I am going to talk some applications of group representations in homotopy theory. Of course I only aim to certain particular questions. The talk will consist of:

1. Two Questions Arising from Homotopy Theory.


1. Questions

One of general ideas in mathematics to classify certain mathematical objects is:
1) decompose an object into ‘smallest factors’ (indecomposable factors) and
2) study these smallest objects.
We asked the following questions in homotopy theory.

**Question 1.** Consider functorial decompositions
\[ \Omega \Sigma X \simeq A(X) \times B(X) \times \cdots, \]
where \( A \) and \( B \) are functors.
How to determine the (mod \( p \)) homology of the factors \( A(X) \), \( B(X) \) and etc?

**Question 2.** Consider functorial decompositions of the \( n \)-fold self-smashes
\[ X^{(n)} \simeq A(X) \vee B(X) \vee \cdots, \]
where \( X \) runs over suspensions.
How to determine the (mod \( p \)) homology of the factors \( A(X) \), \( B(X) \) and etc?

★ These questions are essentially equivalent to the **fundamental problem** of the modular representation theory of the symmetric groups.
2. Algebraic Models

The algebraic model for Question 2 is relatively easier. So we look at Question 2 first.

Let \( X = \Sigma Y \) and let \( S_n \) act on \( X^{(n)} \) by permuting positions. For each \( \sigma \in S_n \), we have a map

\[
\sigma : X^{(n)} \longrightarrow X^{(n)}
\]

and so a map

\[
S_n \longrightarrow [X^{(n)}, X^{(n)}]
\]

which extends a morphism of semi-rings

\[
\theta : \mathbb{Z}(S_n) \longrightarrow [X^{(n)}, X^{(n)}],
\]

where the right head side is a semi-ring because \( X \) is a suspension.

If \( X \) is a \( p \)-local suspension, the map \( \theta \) extends to a morphism of semi-rings

\[
\theta : \mathbb{Z}(p)(S_n) \longrightarrow [X^{(n)}, X^{(n)}].
\]

Let \( V = \bar{H}_*(X) \) be the reduced mod \( p \) homology of \( X \). Then we have the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}(p)(S_n) & \xrightarrow{\theta} & [X^{(n)}, X^{(n)}] \\
\downarrow & & \downarrow \text{H}_* \\
\mathbb{Z}/p(S_n) & \xrightarrow{\theta} & \text{Hom}(V^\otimes n, V^\otimes n),
\end{array}
\]

where \( \theta : \mathbb{Z}/p(S_n) \rightarrow \text{End}(V^\otimes n) \) is the usual tensor product representation of the symmetric groups in the (modular) representation theory.
Diagram (1) is the basic idea for doing functorial decompositions of self-smashes:

Suppose that there is a decomposition of the identity

\[ 1 = \sum_{\alpha} Q_\alpha \]

in terms of orthogonal primitive idempotents in $\mathbb{Z}/p(S_n)$. According to the modular representation theory of the symmetric groups, this decomposition lifts to $\mathbb{Z}(p)(S_n)$ and so we may consider Equation (2) is a decomposition in $\mathbb{Z}(p)(S_n)$. (This is a special property of the symmetric groups.) For each $\alpha$ and any $p$-local suspension $X$, we can construct the space

\[ Q_\alpha(X) = \mathrm{hocolim}_{\alpha} X^{(n)} \]

given by the homotopy colimit of the sequence

\[ X^{(n)} \xrightarrow{Q_\alpha} X^{(n)} \xrightarrow{Q_\alpha} X^{(n)} \rightarrow \cdots \]

This induces a functorial decomposition

\[ X^{(n)} \simeq \bigvee_{\alpha} Q_\alpha(X) \]

with

\[ \bar{H}_*(Q_\alpha(X)) = Q_\alpha(V) = \mathrm{colim}_{\alpha} V^{\otimes^n} = \mathrm{Im}(Q_\alpha : V^{\otimes^n} \to V^{\otimes^n}). \]
According to the representation theory, the primitive idempotents of the symmetric group algebra only depend on the characteristic of the ground field. In other words, we can change the group field $\mathbb{Z}/p$ to any field $k$ of characteristic $p$.

