At Nankai University, 4-6 pm June 8, 2004.

BRAIDS AND HOMOTOPY GROUPS

JIE WU

★ Joint with Fred Cohen (University of Rochester), Jon Berrick (National University of Singapore), Yan Loi Wong (National University of Singapore)

★ Supported in part by the NSF, and the Academic Research Fund of the National University of Singapore

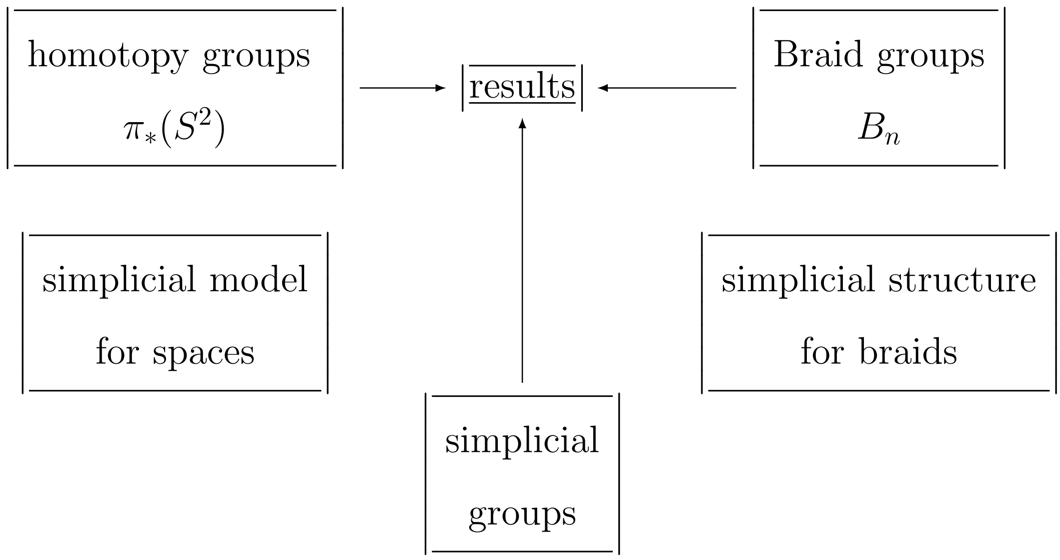
★ Address:
Department of Mathematics
National University of Singapore
Singapore 117543
Republic of Singapore

★ e-mail: matwuj@nus.edu.sg

Reference to this talk:

Jon Berrick, Fred Cohen, Yan Loi Wong and Jie Wu, *Braids, configurations and the homotopy groups*, preprint. (The preprint is available at www.math.nus.edu.sg/∼matwujie.)

The Picture of this Talk



Homotopy Groups

Let X be a pointed topological space. Then the homotopy group

$$\pi_n(X) := [S^n, X],$$

is the set of the (pointed) homotopy classes of (pointed) maps from the n-sphere S^n to X.

draw a picture of how to add in the group by drawing the pinch map for a sphere.

$\pi_0(X)$ is the set of path-connected components of X, which is not a group in general. $\pi_1(X)$ is a group, but non-commutative in general. $\pi_n(X)$ is an abelian group for $n \geq 2$.

1) Čech defined the higher homotopy groups, but abandoned them they are abelian. (1930s)
2) It was originally conjectured that the homotopy groups of spheres are isomorphic to their homology groups. Then Heinz Hopf invented the Hopf map.
3) Many elements are known, but there is still no good way to systematically describe all of the homotopy groups in a computable way. Our theorem (below) gives some global structure.

Applications: Classification of vector bundles, fibre bundles, Algebraic K-theory, deformation theory, physics and etc.

♠$Determinethehomotopygroupsofspheres.$

This is the fundamental and central problem in homotopy theory.

