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**HOMOTOPY GROUPS, CONFIGURATION SPACES AND THE  
BRAID GROUPS**

JIE WU

★ Joint with Fred Cohen (University of Rochester), Jon Berrick (National University of Singapore), Yan Loi Wong (National University of Singapore)

★ Supported in part by the NSF, and the Academic Research Fund of the National University of Singapore

★ Address:

Department of Mathematics

National University of Singapore

Singapore 117543

Republic of Singapore

★ e-mail: [matwuj@nus.edu.sg](mailto:matwuj@nus.edu.sg)

**Reference to this talk:**

Jon Berrick, Fred Cohen, Yan Loi Wong and Jie Wu, *Braids, configurations and the homotopy groups*, preprint. (The preprint is available at [www.math.nus.edu.sg/~matwujie](http://www.math.nus.edu.sg/~matwujie).)

**This talk consists of:**

- 1) homotopy groups;
- 2) configuration spaces and the braid groups;
- 3) Brunnian braids;
- 4) Theorem;
- 5) Ideas and Methods how to obtain the theorem;
- 6) Examples of Brunnian braids over the sphere.

## Homotopy Groups

Let  $X$  be a pointed topological space. Then the homotopy group

$$\pi_n(X) := [S^n, X],$$

is the set of the (pointed) homotopy classes of (pointed) maps from the  $n$ -sphere  $S^n$  to  $X$ .

**draw a picture of how to add in the group by drawing the pinch map for a sphere.**

$\pi_0(X)$  is the set of path-connected components of  $X$ , which is not a group in general.  $\pi_1(X)$  is a group, but non-commutative in general.  $\pi_n(X)$  is an abelian group for  $n \geq 2$ .

- 1) Čech defined the higher homotopy groups, but abandoned them they are abelian. (1930s)
- 2) It was originally conjectured that the homotopy groups of spheres are isomorphic to their homology groups. Then Heinz Hopf invented the Hopf map.
- 3) Many elements are known, but there is still no good way to systematically describe all of the homotopy groups in a computable way. Our theorems give some global structure.

Applications: Classification of vector bundles, fibre bundles, Algebraic  $K$ -theory, deformation theory, mathematical physics and etc.

♠ Determine the homotopy groups of spheres.

This is the fundamental and central problem in homotopy theory.

**Example 0.1.** 1)  $\pi_n(S^1) = 0$  for  $n \neq 1$  and  $\pi_1(S^1) = \mathbb{Z}$ .

2) For  $n > 0$ ,  $\pi_m(S^n) = 0$  for  $m < n$  and  $\pi_n(S^n) = \mathbb{Z}$ .

3) Curtis proved that  $\pi_i(S^5) \neq 0$  for all  $i \geq 5$ .

$\pi_m(S^n)$  for  $m > n$  is not yet well understood for general  $m$  and  $n \geq 2$ , although many non-zero elements are known.

*Main (traditional) methods in calculating  $\pi_*(S^n)$ :* EHP sequence and Toda's brackets, the Adams spectral sequence, Morava  $K$ -theory and periodic elements, and etc.

*Our ideas:*

**Step 1.** Describe the homotopy groups as the derived groups of the braid groups, that is, the quotient of certain subgroup of the braid group by another subgroup.

**Step 2.** Study these special subgroups of the braids by using various methods in other areas of mathematics.

So far the first step seems pretty successful, namely there are good (canonical) descriptions of the homotopy groups as the derived groups of the braids that I am going to talk today.

## Braid Groups

- **ordered configuration spaces:**

$$F(M, n) = \{(x_1, x_2, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Consider the covering  $p: F(M, n) \longrightarrow B(M, n) = F(M, n)/\Sigma_n$  with fibre  $\Sigma_n$ .

- The **braid group**  $B_n(M)$  of  $n$  strings over  $M$  is defined by

$$B_n(M) = \pi_1(B(M, n)).$$

The intuitive description is as follows. Choose a base point  $(q_1, q_2, \dots, q_n)$  for  $F(M, n)$ . Let  $\omega: S^1 \rightarrow B(M, n)$  be a loop. Then there is a lifting path  $\lambda: [0, 1] \rightarrow F(M, n)$  such that

- $\lambda(0) = (q_1, q_2, \dots, q_n)$ ,  $\lambda(1) = (q_{\sigma(1)}, \dots, q_{\sigma(n)})$  for some  $\sigma \in \Sigma_n$  and  $p(\lambda) = \omega$ . Thus
- $\lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))$  with  $\lambda_i(t) \neq \lambda_j(t)$  for  $i \neq j$  and  $0 \leq t \leq 1$ . We obtain  $n$  strings  $\lambda_i(t)$  in the cylinder  $M \times I$  starting at  $q_i$  and ending with  $q_{\sigma(i)}$  for some  $\sigma$ . The multiplication is given by the composition of strings.

