# MA3220 Ordinary Differential Equations

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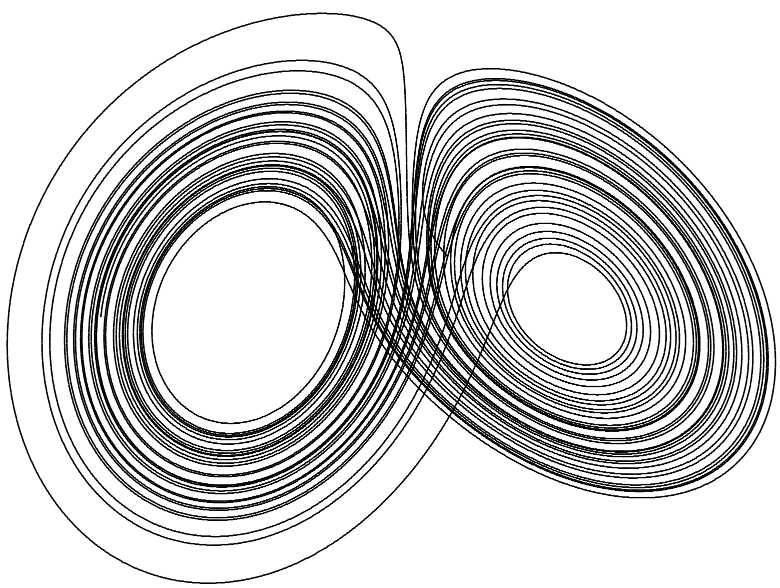


# Contents

# Chapter 1

# First Order Differential Equations

## 1.1 Introduction

**1. Ordinary differential equations.**

An *ordinary differential equation* (ODE for short) is a relation containing one real variable $x$, the real dependent variable $y$, and some of its derivatives $y'$, $y''$, $\cdots$, $y^{(n)}$, $\cdots$, with respect to $x$.

The *order* of an ODE is defined to be the order of the highest derivative that occurs in the equation. Thus, an $n$-th order ODE has the general form

$$F(x, y, y', \cdots, y^{(n)}) = 0. \tag{1.1.1}$$

We shall always assume that (1.1.1) can be solved explicitly for $y^{(n)}$ in terms of the remaining $n+1$ quantities as

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}), \tag{1.1.2}$$

where $f$ is a known function of $x$, $y$, $y'$, $\cdots$, $y^{(n-1)}$.

An $n$-th order ODE is *linear* if it can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = r(x). \tag{1.1.3}$$

The functions $a_j(x)$, $0 \le j \le n$ are called *coefficients* of the equation. We shall always assume that $a_0(x) \not\equiv 0$ in any interval in which the equation is defined. If $r(x) \equiv 0$, (1.1.3) is called a *homogeneous equation*. If $r(x) \not\equiv 0$, (1.1.3) is said to be a *non-homogeneous* equation, and $r(x)$ is called the *non-homogeneous term*.

**2. Solutions.**

A functional relation between the dependent variable $y$ and the independent variable $x$ that satisfies the given ODE in some interval $J$ is called a *solution* of the given ODE on $J$.

A *general solution* of an $n$-th order ODE depends on $n$ arbitrary constants, i.e. the solution $y$ depends on $x$ and $n$ real constants $c_1, \cdots, c_n$.

A first order ODE may be written as

$$F(x, y, y') = 0. \tag{1.1.4}$$

In this chapter we consider only first order ODE. The function $y = \phi(x)$ is called an *explicit solution* of (1.1.4) in the interval $J$ provided

$$F(x, \phi(x), \phi'(x)) = 0 \qquad \text{for all } x \text{ in } J. \tag{1.1.5}$$

A relation of the form $\psi(x, y) = 0$ is said to be an *implicit solution* of (1.1.4) provided it determines one or more functions $y = \phi(x)$ which satisfy (1.1.5). The pair of equations

$$x = x(t), \quad y = y(t) \tag{1.1.6}$$

is said to be a *parametric solution* of (1.1.4) if

$$F\left(x(t), y(t), \frac{\dot{y}(t)}{\dot{x}(t)}\right) = 0.$$

**Example 1.1** Consider the ODE: $x + yy' = 0$ for $x \in (-1, 1)$. $x^2 + y^2 = 1$ is an implicit solution while $x = \cos t, y = \sin t$, $t \in (0, \pi)$ is a parametric solution.