Example 0.1. 1) *$\pi_n(S^1) = 0$ for $n \neq 1$ and $\pi_1(S^1) = \mathbb{Z}$.*

2) *For* $n > 0$*,* $\pi_m(S^n) = 0$ *for* $m < n$ *and* $\pi_n(S^n) = \mathbb{Z}$*.*

3) *Curtis proved that* $\pi_i(S^5) \neq 0$ *for all* $i \geq 5$*.*

$\pi_m(S^n)$ for $m > n$ is not yet well understood for general m and $n \geq 2$, although many non-zero elements are known.

Main methods in calculating $\pi_*(S^n)$*:* EHP sequence and Toda's brackets, the Adams spectral sequence, Morava K-theory and periodic elements, and etc.

Braid Groups

• ordered configuration spaces:

$F(M, n+1) = \{(x_0, x_1, x_2, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}$.

Consider the covering $p\colon F(M,n) \longrightarrow B(M,n) = F(M,n)/\Sigma_n$ with fibre Σ_n.

• The **braid group** $B_n(M)$ of n strings over M is defined by

$$B_n(M) = \pi_1(B(M,n)).$$

The intuitive description is as follows. Choose a base point $(q_1, q_2, \cdots, q_n)$ for $F(M,n)$. Let $\omega\colon S^1 \to B(M,n)$ be a loop. Then there is a lifting path $\lambda\colon [0,1] \to F(M,n)$ such that

• $\lambda(0) = (q_1, q_2, \cdots, q_n)$, $\lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$ and $p(\lambda) = \omega$. Thus

• $\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$ with $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$. We obtain n strings $\lambda_i(t)$ in the cylinder $M \times I$ starting at q_i and ending with $q_{\sigma(i)}$ for some σ. The multiplication is given by the composition of strings.

• The pure braid group $P_n(M)$ is defined by $P_n(M) = \pi_1(F(M,n))$.

The pure braids are n strings $\lambda_i(t)$ in $M \times I$ starting at q_i and ending with q_i.

• When M is the unit disk D^2, $B_n = B_n(D^2)$ is the classical Artin braid group. **Any link** can be obtained by closing up an (Artin) braid.

Combinatorial Models for Spaces (Simplicial Sets)

Let X be a space. Let's review the definition of $H_*(X)$. First we consider a sequence of sets $S_*(X) = \{S_n(X)\}_{n\geq 0}$, where $S_n(X)$ is the set of continuous maps from the n-simplex $\Delta[n]$ to X. The inclusion of the $(i+1)$-th face $d^i\colon \Delta[n-1] \to \Delta[n]$, $0 \leq i \leq n$, induces a function

$$d_i\colon S_n(X) = \mathrm{Map}(\Delta[n], X) \to S_{n-1}(X) = \mathrm{Map}(\Delta[n-1], X).$$

Then we have the differential $\partial = \sum_{i=0}^{n}(-1)^i d_i\colon \mathbb{Z}(S_n(X)) \to \mathbb{Z}(S_{n-1}(X))$ and $H_*(X) = H(\mathbb{Z}(S_n(X)); \partial)$.

• One may ask whether there is a similar *combinatorial* definition of the homotopy groups $\pi_*(X)$, where, by definition, $\pi_n(X)$ is the set of the homotopy classes of pointed map from the sphere S^n to X. The answer is *"Yes"* and it has been much studied since 1950s. People found that the singular simplicial set $S_*(X)$ actually control the homotopy type of the space X in some sense, where one has to add *degeneracies* $s_i\colon S_n(X) \to S_{n+1}(X)$, $0 \leq i \leq n$, which are induced by maps $s^i\colon \Delta[n+1] \to \Delta[n]$. For instance, there are two functions $s^0, s^1\colon \Delta[1] = [0,1] \to \Delta[0]$ given by $s^0(t) = 1$ and $s^1(t) = 0$.

The abstract version of $S_*(X)$ is simplicial set. A simplicial set X means a sequence of sets $X = \{X_n\}_{n\geq 0}$ with faces $d_j\colon X_n \to X_{n-1}$ and degeneracies $s_j\colon X_n \to X_{n+1}$ for $0 \leq j \leq n$ such that "simplicial identities" hold. The difference between simplicial sets and simplicial complexes is that: one needs "degeneracies" for simplicial sets. We also call an abstract simplicial complex a Δ-set.

