A BRAIDED SIMPLICIAL GROUP

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Abstract. By studying the braid group action on Milnor's construction of the 1-sphere, we show that the general higher homotopy group of the 3-sphere is the fixed set of the pure braid group action on certain combinatorially described group.

1. Introduction

In this article, we study the homotopy groups by considering the braid group actions on simplicial groups. The purpose is to establish a relation between the braid group actions on certain combinatorially given groups and the homotopy groups of the 3-sphere. We first recall a combinatorial description of the homotopy groups of the 3-sphere in [16].

Let $F(x_1, \dots, x_n)$ be the free group generated by the letters $x_1, \dots, x_n$. Let $w(x_1, \dots, x_n) = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}$ be a word. Given $a_1, \dots, a_n \in F(x_1, \dots, x_n)$, we write $w(a_1, \dots, a_n) \in F(x_1, \dots, x_n)$ for $a_{i_1}^{\epsilon_1} \cdots a_{i_t}^{\epsilon_t}$. The group $G(n)$ is defined to be the quotient group of $F(x_1, \dots, x_n)$ modulo the following relations:

($\mathcal{R}_1$) the product $x_1 \cdots x_n$;

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($\mathcal{R}_2$) the words $w(x_1, \ldots, x_n)$ that satisfy:

$$w(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 1$$

for $1 \leq i \leq n$.

Relations $\mathcal{R}_2$ consist of all of words that will be trivial if one of the generators is replaced by the identity element 1. The smallest normal subgroup of $F(x_1, \ldots, x_n)$ that contains relations $\mathcal{R}_1$ and $\mathcal{R}_2$ was determined as a subgroup of $F(x_1, \ldots, x_n)$ generated by certain systematic and uniform iterated commutators [16].

Theorem 1.1. [16] *For $n \geq 3$, the homotopy group $\pi_n(S^3)$ is isomorphic to the centre of $G(n)$.*

A natural question that arises from Theorem 1.1 is how to give a group-theoretical approach to the homotopy groups, that is, how to understand the centre of the group $G(n)$. Let B_n be the braid group on n strings. There is a canonical braid group action on $G(n)$ that is induced by the Artin representation of the braid group B_n on the free group $F(x_1, \ldots, x_n)$. (We will go over the definition of B_n and Artin's representation in Section 2.) These actions give a canonical homomorphism from the braid group B_n into the automorphism group of $G(n)$. Since Quillen's plus construction of the classifying space for the stable braid group is (up to homotopy type) the double loop space of the 3-sphere [5], these braid group actions do not appear to be a coincidence. Larry Taylor conjectured that the centre of $G(n)$ is the fixed set of the braid group action on $G(n)$. We solve Taylor's conjecture as follows. Let K_n be the pure braid group, that is, K_n is the kernel of the canonical epimorphism from B_n to the symmetric group Σ_n. In geometry, the group K_n is the fundamental group of the configuration space $F(\mathbb{R}^2, n)$, where

$$F(M,n) = \{(x_1, \dots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}$$

for any manifold M (See [5]). Let $Z(G(n))$ be the centre of $G(n)$.

Theorem 1.2. *Let $n \geq 4$. Then*

1) *the fixed set of the pure braid group action on $G(n)$ is the centre of $G(n)$ and so is $\pi_n(S^3)$;*
2) *the fixed set of the braid group action on $G(n)$ is the subgroup*

$$\{x \in Z(G(n)) | 2x = 0\}.$$

We should point out that the determination of the fixed set of K_n-action on $G(n)$ by (combinatorial) group-theoretic means seems beyond the reach of current techniques. On the other hand, braid group actions have been much studied in several areas such as group theory and low-dimensional topology. Various problems arising from physics are related to braid group actions as well. Theorem 1.2 suggests that the homotopy groups play a certain role for braid group actions. In the range in which $\pi_*(S^3)$ is known ($* \leq 55$, see [9, 15]), by homotopy-theoretic means, we might gain insight into these difficult group-theoretic questions.

The B_n-action on $G(n)$ induces a homomorphism $\theta\colon B_n \to \mathrm{Aut}(G(n))$, where $\mathrm{Aut}(G)$ is the automorphism group of G. Fred Cohen asked what is the kernel of the map θ. We answer Cohen's question by determining the kernel of θ. We find that the kernel of θ is related to so-called Brunnian braids in low-dimensional topology. Recall that a (geometric) braid β on n-strings is called *Brunnian* if β becomes to be a trivial braid when *any* one of the strings is removed. A link obtained by closing up a Brunnian braid is called a *Brunnian link*. An example of a 3-component Brunnian link is the well-known Borromean Ring. Let C_n be the set of Brunnian braids on n strings. Then C_n is a normal subgroup of B_n. Recall that the centre of B_n is isomorphic to the infinite cyclic group $\mathbb{Z}$ for $n \geq 3$. (See [4, Corollary 1.8.4]).

Theorem 1.3. *Let $\theta_n\colon B_n \to \mathrm{Aut}(G(n))$ be the representation map induced by the B_n-action on $G(n)$. Suppose that $n \geq 4$. Then the kernel of θ_n is the subgroup of B_n generated by $Z(B_n)$ and C_n and so*

$$\mathrm{Ker}(\theta_n) \cong Z(B_n) \times C_n \cong \mathbb{Z} \times C_n.$$

The group C_n has been largely studied in low dimensional topology with various important results. There is a 30-year-old problem in low-dimensional topology proposed by J. Birman: how to find a free basis for $R_n \cap C_n$ [4, Problem 23, pp.219], where R_n

is the kernel of $K_n = \pi_1(F(\mathbb{R}^2, n)) \to \pi_1(F(S^2, n))$ induced by the canonical inclusion $F(\mathbb{R}^2, n) \subseteq F(S^2, n)$. Birman's problem still remains open in general. Recently A. J. Berrick, Y. L. Wong and the author showed that the subgroup $R_n \cap C_n$ of B_n is isomorphic to the cycles in dimension $n-1$ of the Milnor construction $F(S^1)$ [3]. This gives a relation between these special Brunnian braids and the homotopy groups. We should point out that the group $R_n \cap C_n$ plays an important role for the Burau and Gassner representations of braids. (See [4].)

It was known that the sequence of pure braid groups, $K = \{K_{n+1}\}_{n \geq 0}$, forms a simplicial group where the i-th face is given by deleting the i-string and the i-degeneracy is given by doubling the i-string. (See for instance [10].) Recently Fred Cohen and the author showed that $F(S^1)$ embeds into K as a simplicial subgroup [6]. From this we obtained certain physical interpretations of the Lambda algebra and the group $G(n)$.

The article is organized as follows. In section 2, we study the braid group action on the Milnor construction of the simplicial 1-sphere. A relation between the simplicial structure and the braid group action is given in Proposition 2.1. After we establish the systematic relations, braided simplicial groups are introduced in this section. Then we discuss some basic properties of braided simplicial groups such as the simplicial loop group and the Moore-Postnikov system. Theorem 2.10 and Proposition 2.11 give a relation between the fixed set of the braid group action and the homotopy groups for a general braided simplicial group. Theorem 1.2 follows from Lemma 2.8 and Theorem 2.14. In section 3, we give a geometrical description of the group $G(n)$. The proof of Theorem 1.3 is given in this section.

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2. BRAID GROUP ACTIONS ON $F(S^1)$

2.1. The braid groups.

In this subsection, we go over some terminology of the braid groups. A reference for braid groups is Birman's book [4].

There are several equivalent definitions of the braid group B_n. A combinatorial definition is as follows: (due to Artin [1])

> *The group* B_n *is generated by letters* $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ *with relations* $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| \geq 2$ *and* $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ *for* $1 \leq i \leq n-1$.