- The pure braid group  $P_n(M)$  is defined by

$$P_n(M) = \pi_1(F(M, n))$$

The pure braids are  $n$  strings  $\lambda_i(t)$  in  $M \times I$  starting at  $q_i$  and ending with  $q_i$ .

- When  $M$  is the unit disk  $D^2$ ,  $B_n = B_n(D^2)$  is the classical Artin braid group. **Any link** can be obtained by closing up an (Artin) braid.

## Brunnian Braids

Consider the coordinate projections

$$d_i: F(M, n+1) \rightarrow F(M, n)$$

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, \hat{x}_i, \dots, x_n).$$

The map  $d_i$  induces, by taking the fundamental group,

- a group homomorphism  $d_i = d_{i*}: P_{n+1}(M) \rightarrow P_n(M)$  and

- a function  $d_i: B_{n+1}(M) \rightarrow B_n(M)$  given by

$$(\lambda_0(t), \dots, \lambda_n(t)) \mapsto (\lambda_0(t), \dots, \hat{\lambda}_i(t), \dots, \lambda_n(t)),$$

that is, deleting the  $(i+1)$ -st string for  $0 \leq i \leq n$ .

A braid  $\beta \in B_{n+1}(M)$  is called **Brunnian** if  $d_i(\beta) = 1$  for all  $0 \leq i \leq n$ .

In other words, the group of Brunnian braids  $\text{Brun}_{n+1}(M)$  is given by

- $\text{Brun}_{n+1}(M) = \bigcap_{i=0}^n \text{Ker}(d_i: B_{n+1}(M) \rightarrow B_n(M)).$

The classical **Borromean Rings** is a link by closing up a Brunnian braid of 3 strings over  $D^2$ .

## Theorem

Our next theorem directly gives connections between the Brunian braids and the homotopy groups.

The canonical embedding  $f: D^2 \subseteq S^2$  induces a group homomorphism  $\text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2)$ .

**★ Theorem.** There is an exact sequence of groups

$$\text{Brun}_{n+1}(S^2) \hookrightarrow \text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2)$$

for  $n \geq 5$ .

• For instance,  $\text{Brun}_5(S^2)$  modulo  $\text{Brun}_5(D^2)$  is  $\pi_4(S^3) = \mathbb{Z}/2$ .

The other low homotopy groups of  $S^3$  are as follows:

$$\pi_5(S^3) = \mathbb{Z}/2, \pi_6(S^3) = \mathbb{Z}/12, \pi_7(S^3) = \mathbb{Z}/2, \pi_8(S^3) = \mathbb{Z}/2, \pi_9(S^3) = \mathbb{Z}/3, \pi_{10}(S^3) = \mathbb{Z}/15, \text{ and etc.}$$

Thus, up to certain range,  $\text{Brun}_{n+1}(S^2)$  modulo  $\text{Brun}_{n+1}(D^2)$  are known by non-trivial calculations of  $\pi_*(S^3)$ .

• Question 23 in the end of Birman's orange book, J. Birman, *Braids, Links and Mapping Class Groups*, Ann. of Math. Studies, vol. 82, Princeton Univ. Press, Princeton, NJ, 1975, essentially she asked to find the free generators of  $\text{Brun}_n(S^2)$ . If her old question were answered, then, together with some of my works, one has the combinatorial determination of the homotopy groups  $\pi_n(S^2)$  by listing generators and relations. Actually, for the purpose of determining generators and relations for  $\pi_n(S^2)$ , we only need a weak version of Birman's question.

**★ Weak Form of Birman's Problem:** Determine a set of generators for  $\text{Brun}_n(S^2)$  for  $n \geq 5$ .

It would be very interesting if one can describe the generators for  $\text{Brun}_n(S^2)$  as certain invariants, say certain link invariants or anything else. One of the ideas might be to construct links in  $S^2 \times S^1$  by closing up Brunnian braids in  $\text{Brun}_n(S^2)$  and then consider certain invariants.