• A simplicial set is in one-to-one correspondence with a cofunctor from finite ordered sets $\mathcal{O}$ to sets. Here the morphisms in $\mathcal{O}$ are function f such that $f(x) \leq f(y)$ whenever $x \leq y$. Objects in $\mathcal{O}$ are given by $\mathbf{n} = \{0, 1, \ldots, n\}$. The coface $d^i \colon \mathbf{n-1} \to \mathbf{n}$ is given by the ordered embedding such that i does not lie the image. The codegeneracy $s^i \colon \mathbf{n+1} \to \mathbf{n}$ is given by $s^i(i) = s^i(i+1) = i$ and maps others in order.

• A Δ-set is in one-to-one correspondence with a cofunctor from finite strictly ordered sets to sets.

A simplicial group G means a sequence of groups $G = \{G_n\}_{n\geq 0}$ with face *homomorphisms* and degeneracy *homomorphisms.* A Δ-group G means a sequence of groups $G = \{G_n\}_{n\geq 0}$ with *only* face homomorphisms.

• The geometric realization $|X|$ of a simplicial set X is a CW-complex. Roughly speaking, the category of simplicial sets is equivalent to the category of CW-complexes. Let X be a space. Then $|S_*(X)|$ is (weak) homotopy equivalent to X. Conversely, if X is a simplicial set, then $X \simeq S_*(|X|)$.

• The geometric realization of a simplicial group is a topological group. Any topological group (or a loop space) is (weak) homotopy equivalent to a topological group which is a CW-complex.

Let $G = \{G_n\}_{n\geq 0}$ be a simplicial group.

1) The Moore complex: $N_nG = \cap_{j=1}^{n} \operatorname{Ker}(d_j \colon G_n \to G_{n-1})$;

2) The Moore cycles: $\mathcal{Z}_nG = \cap_{j=0}^{n} \operatorname{Ker}(d_j \colon G_n \to G_{n-1})$;

3) The Moore boundaries: $\mathcal{B}_n G = d_0(N_{n+1}G)$.

The sequence of groups $NG = \{N_n G\}$ with d_0 is a (*non-commutative in general*) chain complex. The classical theorem due to John Moore is

Theorem 0.2. *Let G be a simplicial group and let $|G|$ be the geometric realization of G. Then $\pi_n(|G|) \cong H_n(NG; d_0) = \mathcal{Z}_n(G)/\mathcal{B}_n(G)$ for each $n \geq 0$.*

• If G is an abelian simplicial group, then $\pi_*(G) = H_*(NG; d_0) \cong H_*(G; \partial)$, where $\partial = \sum_i (-1)^i d_i$. In other words, the homotopy groups are non-commutative version of the homology groups in this sense.

• Let G be any simplicial group. The Hurewicz homomorphism $\pi_*(G) \to \pi_*(\mathbb{Z}(G)) = H_*(\mathbb{Z}(G); \partial)$ is induced by $G \to \mathbb{Z}(G) \quad x \mapsto x - 1$.

If G is a Δ-group, then one still has NG, $\mathcal{Z}(G)$ and $\mathcal{B}(G)$ defined in the same way and then $\pi_n(G)$ is defined to be the coset of $\mathcal{Z}(G)$ by $\mathcal{B}(G)$.

Simplicial Group Models for Loop Spaces

Let X be a finite complex. Then $S_*(X)$ is usually "too" large (at least uncountable). So one wants to find relatively smaller simplicial group model G for ΩX, that is $|G| \simeq \Omega X$ and "hopefully" $\pi_*(X)$ could be understood. This has been studied much. First if X is finite complex, then there is simplicial set S such that each S_n is finite and $|S| \simeq X$. Then one can do group-theoretical construction on S.