**3. Integral curves.**

The solutions of a first order ODE

$$y' = f(x, y) \tag{1.1.7}$$

represent a one-parameter family of curves in the $xy$-plane. These are called *integral curves*.

In other words, if $y = y(x)$ is a solution to (1.1.7), then vector field $\mathbf{F}(x, y) = \langle 1, f(x, y)\rangle$ is tangent to the curve $\mathbf{r}(x) = \langle x, y(x)\rangle$ at every point $(x, y)$ since $\mathbf{r}'(x) = \mathbf{F}(x, y)$.

**4. Elimination of constants: formation of ODE**.

Given a family of functions parameterized by some constants, a differential equation can be formed by eliminating the constants of this family and its derivatives.

**Example 1.2** The family of functions $y = Ae^x + B\sin x$ satisfies the ODE: $y'''' - y = 0$ when the constants $A$ and $B$ are eliminated using the derivatives.

**Exercise 1.1** Find the differential equation satisfied by the family of functions $y = x^c$ for $x > 0$, where $c$ is a parameter.

Ans: $y' = \frac{y \ln y}{x \ln x}$.

**5. Separable equations.**

Typical separable equation can be written as

$$y' = \frac{f(x)}{g(y)}, \qquad \text{or} \qquad g(y)dy = f(x)dx. \tag{1.1.8}$$

The solution is given by

$$\int g(y)dy = \int f(x)dx + c.$$

**Exercise 1.2** Solve $y' = -2xy$, $y(0) = 1$.

Ans: $y = e^{-x^2}$.

The equation $y' = f(\frac{y}{x})$ can be reduced to a separable equation by letting $u = \frac{y}{x}$, i.e. $y = xu$. So $f(u) = y' = u + xu'$,

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x} + c.$$

**Exercise 1.3** Solve $2xyy' + x^2 - y^2 = 0$.

Ans: $x^2 + y^2 = cx$.

**6. Homogeneous equations.**

A function is called *homogeneous of degree* $n$ if $f(tx, ty) = t^n f(x, y)$ for all $x, y, t$.
For example $\sqrt{x^2 + y^2}$ and $x + y$ are homogeneous of degree 1, $x^2 + y^2$ is homogeneous of degree 2 and $\sin(x/y)$ is homogeneous of degree 0.
The ODE $M(x, y) + N(x, y)y' = 0$ is said to be *homogeneous of degree* $n$ if both $M(x, y)$ and $N(x, y)$ are homogeneous of degree $n$.
If we write the above DE as $y' = f(x, y)$, where $f(x, y) = -M(x, y)/N(x, y)$. Then $f(x, y)$ is homogeneous of degree 0. To solve the DE

$$y' = f(x, y),$$

where $f$ is homogeneous of degree 0, we use the substitution $y = zx$. Then

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

Thus the DE becomes

$$z + x\frac{dz}{dx} = f(x, zx) = x^0 f(1, z) = f(1, z).$$

Consequently, the variables can be separated to yield

$$\frac{dz}{f(1, z) - z} = \frac{dx}{x},$$

and integrating both sides will give the solution.

**Exercise 1.4** Solve $y' = \frac{x+y}{x-y}$.

Ans: $\tan^{-1}(y/x) = \ln\sqrt{x^2 + y^2} + c$.

**Example 1.3** An equation in the form

$$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}.$$

can be reduced to a homogeneous equation by a suitable substitution $x = z + h, y = w + k$ when $a_1 b_2 \neq a_2 b_1$, where $h$ and $k$ are solutions of the system of linear equations $a_1 h + b_1 k + c_1 = 0, a_2 h + b_2 k + c_2 = 0$.