For geometric reasons, B_n is called the *braid group on* n *strings.* Recall that the symmetric group Σ_n is the quotient group of B_n by the additional relation that $\sigma_i^2 = 1$ for each i. Let $F_n = F(x_1, \ldots, x_n)$ be the free group generated by letters $x_1, \ldots, x_n$.

The Artin representation of B_n is a homomorphism ϕ from B_n to $\mathrm{Aut}(F_n)$, where

$$\phi(\sigma_i)(x_j) = \begin{cases} x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1} x_i x_{i+1} & \text{if } j = i+1, \\ x_j & \text{otherwise}, \end{cases} \tag{1}$$

for $1 \leq i \leq n-1$. By Artin's theorem [2], the map ϕ is a monomorphism. Usually we still write σ_i for $\phi(\sigma_i)$ as an element in $\mathrm{Aut}(F_n)$. This gives another definition of B_n, that is, B_n is the subgroup of $\mathrm{Aut}(F_n)$ generated by σ_i for $1 \leq i \leq n-1$. Clearly the product element $x_1 x_2 \cdots x_n$ is a fixed point of the B_n-action on F_n. Thus the Artin representation induces a B_n-action on $\hat{F}_n$, where $\hat{F}_n$ is the quotient group of F_n modulo the single relation $x_1 x_2 \cdots x_n = 1$. As a group, $\hat{F}_n$ is a free group of rank $n-1$.

2.2. **Braided simplicial groups.** We refer to [8, 12] for terminology of simplicial sets and simplicial groups. Let Δ be the category of finite ordered sets and ordered functions, where a function f is *ordered* if $f(x) \leq f(y)$ when $x \leq y$. Let $\mathcal{C}$ be any category. The category Δ has objects $[n] = \{0, \cdots, n\}$ for $n \geq 0$ and the morphisms in $\mathcal{C}$ are generated by the face functions $d^i \colon [n-1] \to [n]$ (misses i) and the degeneracy functions $s^i \colon [n] \to [n+1]$ (hits i twice) for $0 \leq i \leq n$. Recall that a simplicial object X over $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. In other words, $X = \{X_n\}_{n \geq 0}$, where $X_n = X([n])$. The face $d_i \colon X_n \to X_{n-1}$ is given by $d_i = X(d^i)$

and the degeneracy $s_i\colon X_n \to X_{n+1}$ is given by $s_i = X(s^i)$ for $0 \leq i \leq n$. The simplicial identities:

1) $d_jd_i = d_id_{j+1}$ for $j \geq i$;
2) $s_js_i = s_{i+1}s_j$ for $j \leq i$;
3) $d_js_i = \begin{cases} s_{i-1}d_j & \text{for} \quad j < i \\ \mathrm{id} & \text{for} \quad j = i, i+1 \\ s_id_{j-1} & \text{for} \quad j > i \end{cases}$

follow from the well-known formulas for functions d^i and s^j in the category Δ. A simplicial object over sets (resp. monoids, groups, Lie algebras, spaces and etc) is called a simplicial set (resp. monoid, group, Lie algebra, space and etc.)

Let X be a pointed set. We write $J(X)$ (resp. $F(X)$) for the free monoid (resp. free group) generated by X with the single relation that the base-point $* = 1$. The functor J (resp. F) is the coadjoint functor of the forgetful functor from monoids (resp. groups) to pointed sets. Let X be a pointed simplicial set. The simplicial monoid $J(X) = J \circ X$ is called the *James construction* of X and the simplicial group $F(X) = F \circ X$ is called the *Milnor construction* of X. The geometric realization $|F(X)|$ of $F(X)$ is a free group generated by $|X|$ with the single relation that the base-point $* = 1$ and with compactly generated (weak) topology. As a space, there is a homotopy equivalence $|F(X)| \simeq \Omega\Sigma|X|$. The geometric realization of $J(X)$ is the classical James construction on the pointed space $|X|$. We keep in mind that $F(S^n)$ is a simplicial group model of ΩS^{n+1}.

Let $F(S^1)$ be Milnor's construction of the simplicial 1-sphere S^1. Then

$$F(S^1)_{n+1} = F(y_0, \dots, y_n)$$

is a free group generated by letters $y_0, \dots, y_n$ with the following simplicial structure:

$$d_j y_k = \begin{cases} y_{k-1} & \text{for} \quad j \leq k, \\ 1 & \text{for} \quad j = k+1, \\ y_k & \text{for} \quad j > k+1, \end{cases}$$

and

$$s_j y_k = \begin{cases} y_{k+1} & \text{for} \quad j \leq k, \\ y_k y_{k+1} & \text{for} \quad j = k+1, \\ y_k & \text{for} \quad j > k+1, \end{cases}$$

for $0 \leq j \leq n+1$, where $y_{-1} = (y_0 \cdots y_{n-1})^{-1}$ in $F(S^1)_n$ (see [16, Lemma 4.1]).

Let $F_{n+2} = F(y_{-1}, y_0, \dots, y_n)$ be the free group generated by the letters $y_{-1}, y_0, \dots, y_n$. Let B_{n+2} act on F_{n+2} in Artin's sense. We relabel the generators for B_{n+2} by $\sigma_{-1}, \sigma_0, \dots, \sigma_{n-1}$. Observe that the composite

$$\begin{tikzcd} F(y_0, y_1, \dots, y_n) \arrow[r, hook] & F_{n+2} = F(y_{-1}, y_0, \dots, y_n) \arrow[r, two heads] & \hat{F}_{n+2} \end{tikzcd}$$

is an isomorphism. Thus the B_{n+2}-action on F_{n+2} induces a B_{n+2}-action on $F(y_0, \dots, y_n)$, where the map $\sigma_i \colon F(y_0, \dots, y_n) \to F(y_0, \dots, y_n)$ is given by the equation (1) for $i \geq 0$ and σ_{-1} has to be reformulated by

$$\sigma_{-1}(y_0) = y_0^{-1} y_{-1} y_0 = y_0^{-1} y_n^{-1} \cdots y_1^{-1}, \quad \sigma_{-1}(y_j) = y_j \quad \text{for} \quad j > 0.$$

By direct calculation, we have

Proposition 2.1. *The following identities hold for the braid groups action on $F(S^1)$:*

$$d_j \sigma_k = \begin{cases} \sigma_{k-1} d_j & j \leq k, \\ d_{k+2} & j = k+1, \\ d_{k+1} & j = k+2, \\ \sigma_k d_j & j > k+2; \end{cases} \tag{2}$$

$$s_j \sigma_k = \begin{cases} \sigma_{k+1} s_j & j \leq k, \\ \sigma_{k+1} \circ \sigma_k \circ s_{k+2} & j = k+1, \\ \sigma_k \circ \sigma_{k+1} \circ s_{k+1} & j = k+2, \\ \sigma_k s_j & j > k+2. \end{cases} \tag{3}$$

Motivated by this proposition, we give the following definition.

Definition 2.2. A *graded* group means a sequence of groups $G = \{G_n\}_{n \geq 0}$. A graded group G is called *braided* if each G_n admits a B_{n+1}-representation, that is, a group homomorphism $B_{n+1} \to \mathrm{Aut}(G_n)$. Let G and H be braided (graded) groups. A *braided homomorphism* $f \colon G \to H$ means a sequence of homomorphisms $f_n \colon G_n \to H_n$ such that each f_n is B_{n+1}-equivariant. A *braided simplicial group* is a braided group that in addition satisfies the relations spelled out in Proposition 2.1.

Note. It was known [10] that the sequence of the braid groups $B = \{B_{n+1}\}_{n\geq 0}$ is a *crossed* simplicial group. Recently F. Cohen and the author observed that a braided simplicial group G means a simplicial group G that admits a *crossed* B-representation [6], that is, the action $B \times G \to G$ is not a simplicial map but satisfies certain "crossed" conditions. Roughly speaking, this action twists faces and degeneracies.