• Kan's construction $G(S)$ is a simplicial group, where $G(S)_n$ is the free group generated by S_n modulo certain relations. The geometric realization of $G(S)$ is ΩX.

• Milnor considered pointed simplicial sets K and let $F(K)_n$ be free group generated by K_n subject to the single relation that the base-point $* = 1$. The geometric realization of $F(K)$ is $\Omega\Sigma K$. In fact, $|F(K)|$ is the free (topological) group generated by $|K|$ modulo the single relation $* = 1$.

• Given a (simplicial or topological) group G and a pointed simplicial set X. Carlsson considered $F^G(X)_n$ which is the free product of G with indexed in $X_n \setminus \{*\}$. Then one obtains a simplicial group $F^G(X)$ such that $|F^G(X)| \simeq \Omega(BG \wedge X)$, where BG is the classifying space of G. This construction helps to understand the homology of $\Omega(\mathbb{R}\mathrm{P}^\infty \wedge X)$, $\Omega(\mathbb{C}\mathrm{P}^\infty \wedge X)$ and $\Omega(\mathbb{H}\mathrm{P}^\infty \wedge X)$.

The Δ and Simplicial-Structure on Configurations

• Let M be any space. The ordered **configuration space** $F(M, n+1)$ by definition:

$$F(M, n+1) = \{(x_0, x_1, x_2, \ldots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

• Consider the sequence of spaces $\{F(M, n+1)\}_{n\geq 0}$ with coordinate projections: $d_i\colon F(M, n+1) \to F(M, n)$

$(x_0, x_1, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

• Let A be any pointed space. Assume that M has a *good* base-point (roughly speaking there is an embedding of $\mathbb{R}^+ = [0, \infty)$ into M), for instance, M has a whisker. Then

★ the sequence of sets $\{[A, F(M, n+1)]\}_{n\geq 0}$ forms a Δ-set with faces d_i induced by the above maps, namely $d_j d_i = d_i d_{j+1}$ for $i \leq j$.

• **Note.** The only point is that coordinate projections do not preserve base-points, but this base-point trouble can be overcome by assuming that M has a good base-point.

★ If A is a cogroup, then $\{[A, F(M, n+1)]\}_{n\geq 0}$ is a Δ-group. In particular, the sequence of fundamental groups $\{\pi_1(F(M, n+1))\}_{n\geq 0}$ is a Δ-group.

• **Note.** For unordered **configuration space** $B(M, n) = F(M, n)/\Sigma_n$, the sequence of groups $\{\pi_1(B(M, n+1))\}_{n\geq 0}$ is a *crossed* Δ-group, roughly speaking, faces are only functions (not group homomorphisms) satisfying certain *crossed* conditions.

• Under certain conditions, $\{[A, F(M, n+1)]\}_{n\geq 0}$ can be a simplicial set (simplicial group if A is a cogroup). Our idea is to construct degeneracy $s_i\colon F(M, n+1) \to F(M, n+2)$ given something like:

$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n)$,

where, roughly speaking, x_i' is a point very close to x_i but different from $\{x_0, \ldots, x_n\}$, and the function $x_i'(x_0, \ldots, x_n)$ should be continuous.

We consider the case that M is metric space with a so-called *steady flow*. In the case that M is a differentiable manifold, our condition is equivalent to that M has a (continuous) nonvanishing vector field (or equivalently M has zero Euler characteristic).

The most interesting examples in our work are:

• $\{\pi_1(F(S^2, n+1))\}_{n\geq 0}$ is a Δ-group (but not simplicial group).

• $\{\pi_1(F(D^2, n+1))\}_{n\geq 0}$ is a simplicial group.

I will explain the relations between these examples and the general homotopy groups of the sphere.

• **Note.** One could compare these ideas with Cohen groups, where roughly speaking the Cohen groups are obtained from the equalizers of the faces of the Δ-group $\{[X^{n+1}, \Omega Y]\}_{n\geq 0}$ and where the Δ-structure is obtained by considering coordinate inclusions $X^n \to X^{n+1}$. If one also consider the coordinate projections of X's, then one gets Δ and co-Δ on $\{[X^{n+1}, \Omega Y]\}_{n\geq 0}$ with relations between faces and cofaces. From this, Hopf invariants can be obtained combinatorially by working out formulae on cofaces.