**Exercise 1.5** Solve $y' = \frac{x+y-2}{x-y}$.

Ans: $\tan^{-1}\left(\frac{y-1}{x-1}\right) = \ln\sqrt{(x-1)^2+(y-1)^2} + c$.

**Exercise 1.6** Solve $(x+y+1) + (2x+2y+1)y' = 0$.

Ans: $x + 2y + \ln|x+y| = c, x+y = 0$.

## 1.2 Exact Equations, Integrating Factors

**1. Exact equations.**

We can write a first order ODE in the following form

$$M(x,y)dx + N(x,y)dy = 0. \tag{1.2.1}$$

(1.2.1) is called *exact* if there exists a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Once (1.2.1) is exact, the general solution is given by $u(x,y) = c$, where $c$ is an arbitrary constant.

**Theorem 1.1** *Assume $M$ and $N$ together with their first partial derivatives are continuous in the rectangle $S$: $|x-x_0| < a$, $|y-y_0| < b$. A necessary and sufficient condition for (1.2.1) to be exact is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{for all } (x,y) \text{ in } S. \tag{1.2.2}$$

*When (1.2.2) is satisfied, a general solution of (1.2.1) is given by $u(x,y) = c$, where*

$$u(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt \tag{1.2.3}$$

*and $c$ is an arbitrary constant.*

**Proof.** Let $u(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt$. We have to show that $\frac{\partial u}{\partial x} = M(x,y)$ and $\frac{\partial u}{\partial y} = N(x,y)$. The first equality is immediate by the fundamental theorem of calculus. For the second equality, we have $\frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial}{\partial y}M(s,y)ds + N(x_0,y) = \int_{x_0}^{x} \frac{\partial}{\partial s}N(s,y)ds + N(x_0,y) = N(x,y) - N(x_0,y) + N(x_0,y) = N(x,y)$.

**Remark.** In Theorem 1.1, the rectangle $S$ can be replaced by any region which does not include any "hole". In that case, the proof is by Green's theorem.

**Exercise 1.7** Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Ans: $x^4 + 6x^2y^2 + y^4 = c$.

**2. Integrating factors**.

A non-zero function $\mu(x, y)$ is an *integrating factor* of (1.2.1) if the equivalent differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \tag{1.2.4}$$

is exact.
If $\mu$ is an integrating factor of (1.2.1) then $(\mu M)_y = (\mu N)_x$, i.e.

$$N\mu_x - M\mu_y = \mu(M_y - N_x). \tag{1.2.5}$$

One may look for an integrating factor of the form $\mu = \mu(v)$, where $v$ is a known function of $x$ and $y$. Plugging into (1.2.5) we find

$$\frac{1}{\mu}\frac{d\mu}{dv} = \frac{M_y - N_x}{Nv_x - Mv_y}. \tag{1.2.6}$$

If $\frac{M_y - N_x}{Nv_x - Mv_y}$ is a function of $v$ alone, say, $\phi(v)$, then

$$\mu = e^{\int^v \phi(v)dv}$$

is an integrating factor of (1.2.1).

Let $v = x$. If $\frac{M_y - N_x}{N}$ is a function of $x$ alone, say, $\phi_1(x)$, then $e^{\int^x \phi_1(x)dx}$ is an integrating factor of (1.2.1).

Let $v = y$. If $-\frac{M_y - N_x}{M}$ is a function of $y$ alone, say, $\phi_2(y)$, then $e^{\int^y \phi_2(y)dy}$ is an integrating factor of (1.2.1).

Let $v = xy$. If $\frac{M_y - N_x}{yN - xM}$ is a function of $v = xy$ alone, say $\phi_3(xy)$, then $e^{\int^{xy} \phi_3(v)dv}$ is an integrating factor of (1.2.1).

**Exercise 1.8** Solve $(x^2y + y + 1) + x(1 + x^2)y' = 0$.

Ans: $xy + \tan^{-1} x = c$.

**Exercise 1.9** Solve $(y - y^2) + xy' = 0$

Ans: $y = (1 - cx)^{-1},\ y = 0$.