Let G be a simplicial group and let NG be the Moore chain complex of G, that is,

$$NG_n = \{x \in G_n | d_j x = 1 \quad \text{for} \quad j > 0\}$$

with differential $d_0 \colon NG_n \to NG_{n-1}$, where $d_0^2 = 1$ follows from the simplicial identity $d_i d_j = d_j d_{i+1}$ for $i \geq j$. Let $\mathcal{Z}(G)$ and $\mathcal{B}(G)$ be the sets of cycles and boundaries of G, respectively, that is,

$$\mathcal{Z}_n(G) = \{x \in G_n | d_j x = 1 \quad \text{for} \quad j \geq 0\},$$

$$\mathcal{B}_n(G) = \{d_0 x | x \in N_{n+1}(G)\}.$$

By Moore's classical theorem [14], $\pi_n(G) = \mathcal{Z}_n(G)/\mathcal{B}_n(G)$, where $\pi_n(G)$ means the n-th homotopy group of the geometric realization $|G|$ of G. Let G be a braided simplicial group. A subgroup H_n of G_n is called a *braided subgroup* if $\beta(H_n) \subseteq H_n$ for any $\beta \in B_{n+1}$, that is, H_n is invariant under the B_{n+1}-action.

Proposition 2.3. *Let G be a braided simplicial group. Then the subgroups $\mathcal{Z}(G)$ and $\mathcal{B}(G)$ of G are braided.*

Proof. By Proposition 2.1(2), $\mathcal{Z}(G)$ is a braided subgroup. Now let $x = d_0 y \in \mathcal{B}(G)_n$, where $y \in NG_{n+1}$. By Proposition 2.1(2), we have

$$\sigma_k x = \sigma_k d_0 y = d_0 \sigma_{k+1} y$$

for each $k \geq -1$ and

$$d_j \sigma_{k+1} y = 1$$

for each $j > 0$ and $k \geq -1$. Thus $\sigma_k x \in \mathcal{B}(G)_n$ for each $k \geq -1$ and so $\mathcal{B}(G)$ is a braided subgroup, which is the assertion. □

Note: NG_n is invariant under the subgroup of B_{n+1} generated by σ_k with $k \geq 0$. But it is not invariant under σ_{-1}.

Since there are relations

$$d_0\sigma_{-1} = d_1, \quad d_1\sigma_{-1} = d_0, \quad \text{and} \quad d_j\sigma_{-1} = \sigma_{-1}d_j \quad \text{for} \quad j > 1,$$

we have

Proposition 2.4. *Let G be a braided simplicial group. Then*

$$\mathcal{Z}(G)_n = NG_n \cap \sigma_{-1}(NG_n).$$

for each n.

Now we study the braid group actions on the Postnikov system of a braided simplicial group G. Let $I = (i_1, \ldots, i_k)$ be a sequence of non-negative integers and let d_I denote the composite of face homomorphisms

$$d_I = d_{i_1} \cdots d_{i_k}.$$

Given a simplicial group G, the simplicial sub groups R_nG and $\bar{R}_n(G)$ are defined as follows:

$$R_nG_q = \{x \in G_q | d_I(x) = 1 \text{ for any } I = (i_1, \ldots, i_{q-n})\},$$

$$\bar{R}_nG_q = \{x \in G_q | d_I(x) \in \mathcal{B}(G)_n \text{ for any } I = (i_1, \ldots, i_{q-n})\}.$$

Let $P_nG = G/R_nG$ and $\bar{P}_nG = G/\bar{R}_nG$. Then $\{P_nG\}$ is the Postnikov system of G (see [8, 14]). The quotient homomorphism $P_nG \to \bar{P}_nG$ is a homotopy equivalence (see [16]). The tower

$$\cdots \to P_nG \to \bar{P}_nG \to P_{n-1}G \to \cdots$$

is called a *modified Postnikov system* of G. An important property of the modified Postnikov system is that the short exact sequence of simplicial groups

$$0 \to K(\pi_n(G), n) \to \bar{P}_n G \to P_{n-1} G \to 1$$

is a central extension for $n \geq 1$ [16, Theorem 2.12].

By Propositions 2.1 and 2.3, we have

Theorem 2.5. *Let G be a braided simplicial group. Then, for each n, the simplicial quotient groups $P_n G$ and $\bar{P}_n G$ are braided. Thus the modified Postnikov system of G is braided. In particular, there is a braided central extension*

$$0 \to K(\pi_n(G), n) \to \bar{P}_n(G) \to P_{n-1}(G) \to 1$$

for $n \geq 1$.

Let G be a braided simplicial group. Then $R_0 G$ is a braided simplicial subgroup of G by the theorem above. Recall that the simplicial loop group ΩG of G is defined by $\Omega G_n = \mathrm{Ker}(d_0) \cap R_0 G_{n+1}$ with $d_j(\Omega G) = d_{j+1}(G)$ and $s_j(\Omega G) = s_{j+1}(G)$ (see [8]). By Proposition 2.1, $\mathrm{Ker}(d_0) \cap R_0 G_{n+1}$ is invariant under the action of σ_j for $0 \leq j \leq n-1$. Let B'_{n+1} be the subgroup of B_{n+2} generated by $\sigma_0, \ldots, \sigma_{n-1}$. Then $B'_{n+1} \cong B_{n+1}$ under the canonical isomorphism that sends σ_j to σ_{j-1} for $0 \leq j \leq n-1$. Thus we obtained the following theorem.

Theorem 2.6. *Let G be a braided simplicial group. Then the simplicial loop group ΩG is braided. Thus any iterated simplicial loop groups of G are braided.*

Corollary 2.7. *The loops and the modified Postnikov system of $F(S^1)$ are braided.*

2.3. **Fixed Sets of Braided Actions.** Let G be a simplicial group and let $x, y \in G_n$ be two elements. We say that x is *homotopic* to y, which is denoted by $x \simeq y$, if $xy^{-1} \in \mathcal{B}(G)_n$. (**Note.** If $x \simeq y$, then $d_j x = d_j y$ for each j.)

Lemma 2.8. *Let G be a braided simplicial group and let $x \in \mathcal{Z}(G)_n$ be a cycle with $n \geq 1$. Then*

$$\sigma_k(x) \simeq x^{-1}$$

for each $k \geq -1$.

Proof. For each $k \geq -1$, consider the element $\sigma_{k+1}s_{k+1}x$. By the identities (2) and (3) of Proposition 2.1, we have

$$d_j\sigma_{k+1}s_{k+1}x = \begin{cases} \sigma_k d_j s_{k+1}x = 1 & \text{for} \quad j < k+1 \\ \sigma_k x & \text{for} \quad j = k+1 \\ d_{k+3}s_{k+1}x = 1 & \text{for} \quad j = k+2 \\ d_{k+2}s_{k+1}x = x & \text{for} \quad j = k+3 \\ \sigma_{k+1}d_j s_{k+1}x = 1 & \text{for} \quad j > k+3. \end{cases}$$

By the Homotopy Addition Theorem [8, Theorem 2.4], one gets $\sigma_k x \simeq x^{-1}$ and hence the result. □

Let G be a braided simplicial group and let S be a subset of G_n. Define

$$\mathrm{hStab}_{B_{n+1}}(S) = \{\sigma \in B_{n+1} | \sigma x \simeq x \quad \text{for} \quad \text{all} \quad x \in S\}.$$

Since $\mathcal{B}(G)$ is invariant under the braid group action, $\mathrm{hStab}_{B_{n+1}}(S)$ is a subgroup of B_{n+1}. Let $\tilde{B}_{n+1}$ be the kernel of the composite

$$\begin{tikzcd} B_{n+1} \arrow[r] & \Sigma_{n+1} \arrow[r, "\text{sign}"] & \mathbb{Z}/2, \end{tikzcd}$$

that is, $\tilde{B}_{n+1}$ is the pre-image of the alternating group A_{n+1}.