Brunnian Braids

Consider the coordinate projections

$$d_i\colon F(M,n+1)\to F(M,n)\quad (x_0,x_1,\cdots,x_n)\mapsto (x_0,x_1,\ldots,\hat{x}_i,\ldots,x_n).$$

The map d_i induces, by taking the fundamental group,

- a group homomorphism $d_i=d_{i*}\colon P_{n+1}(M)\to P_n(M)$ and
- a function $d_i\colon B_{n+1}(M)\to B_n(M)$ given by

$(\lambda_0(t),\cdots,\lambda_n(t))\mapsto(\lambda_0(t),\cdots,\hat{\lambda}_i(t),\cdots,\lambda_n(t))$,

that is, deleting the $(i+1)$-th string for $0\leq i\leq n$.

A braid $\beta\in B_{n+1}(M)$ is called **Brunnian** if $d_i(\beta)=1$ for all $0\leq i\leq n$.

In other words, the group of Brunnian braids $\mathrm{Brun}_{n+1}(M)$ is given by

- $\mathrm{Brun}_{n+1}(M)\colon=\bigcap_{i=0}^{n}\mathrm{Ker}(d_i\colon B_{n+1}(M)\to B_n(M))$.

The classical **Borromean Rings** is a link by closing up a Brunnian braid of 3 strings over D^2.

Moore Cycles and Brunnian Braids

Consider the sequence of groups $\{B_{n+1}(M)=\pi_1(F(M,n+1))\}_{n\geq 0}$. Observe that the group of Brunnian braids is given by

• $\mathrm{Brun}_{n+1}(M)\colon=\bigcap_{i=0}^{n}\mathrm{Ker}(d_i\colon B_{n+1}(M)\to B_n(M))$.

• **Lemma.** Let β be a Brunnian braid of n strings over M. If $n\geq 3$, then β is a pure braid. Thus

★ For $n\geq 2$, $\mathrm{Brun}_{n+1}(M)\colon=\bigcap_{i=0}^{n}\mathrm{Ker}(d_i\colon P_{n+1}(M)\to P_n(M))$ is the **Moore cycles of the Δ-group** $\{P_{n+1}(M)=\pi_1(F(M,n+1))\}_{n\geq 0}$.

In other words, Brunnian braids are essentially Moore cycles.

• The braided interpretation of boundaries seems unclear. As we know that the Moore homotopy groups are certain *derived* groups of Δ-groups. The Moore homotopy groups of $\{P_{n+1}(M)\}_{n\geq 0}$ MIGHT be certain *invariants* on braids. When $M=S^2$, the following theorem gives a connection with the homotopy groups of the sphere.

★ **Theorem A.** Let $\mathcal{F}(S^2)^{\pi_1}=\{P_{n+1}(S^2)\}_{n\geq 0}$ be the Δ-group defined above. Then for each $n\geq 1$ $\pi_n(\mathcal{F}(S^2)^{\pi_1})$ is a group, and there is an isomorphism of groups

$$\pi_n(\mathcal{F}(S^2)^{\pi_1})\cong\pi_n(S^2)\cong\pi_n(S^3)$$

for $n\geq 4$.

Main Results

In addition to Theorem A. Our next theorem directly gives connections between the Brunnian braids and the homotopy groups.

The canonical embedding $f\colon D^2 \subseteq S^2$ induces a group homomorphism $\mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2)$.

★ **Theorem B.** There is an exact sequence of groups

$$1 \longrightarrow \mathrm{Brun}_{n+1}(S^2) \longrightarrow \mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2) \longrightarrow \pi_{n-1}(S^2) \longrightarrow 1$$

for $n \geq 5$.