**Exercise 1.10** Solve $(xy^3 + 2x^2y^2 - y^2) + (x^2y^2 + 2x^3y - 2x^2)y' = 0$

Ans: $e^{xy}(1/x + 2/y) = c,\ y = 0$.

**3. Find integrating factors by inspection.**
The followings are some differential formulas that are often useful.

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$$

$$d(xy) = xdy + ydx$$

$$d(x^2 + y^2) = 2xdx + 2ydy$$

$$d(\tan^{-1}\frac{x}{y}) = \frac{ydx - xdy}{x^2 + y^2}$$

$$d(\ln|\frac{x}{y}|) = \frac{ydx - xdy}{xy}$$

We see that the very simple ODE $ydx - xdy = 0$ has $1/x^2, 1/y^2, 1/(x^2+y^2)$ and $1/xy$ as integrating factors.

**Example 1.4** Solve $xdy + ydx = x\cos xdx$.

**Solution**. The differential $xdy + ydx$ can be written as $d(xy)$. Thus the DE can be written as $d(xy) = x\cos xdx$. Integrating, we have $xy = x\sin x + \cos x + c$.

**Exercise 1.11** Solve $(x^2y^3 + y)dx + (x - x^3y^2)dy = 0$.

Ans: $\ln|x/y| = 1/(2x^2y^2) + c$ and $y = 0$.

## 1.3 First Order Linear Equations

**1. Homogeneous equations.**

A first order homogeneous linear equation is of the form

$$y' + p(x)y = 0, \tag{1.3.1}$$

where $p(x)$ is a continuous function on an interval $J$. Let $P(x) = \int_a^x p(s)ds$. Multiplying (1.3.1) by $e^{P(x)}$, we get

$$\frac{d}{dx}[e^{P(x)}y] = 0,$$

so $e^{P(x)}y = c$. The general solution of (1.3.1) is given by

$$y(x) = ce^{-P(x)}, \qquad \text{where} \quad P(x) = \int_a^x p(s)ds. \tag{1.3.2}$$

**2. Non-homogeneous equations.**

Now consider a first order non-homogeneous linear equation

$$y' + p(x)y = q(x), \tag{1.3.3}$$

where $p(x)$ and $q(x)$ are continuous functions on an interval $J$. Let $P(x) = \int_a^x p(s)ds$. Multiplying (1.3.3) by $e^{P(x)}$ we get

$$\frac{d}{dx}[e^{P(x)}y] = e^{P(x)}q(x).$$

Thus

$$e^{P(x)}y(x) = \int_a^x e^{P(t)}q(t)dt + c.$$

The general solution is given by

$$\begin{aligned} y(x) &= e^{-P(x)}[\int_a^x e^{P(t)}q(t)dt + c], \qquad \text{where} \\ P(x) &= \int_a^x p(s)ds. \end{aligned} \tag{1.3.4}$$

**Exercise 1.12** Solve $y' - y = e^{2x}$.

Ans: $y = ce^x + e^{2x}$.

**3. The Bernoulli equation.**

An ODE in the form

$$y' + p(x)y = q(x)y^n, \tag{1.3.5}$$

where $n \neq 0, 1$, is called the *Bernoulli equation*. The functions $p(x)$ and $q(x)$ are continuous functions on an interval $J$.

Let $u = y^{1-n}$. Substituting into (1.3.5) we get

$$u' + (1-n)p(x)u = (1-n)q(x). \tag{1.3.6}$$

This is a first order linear ODE.

**Exercise 1.13** Solve $xy' + y = x^4y^3$.

Ans: $\frac{1}{y^2} = -x^4 + cx^2$, or $y = 0$.

**4. The Riccati equation.**

An ODE of the form

$$y' = P(x) + Q(x)y + R(x)y^2 \tag{1.3.7}$$

is called the *Riccati equation*. The functions $P(x)$, $Q(x)$, $R(x)$ are continuous on an interval $J$. In general, the Riccati equation cannot be solved by a sequence of integrations. However, if a particular solution is known, then (1.3.7) can be reduced to a linear equation, and thus is solvable.