Lemma 2.9. *Any normal subgroup of B_{n+1} which contains all elements of the form $\sigma_i\sigma_j$ is either B_{n+1} or $\tilde{B}_{n+1}$.*

Proof. Let P be a normal subgroup of B_{n+1} which contains all elements of the form $\sigma_i\sigma_j$. According to [11, Theorem N7, pp.173], the kernel K_{n+1} of the mapping $B_{n+1} \to \Sigma_{n+1}$ is the normal subgroup generated by σ_1^2. Thus $K_{n+1} \subseteq P$. Since the image of P in Σ_{n+1} contains all even permutations, we have $\mathrm{Im}(P \to \Sigma_{n+1}) \supseteq A_{n+1}$ and hence the result. □

Theorem 2.10. *Let G be a braided simplicial group. Then*

1) $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n) = \tilde{B}_{n+1}$ *or* B_{n+1}*;*

2) $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n) = B_{n+1}$ *if and only if* $2 \cdot \pi_n(G) = 0$.

Proof. Let $\sigma \in \mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n)$ and let $x \in \mathcal{Z}(G)_n$. By Lemma 2.8, we have

$$(\sigma_k^{-1}\sigma\sigma_k)(x) \simeq (\sigma_k\sigma)(x^{-1}) \simeq \sigma_k(x^{-1}) \simeq x$$

for each $k \geq -1$. Thus $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n)$ is a normal subgroup of B_{n+1}. By Lemma 2.8, we have

$$\sigma_s\sigma_t \in \mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n)$$

for any $s, t \geq -1$. It follows that $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n) = \tilde{B}_{n+1}$ or B_{n+1}, which is the assertion 1.

If $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n) = B_{n+1}$, then by Lemma 2.8

$$x \simeq x^{-1}$$

for any $x \in \mathcal{Z}(G)_n$. Thus $2 \cdot \pi_n(G) = 0$. Conversely, if $2 \cdot \pi_n(G) = 0$, then

$$\sigma_k(x) \simeq x$$

for any $x \in \mathcal{Z}(G)_n$ and $k \geq -1$. Thus $\mathrm{hStab}_{B_{n+1}}(\mathcal{Z}(G)_n) = B_{n+1}$. This shows assertion 2. □

Let G be a braided simplicial group and let H be a subgroup of B_{n+1}. Define

$$(G_n)^{\mathrm{h}\,H} = \{x \in G_n | \sigma(x) \simeq x \quad \text{for} \quad \text{all} \quad \sigma \in H\},$$

that is, $(G_n)^{\mathrm{h}\,H}$ is the "homotopy" fixed set of H. Then $(G_n)^{\mathrm{h}\,H}$ is a subgroup of G_n. By Theorem 2.10, we have that

$$(G_n)^{\mathrm{h}\,\tilde{B}_{n+1}} \supseteq \mathcal{Z}(G)_n.$$

Proposition 2.11. *Let G be a braided simplicial group and let $x \in G_n^{\mathrm{h}\,\tilde{B}_{n+1}}$ with $n \geq 2$. Then*

1) *$d_j(x) = d_{j+2}(x)$ for each j;*
2) *$\sigma_k d_j(x) = d_{j+1}(x)$ for each j, k;*
3) *$d_j(x)$ is a fixed point of $\tilde{B}_n$ for each j.*

In particular, if $d_j x = 1$ for some j, then $x \in \mathcal{Z}(G)_n$.

Proof. Since $x \in (G_n)^{\mathrm{h}\,\tilde{B}_{n+1}}$, we have

$$\sigma_s\sigma_t(x) \simeq x$$

for any $s, t \geq -1$. It follows that

$$\sigma_{-1}x \simeq \sigma_0 x \simeq \sigma_1 x \simeq \ldots \simeq \sigma_{n-2}x.$$

Now for each $-1 \leq k \leq n-3$, we have

$$d_{k+1}x = d_{k+2}\sigma_k x = d_{k+2}\sigma_{k+1}x = d_{k+3}x$$

and so assertion 1 follows.

Now for each $0 \leq s \leq n-2$, we have

$$d_{s+1}(x) = d_s\sigma_{s-1}(x) = d_s\sigma_s(x) = \sigma_{s-1}d_s(x).$$

Let j and k be integers with $0 \leq j \leq n-1$ and $-1 \leq k \leq n-3$. If $j \equiv k+1 \mod 2$, then

$$\sigma_k d_j(x) = \sigma_k d_{k+1}x = d_{k+2}x = d_{j+1}x$$

by assertion 1. If $j \equiv k \mod 2$ with $k \geq 0$, then

$$\sigma_k d_j(x) = \sigma_k d_k(x) = d_k\sigma_{k+1}(x) = d_k\sigma_{k-1}(x) = d_{k+1}(x) = d_{j+1}(x)$$

by assertion 1. Assume that $j \equiv k \mod 2$ with $k = -1$. Then

$$\sigma_{-1}d_j x = \sigma_{-1}d_1 x = \sigma_{-1}d_0\sigma_{-1}x = d_0(\sigma_0\sigma_{-1}(x)) = d_0 x = d_{j+1}x.$$

Assertion 2 follows.

For any $s, t \geq -1$, we have

$$\sigma_s(\sigma_t d_0(x)) = \sigma_s(d_0(\sigma_{t+1}x)) = d_0(\sigma_{s+1}\sigma_{t+1}x) = d_0 x.$$

Thus

$$\sigma_t\sigma_s(d_0(x)) = d_0(x)$$

for any $s, t \geq -1$ and so $d_0(x)$ is a fixed point of $\tilde{B}_n$. By assertions 1 and 2, we have

$$\sigma_t(\sigma_s(d_1(x))) = \sigma_t(d_2(x)) = \sigma_t(d_0(x)) = d_1(x)$$

and so $d_1(x)$ is a fixed point of $\tilde{B}_n$. Assertion 3 follows. $\square$

Let B'_n be the subgroup of B_{n+1} generated by σ_j with $j \geq 0$. Let $\tilde{B}'_n$ be the normal subgroup of B'_n generated by $\sigma_i\sigma_j$ for $i, j \geq 0$. By inspecting the proof, we have

Proposition 2.12. *Let G be a braided simplicial group and let $x \in (G_n)^{\mathrm{h}\tilde{B}'_n}$ with $n \geq 2$. Then*

1) *$d_j(x) = d_{j+2}(x)$ for each $j \geq 1$;*
2) *$\sigma_k d_j(x) = d_{j+1}(x)$ for each $j, k \geq 1$;*
3) *$d_j(x)$ is a fixed point of $\tilde{B}'_{n-1}$ for each j;*
4) *$d_0(x)$ is a fixed point of $\tilde{B}_n$.*

In particular, if $d_0x = 1$ and $d_jx = 1$ for some $j \geq 1$, then $x \in \mathcal{Z}(G)_n$.

The following lemma seems well-known, but we could not find a reference and so we provide a proof here. Let S be a subset of a group G. We write $\langle S\rangle$ for the subgroup of G generated by S. Let G be a group with $w \in G$. We write $\chi_w \colon G \to G,\ x \mapsto w^{-1}xw$ for the conjugation map.