• For instance, $\mathrm{Brun}_5(S^2)$ modulo $\mathrm{Brun}_5(D^2)$ is $\pi_4(S^3) = \mathbb{Z}/2$. The other low homotopy groups of S^3 are as follows:

$\pi_5(S^3) = \mathbb{Z}/2$, $\pi_6(S^3) = \mathbb{Z}/12$, $\pi_7(S^3) = \mathbb{Z}/2$, $\pi_8(S^3) = \mathbb{Z}/2$, $\pi_9(S^3) = \mathbb{Z}/3$, $\pi_{10}(S^3) = \mathbb{Z}/15$, and etc.

Thus, up to certain range, $\mathrm{Brun}_{n+1}(S^2)$ modulo $\mathrm{Brun}_{n+1}(D^2)$ are known by non-trivial calculations of $\pi_*(S^3)$.

• Question 23 in the end of Birman's red book, J. Birman, *Braids, Links and Mapping Class Groups*, Ann. of Math. Studies, vol. 82, Princeton Univ. Press, Princeton, NJ, 1975, essentially she asked to find the free generators of $\mathrm{Brun}_n(S^2)$. If her old question were answered, then, together with some of my works, one has the combinational determination of the homotopy groups $\pi_n(S^2)$ by listing generators and relations. Actually, for the purpose of determining generators and relations for $\pi_n(S^2)$, we only need a weak version of Birman's question.

★ **Weak Form of Birman's Problem:** Determine a set of generators for $\mathrm{Brun}_n(S^2)$ for $n \geq 5$.

It would be very interesting if one can describe the generators for $\mathrm{Brun}_n(S^2)$ as certain invariants, say certain link invariants or anything else. One of the ideas might be to construct links in $S^2 \times S^1$ by closing up Brunnian braids in $\mathrm{Brun}_n(S^2)$ and then consider certain invariants.

Our next result gives connections between the classical braid groups and the homotopy groups. Since the disk D^2 admits a nonvanishing vector field, $\{P_{n+1}(D^2)\}_{n\geq 0}$ is a (contractible) simplicial group. Our idea is to add *one more canonical face*, in addition to coordinate projections, such that $\{P_n(D^2)\}_{n\geq 0}$ is a Δ-group with non-trivial Moore homotopy groups.

Let $B_n = B_n(D^2)$ be the classical braid groups and let $P_n = P_n(D^2)$. First we describe an operation $\tilde{\partial}\colon B_{n+1} \to B_n$ as follows.

Let $\delta\colon F(\mathbb{C}, n+1) \longrightarrow F(\mathbb{C}, n)$ be the map defined by

$$\delta(z_0, z_1, \ldots, z_n) = \left(\frac{1}{\bar{z}_1 - \bar{z}_0}, \frac{1}{\bar{z}_2 - \bar{z}_0}, \ldots, \frac{1}{\bar{z}_n - \bar{z}_0}\right),$$

corresponding geometrically to the reflection map in $\mathbb{C}$ about the unit circle centered at z_0.

We can show that on fundamental groupoids δ induces a function $\tilde{\partial}\colon B_{n+1} \to B_n$ that restricts to a group homomorphism from P_{n+1} to P_n and from $\mathrm{Brun}_{n+1}(D^2)$ to $\mathrm{Brun}_n(D^2)$.

From the braid relations, there is a canonical involution homomorphism $\chi\colon B_n \longrightarrow B_n$ that sends each standard generator to its inverse. Likewise it restricts to a group homomorphism from P_n to P_n and from $\mathrm{Brun}_n(D^2)$ to $\mathrm{Brun}_n(D^2)$.

Composing χ with $\tilde{\partial}$ gives a homomorphism ∂ on $\mathrm{Brun}_{n+1}(D^2)$ that maps into $\mathrm{Brun}_n(D^2)$ and has the further property that $\partial \circ \partial$ is trivial.