## Ideas and Methods

Observe that the coordinate projections

$$d_i: F(M, n+1) \rightarrow F(M, n)$$

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, \hat{x}_i, \dots, x_n).$$

induces, by taking the fundamental group,

- a group homomorphism  $d_i = d_{i*}: P_{n+1}(M) \rightarrow P_n(M)$  and

- a function  $d_i: B_{n+1}(M) \rightarrow B_n(M)$  given by

$$(\lambda_0(t), \dots, \lambda_n(t)) \mapsto (\lambda_0(t), \dots, \hat{\lambda}_i(t), \dots, \lambda_n(t)),$$

that is, deleting the  $(i+1)$ -st string for  $0 \leq i \leq n$ .

The functions  $d_i: B_{n+1}(M) \rightarrow B_n(M)$  and group homomorphisms  $d_i: P_{n+1}(M) \rightarrow P_n(M)$  satisfy the following identity:

$$d_j d_i = d_i d_{j+1} \text{ for } i \leq j.$$

A  **$\Delta$ -set** ( **$\Delta$ -group**) means a sequence of sets (groups)  $S = \{S_n\}_{n \geq 0}$  with face functions (face homomorphisms)  $d_i: S_n \rightarrow S_{n-1}$  such that the above identity holds.

★ The theorem is obtained by studying the  $\Delta$ -groups from the braids and the simplicial group models for loop spaces.

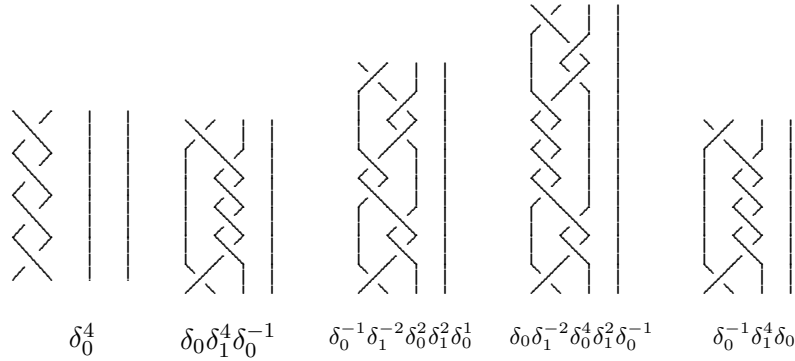
A **simplicial group** means a  $\Delta$ -group  $G = \{G_n\}_{n \geq 0}$  together with *degeneracy homomorphisms*  $s_i: G_n \rightarrow G_{n+1}$  such that so-called simplicial identities hold.

### Examples

Let  $\sigma_i$  denote the usual generator for Artin braid group  $B_{n+1} = \pi_1(B(D^2, n+1))$  for  $0 \leq i \leq n-1$ . (**Note.** Our counting always starts from 0. So  $\sigma_i$  really means  $\sigma_{i+1}$  in Birman's book.) Write  $\delta_i$  for the image of  $\sigma_i$  in  $\pi_1(B(S^2, n+1))$ .

★ The Brunnian group  $\text{Brun}_4(S^2)$  is the free group of rank 5 generated by the braids  $\delta_0^4$ ,  $\delta_0\delta_1^4\delta_0^{-1}$ ,  $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^{-1}$ ,  $\delta_0\delta_1^{-2}\delta_0^4\delta_1^2\delta_0^{-1}$  and  $\delta_0^{-1}\delta_1^4\delta_0$ .

The pictures of these braids are as follows.



• **Remark.** By deleting the last trivial string of the 4-string braid  $\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^{-1}$  over  $S^2$  we obtain the 3-string braid  $\sigma_0^{-1}\sigma_1^{-2}\sigma_0^2\sigma_1^2\sigma_0^{-1}$  over  $D^2$ . In turn, closing up this 3-string braid gives a link that is readily seen to be the Borromean rings. This link corresponds to a Moore cycle in  $F[S^1]_2$ , where the Milnor construction  $F[S^1]$  is the simplicial group model for  $\Omega S^2$ , that represents the generator  $\eta_2$  for  $\pi_2(\Omega S^2) = \pi_3(S^2)$ . In other words, the Hopf map  $\eta_2 : S^3 \rightarrow S^2$  corresponds to the Borromean rings in this way.