If the group field $k$ has infinite elements and $V$ is an ungraded module (that is, $p = 2$, or $p > 2$ with $V_{\text{odd}} = 0$, then
\[
\text{Im}(\theta: k(S_n) \to \text{End}(V^\otimes n)) = \text{End}_{\text{GL}(V)}(V^\otimes n).
\]

Under the above assumptions, the determination of $Q_\alpha(V)$ is equivalent to the decompositions of $V^\otimes n$ over the general linear group $\text{GL}(V)$.

If $k$ is a small field (say $\mathbb{Z}/p$), then the above general linear group $\text{GL}(V)$ is replaced by so-called hyperalgebra over $V$, and so $Q_\alpha(V)$ is equivalent to the decompositions of $V^\otimes n$ over the hyperalgebra. (Roughly speaking the hyperalgebra is the cofree cocommutative Hopf covering of the algebra $\text{End}(V)$, which seems close to the Steenrod algebra.)
★ **Theorem** Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{K}$ and let $X = \mathbb{F}P^2$. Let $V = \tilde{H}_*(X)$ with a basis $\{u, v\}$. Let $P_{2s}(V)$ be the module with a basis $\{u^{2^s}, v^{2^s}\}$. Let $s(n)$ be the integer such that $2^{s(n)} - 1 \leq n \leq 2^{s(n)+1} - 2$ and let

$$n - 2^{s(n)} + 1 = 2^{a(1;n)} + 2^{a(2;n)} + \cdots + 2^{a(l(n);n)}$$

with $0 \leq a(1;n) < a(2;n) < \cdots < a(l(n);n)$. Then there is a sequence of spaces $\mathbb{F}Q^k$ for $k \geq 1$ with the following properties:

1) $X^{(n)} \simeq \mathbb{F}Q^n \vee \bigvee_{i=1}^{[n-1/2]} \bigvee_{j=1}^{\frac{n-1}{2}} \Sigma(|u|+|v|)i \mathbb{F}Q^{n-2i}$ localized at 2.

2) $\mathbb{F}Q^{2n} \simeq \mathbb{F}Q^{2n-1} \wedge \mathbb{F}P^2$.

3) There is an isomorphism of modules over the Steenrod algebra

$$\tilde{H}_*(\mathbb{F}Q^n) \cong \bigotimes_{j=0}^{s(n)-1} P_{2j}(V) \otimes \bigotimes_{j=1}^{l(n)} P_{2a(j;n)}(V);$$

4) $\mathbb{F}Q^n$ is an atomic space for each $n$.

★ **Examples.**

- $\mathbb{R}Q^3 \simeq \mathbb{C}P^2 \wedge \mathbb{R}P^2$,
- $\mathbb{R}Q^5 \simeq \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$,
- $\mathbb{R}Q^7 \simeq \mathbb{H}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{R}P^2$,
- $\mathbb{C}Q^3 \simeq \mathbb{H}P^2 \wedge \mathbb{C}P^2$. 
Now we consider Question 1. We start with an algebraic problem.

Consider the tensor algebra functor $T, V \mapsto T(V)$, from modules to Hopf algebras, where $T(V)$ is a Hopf algebra by saying $V$ primitive. Let $\mathcal{H} = \text{coalg}(T, T)$ denote the set of natural coalgebra self-transformations of $T$. Then $\mathcal{H}$ is a group under the convolution product

$$T(V) \xrightarrow{\psi} T(V) \otimes T(V) \xrightarrow{f_V \otimes g_V} T(V) \otimes T(V) \xrightarrow{\mu} T(V).$$

Filter $T(V)$ by

$$\text{Fil}_n T(V) = \bigoplus_{j=0}^{n} T_j(V)$$

Then each $\text{Fil}_n T(V)$ is a subcoalgebra of $T(V)$ and so a tower of groups

$$\mathcal{H} = \text{coalg}(T, T) \longrightarrow \cdots \longrightarrow \text{coalg}(\text{Fil}_n T, T) \longrightarrow \text{coalg}(\text{Fil}_{n-1} T, T) \longrightarrow \cdots.$$