**Theorem 1.2** *Let $y = y_0(x)$ be a particular solution of the Riccati equation (1.3.7). Set*

$$\begin{aligned} H(x) &= \int_{x_0}^x [Q(t) + 2R(t)y_0(t)]dt, \\ Z(x) &= e^{-H(x)}\Big[c - \int_{x_0}^x e^{H(t)}R(t)dt\Big], \end{aligned} \tag{1.3.8}$$

*where $c$ is an arbitrary constant. Then the general solution is given by*

$$y = y_0(x) + \frac{1}{Z(x)}. \tag{1.3.9}$$

**Proof.** In (1.1.7) we let $y = y_0(x) + u(x)$ to get

$$y_0' + u' = P + Q(y_0 + u) + R(y_0 + u)^2.$$

Since $y_0$ satisfies (1.3.7), we have

$$y_0' = P + Qy_0 + Ry_0^2.$$

From these two equalities we get

$$u' = (Q + 2Ry_0)u + Ru^2. \tag{1.3.10}$$

This is a Bernoulli equation with $n = 2$. Set $Z = u^{-1}$ and reduce (1.3.10) to

$$Z' + (Q + 2Ry_0)Z = -R. \tag{1.3.11}$$

(1.3.11) is a linear equation and the solution is given by (1.3.8). □

**Exercise 1.14** Solve $y' = y/x + x^3y^2 - x^5$. Note $y_0 = x$ is a solution.

Ans: $ce^{2x^5/5} = \frac{y-x}{y+x}$.

From (1.3.8), (1.3.9), the general solution $y$ of the Riccati equation (1.3.7) can be written as

$$y = \frac{cF(x) + G(x)}{cf(x) + g(x)}, \tag{1.3.12}$$

where

$$\begin{aligned} &f(x) = e^{-H(x)}, \\ &g(x) = -e^{-H(x)} \int_{x_0}^{x} e^{H(t)} R(t)dt, \\ &F(x) = y_0(x)f(x), \quad G(x) = y_0 g(x) + 1. \end{aligned}$$

Given four distinct functions $p(x)$, $q(x)$, $r(x)$, $s(x)$, we define the cross ratio by

$$\frac{(p-q)(r-s)}{(p-s)(r-q)}.$$

**Property 1.** The cross ratio of four distinct particular solutions of a Riccati equation is independent of $x$.

**Proof.** From (1.3.12), the four solutions can be written as

$$y_j(x) = \frac{c_jF(x) + G(x)}{c_jf(x) + g(x)}.$$

Computations show that

$$\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)} = \frac{(c_1 - c_2)(c_3 - c_4)}{(c_1 - c_4)(c_3 - c_2)}.$$

The right-hand side is independent of $x$. □

As a consequence we get

**Property 2.** Suppose $y_1$, $y_2$, $y_3$ are three distinct particular solutions of a Riccati equation (1.3.7). Then the general solution is given by

$$\frac{(y_1 - y_2)(y_3 - y)}{(y_1 - y)(y_3 - y_2)} = c, \tag{1.3.13}$$

where $c$ is an arbitrary constant.

**Property 3.** Suppose that $y_1$ and $y_2$ are two distinct particular solutions of a Riccati equation (1.3.7), then its general solution is given by

$$\ln\left|\frac{y - y_1}{y - y_2}\right| = \int [y_1(x) - y_2(x)]R(x)dx + c, \tag{1.3.14}$$

where $c$ is an arbitrary constant.
**Proof.** $y$ and $y_j$ satisfy (1.3.7). So

$$\begin{aligned} y' - y_j' &= (y - y_j)[Q + R(y + y_j)], \\ \frac{y' - y_j'}{y - y_j} &= Q + R(y + y_j). \end{aligned}$$

Thus

$$\frac{y' - y_1'}{y - y_1} - \frac{y' - y_2'}{y - y_2} = R(y_1 - y_2).$$

Integrating yields (1.3.14). □

**Exercise 1.15** Solve $y' = e^{-x}y^2$. Note $y_1 = e^x$ and $y_2 = 0$ are 2 solutions.

Ans: $y = \frac{e^x}{1-Ae^x}$, $y = 0$.

