This talk will consist of:

1. The Exponent Problem.  
2. The Cohen Groups.  
3. Shuffle Relations.  
4. Configuration Spaces
1. The Exponent Problem

It is not clear at least to me whether the answer to the Barratt conjecture is yes or no, but I will talk some progress along this direction. These progress seem interesting by themselves.

**Barratt Conjecture.** Let $X$ be a path-connected co-$H$-space such that the degree map $[p^r] : X \to X$ is null homotopic. Then $p^{r+1} \pi_*(X) = 0$.

**Barratt Conjecture for Maps.** Let $X$ be a path-connected co-$H$-space and let $f : X \to Y$ be a map such that $[f]$ is of order $p^r$ in $[X,Y]$. Then $p^{r+1} \text{Im}(f_* : \pi_*(X) \to \pi_*(Y)) = 0$.

The first statement follows from the second statement because one can choose $Y = X$ and $f = \text{id}_X$.

So far essentially only knew that the Barratt conjecture holds for the cases when $X$ is a mod $p^r$ Moore space for $p > 2$. For $p = 2$ and $r > 1$, it was known by F. Cohen that $\pi_*(P^n(2^r))$ has a bounded exponent. For mod 2 Moore spaces, it is even not clear where $\pi_*(P^n(2))$ has a bounded exponent. It was
known that $\mathbb{Z}/8$ occurs as a summand of $\pi_*(P^n(2))$ for infinite times in different dimensions for $n \geq 3$.

- For technical reasons, we assume that $X$ is a double (or triple or even higher) suspension in case.
- From the cofibre sequence

$$S^n \xrightarrow{[p^r]} S^n \xrightarrow{j} P^{n+1}(p^r),$$

the Barratt conjecture (for double suspensions) is equivalent to that

$$p^{r+1} : \text{Im}((\text{id}_Y \land j)_* : \pi_*(Y \land S^n) \to \pi_*(Y \land P^{n+1}(p^r)))$$

for $n \geq 2$ and any $Y$ because a $f : Y \land S^n \to Z$ is of order $p^r$ in $[Y \land S^n, Z]$ if and only if it factors through $Y \land P^{n+1}(p^r)$, and $\text{id}_Y \land j : Y \land S^n \to Y \land P^{n+1}(p^r)$ is of order $p^r$. 
★ **Basic Ideas** (due to F. Cohen): Assume that $X = \Sigma X'$ is a suspension. Given a map $f: \Sigma X' \to Y$, let $f': X' \to \Omega Y$ be the adjoint map of $f$. Observe that $\Omega f: \Omega X = \Omega \Sigma' X \to \Omega Y$ is the (unique up to homotopy) $H$-map such that $\Omega f|_{X'} \simeq f'$.

- **Step 1.** Construct certain group $\mathfrak{H}$ (we will go through how to construct $\mathfrak{H}$) with a group homomorphism

$$\theta: \mathfrak{H} \to [\Omega \Sigma X', \Omega Y].$$

The group $\mathfrak{H}$ is *noncommutative* and the image of $\theta$ contains $[\Omega f]$ and many other homotopy classes given by composite of Hopf invariants and Whitehead products. Roughly speaking $\mathfrak{H}$ should be *big enough* such that we can do something for making relations between the powers of $[\Omega f]$ and other types of maps.

- **Step 2.** Consider the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{H} & \xrightarrow{\theta} & [\Omega \Sigma X', \Omega Y] \\
\downarrow & & \downarrow_{\Omega} \\
\tilde{\mathfrak{H}} & \xrightarrow{-} & [\Omega^2 \Sigma X', \Omega^2 Y],
\end{array}
\]
where $\tilde{\mathcal{H}}$ is certain extension of certain quotient of $\mathcal{H}$ by adding possible relations (for instance since $[\Omega^2 \Sigma' X, \Omega^2 Y]$ is abelian, the composite $\Omega \circ \theta$ factors through the abelianization of $\mathcal{H}$) and, if possible, adding new elements.

Assume that the answer to the Barratt conjecture would be positive. We wish that $\tilde{\mathcal{H}}$ has a bounded exponent by adding enough relations to $\mathcal{H}$. (Certainly one could consider triple loop spaces if there are good relations occur after looping twice or more.)