Lemma 2.13. *Let $w \in F(y_0, \ldots, y_n)$ with $n \geq 0$. Suppose that there is a positive integer k such that $\sigma_j^k(w) = w$ for $0 \leq j \leq n-1$. Then w lies in the subgroup generated by $y_0y_1\cdots y_n$. In addition, if $\sigma_{-1}^k w = w$ and $n \geq 1$, then $w = 1$.*

Proof. The proof is given by induction on n. The assertion is trivial for $n = 0$. Let $n = 1$. We may assume that $k = 2t$ is an even integer. Let $x_0 = y_1^{-1}$ and let $x_1 = y_0y_1$. Then $F(y_0, y_1) = F(x_0, x_1)$. Since $\sigma_0(y_0) = y_1$ and $\sigma_0(y_1) = y_1^{-1}y_0y_1$, we have $\sigma_0^2 = \chi_{x_1}$ and so

$$\sigma_0^k = \chi_{x_1}^t = \chi_{x_1^t}.$$

We can write w as a reduced word in $F(y_0, y_1) = F(x_0) * F(x_1)$. Then

$$w = x_0^{n_1}x_1^{l_1} \cdots x_0^{n_s}x_1^{l_s},$$

where $n_j \neq 0$ for $2 \leq j \leq s$ and $l_j \neq 0$ for $1 \leq j \leq s - 1$. Suppose that $w \notin \langle x_1 \rangle$. There are two cases: $n_1 \neq 0$ or $n_1 = 0$. If $n_1 \neq 0$, then $x_1^t w \neq wx_1^t$. This contradicts the assumption that $\chi_{x_1^t}(w) = w$. Otherwise, $n_1 = 0$ and $s > 1$. Then $w = x_1^{l_1}x_0^{n_2}w'$ and $x_1^{t+l_1}x_0^{n_2}w' \neq x_1^{l_1}x_0^{l_2}w'x_1^t$. This contradicts the assumption that $\chi_{x_1^t}(w) = w$. Hence $w \in \langle x_1 \rangle = \langle y_0y_1 \rangle$.

Now suppose that the assertion holds for $n - 1$ with $n > 1$. Since

$$F(y_0, \dots, y_n) = F(y_0, \dots, y_{n-1}) * F(y_n)$$

is a free product, we can write w as a word

$$w = y_n^{l_0}w_1y_n^{l_1} \cdots w_ty_n^{l_t},$$

where $w_j \neq 1 \in F(y_0, \ldots, y_{n-1)}$ and $l_j \neq 0$ for $1 \leq j \leq t-1$. Because $\sigma_j(y_n) = y_n$ for $j < n-1$, we have

$$\sigma_j^k(w_i) = w_i$$

for $1 \leq i \leq t$ and $0 \leq j \leq n-2$. Let $x = y_0 y_1 \cdots y_{n-1}$. By induction, we have

$$w_i \in \langle x \rangle$$

for $1 \leq i \leq t$ and so

$$w \in \langle x, y_n \rangle.$$

Let $q\colon F(y_0, y_1, \ldots, y_n) \to F(y_{n-1}, y_n)$ be the projection defined by $q(y_j) = 1$ for $j < n-1$ and $q(y_j) = y_j$ for $j \geq n-1$. Then

$$q \circ \sigma_{n-1} = \sigma_{n-1} \circ q.$$

Since $\sigma_{n-1} w = w$, we have $\sigma_{n-1}(q(w)) = q(w)$ and so

$$q(w) \in \langle y_{n-1} \cdot y_n \rangle.$$

Because the restriction

$$q|_{\langle x, y_n \rangle}\colon \langle x, y_n \rangle \to F(y_{n-1}, y_n)$$

is an isomorphism, we have

$$w \in \langle x \cdot y_n \rangle = \langle y_0 y_1 \cdots y_n \rangle$$

and hence the result. □

Let K_n be the pure braided group, that is, K_n is the kernel of the canonical epimorphism $B_n \to \Sigma_n$. Recall that the group B'_n is defined as the subgroup of B_{n+1} generated by σ_i for $i \geq 0$. There is a canonical isomorphism $\phi\colon B'_n \to B_n$ with $\phi(\sigma_i) = \sigma_{i-1}$. We write K'_n for $\phi^{-1}(K_n)$. Recall that

$$\pi_{n+1}(F(S^1)) = \mathcal{Z}(F(S^1))_{n+1}/\mathcal{B}(F(S^1))_{n+1}.$$

Consider the actions of two groups B_{n+2} and B'_{n+1} on $F(S^1)_{n+1}/\mathcal{B}(F(S^1))_{n+1}$. We have

Theorem 2.14. *If $n \geq 2$, then in $F(S^1)_{n+1}/\mathcal{B}(F(S^1))_{n+1}$,*

1) *the fixed set of the pure braided group K'_{n+1}-action is*

$$\mathbb{Z} \times \pi_{n+1}(F(S^1));$$

2) *the fixed set of the K_{n+2}-action is*

$$\pi_{n+1}(F(S^1)).$$

Proof. (1). Let $y_{-1} = (y_0 y_1 \cdots y_n)^{-1}$. (1). We show that the homotopy fixed set of K'_{n+1} on $F(S^1)_{n+1}$ is generated by y_{-1} and $\mathcal{Z}(F(S^1))_{n+1}$. Assertion 1 will follow from this statement.

Let w be a homotopy fixed point of the K'_{n+1}-action on $F(S^1)_{n+1}$. Since

$$\sigma_k^2 d_0 = d_0 \sigma_{k+1}^2$$

for $k \geq -1$, we have

$$\sigma_k^2 d_0(w) = d_0(w)$$

for each $k \geq -1$. By Lemma 2.13, we have

$$d_0(w) = 1.$$

Now for each $1 \leq j \leq n+1$, we have

$$\sigma_k^2 d_j = \begin{cases} d_j \sigma_{k+1}^2 & \text{if} \quad j \leq k+1; \\ d_j \circ \sigma_{j-1}^{-1} \circ \sigma_{j-2}^2 \circ \sigma_{j-1} & \text{if} \quad j = k+2; \\ d_j \sigma_k^2 & \text{if} \quad j > k+2. \end{cases}$$

By Lemma 2.13, there exists integers $k_1, k_2, \ldots, k_{n+1}$ such that

$$d_j(w) = y_{-1}^{k_j}$$

for $1 \leq j \leq n+1$. Since $d_k y_{-1} = y_{-1}$ for $k > 0$, we have

$$y_{-1}^{k_j} = d_j(y_{-1}^{k_j}) = d_j d_j w = d_j d_{j+1} w = d_j(y_{-1}^{k_{j+1}}) = y_{-1}^{k_{j+1}}$$

for $1 \leq j \leq n$ and so

$$k_1 = k_2 = \cdots = k_{n+1}.$$

Let $w' = y_{-1}^{-k_1} w$. Then

$$d_j(w') = 1$$

for each $0 \leq j \leq n+1$ and $w' \in \mathcal{Z}(F(S^1))_{n+1}$. This shows that w lies in the subgroup generated by y_{-1} and cycles $\mathcal{Z}(F(S^1))_{n+1}$. This finishes the proof of the statement and hence assertion 1 holds.