**Exercise 1.16** A natural generalization of Riccati's equation is Abel's equation

$$y' = P(x) + Q(x)y + R(x)y^2 + S(x)y^3,$$

where $P(x), Q(x), R(x)$ and $S(x)$ are continuous functions of $x$ on an interval $J$. Using the substitution $z = y/x$, solve the equation

$$y' = \frac{y}{x} + xy^2 - 3y^3.$$

Ans: $x/y + 3\ln|x/y - 3| = -\frac{x^3}{3} + c$, $y = x/3$ and $y = 0$.

## 1.4 First Order Implicit Equations

In the above we discussed first order explicit equations, i.e. equations in the form $y' = f(x, y)$. In this section we discuss solution of some first order explicit equations

$$F(x, y, y') = 0 \tag{1.4.1}$$

which are not solvable in $y'$.

**1. Method of differentiation.**
Consider an equations solvable in $y$:

$$y = f(x, y'). \tag{1.4.2}$$

Let $p = y'$. Differentiating $y = f(x, p)$ we get

$$[f_x(x, p) - p]dx + f_p(x, p)dp = 0. \tag{1.4.3}$$

This is a first order explicit equation in $x$ and $p$. If $p = \phi(x)$ is a solution of (1.4.3), then

$$y = f(x, \phi(x))$$

is a solution of (1.4.2).

**Example 1.5** Clairaut's equation is the equation of the form

$$y = xy' + f(y'), \tag{1.4.4}$$

where $f$ has continuous first order derivative.

Let $p = y'$. We have $y = xp + f(p)$. Differentiating we get

$$[x + f'(p)]p' = 0.$$

When $p' = 0$ we have $p = c$ and (1.4.4) has a general solution

$$y = cx + f(c).$$

When $x + f'(p) = 0$ we get a solution of (1.4.4) given by parameterized equations

$$x = -f'(p), \quad y = -pf'(p) + f(p).$$

**Exercise 1.17** Solve Clairaut's equation $y = xy' - y'^2/4$.

Ans: $y = cx - c^2/4, y = x^2$.

**Exercise 1.18** Let $C$ be the curve with parametric equation

$$x = -f'(p),\ y = -pf'(p) + f(p).$$

Show that the tangent to $C$ at the point $p = c$ has the equation $y = cx + f(c)$.

**2. Method of parameterization.**
This method can be used to solve equations where either $x$ or $y$ is missing. Consider

$$F(y, y') = 0, \tag{1.4.5}$$

where $x$ is missing. Let $p = y'$ and write (1.4.5) as

$$F(y, p) = 0.$$

It determines a family of curves in $yp$ plane. Let $y = g(t)$, $p = h(t)$ be one of the curves, i.e. $F(g(t), h(t)) = 0$. Since

$$dx = \frac{dy}{y'} = \frac{dy}{p} = \frac{g'(t)dt}{h(t)},$$

we have $x = \int_{t_0}^{t} \frac{g'(t)}{h(t)} dt + c$. The solutions of (1.4.5.) are given by

$$x = \int_{t_0}^{t} \frac{g'(t)}{h(t)} dt + c, \quad y = g(t).$$

This method can also be applied to the equations $F(x, y') = 0$, where $y$ is missing.

**Exercise 1.19** Solve $y^2 + y'^2 - 1 = 0$.

Ans: $y = \cos(c - x)$.

For the equation $F(y, y') = 0$ or $F(x, y') = 0$, if $y'$ can be solved in terms of $y$ or $x$, then the equation becomes explicit.

**Exercise 1.20** Solve $y'^2 - (2x + e^x)y' + 2xe^x = 0$.

Ans: $y = x^2 + c_1, y = e^x + c_2$.