We therefore obtain a 'chain complex' of nonabelian groups

$(\mathrm{Brun}(D^2), \partial) : \cdots \to \mathrm{Brun}_{n+1}(D^2) \xrightarrow{\partial} \mathrm{Brun}_n(D^2) \xrightarrow{\partial} \mathrm{Brun}_{n-1}(D^2) \to \cdots$.

The homology of this chain complex is a very pleasant surprise ...

★ **Theorem C.** For all n there is an isomorphism of groups

$$H_n(\mathrm{Brun}(D^2)) \cong \pi_n(S^2).$$

• Let $\Gamma = \{\Gamma_n\}_{n\geq 0}$ be the sequence of groups defined by $\Gamma_0 = 1$ and, for $n \geq 1$, $\Gamma_n = P_n$ with the faces $d_0 = \partial$, and, for $1 \leq i \leq n$, d_i given by deleting the i th string. Then Γ is a Δ-group.

★ **Theorem D.** $\pi_*(\Gamma) = \pi_*(S^2)$.

• Note that S^2 is NOT an H-space. There is NO simplicial group model for S^2. (The geometric realization of a simplicial group is always a loop space.) This result says that there is a Δ-group model for S^2, and these groups are just given by **Artin pure braid groups**!

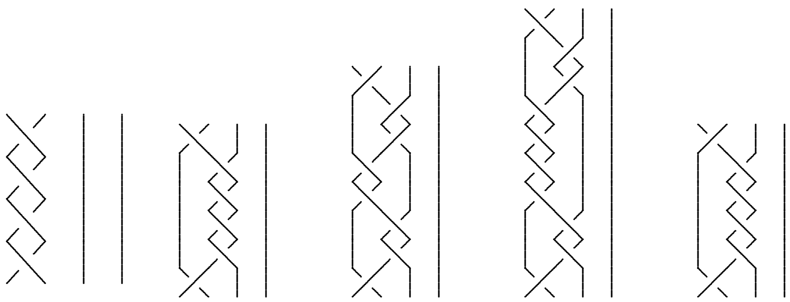
• Note P_n is a semi-direct product of $F_{n-1}, F_{n-2}, \ldots, F_1$, where F_k is the free group of rank k. As a sequence of sets, Γ looks like $\bar{W}F[S^1]$, the classifying space of Milnor's construction on S^1.

Examples

Let σ_i denote the usual generator for Artin braid group $B_{n+1} = \pi_1(B(D^2, n+1))$ for $0 \leq i \leq n-1$. (**Note.** Our counting always starts from 0. So σ_i really means σ_{i+1} in Birman's book.) Write δ_i for the image of σ_i in $\pi_1(B(S^2, n+1))$.

★ The Brunnian group $\mathrm{Brun}_4(S^2)$ is the free group of rank 5 generated by the braids δ_0^4, $\delta_0\delta_1^4\delta_0^{-1}$, $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^{-1}$, $\delta_0\delta_1^{-2}\delta_0^4\delta_1^2\delta_0^{-1}$ and $\delta_0^{-1}\delta_1^4\delta_0$.

The pictures of these braids are as follows.



δ_0^4 $\quad$ $\delta_0\delta_1^4\delta_0^{-1}$ $\quad$ $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^1$ $\quad$ $\delta_0\delta_1^{-2}\delta_0^4\delta_1^2\delta_0^{-1}$ $\quad$ $\delta_0^{-1}\delta_1^4\delta_0$

• **Remark.** By deleting the last trivial string of the 4-string braid $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^{-1}$ over S^2 we obtain the 3-string braid $\sigma_0^{-1}\sigma_1^{-2}\sigma_0^2\sigma_1^2\sigma_0^{-1}$ over D^2. In turn, closing up this 3-string braid gives a link that is readily seen to be the Borromean rings. This link corresponds to a Moore cycle in $F[S^1]_2$, where the Milnor construction $F[S^1]$ is the simplicial group model for ΩS^2, that represents the generator η_2 for $\pi_2(\Omega S^2) = \pi_3(S^2)$. In other words, the Hopf map $\eta_2 : S^3 \to S^2$ corresponds to the Borromean rings in this way.