(2). Let w be a homotopy fixed point of K_{n+2}. By assertion 1, w lies in the subgroup generated y_{-1} and $\mathcal{Z}(F(S^1))_{n+1}$. Since $\pi_{n+1}(F(S^1)) = \mathcal{Z}(F(S^1))_{n+1}/\mathcal{B}(F(S^1))_{n+1}$ is the centre of $F(S^1)_{n+1}/\mathcal{B}(F(S^1))_{n+1}$, we have

$$w \simeq y_{-1}^a w'$$

for some $a \in \mathbb{Z}$ and $w' \in \mathcal{Z}(F(S^1))_{n+1}$. Recall that $\sigma_{-1}(y_{-1}) = y_0$. By Lemma 2.8, we have $\sigma_{-1}^2(w') \simeq w'$ and so

$$y_{-1}^a w' \simeq w \simeq \sigma_{-1}^2(w) \simeq \sigma_{-1}^2(y_{-1}^a) w' \simeq y_0^{2a} w'.$$

Thus

$$(y_0 y_1 \cdots y_{n-1})^a = (d_1(y_{-1})^a = d_1(y_{-1}^a w') = d_1(y_0^{2a} w') = 1.$$

It follows that $a = 0$ and hence assertion 2 holds. $\square$

Note: In $F(S^1)_{n+1}/\mathcal{B}_{n+1}$, since any element in the homotopy group is a homotopy fixed point of $\tilde{B}_{n+2}$, the fixed set of $\tilde{B}'_{n+1}$ is $\mathbb{Z} \times \pi_{n+1}(F(S^1))$ and the fixed set of $\tilde{B}_{n+2}$ is $\pi_{n+1}(F(S^1))$.

2.4. **Some Remarks on the simplicial group $F(S^n)$.** In the beginning of this section, we use a consequence of [16, Lemma 4.1] to get a simplicial group model for ΩS^2 by using Milnor's construction $F(S^1)$. Below we provide a method to test certain simplicial free group models for ΩS^{n+1} by counting rank. The simplicial group $F(S^n)$ has the following uniqueness property.

Let G be a (simplicial) group. We write $\{\Gamma_k G\}_{k\geq 1}$ for the descending central series of G starting with $\Gamma_1 G = G$. The completion of G with respect to the descending central series is the (simplicial) pro-group $G^{\wedge} = \lim_k G/\Gamma_k(G)$. Let

$$L(G) = \bigoplus_{k\geq 1} \Gamma_k(G)/\Gamma_{k+1}(G)$$

be the (simplicial) Lie algebra induced by the (simplicial) group G. We write $\{\Gamma_k^{(p)}(G)\}_{k\geq 1}$ for the mod p descending central series of G. The induced (simplicial) restricted Lie algebra is denoted by

$$L^{(p)}(G) = \bigoplus_{k\geq 1} \Gamma_k^{(p)}(G)/\Gamma_{k+1}^{(p)}(G).$$

The p-completion of G is the (simplicial) pro-group $G^{\wedge_p} = \lim_k G/\Gamma_k^{(p)}(G)$.

Proposition 2.15. *Let G be a simplicial group. Suppose that*

1) *$G_q = 1$ for $q < n$ and*
2) *G_{n+k} is a free group of rank $\binom{n+k}{n}$ for $k \geq 0$.*

Then there is a simplicial monomorphism $\phi\colon F(S^n) \to G$ such that

1) *ϕ is a simplicial monomorphism;*

2) *ϕ is a homotopy equivalence;*

3) *the map $\phi^\wedge\colon F(S^n)^\wedge \to G^\wedge$ is an isomorphism;*

4) *the map $\phi^\wedge\colon F(S^n)^{\wedge_p} \to G^{\wedge_p}$ is an isomorphism for any prime p.*

Proof. Let z be the generator for the group $G_n \cong \mathbb{Z}$ and let $f_z\colon \Delta[n] \to G$ be the representing map of the element z, where $\Delta[n]$ is the standard n-simplex. Since $G_{n-1} = 1$, we have $d_j z = 1$ for each j and so the map f_z factors through the n-sphere S^n. Let $g_z\colon S^n \to G$ be the resulting map. By the universal property of the functor F, there is a (unique) simplicial homomorpism $\phi\colon F(S^n) \to G$ such that $\phi|_{S^n} = g_z$. Let $\phi_{\mathrm{ab}}\colon F(S^n)_{\mathrm{ab}} = K(\mathbb{Z}, n) \to G_{\mathrm{ab}}$ be the induced map of the abelianization of simplicial groups. By assumption on G, the abelianization G_{ab} is the minimal simplicial group $K(\mathbb{Z}, n)$ and so the map ϕ_{ab} is an isomorphism. Since each G_n is a free group, $L(G)$ is the (simplicial) free Lie algebra generated by G_{ab}. Thus

$$L(\phi)\colon L(F(S^n)) \to L(G) \tag{4}$$

is an isomorphism. It follows that ϕ is a monomorphism and $\phi^\wedge\colon F(S^n)^\wedge \to G^\wedge$ is an isomorphism, which are assertions 1 and 3. By Equation (4), Curtis' convergence theorem [7] holds for G and so ϕ is a homotopy equivalence by the Adams spectral sequence [8], which is assertion 2. Similarly,

$$L^{(p)}(\phi)\colon L^{(p)}(F(S^n)) \to L^{(p)}(G)$$

is an isomorphism and hence assertion 4. □

Corollary 2.16. *Let G be a simplicial group such that each G_n is a free group of rank n. Then G is a simplicial group model for ΩS^2.*

Note. The point in Proposition 2.15 is that for any given simplicial free group G, we may just count the rank of each G_n to see whether we obtain a simplicial group model for ΩS^{n+1}.

Note. By looking at Proposition 2.1, the braid group actions on $F(S^1)$ are certain automorphisms that twist the faces and degeneracies. It seems interesting to have a generalization of these results on $F(S^n)$ for general n.

3. The Geometry of the Group $G(n)$ and the Proof of Theorem 1.3

In this section we give a proof of Theorem 1.3. First we need to represent the group $G(n)$ as the fundamental group of a certain space X_n. We will show that the braid group action on $G(n)$ is induced by certain self homeomorphisms of X_n.

3.1. The Geometry of the Group $G(n)$.

Let $F_n = F(x_1, \dots, x_n)$ and let $\hat{F}_n$ be the quotient group of F_n by the single relation $x_1 x_2 \cdots x_n = 1$. Let B_n act on F_n in

Artin's sense (see Subsection 2.1 above) and let $\hat{B}_n$ be the image of B_n in $\mathrm{Aut}(\hat{F}_n)$. We recall a geometric interpretation of the B_n-action on F_n and $\hat{F}_n$.

Let D^2 be the unit disc and let $Q_n = \{q_1, \ldots, q_n\}$ be a set of distinct fixed points of $D^2 \setminus \partial D^2$. Then $\pi_1(D^2 \setminus Q_n)$ is a free group of rank n. Let $x_1, \ldots, x_n$ be a basis for $\pi_1(D^2 \setminus Q_n)$, where x_i is represented by a simple loop which encloses the boundary point q_i, but no boundary point q_j for $j \neq i$. According to [4, Theorem 1.10], the braid group B_n is precisely the group of automorphisms of $\pi_1(D^2 \setminus Q_n)$ that are induced by self homeomorphisms of $D^2 \setminus Q_n$ that keep the boundary of D^2 fixed pointwise. (This result is due to Artin as well and can be regarded as a third definition of the group B_n.) Let $D^2 \cong S^2_- \subseteq S^2$ be the canonical embedding of the lower hemisphere. Then $\pi_1(S^2 \setminus Q_n) = \hat{F}_n$ and $\hat{B}_n$ is precisely the group of automorphisms of $\phi_1(S^2 \setminus Q_n)$ that are induced by self homeomorphisms of $S^2 \setminus Q_n$ that keep the upper hemisphere $S^2 \setminus S^2_-$ fixed pointwise.