**3. Reduction of order.**
Consider the equation

$$F(x, y', y'') = 0, \tag{1.4.6}$$

where $y$ is missing. Let $p = y'$. Then $y'' = p'$. Write (1.4.6) as

$$F(x, p, p') = 0. \tag{1.4.7}$$

It is a first order equation in $x$ and $p$. If $p = \phi(x, c_1)$ is a general solution of (1.4.7), then the general solution of (1.4.6) is

$$y = \int_{x_0}^{x} \phi(t, c_1)dt + c_2.$$

**Exercise 1.21** Solve $xy'' - y' = 3x^2$.

Ans: $y = x^3 + c_1x^2 + c_2$.

Consider the equation

$$F(y, y', y'') = 0, \tag{1.4.8}$$

where $x$ is missing. Let $p = y'$. Then $y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p$. Write (1.4.8) as

$$F(y, p, p\frac{dp}{dy}) = 0. \tag{1.4.9}$$

It is a first order equation in $y$ and $p$. If $p = \psi(y, c_1)$ is a general solution of (1.4.9), then we solve the equation

$$y' = \psi(y, c_1)$$

to get a general solution of (1.4.8).

**Exercise 1.22** Solve $y'' + k^2y = 0$, where $k$ is a positive constant.

Ans: $y = c_1 \sin(kx) + c_2 \cos(kx)$.

## 1.5 Further Exercises

**Exercise 1.23** Solve the differential equation $y^2dx + (xy - x^3)dy = 0$ by using the substitution $u = x^{-2}$.

Ans: $3y = 2x^2 + cx^2y^3$ and $y = 0$.

**Exercise 1.24** Solve the differential equation $y^2dx + (xy + \tan(xy))dy = 0$ by using (i) the substitution $u = xy$, (ii) the integrating factor $\cos(xy)$.

Ans: $y\sin(xy) = c$.

**Exercise 1.25** Solve the differential equation $(x^2 + 3\ln y)ydx - xdy = 0$ (i) by using the substitution $y = e^z$, (ii) by finding an integrating factor using the formula (1.2.6).

Ans: $\ln y = -x^2 + cx^3$.

**Exercise 1.26** Solve $y'' + y = \begin{cases} 0 & \text{if } x \le 1 \\ 1 - x & \text{if } x > 1 \end{cases}$, $y(0) = 0,\ y'(0) = 0$.

Ans: $y = \begin{cases} 0 & \text{if } x \le 1 \\ \sin(x-1) + 1 - x & \text{if } x > 1. \end{cases}$

# Chapter 2

# Linear Differential Equations

## 2.1 General Theory

Consider $n$-th order linear equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x), \tag{2.1.1}$$

where $y^{(k)} = \frac{d^k y}{dx^k}$. Throughout this section we assume that $a_j(x)$'s and $f(x)$ are continuous functions defined on the interval $(a, b)$. When $f(x) \not\equiv 0$, (2.1.1) is called a non-homogeneous equation. The associated homogeneous equation is

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{2.1.2}$$

Let us begin with the initial value problem:

$$\begin{cases} y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x), \\ y(x_0) = y_0, \\ y'(x_0) = y_1, \\ \cdots\cdots\cdots \\ y^{(n-1)}(x_0) = y_{n-1}. \end{cases} \tag{2.1.3}$$

**Theorem 2.1 (Existence and Uniqueness Theorem)** *Assume that $a_1(x)$, $\cdots$, $a_n(x)$ and $f(x)$ are continuous functions defined on the interval $(a, b)$. Then for any $x_0 \in (a, b)$ and for any numbers $y_0$, $\cdots$, $y_{n-1}$, the initial value problem (2.1.3) has a unique solution defined on $(a, b)$.*
*Especially if $a_j(x)$'s and $f(x)$ are continuous on $\mathbb{R}$ then for any $x_0$ and $y_0$, $\cdots$, $y_{n-1}$, the initial value problem (2.1.3) has a unique solution defined on $\mathbb{R}$.*

Proof of this theorem will be given in later chapter.