- I have no ideas what $\tilde{\mathcal{H}}$ looks like. But I will talk certain relations to $\mathcal{H}$. These relations do not seem good enough for answering the Barratt conjecture, but they are canonical in some sense.
2. The Cohen Groups

Let $X$ be a path-connected space. Recall that the James construction $J(X) \simeq \Omega \Sigma X$. From the James filtration

$$X^n \xrightarrow{q_n} X = J_1(X) \subseteq \cdots \subseteq J_n(X) \subseteq \cdots \subseteq J(X),$$

there is a tower of groups

$$[J(X), \Omega Y] \twoheadrightarrow \cdots \twoheadrightarrow [J_n(X), \Omega Y] \twoheadrightarrow \cdots \twoheadrightarrow [X, \Omega Y]$$

by suspension splitting theorem, that is,

$$[J(X), \Omega Y] = \lim_n [J_n(X), \Omega Y]$$

with representation

$$q_n^*: [J_n(X), \Omega Y] \hookrightarrow [X^n, \Omega Y].$$

The coordinate inclusions and projections

$$d_i: X^{n+1} \to X^n \quad (x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$d^i: X^n \to X^{n+1} \quad (x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{i-1}, *, x_i, \ldots, x_{n-1})$$
induce faces $d_i = d^i$ and cofaces $d^i = d^*_i$ on $G = \{[X^{n+1}, \Omega Y]\}$ such that $G$ is a $\Delta$-group and a co-$\Delta$-group with the relation

$$d_j d^i = \begin{cases} 
  d^{i-1} d_j & \text{for } j < i, \\
  \text{id} & \text{for } j = i, \\
  d^i d_{j-1} & \text{for } j > i.
\end{cases}$$

This relation is different from the third identity of simplicial sets, and we call this kind of objects $bi$-$\Delta$-groups. (I do not know any references for studying these objects.)

- The faces are used by obtaining $[J_n(X), \Omega Y]$ by the following lemma.

**Lemma.** Let $X$ and $Z$ be path-connected spaces. Suppose that $X$ is a co-$H$-space or $Z$ is an $H$-space. Then $[J_{n+1}(X), Z]$ is the equalizer of the faces $d_i = d^i_* : [X^{n+1}, Z] \to [X^n, Z]$ for $0 \leq i \leq n$.

- The cofaces can be used as operations for constructing Hopf invariants combinatorially. It turns out that many classical results such as the distributivity law hold for any bi-$\Delta$-group.

- There are two ways to obtain the Cohen group $\mathfrak{H}$. The first way was given by F. Cohen. Given a map $f : X \to \Omega Y$, he
constructed a group homomorphism from the free group

$$\theta: F_{n+1} \rightarrow [X^{n+1}, \Omega Y]$$

by sending the generators to

$$X^{n+1} \xrightarrow{\text{coordinate projections}} X \xrightarrow{f} \Omega Y.$$ 

When $X$ is a suspension, he discovered certain commutator relations (the iterated commutators on generators such that one of the generators occurs at least twice) and so the map $\theta$ factors through a quotient group $K_{n+1}$ of $F_{n+1}$. Then he took the equalizer of faces to obtain a group $\mathcal{H}_n$ with a group homomorphism

$$\theta: \mathcal{H}_n \rightarrow [J_n(X), \Omega Y]$$

for any suspension $X$ and any given map $f: X \rightarrow \Omega Y$. By taking limits, then he obtained a group homomorphism

$$\theta: \mathcal{H} = \lim_n \mathcal{H}_n \rightarrow [J(X), \Omega Y]$$

for any suspension $X$ and any given map $f: X \rightarrow \Omega Y$.

One important property of $\mathcal{H}_n$ is that:
Cohen Theorem. $\mathcal{H}$ is a progroup and the kernel of $\mathcal{H}_n \to \mathcal{H}_{n-1}$ is isomorphic to $\text{Lie}(n)$, where $\text{Lie}(n)$ over a ring $R$ (in this case $R = \mathbb{Z}$) is given as follows:

Let $V$ be a free $n$-dimensional $R$-module with a basis $\{x_1, \ldots, x_n\}$. Then $\text{Lie}(n)$ is the $R$-submodule of $V^\otimes n$ spanned by Lie elements

$$[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)}]$$

for $\sigma \in S_n$. The symmetric group $S_n$ acts on $\text{Lie}(n)$ by permuting letters $\{x_1, \ldots, x_n\}$.

- The elements in $\mathcal{H}$ are given by (possibly infinite) products of composites

$$J(X) \xrightarrow{H_n} J(X^{(n)}) \xrightarrow{J(\alpha)} J(X^{(n)}) \xrightarrow{\Omega W_n} J(X) \xrightarrow{\tilde{f}} \Omega Y,$$

where $H_n$ is the James-Hopf map, $W_n$ is the Whitehead product, $\tilde{f}$ is the $H$-map such that $\tilde{f}|_X \simeq f$ and $\alpha$ is a linear combination of permutations.
3. Shuffle Relations

• Observe that \( \Omega: [X, Y] \to [\Omega X, \Omega Y] = [\Sigma \Omega X, Y] \) is induced by the evaluation map \( \sigma: \Sigma \Omega X \to X \). Recall that there is a homotopy pull-back diagram

\[
\begin{array}{ccc}
\Sigma \Omega X & \xrightarrow{\sigma} & X \\
\downarrow & & \downarrow \Delta \\
X \vee X & \xleftarrow{} & X \times X
\end{array}
\]

for any space \( X \). In particular, the reduced diagonal \( \bar{\Delta}: X \to X \wedge X \) is null homotopic after looping.

