FROM CALCULUS TO TOPOLOGY

J. WU

In this topic, I explain some basic ideas in general topology. I assume that you have been to take a calculus course. Let's recall the definition of continuity: A function $f(x)$ is called *continuous* at a point x_0 if for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. This is so-called $\epsilon - \delta$ language which describes the intuitive idea that $f(x)$ is continuous at x_0 just means that $f(x)$ should approach to $f(x_0)$ when x approaches to x_0, in other words the graph of $f(x)$ must be *continuous* in usual sense.

Now let us do some observations. First of all a function $f(x)$ means that a function from its domain, which is a subset of $\mathbb{R}$ in calculus we may assume that its domain is an interval, to real numbers $\mathbb{R}$. In the real world, a function may have more than one variable, for instance the temperature depends on three variables, x, y, z, because our space is 3-dimensional. If you want to measure certain economic event, this event may depends on many factors. Each factor may be considered as one variable, then you get a function which has many variables. On the other hand, the *values* of a function may not be real numbers. For instance, the projection of a flight on the ground could be considered as a function from points in the sky to points on the surface of our earth. This function takes values in a sphere (our earth). Thus, mathematically, it is very important to understand the continuity of more general functions. Now let's take this project and let's keep $\epsilon - \delta$ language in mind.

We assume that X and Y are JUST sets. Let $f\colon X \to Y$ be a function and let x_0 be a point in X, that is, x_0 is an element in X. Our intuitive idea for the continuity is that $f(x)$ must be sufficiently close to $f(x_0)$ when x is sufficiently close to x_0. Suppose that the elements in X and Y can be measured by *'distance'*. Then we can apply our $\epsilon - \delta$ language. More precisely, let $d(x, y)$ denote the distance between x and y. We have the definition of continuity: $f(x)$ is called *continuous* at x_0 for any $\epsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon$ whenever $d(x, x_0) < \delta$. Now what is *distance*? We need some basic (obvious) rules: $d(x, y) = d(y, x) \geq 0$, $d(x, z) \leq d(x, y) + d(y, z)$ and $d(x, y) = 0$ if and only if $x = y$. A set X together with a distance $d(-,-)$ is called *a metric space*. People then found out that most theorems about continuity in calculus actually hold for functions from a metric space to another metric space. One of the theorems states that

A function $f\colon X \to Y$ between metric spaces is continuous if and only if the preimage of any open set in Y under f is open in X.

A subset U of a metric space X is called *open* if for any point x_0 in U there is a small open disc centered at x_0, $\{y|d(y,x_0) < \epsilon\}$, contained in U. For instance, an open subset of $\mathbb{R}$ is a disjoint union of open intervals.

This theorem help us to understand *continuity* in even more general situation, namely in the cases where *distance* may not be defined. Topology is introduced: Let X be a set. A topology on X means a collection of subsets, so-called *open sets*, in X that satisfies: 1) the empty set $\emptyset$ and the total set X are open; 2) any union of open sets, including a union of infinitely members, is open; and 3) the intersection of two open sets is open. A *topological space* X means a set X with a topology. The theorem above is now used as the definition of a continuous function between topological spaces, namely, a function $f\colon X \to Y$ between topological spaces is called *continuous* if the preimage of any open set in Y under f is open in X, that is, $f^{-1}(U)$ is open for any open set U in Y. In homotopy theory, usually a *space* means a topological space and a *map* means a continuous function from a topological space to a topological space. Again most results about continuity in calculus hold for functions between topological spaces.

By using the terminology of topology, we can understand some “difficult” spaces much better and we may also find new “spaces”. Below I just give few examples. First let me explain so-called *quotient topology.* Let X be a topological space and let Y be just a set with a onto function $f\colon X \to Y$. The set Y can be regarded as a *quotient* of X by making identification of elements in the preimage $f^{-1}(y)$ to the one point y for each y in Y. Starting with X we can do many identification and so we may obtain many quotient sets of X. Now the *quotient topology* on Y is: a subset U of Y is called open if $f^{-1}(U)$ is open. This defines the *largest* topology on Y such that the function $f\colon X \to Y$ is continuous. In other words any quotient of a topological space is a topological space in a canonical way.

Now consider an example $X = [0,1]$, the unit closed interval. Let Y be the quotient space of X by making identification of 0 with 1. Intuitively we obtain a circle because the two ending points of X are putting together. As an exercise, one can prove that the topological space Y is homeomorphic to the unit circle S^1, where a space X is called homeomorphic to Y is there is a one-to-onto continuous function $f\colon X \to Y$ such that the inverse f^{-1} is also continuous. Let's consider a similar example. Let $X = D^2$ be the unit disc in the plane. We can make the identification of the boundary of D^2, which is the unit circle, to be one point. Then we obtain a quotient space of D^2. Again as intuitive observation this quotient space is homeomorphic to the unit sphere S^2 in $\mathbb{R}^3$. Now we make the identification of the boundary S^2 of the unit ball D^3 to be a point. The quotient space is then homeomorphic to the 3-dimensional sphere S^3. Although it might be difficult to image the 3-dimensional sphere S^3 in our 3-dimensional space, we may just think that S^3 is the quotient of the ball D^3 by identifying the boundary to be a single point. The picture named Klein bottle in the previous section is obtained by making an identification on the rectangle, where one of two parallel sides is identified each other in the *same* direction and another is identified in the *opposite* direction. The Klein bottle is NOT realizable in our 3-dimensional space in the sense that we could not draw it, but we may simply consider the Klein bottle as a quotient of a rectangle described above. There are many important topological spaces that might be out of intuition. For instance, the projective plane is the space of all lines in $\mathbb{R}^3$ through the origin. Similarly there are higher dimensional projective spaces. Given two topological spaces X and Y we obtain a topological space consisting of ALL continuous functions from X to Y. This space is called a *mapping space*. It is really a large space, isn't it?

Department of Mathematics, National University of Singapore, Singapore 119260, Republic of Singapore, matwujie@math.nus.edu.sg