Now let $Q_{n,i} = Q_n \backslash \{q_i\}$ for each $1 \leq i \leq n$. The n-fold diagonal map $\Delta_n \colon D^2 \to (D^2)^n$ induces the map

$$\psi_n = \Delta_n|_{D^2 \setminus Q_n} \colon D^2 \setminus Q_n \to (D^2 \setminus Q_{n,1}) \times \ldots \times (D^2 \setminus Q_{n,n}), \quad x \mapsto (x, \ldots, x).$$

It follows that there is a (unique) covering space Y_n of $\prod_{i=1}^{n} D^2 \setminus Q_{n,i}$ such that

$$\pi_1(Y_n) = \mathrm{Im}(\psi_{n*} \colon \pi_1(D^2 \setminus Q_n) \to \pi_1(\prod_{i=1}^{n} D^2 \setminus Q_{n,i}))$$

and the map ψ_n lifts to Y_n (uniquely). Let $q_n\colon Y_n \to \prod_{i=1}^{n} D^2 \setminus Q_{n,i}$ be the covering map and let $\tilde{\psi}_n\colon D^2 \to Y_n$ be the lifting of ψ_n, that is $q_n \circ \tilde{\psi}_n = \psi_n$. Since the map ψ_n is an embedding, so is $\tilde{\psi}_n$. Observe that $\pi_1(Y_n)$ is the quotient group of $F(x_1, \dots, x_n)$ modulo Relation $(\mathcal{R}_2)$ in the definition of the group $G(n)$. Let $X_n = Y_n \bigcup_{D^2\setminus Q_n} S^2 \setminus Q_n$. By the Seifert-van Kampen theorem, we have

Lemma 3.1. *The fundamental group $\pi_1(X_n)$ is isomorphic to $G(n)$.*

Note. The space Y_n is a $K(\pi, 1)$ and $\pi_1(Y_n)$ is isomorphic to $F(S^1)_n/NF(S^1)_n$, the quotient group of $F(S^1)_n$ modulo the Moore chains. On the other hand, X_n is *not* a $K(\pi, 1)$ in general because X_n is an n-dimensional CW-complex and $\pi_1(X_n) = G(n)$ contains finite subgroups in general. According to [16], the maximal finite subgroup of $G(n)$ is precisely the centre $Z(G(n)) \cong \pi_n(S^2)$ for $n \geq 4$. Fred Cohen asked how to make X_n into a $K(\pi, 1)$ by considering the Borel construction on a certain group action on X_n. Cohen's problem remains open.

Let β be an element in the braid group B_n. By the third definition of the group B_n [4, Theorem 1.10], there is a homeomorphism $h\colon D^2 \setminus Q_n \to D^2 \setminus Q_n$ such that h keeps the boundary S^1 of D^2 fixed pointwise and

$$\beta = h_*\colon \pi_1(D^2 \setminus Q_n) \to \pi_1(D^2 \setminus Q_n).$$

The self homeomorphism h of $D^2 \setminus Q_n$ has a unique extension $\bar{h}$ to D^2 which permutes the points of Q_n. Thus there exists a (unique) element μ_β in Σ_n such that the diagram

$$\begin{tikzcd}
D^2 \setminus Q_n \arrow[r, hook] \arrow[d, "\cong"', "h"] & D^2 \setminus Q_{n,i} \arrow[d, "\cong"', "d_i(h) = \bar{h}|_{D^2 \setminus Q_{n,i}}"] \\
D^2 \setminus Q_n \arrow[r, hook] & D^2 \setminus Q_{n,\mu_\beta(i)}
\end{tikzcd}$$

commutes. We write $d_i(\beta)$ for the isomorphism

$$d_i(h)_*\colon \pi_1(D^2 \setminus Q_{n,i}) \to \pi_1(D^2 \setminus Q_{n,\mu_\beta(i)}).$$

As a geometric braid, $d_i(\beta)$ is the braid on $n-1$ strings obtained from β by deleting the i-th string.

Note. $\beta \in K_n$ if and only if $\mu_\beta = 1$. In this case $d_i(h)$ is an self homeomorphism of $D^2 \setminus Q_{n,i}$.

Consider the commutative diagram

$$\begin{tikzcd}
D^2 \setminus Q_n \arrow[r, hook, "\psi_n"] \arrow[d, "\cong"', "h"] & \prod_{i=1}^{n} D^2 \setminus Q_{n,i} \arrow[d, "\cong"', "\prod_{i=1}^n d_i(h)"] \\
D^2 \setminus Q_n \arrow[r, hook, "\psi_n"] & \prod_{i=1}^{n} D^2 \setminus Q_{n,i}.
\end{tikzcd}$$

The self homeomorphism $\prod_{i=1}^{n} d_i(h)$ of $\prod_{i=1}^{n} D^2 \setminus Q_{n,i}$ induces a unique self homeomorphism $\tilde{h}$ of Y_n such that $\tilde{h} \circ \tilde{\psi}_n = \tilde{\psi}_n \circ h$ and $q_n \circ \tilde{h} = (\prod_{i=1}^{n} h_i) \circ q_n$. Let h' be an self homeomorphism of $S^2 \setminus Q_n$ such that $h'|_{D^2\setminus Q_n} = h$ and $h'|_{S^2\setminus D^2}$ is the identity map. Then the maps $\tilde{h}$ and h' define an self homeomorphism $h'' = \tilde{h} \cup h'$ of X_n. Clearly the map $h''_*\colon \pi_1(X_n) \to \pi_1(X_n)$ is the automorphism induced by

$$\beta\colon \pi_1(D^2 \setminus Q_n) \to \pi_1(D^2 \setminus Q_n).$$

Thus we have

Proposition 3.2. *The B_n-action on $G(n)$ is induced by certain self homeomorphisms of X_n.*

3.2. **Proof of Theorem 1.3.** Recall that the centre of B_n is the infinite cyclic subgroup generated by $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n$ for $n \geq 3$. (See [4, Corollary 1.8.4].) Clearly the element $(\sigma_1\cdots\sigma_{n-1})^n$ acts trivially on $\hat{F}_n$. Thus $\hat{B}_n$ is a quotient group of $B_n/Z(B_n)$, where $Z(G)$ denotes the centre of a group G.

Lemma 3.3. *Let $n \geq 3$. Then the group $\hat{B}_n$ is isomorphic to $B_n/Z(B_n)$.*

Proof. By [4, Lemma 3.17.2], the kernel of $K_n \to \mathrm{Aut}(\hat{F}_n)$ is $Z(K_n) = Z(B_n)$. Let G be the kernel of $B_n \to \mathrm{Aut}(\hat{F}_n)$. By the commutative diagram

$$\begin{tikzcd}
\Sigma_n \arrow[r, hook] & \mathrm{Aut}(\hat{F}_n/\Gamma^2\hat{F}_n) \cong \mathrm{GL}(n-1,\mathbb{Z}) \\
B_n \arrow[u, two heads] \arrow[r] & \mathrm{Aut}(\hat{F}_n), \arrow[u]
\end{tikzcd}$$

we have $G = Z(B_n)$ and hence the result. $\square$

Let

$$C_n = \{\beta \in B_n | d_i(\beta) = 1 \text{ for } 1 \leq i \leq n\}.$$

Then C_n is a normal subgroup of B_n. Let $\delta_i\colon F_n = \pi_1(D^2 \backslash Q_n) \to F_{n,i} = \pi_1(D^2 \backslash Q_{n,i})$ be induced by the inclusion. Observe that the group $F_{n,i}$ is the free group generated by $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ and the homomorphism δ_i is given by $\delta_i(x_i) = 1$ and $\delta_i(x_j) = x_j$ for $j \neq i$. There is commutative diagram

$$\begin{tikzcd}
F_n \arrow[r, "\delta_i"] \arrow[d, "\beta"] & F_{n,i} \arrow[d, "d_i(\beta)"] \\
F_n \arrow[r, "\delta_{\mu_\beta(i)}"] & F_{n,\mu_\beta(i)}
\end{tikzcd}$$

for any $\beta \in B_n$ by the definition of $d_i(\beta)$.