The shuffle relations are essentially obtained by considering elements in \( \mathcal{H} \) that occur as composite

\[
J(X) \xrightarrow{\bar{\Delta}} J(X) \wedge J(X) \xrightarrow{\text{any map}} \Omega Y.
\]

These maps are null homotopic after looping because \( \Omega \bar{\Delta} \simeq * \).

\textbf{Lemma.} If \( X \) is a co-\( H \)-space, then \( \bar{\Delta}: J(X) \to J(X) \wedge J(X) \) preserves the filtration up to homotopy, where the filtration on \( J(X) \wedge J(X) \) is given by the product filtration.

The proof is almost straightforward. Let \( \mu': X \to X \vee X \) be the comultiplication. Then the diagonal \( \Delta: X^n \to X^n \times X^n = X^{2n} \) preserves the filtration up to homotopy, namely (up to
homotopy) $\Delta$ maps into the subspace of $X^{2n}$ that has at most $n$ coordinates different from the basepoint. By taking quotients, one get that (up to homotopy) $\Delta: J_n(X) \to J_n(X) \times J_n(X)$ maps into the subspace $\bigcup_{i+j \leq n} J_i(X) \times J_j(X)$, and so does the reduced diagonal.

- Let $\tilde{\psi}: J(X) \to J(X) \wedge J(X)$ be the filtered map with $\tilde{\psi} \simeq \bar{\Delta}$. Then there is a commutative diagram of cofibre sequences

\[
\begin{array}{ccc}
J_{n-1}(X) & \xrightarrow{\tilde{\psi}} & J_n(X) \\
\downarrow{\tilde{\psi}} & & \downarrow{\tilde{\psi}} \\
\bigcup_{i+j \leq n-1} J_i(X) \wedge J_j(X) & \xrightarrow{\bar{\psi}} & \bigcup_{i+j \leq n} J_i(X) \wedge J_j(X) \\
& & \xrightarrow{\text{shuffles}} \bigvee_{i=1}^{n-1} X^{(i)} \wedge X^{(n-i)}
\end{array}
\]

**Lemma.** Let $X$ be a path-connected co-$H$-space. Then the composite

$$J_n(X) \to X^{(n)} \xrightarrow{\tilde{\psi}} \bigvee_{i=1}^{n-1} X^{(i)} \wedge X^{(n-i)} \xrightarrow{\text{map}} \Omega Y$$

is null homotopic after looping.

- Let $\mathcal{R}$ be the quotient group of $\mathfrak{H}$ by adding the shuffle relations. Then the composite $\mathfrak{H} \to [J(X), \Omega Y] \to [\Omega J(X), \Omega^2 Y]$ factors through $\mathcal{R}$. The group $\mathcal{R}$ has the following properties:

  - $\mathcal{R}$ is abelian.
• There is commutative diagram of the tower of groups

\[
\begin{array}{c}
\mathcal{H} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}_n \\
\downarrow \\
\mathcal{R} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{R}_n \\
\downarrow \\
\mathcal{R}_1
\end{array}
\]

\[
\begin{array}{c}
\mathcal{H} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}_1 \\
\downarrow \\
\mathcal{R} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{R}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{R}_1
\end{array}
\]

**Theorem.** Let Ker\(_n\) be the kernel of \(\mathcal{R}_n \to \mathcal{R}_{n-1}\). Then there is an exact sequence of groups

\[
0 \to \text{Ext}^{-1}_{R(S_n)}(\text{Lie}(n), \text{Lie}(n)) \to \text{Ker}_n \to \text{End}_{R(S_n)}(\text{Lie}(n)),
\]

where the ground ring \(R = \mathbb{Z}, \mathbb{Z}_p\) (if \(X\) is a \(p\)-local space) or \(\mathbb{Z}/p^r\) (if \(f: X \to \Omega Y\) is of order \(p^r\)).

• The composition operation on \([J(X), J(X)]\) induces an operation on \(\mathcal{H}\) such that \(\mathcal{H}\) is a semi-ring and \(\theta: \mathcal{H} \to [J(X), J(X)]\) corresponding the \(f = \text{id}_{\Sigma X}\) is the morphism of semi-rings. The abelian group \(\mathcal{R}\) has the better property that \(\mathcal{R}\) is a ring and the quotient \(\mathcal{H} \to \mathcal{R}\) is a morphism of semi-rings.