Let $\bar{V}$ be a fixed free $R$-module of rank $n$ with a basis $\{x_1, \ldots, x_n\}$. Let $\text{Lie}(n)$ be the submodule of $\bar{V}^\otimes n$ spanned by the commutators

$$\{[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)}] \mid \sigma \in S_n\}.$$ 

Let $S_n$ act on $\text{Lie}(n)$ by permuting the letters.

\begin{itemize}
  \item \textbf{Theorem.} Each
    
    $$\text{coalg}(\text{Fil}_n T, T) \longrightarrow \text{coalg}(\text{Fil}_{n-1} T, T)$$
    
    is a group epimorphism with the kernel given by $\text{Lie}(n)$.
  
    Write $\mathcal{H}_n$ (or $\mathcal{H}_n^R$) for $\text{coalg}(\text{Fil}_n T, T)$. We have the progroup
    
    $$\mathcal{H} \longrightarrow \cdots \longrightarrow \mathcal{H}_n \longrightarrow \mathcal{H}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{H}_1 = R$$
\end{itemize}
Now we come back to consider Question 1. Let $X$ be any suspension and let $J(X) \simeq \Omega \Sigma X$ be the James construction of $X$ with the James filtration

$$J_0(X) = \ast \subseteq J_1(X) = X \subseteq \cdots \subseteq J_n(X) \subseteq \cdots.$$ 

Let $q_n : X^n \to J_n(X)$ be the quotient map. By the suspension splitting theorem,

1. There is a tower of group epimorphisms

$$[J_n, \Omega \Sigma] \longrightarrow \cdots \longrightarrow [J_1, \Omega \Sigma] \longrightarrow [J_0, \Omega \Sigma] \longrightarrow \cdots$$

for any $X$.

2. There is a group monomorphism

$$q^*_n : [J_n(X), \Omega \Sigma X] \hookrightarrow [X^n, \Omega \Sigma X].$$

F. Cohen considered the group homomorphism from the free group $F_n$

$$\theta : F_n = F(x_1, \cdots, x_n) \longrightarrow [X^n, \Omega \Sigma X]$$

sending the generator $x_i$ to the homotopy class represented by

$$X^n \xrightarrow{\pi_i} X \xleftarrow{\text{proj.}} \Omega \Sigma X.$$

If $X$ is a suspension, the map $\theta$ factors through the quotient group $K_n$ of $F_n$ by the commutator relations

$$[[x_{i_1}, x_{i_2}], \cdots, x_{i_t}] = 1$$

if $i_s = i_t$ for some $s < t$.

By pulling back along $q^*_n$, he obtained a representation

$$\theta : \mathcal{H}_n^Z \longrightarrow [J_n(X), \Omega \Sigma X]$$

for any suspension $X$. 
This gives a morphism of progroups
\[ \theta: \mathcal{H}^\mathbb{Z} \longrightarrow [J(X), \Omega \Sigma X] \]
for any suspension \( X \).

A modification of his work gives a morphism of progroups
\[ \theta: \mathcal{H}^\mathbb{Z}(p) \longrightarrow [J(X), \Omega \Sigma X] \]
for any suspension \( X \).

Our basic idea for working on Question 1 is the following commutative diagram of progroups
\[
\begin{array}{ccc}
\mathcal{H}^\mathbb{Z}(p) & \xrightarrow{\theta} & [J, J] \\
\downarrow & & \downarrow H_* \\
\mathcal{H}^\mathbb{Z}/p & \cong & \text{coalg}(T, T),
\end{array}
\]
where \( J \) is the functor from the \( p \)-local suspensions to spaces and the group ring for \( T \) is \( \mathbb{Z}/p \).
Some results.

• Let $f(V): T(V) \to T(V)$ be a functorial coalgebra map and let $W$ be a connected graded module. Then there is a functorial coalgebra map $\tilde{f}(W): T(W) \to T(W)$ such that $\tilde{f}(W) = f(W)$ when $W_{\text{odd}} = 0$. In other words, all functorial coalgebra self maps of tensor algebras on “ungraded” modules canonically extends to functorial coalgebra self maps of tensor algebras on graded modules. We will still write $f$ for $\tilde{f}$.