Lemma 3.4. *The kernel of the representation* $B_n \to \mathrm{Aut}(\pi_1(Y_n))$ *is precisely the subgroup* C_n *for each* n*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
F_n \arrow[r, two heads] \arrow[d, "\beta"] & \pi_1(Y_n) \arrow[r, hook] \arrow[d] & \prod_{i=1}^{n} F_{n,i} \arrow[d, "\prod_{i=1}^{n} d_i(\beta)"] \\
F_n \arrow[r, two heads] & \pi_1(Y_n) \arrow[r, hook] & \prod_{i=1}^{n} F_{n,i}.
\end{tikzcd}$$

If $\beta \in C_n$, then each $d_i(\beta) = 1$. Thus

$$C_n \subseteq \operatorname{Ker}(B_n \to \operatorname{Aut}(\pi_1(Y_n)))$$

Conversely, suppose that $\beta \in \operatorname{Ker}(B_n \to \operatorname{Aut}(\pi_1(Y_n)))$. By the commutative diagram

$$\begin{tikzcd}
F_n \arrow[r, "\beta"] \arrow[d] & F_n \arrow[d] \\
\pi_1(Y_n) \arrow[r, "\prod_{i=1}^n d_i(\beta)|_{\pi_1(Y_n)}"] \arrow[d] & \pi_1(Y_n) \arrow[d] \\
F_{ni} \arrow[r, "d_i(\beta)"] & F_{n\mu_\beta(i)},
\end{tikzcd}$$

we have $d_i(\beta) = 1$ for all i and so $\beta \in C_n$. This finishes the proof. □

Consider the map

$$\psi'_n \colon S^2 \setminus Q_n \longrightarrow \prod_{i=1}^n S^2 \setminus Q_{n,i}, \quad , x \mapsto (x, \dots, x).$$

There is a (unique) covering space Z_n of $\displaystyle\prod_{i=1}^n S^2 \setminus Q_{n,i}$ such that the map ψ'_n lifts to Z_n and $\pi_1(Z_n)$ is given by the image of ψ'_{n*}.

Note. The space Z_n is a $K(\pi, 1)$ and $\pi_1(Z_n)$ is isomorphic to $F(S^1)_{n-1}/\mathcal{Z}(F(S^1))_{n-1}$, the quotient group of $F(S^1)_{n-1}$ modulo the cycles.

Let $\hat{F}_{n,i} = \pi_1(S^2 \setminus Q_{n,i})$ and let $\hat{\delta}_i \colon \hat{F}_n \to \hat{F}_{n,i}$ be induced by the inclusion $S^2 \setminus Q_n \subseteq S^2 \setminus Q_{n,i}$. Let h be an self homeomorphism of $D^2 \setminus Q_n$ that keeps the boundary of D^2 fixed pointwise. We write $\hat{d}_i(h)$ for the homeomorphism $S^2 \setminus Q_{n,i} \to S^2 \setminus Q_{n,\mu_h(i)}$ induced by the canonical extension of h to S^2. Let $\beta \in B_n$ be represented by h. We write $\hat{d}_i(\beta)$ for the isomorphism $\hat{F}_{n,i} \to \hat{F}_{n,\mu_\beta(i)}$ induced by the map $\hat{d}_i(h)$. Then there is a commutative diagram

$$\begin{tikzcd}
\hat{F}_n \arrow[r, "\hat{\delta}_i"] \arrow[d, "\beta"] & \hat{F}_{n,i} \arrow[d, "\hat{d}_i(\beta)"'] \\
\hat{F}_n \arrow[r, "\hat{\delta}_{\mu_\beta(i)}"] & \hat{F}_{n,\mu_\beta(i)}
\end{tikzcd}$$

and so the B_n-action on $\hat{F}_n$ induces a B_n-action on $\pi_1(Z_n)$. Let

$$D_n = \{\beta \in B_n | \hat{d}_i(\beta) = 1 \text{ for } 1 \leq i \leq n\}.$$

Clearly D_n is a normal subgroup of B_n. By the proof of Lemma 3.4, we have

Lemma 3.5. *The kernel of the representation* $B_n \to \mathrm{Aut}(\pi_1(Z_n))$ *is precisely the subgroup* D_n.

Proof of Theorem 1.3. The canonical inclusions of Y_n and $S^n \setminus Q_n$ into Z_n induce an inclusion of X_n into Z_n. Let $\bar{D}_n$ be the kernel of the representation

$$B_n \to \mathrm{Aut}(G(n)) = \mathrm{Aut}(\pi_1(X_n)).$$

By Lemmas 3.3-3.5, we have

$$\mathbb{Z} \times C_n \cong Z(B_n) \cdot C_n \subseteq D'_n \subseteq D_n.$$

It suffices to show that $D_n \subseteq Z(B_n) \cdot C_n$. First we show that $K_n \cap D_n \subseteq Z(B_n) \cdot C_n$. Suppose that $\beta \in K_n \cap D_n$. Let $x_0 = x_1 \ldots x_n \in F_n$ and let $\alpha = (\sigma_1 \cdots \sigma_{n-1})^n$ be the generator for $Z(B_n)$. Then $\alpha = \chi_{x_0}\colon F_n \to F_n$, where $\chi_w(x) = w^{-1}xw$ is the conjugation map. Observe that $d_i(\beta)\colon F_{n,i} \to F_{n,i}$ is a braid for each i, that is, $d_i(\beta)$ lies in the braid group B_{n-1} as a subgroup of $\mathrm{Aut}(F_{n,i})$. Since $\hat{d}_i(\beta)\colon \hat{F}_{n,i} \to \hat{F}_{n,i}$ is the identity for each i, the element $d_i(\beta)$ lies in the kernel of $B_{n-1} \to \mathrm{Aut}(\hat{F}_{n,i})$ and so there exists an integer l_i such that $d_i(\beta) = \chi_{(\delta_i(x_0))^{l_i}}$ for each i. For each $1 \leq i < j \leq n$, there exists a unique automorphism $d_{ij}(\beta_{ij})\colon F_{n,ij} \to F_{n,ij}$ such that the diagram

$$\begin{tikzcd}
F_n \arrow[r, "\delta_{ij}"] \arrow[d, "\beta"] & F_{n,ij} \arrow[d, "d_{ij}(\beta)"] \\
F_n \arrow[r, "\delta_{ij}"] & F_{n,ij}
\end{tikzcd}$$

commutes, where $F_{n,ij} = F(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$ and the homomorphism δ_{ij} is given by $\delta_{ij}(x_k) = x_k$ for $k \neq i, j$ and $\delta_{ij}(x_i) = \delta_{ij}(x_k) = 1$. It follows that

$$\chi_{\delta ij(x_0)^{l_i}} = d_j \circ d_i(\beta) = d_{ij}(\beta) \circ \delta_{ij} = d_i \circ d_j(\beta) = \chi_{\delta_{ij}(x_0)^{l_j}}.$$

Since $F_{n,ij}$ is a free group of rank at least 2, $F_{n,ij}$ embeds into $\mathrm{Aut}(F_{n,ij})$ as inner automorphisms and so $l_i = l_j$ for $i < j$. Thus $\beta\alpha^{-l_1} \in C_n$. In other words,

$$K_n \cap D_n \subseteq Z(B_n) \cdot C_n.$$

By the commutative diagram

$$\begin{tikzcd}
\hat{B}_n \arrow[r, two heads] \arrow[d] & \Sigma_n \arrow[d, hook] \\
\mathrm{Aut}(\pi_1(Z_n)) \arrow[r] & \mathrm{Aut}(\pi_1(Z_n)/\Gamma^2\pi_1(Z_n)) \cong \mathrm{GL}(n-1;\mathbb{Z}),
\end{tikzcd}$$

we have $D_n = K_n \cap D_n$ and hence the result. $\square$

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