• Paul Selick and I have been to introduce a functor \(A^\text{min}(X)\) which roughly speaking is the functorial smallest retract of \(J(X)\) that contains the bottom cell. There is a subring \(\mathcal{R}_\text{min}\) of \(\mathcal{R}\) such that \(\mathcal{R}_\text{min}\) is a local ring with a representation \(\mathcal{R}_\text{min} \to [\Omega A^\text{min}(X), \Omega A^\text{min}(X)]\).
4. Configuration Spaces

Now we are going to give some remarks on using configuration spaces for understanding the evaluation map $\sigma: \Sigma\Omega^2\Sigma^2 X \to \Omega\Sigma^2 X$. Let $(M, M_0)$ be a pair of manifolds. Recall that the configuration space

$$C(M, M_0; X) = \prod_{n=1}^{\infty} F(M, n) \times S^n X^n / \approx$$

filtered by configuration length with

$$D_n(M, M_0; X) = C_n/C_{n-1} = F(M, n)/F(M|M_0, n) \land S^n X^{(n)},$$

where $F(M|M_0, n)$ is the subspace of $F(M, n)$ consisting of all configurations with at least one coordinate in $M_0$.

We use the following basic property of configuration spaces:

$\star$ Let $M$ be a smooth compact manifold and let $M_0$ and $N$ be the smooth compact submanifolds of $M$ with $\text{codim} N = 0$. If $N/M_0 \cap N$ or $X$ is path connected, then

$$C(N, N \cap M_0; X) \to C(M, M_0; X) \to C(M, N \cup M_0; X)$$

is a quasifibration.
• Let $E$ be the pull-back of the diagram

$$
\begin{array}{ccc}
E & \rightarrow & C(M \times (I, I_0 \cup I_1); X) \\
\downarrow & & \downarrow q \\
C(M \times (I, I_0 \cup I_{0.5}); X) \vee C(M \times (I, I_{0.5} \cup I_1); X) & \hookrightarrow & C(M \times (I, I_0 \cup I_{0.5} \cup I_1); X),
\end{array}
$$

where $I = [0, 1]$ and $I_t$ is a small neighborhood of $t$ for $t \in I$.

Since $q$ is a quasi-fibration and $q$ is homotopic to the diagonal map, the space

$$
E \simeq \Sigma \Omega C(M \times (I, I_0 \cup I_1); X) \simeq \Sigma C(M \times I; X)
$$

and the map

$$
\sigma : E \rightarrow C(M \times (I, I_0 \cup I_1); X)
$$

is an evaluation map. It follows that the evaluation map

$$
\sigma : \Sigma C(M \times I; X) \rightarrow C(M \times (I, \partial I); X) \simeq C(M; \Sigma X)
$$

preserves that configuration filtration and so

$$
\begin{array}{ccc}
\Sigma C_{n-1}(M \times I; X) & \rightarrow & \Sigma C_n(M \times I; X) \\
\downarrow \sigma_{n-1} & & \downarrow \sigma_n \\
C_{n-1}(M; \Sigma X) & \rightarrow & C_n(M; \Sigma X) \rightarrow D_n(M; \Sigma X)
\end{array}
$$
★ Theorem. The map

$$\bar{\sigma}_n: \Sigma D_n(M \times I; X) \longrightarrow D_n(M; \Sigma X)$$

is homotopic to the suspension of the quotient map

$$F(M \times I; n)^+ \wedge S_n X^{(n)} \longrightarrow F(M \times I; n)/A_n \wedge S_n X^{(n)},$$

where $A_n$ is the subspace of $F(M \times I; n)$ consisting of all configurations with the property that at least one coordinate lies in $M \times I_0$ and at least one coordinate lies in $M \times I_1$.

• Roughly speaking the $S_n$-equivariant map

$$(F(M \times I, n), \emptyset) \longrightarrow (F(M \times I, n), A_n)$$

gives the evaluation map in certain sense.

• For the case $M = I$, the composite

$$\Sigma D_n(\mathbb{R}^2; X) \xrightarrow{\bar{\sigma}_n} (\Sigma X)^{(n)} \xrightarrow{\psi} \bigvee_{i=1}^{n-1} (\Sigma X)^{(i)} \wedge (\Sigma X)^{(n-i)}$$

is null homotopic.
• For the Barratt conjecture, one might consider

\[
\Sigma C_n(\mathbb{R}^2; X) \rightarrow \Sigma D_n(\mathbb{R}^2; X) \rightarrow \Sigma^2 C_{n-1}(\mathbb{R}^2; X)
\]

\[
J_n(\Sigma X) \rightarrow (\Sigma X)^{(n)} \rightarrow \Omega Y
\]