• Let $T(V) \cong A(V) \otimes B(V)$ be a functorial coalgebra decompositions and let $W$ be any connected graded module. Then there is a functorial coalgebra decomposition $T(W) \cong \tilde{A}(W) \otimes \tilde{B}(W)$ such that $\tilde{A}(W) = A(W)$ and $\tilde{B}(W) = B(W)$ when $W_{\text{odd}} = 0$. We will still write $A$ for $\tilde{A}$.

Let $X$ be any path-connected space. Recall that $H_*(\Omega \Sigma X)$ is isomorphic to $T(\bar{H}_*(X))$ as an algebra. By taking the augmentation ideal filtration, we have $\text{Gr}(H_*(\Omega \Sigma X)) \cong T(V)$ as Hopf algebras, where $V = \bar{H}_*(X)$ as a module and $T(V)$ is Hopf by saying $V$ primitive.

• Let $f: T(V) \to T(V)$ be any functorial coalgebra map and let $X$ be any path-connected $p$-local (or rational) space. Then there is a functorial map $\phi: \Omega \Sigma X \to \Omega \Sigma X$ such that

$$\text{Gr}(\phi_*) = f(\bar{H}_*(X)): \text{Gr}(H_*(\Omega \Sigma X)) = T(\bar{H}_*(X)) \to \text{Gr}(H_*(\Omega \Sigma X)) = T(\bar{H}_*(X)).$$

• Let $T(V) \cong A(V) \otimes B(V)$ be any functorial decomposition of $T(V)$ and let $X$ be any path-connected $p$-local (or rational) space. Then there are homotopy functors $A$ and $B$ from spaces to spaces such that (1). $\Omega \Sigma X \simeq A(X) \times B(X)$, (2). $\text{Gr } H_*(A(X)) = A(\bar{H}_*(X))$ and (3). $\text{Gr } H_*(B(X)) \cong B(\bar{H}_*(X))$.

Thus the geometric problem on functorial decompositions of $\Omega \Sigma$ reduces to the corresponding algebraic question.
3. The functor $A^{\text{min}}$ and the Functorial Version of Poincaré-Birkhoff-Witt Theorem

The PBW theorem says that $T(V) \cong \Lambda(L(V))$, where $\Lambda(W)$ is the free commutative algebra generated by $W$. Unfortunately the PBW theorem is not functorial in modular case. In other words, this isomorphism depends on a choice of basis for $V$.

**Note.** If $k$ is of characteristic 0, then there is a functorial coalgebra isomorphism $T(V) \cong \Lambda(L(V))$. In other words, the PBW theorem holds functorially in rational case.

Consider the modular case, that is, $k$ is of characteristic $p$ now. Let $\text{Lie}^{\text{max}}(n)$ denote a maximal (largest) projective $k(\Sigma_n)$-submodule of $\text{Lie}(n)$. For instance,

$\text{Lie}(n)$ is projective if and only if $n \not\equiv 0 \mod p$. Thus

$$\text{Lie}^{\text{max}}(n) = \text{Lie}(n)$$

if and only if $n \not\equiv 0 \mod p$.

Let $V$ be any (graded) module and let $\Sigma_n$ act on $V^\otimes n$ by permuting positions in graded sense. For instance $n = 2$, then $\tau = (12) : V^\otimes 2 \rightarrow V^\otimes 2$ is given by $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$. Let

$$L_n^{\text{max}}(V) = V^\otimes n \otimes_{k(\Sigma_n)} \text{Lie}^{\text{max}}(n).$$

This defines a functor $L_n^{\text{max}}$ with $L_n^{\text{max}}(V) \subseteq L_n(V)$. 
Let $B_{\text{max};n}(V)$ be the subalgebra of $T(V)$ generated by $L_m^{\text{max}}(V)$ for $m \geq n$. Then we obtain a sequence of functorial sub-Hopf algebras of $T(V)$
\[ \cdots \subseteq B_{\text{max};n}(V) \subseteq \cdots \subseteq B_{\text{max};2}(V) \subseteq B_{\text{max};1}(V) = T(V). \]

Let $A_{\text{min};n} = k \otimes_{B_{\text{max};n+1}(V)} B_{\text{max};n}(V)$. Then $A_{\text{min};n}(V)$ is a “smallest” coalgebra retract of $T(V)$ which contains $L_n^{\text{max}}(V)$.

