:**
  \[ Z_n G = \bigcap_{i=0}^n \ker(d_i : G_n \to G_{n-1}) \]

- **Moore boundaries:**
  \[ B_n G = d_0(N_{n+1} G) \]

- **Moore homotopy groups:**
  \[ \pi_n(G) = Z_n G / B_n G = H_n(N G, d_0) \]

- **Note.** For a \(\Delta\)-group \(G\), the homotopy \(\pi_n(G)\) need not be a group, that is \(B_n G\) need not be normal in \(Z_n G\) in general, (and need not be commutative when \(n \geq 1\)). But some classical results on simplicial groups also hold for *fibrant* \(\Delta\)-groups.

- **Note.** If \(G\) is a simplicial group, then, by the classical Moore Theorem, \(\pi_n(G)\) are the same as the homotopy groups of its geometric realization \(|G|\).
Moore Cycles and Brunnian Braids

Consider the sequence of groups \( \{B_{n+1}(M) = \pi_1(F(M, n+1))\}_{n \geq 0} \).

Observe that the group of Brunnian braids is given by

- \( \text{Br}_{n+1}(M) : = \bigcap_{i=0}^{n} \ker (d_i : B_{n+1}(M) \to B_n(M)). \)

- **Lemma.** Let \( \beta \) be a Brunnian braid of \( n \) strings over \( M \). If \( n \geq 3 \), then \( \beta \) is a pure braid. Thus

  \( \star \) For \( n \geq 2 \), \( \text{Br}_{n+1}(M) : = \bigcap_{i=0}^{n} \ker (d_i : P_{n+1}(M) \to P_n(M)) \) is the

  **Moore cycles of the \( \Delta \)-group** \( \{P_{n+1}(M) = \pi_1(F(M, n+1))\}_{n \geq 0} \).

  In other words, Brunnian braids are essentially Moore cycles.

- The braided interpretation of boundaries seems unclear. As we know that the Moore homotopy groups are certain derived groups of \( \Delta \)-groups. The Moore homotopy groups of \( \{P_{n+1}(M)\}_{n \geq 0} \) MIGHT be certain invariants on braids. When \( M = S^2 \), the following theorem gives a connection with the homotopy groups of the sphere.

  \( \star \) **Theorem A.** Let \( \mathcal{F}(S^2)^{\pi_1} = \{P_{n+1}(S^2)\}_{n \geq 0} \) be the \( \Delta \)-group defined above. Then for each \( n \geq 1 \) \( \pi_n(\mathcal{F}(S^2)^{\pi_1}) \) is a group, and there is an isomorphism of groups

  \[ \pi_n(\mathcal{F}(S^2)^{\pi_1}) \cong \pi_n(S^2) \cong \pi_n(S^3) \]

  for \( n \geq 4 \).
Main Results

In addition to Theorem A. Our next theorem directly gives connections between the Brunnian braids and the homotopy groups.

The canonical embedding $f: D^2 \subseteq S^2$ induces a group homomorphism $\text{Br}_n(D^2) \xrightarrow{f_*} \text{Br}_n(S^2)$.

★ Theorem B. There is an exact sequence of groups

$$1 \longrightarrow \text{Br}_{n+1}(S^2) \longrightarrow \text{Br}_n(D^2) \xrightarrow{f_*} \text{Br}_n(S^2) \longrightarrow \pi_{n-1}(S^2) \longrightarrow 1$$

for $n \geq 5$.

- For instance, $\text{Br}_5(S^2)$ modulo $\text{Br}_5(D^2)$ is $\pi_4(S^3) = \mathbb{Z}/2$. The other low homotopy groups of $S^3$ are as follows:

  $$\pi_5(S^3) = \mathbb{Z}/2, \pi_6(S^3) = \mathbb{Z}/12, \pi_7(S^3) = \mathbb{Z}/2, \pi_8(S^3) = \mathbb{Z}/2, \pi_9(S^3) = \mathbb{Z}/3, \pi_{10}(S^3) = \mathbb{Z}/15,$$

  and etc.

  Thus, up to certain range, $\text{Br}_{n+1}(S^2)$ modulo $\text{Br}_{n+1}(D^2)$ are known by non-trivial calculations of $\pi_*(S^3)$.
Question 23 in the end of Birman’s red book, J. Birman, *Braids, Links and Mapping Class Groups*, Ann. of Math. Studies, vol. 82, Princeton Univ. Press, Princeton, NJ, 1975, essentially she asked to find the free generators of $Br_n(S^2)$. If her old question were answered, then, together with some of my works, one has the combinational determination of the homotopy groups $\pi_n(S^2)$ by listing generators and relations. Actually, for the purpose of determining generators and relations for $\pi_n(S^2)$, we only need a weak version of Birman’s question.

★ **Weak Form of Birman’s Problem:** Determine a set of generators for $Br_n(S^2)$ for $n \geq 5$.