★ Theorem. There is a functorial coalgebra decomposition
\[
T(V) \cong \bigotimes_{n=1}^{\infty} A_{\text{min};n}(V) \cong B_{\text{max};n+1}(V) \otimes \bigotimes_{j=1}^{n} A_{\text{min};j}(V).
\]

This decomposition is the functorial version of the PBW theorem because, in rational case, $L_{n}^{\text{max}}(V) = L_n(V)$ for each $n$ and $A_{\text{min};n}(V) = \Lambda(L_n(V))$.

By taking the geometric realization, we have
\[
\Omega \Sigma X \simeq \prod_{n=1}^{\infty} A_{\text{min};n}(X) \cong B_{\text{max};n+1}(X) \times \prod_{j=1}^{n} A_{\text{min};j}(X)
\]
for any path-connected $p$-local space $X$. Furthermore, we know that
\[ B_{\text{max};n}(X) \simeq \Omega \Sigma Q_{\text{max};n}(X) \]
functorially for some functor $Q_{\text{max};n}$. 
The most interesting factor is $A_{\text{min};1}(X)$. Roughly speaking this is a the smallest functorial retract of $\Omega \Sigma X$ which contains the bottom cell. We write $A_{\text{min}}(X)$ for $A_{\text{min};1}(X)$. Rationally, $A_{\text{min}}(V) = \Lambda(V)$. In modular case, $A_{\text{min}}(V)$ has at least sub-exponential growth and so it is not so small. The decomposition above says that

$$\Omega \Sigma X \simeq \Omega \Sigma (Q_{\text{max};2}(X)) \times A_{\text{min}}(X).$$

Now the question is how to determine $A_{\text{min}}(V)$. As we see above, $B_{\text{max};2}(V)$ is the subalgebra of $T(V)$ generated by $L_{\text{max}}^n(V) = V^{\otimes n} \otimes_{k(\Sigma_n)} \text{Lie}_{\text{max}}^n(n)$ for $n \geq 2$. Thus one may want to know what is $\text{Lie}_{\text{max}}^n(n)$. It turns out that this is very difficult question in modular representation theory of symmetric groups. In some sense, this is exactly the fundamental problem in modular representation theory. We are only able to determine an upper bound for $A_{\text{min}}$.

Let $A_{\text{min}}^n(V) = T_n(V) \cap A_{\text{min}}(V)$, that is, the homogeneous component corresponds to $T_n(V) = V^{\otimes n}$. Let $P(A_{\text{min}}(V))$ be the set of primitive elements. We have the following theorem

★ Theorem. $P(A_{\text{min}}^n(V)) = 0$ if $n$ is not a power of $p$.

This answers a conjecture due to Fred Cohen. In other words, the primitive elements in $A_{\text{min}}(V)$ has only tensor length 1, $p$, $p^2$, $p^3$, \ldots.

Example Let $X$ be a finite path-connected $p$-local space such that $\bar{H}_{\text{even}}(X) = 0$ and $\text{dim } \bar{H}_*(X) \leq p - 1$. Then $H_*(A_{\text{min}}(X))$ is isomorphic to the exterior algebra generated by $\bar{H}_*(X)$. 