It would be very interesting if one can describe the generators for $Br_n(S^2)$ as certain invariants, say certain link invariants or anything else. One of the ideas might be to construct links in $S^2 \times S^1$ by closing up Brunnian braids in $Br_n(S^2)$ and then consider certain invariants.
Our next result gives connections between the classical braid groups and the homotopy groups. Since the disk $D^2$ admits a nonvanishing vector field, $\{P_{n+1}(D^2)\}_{n \geq 0}$ is a (contractible) simplicial group. Our idea is to add one more canonical face, in addition to coordinate projections, such that $\{P_n(D^2)\}_{n \geq 0}$ is a $\Delta$-group with non-trivial Moore homotopy groups.

Let $B_n = B_n(D^2)$ be the classical braid groups and let $P_n = P_n(D^2)$. First we describe an operation $\tilde{\partial}: B_{n+1} \to B_n$ as follows.

Let $\delta: F(\mathbb{C}, n+1) \to F(\mathbb{C}, n)$ be the map defined by

$$\delta(z_0, z_1, \ldots, z_n) = \left( \frac{1}{\bar{z}_1 - \bar{z}_0}, \frac{1}{\bar{z}_2 - \bar{z}_0}, \ldots, \frac{1}{\bar{z}_n - \bar{z}_0} \right),$$

corresponding geometrically to the reflection map in $\mathbb{C}$ about the unit circle centered at $z_0$.

We can show that on fundamental groupoids $\delta$ induces a function $\tilde{\partial}: B_{n+1} \to B_n$ that restricts to a group homomorphism from $P_{n+1}$ to $P_n$ and from $\text{Br}_{n+1}(D^2)$ to $\text{Br}_n(D^2)$.

From the braid relations, there is a canonical involution homomorphism $\chi: B_n \to B_n$ that sends each standard generator to its inverse. Likewise it restricts to a group homomorphism from $P_n$ to $P_n$ and from $\text{Br}_n(D^2)$ to $\text{Br}_n(D^2)$. 
Composing $\chi$ with $\tilde{\partial}$ gives a homomorphism $\partial$ on $\text{Br}_{n+1}(D^2)$ that maps into $\text{Br}_n(D^2)$ and has the further property that $\partial \circ \partial$ is trivial.

We therefore obtain a ‘chain complex’ of nonabelian groups

$$(\text{Br}(D^2), \partial) : \cdots \to \text{Br}_{n+1}(D^2) \xrightarrow{\partial} \text{Br}_n(D^2) \xrightarrow{\partial} \text{Br}_{n-1}(D^2) \to \cdots.$$ 

The homology of this chain complex is a very pleasant surprise ...

★ Theorem C. For all $n$ there is an isomorphism of groups

$$H_n(\text{Br}(D^2)) \cong \pi_n(S^2).$$

- Let $\Gamma = \{\Gamma_n\}_{n \geq 0}$ be the sequence of groups defined by $\Gamma_0 = 1$ and, for $n \geq 1$, $\Gamma_n = P_n$ with the faces $d_0 = \partial$, and, for $1 \leq i \leq n$, $d_i$ given by deleting the $i$th string. Then $\Gamma$ is a $\Delta$-group.

★ Theorem D. $\pi_*(\Gamma) = \pi_*(S^2)$.

- Note that $S^2$ is NOT an $H$-space. There is NO simplicial group model for $S^2$. (The geometric realization of a simplicial group is always a loop space.) This result says that there is a $\Delta$-group model for $S^2$, and these groups are just given by **Artin pure braid groups**!

- Note $P_n$ is a semi-direct product of $F_{n-1}, F_{n-2}, \ldots, F_1$, where $F_k$ is the free group of rank $k$. As a sequence of sets, $\Gamma$ looks like $\widetilde{WF}[S^1]$, the classifying space of Milnor’s construction on $S^1$. 


Examples

Let $\sigma_i$ denote the usual generator for Artin braid group $B_{n+1} = \pi_1(B(D^2, n + 1))$ for $0 \leq i \leq n - 1$. (Note. Our counting always starts from 0. So $\sigma_i$ really means $\sigma_{i+1}$ in Birman’s book.) Write $\delta_i$ for the image of $\sigma_i$ in $\pi_1(B(S^2, n + 1))$.

★ The Brunnian group $\text{Br}_4(S^2)$ is the free group of rank 5 generated by the braids $\delta_0^4$, $\delta_0\delta_1\delta_0^{-1}$, $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^{-1}$, $\delta_0\delta_1^{-2}\delta_0^2\delta_1\delta_0^{-1}$ and $\delta_0^{-1}\delta_1^4\delta_0$.

The pictures of these braids are as follows.

![](image)

- Remark. By deleting the last trivial string of the 4-string braid $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1\delta_0^{-1}$ over $S^2$ we obtain the 3-string braid $\sigma_0^{-1}\sigma_1^{-2}\sigma_0^2\sigma_1\sigma_0^{-1}$ over $D^2$. In turn, closing up this 3-string braid gives a link that is readily seen to be the Borromean rings. This link corresponds to a Moore cycle in $F[S^1]_2$, where the Milnor construction $F[S^1]$ is the simplicial group model for $\Omega S^2$, that represents the generator $\eta_2$ for $\pi_2(\Omega S^2) = \pi_3(S^2)$. In other words, the Hopf map $\eta_2 : S^3 \to S^2$ corresponds to the Borromean rings in this way.