Generalizations. We first considered the functorial decompositions of $\Omega \Sigma X$ for $p$-local suspensions $X$. Later we extended the results on $A^{\min}(X)$ to $\Omega \Sigma X$ for any $p$-local path-connected $X$. It seems that this result can be generalized as follows:

Let $X$ be any (path-connected) $p$-local co-$H$-space and let $s: X \to \Sigma \Omega X$ be a cross-section of the evaluation map $\sigma: \Sigma \Omega X \to X$. Let $A^{\min}(s)$ be the homotopy colimit of the composite

$$
\Omega X \xrightarrow{\Omega s} \Omega \Sigma \Omega X \xrightarrow{\text{retraction}} A^{\min}(\Omega X) \xrightarrow{\text{inclusion}} \Omega \Sigma X \xrightarrow{\Omega \sigma} \Omega X.
$$

Then $A^{\min}(s)$ is a retract of $\Omega X$, which is functorial with respect to the cross-sections $s$. (Paul Selick, Stephen Theriault and I did the case where $X$ is a coassociative co-$H$-space. In this case $s$ can be chosen a co-$H$-map and so $\Omega s$ is a Hopf map in homology.)

- I do not know how to do decompositions of $\Omega X$ when $X$ is of LS category 2 or more. The first interesting case might be $X = Y \cup e^n$, where $Y$ is a co-$H$-space. (This case includes some closed manifolds.)

- In theory, given any map $f: \Omega X \to \Omega \Sigma \Omega X$, the homotopy colimit of the composite

$$
\Omega X \xrightarrow{f} \Omega \Sigma \Omega X \xrightarrow{\text{retraction}} A^{\min}(\Omega X) \xrightarrow{\text{inclusion}} \Omega \Sigma X \xrightarrow{\Omega \sigma} \Omega X
$$

is a retract of $\Omega X$ under some conditions (say $X$ is $p$-completed of finite type). Somehow a good choice of $f$ may give a decomposition of $\Omega X$ by computing the homology, but I do not know the good examples of non-co-$H$-spaces $X$ that we can try to decompose $\Omega X$ with possible applications.
Questions. Our decompositions on self-smashes $X^{(n)}$ of a sus-
pension $X$ are obtained by considering the symmetric group $S_n$-
action on $X^{(n)}$. When $X$ is a projective plane, we know that this
gives a complete decomposition of $X^{(n)}$ with explicit information
on the homology of $Q_\alpha(X)$.

Assume that $X$ is a suspension of a projective space. Can we
determine $\bar{H}_*(Q_\alpha(X))$? If so, are they atomic?

Let $G$ be a (finite) group. Assume that $X$ is a suspension with
a (pointed) $G$-action. Then, in addition to the symmetric group
$S_n$, $G^n$ acts on $X^{(n)}$ and so does $G \wr S_n$. Would $G \wr S_n$-action give
some new decompositions of $X^{(n)}$?

Assume that $X$ is a suspension of a projective plane. Can we
get the explicit information on the homology $A^{\text{min}}(X)$? This may
help to understand the homotopy theory of mod 2 Moore spaces.

The decompositions on the single loop suspension
$$\Omega \Sigma X \simeq A(X) \times B(X) \times \cdots$$
induce decompositions
$$\Omega^2 \Sigma X \simeq \Omega A(X) \times \Omega B(X) \times \cdots.$$

Recall that there are configuration spaces models for $\Omega^q \Sigma^q X \simeq
C(\mathbb{R}^q; X)$, where
$$C(M; X) = \prod_{n=1}^{\infty} F(M, n) \times S_n X^n/ \simeq, \text{ with}$$
$$F(M, n) = M^n \setminus \text{flat diagonal}.$$

It seems that these decompositions give certain canonical product
decompositions on $C(\mathbb{R}^q; X)$ and wedge decompositions on the
$n$-adic construction $F(\mathbb{R}^q, n)^+ \wedge S_n X^{(n)}$. Assume that $X$ is a
suspension and require that the decompositions are functorial. It seems that this is related to (stable) $S_n$-equivariant decomposition of ordered configuration spaces $F(\mathbb{R}^q, n)$.

So possibly we can ask how to give (stable) $S_n$-equivariant decompositions of $F(M, n)$ and how to give product decompositions of $C(M; X)$?

**Note.** For functorial self-smashes, we can use $X^{(n)} = S_n^+ \wedge_{S_n} X^{(n)}$ and then do equivariant stable decompositions on $S_n$.

**Some references for this talk:** (available through my website http://www.math.nus.edu.sg/~matwujie)