**Corollary 2.2** *Let $y = y(x)$ be a solution of the homogeneous equation (2.1.2) in an interval $(a, b)$. Assume that there exists $x_0 \in (a, b)$ such that*

$$y(x_0) = 0,\ y'(x_0) = 0,\ \cdots,\ y^{(n-1)}(x_0) = 0. \tag{2.1.4}$$

*Then $y(x) \equiv 0$ on $(a, b)$.*

**Proof.** $y$ is a solution of the initial value problem (2.1.2), (2.1.4). From Theorem 2.1, this problem has a unique solution. Since $\phi(x) \equiv 0$ is also a solution of the problem, we have $y(x) \equiv 0$ on $(a, b)$. □

In the following we consider the general solutions of (2.1.1) and (2.1.2).
Given continuous functions $a_j(x)$, $j = 0, 1, \cdots, n$ and $f(x)$, define an operator $L$ by

$$L[y] = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y. \tag{2.1.5}$$

**Property 1.** $L[cy] = cL[y]$ for any constant $c$.
**Property 2.** $L[u + v] = L[u] + L[v]$.

**Proof**. Let's verify Property 2.

$$\begin{aligned} & L[u+v] \\ = \; & a_0(x)(u+v)^{(n)} + a_1(x)(u+v)^{(n-1)} + \cdots + a_n(x)(u+v) \\ = \; & \big(a_0(x)u^{(n)} + a_1(x)u^{(n-1)} + \cdots + a_n(x)u\big) + \big(a_0(x)v^{(n)} + a_1(x)v^{(n-1)} + \cdots + a_n(x)v\big) \\ = \; & L[u] + L[v]. \end{aligned}$$

**Definition.** An operator satisfying Properties 1 and 2 is called a *linear operator*.

The differential operator $L$ defined in (2.1.3) is a linear operator.
Note that (2.1.1) and (2.1.2) can be written as

$$L[y] = f(x), \tag{2.1.1'}$$

and

$$L[y] = 0. \tag{2.1.2'}$$

From Properties 1 and 2 we get the following conclusion.

**Theorem 2.3** *(1) If $y_1$ and $y_2$ are solutions of the homogeneous equation (2.1.2) in an interval $(a, b)$, then for any constants $c_1$ and $c_2$,*

$$y = c_1y_1 + c_2y_2$$

*is also a solution of (2.1.2) in the interval $(a, b)$.*
*(2) If $y_p$ is a solution of (2.1.1) and $y_h$ is a solution of (2.1.2) on an interval $(a, b)$, then*

$$y = y_h + y_p$$

*is also a solution of (2.1.1) in the interval $(a, b)$.*

**Proof.** (1) As $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2] = 0$, we see that $c_1y_1 + c_2y_2$ is also a solution of (2.1.2).
(2) $L[y_h + y_p] = L[y_h] + L[y_p] = 0 + f(x)$, we see that $y = y_h + y_p$ is a solution of (2.1.1).

In order to discuss structures of solutions, we need the following definition.

**Definition.** Functions $\phi_1(x), \cdots, \phi_k(x)$ are *linearly dependent* on $(a,b)$ if there exists constants $c_1$, $\cdots$, $c_k$, not all zero, such that

$$c_1\phi_1(x) + \cdots + c_k\phi_k(x) = 0$$

for all $x \in (a,b)$. A set of functions are *linearly independent* on $(a,b)$ if they are not linearly dependent on $(a,b)$.

**Lemma 2.4** *Functions $\phi_1(x), \cdots, \phi_k(x)$ are linearly dependent on $(a,b)$ if and only if the following vector-valued functions*

$$\begin{pmatrix} \phi_1(x) \\ \phi_1'(x) \\ \cdots \\ \phi_1^{(n-1)}(x) \end{pmatrix}, \cdots, \begin{pmatrix} \phi_k(x) \\ \phi_k'(x) \\ \cdots \\ \phi_k^{(n-1)}(x) \end{pmatrix} \tag{2.1.6}$$

*are linearly dependent on $(a,b)$.*

**Proof.** "$\Longleftarrow$" is obvious. To show "$\Longrightarrow$", assume that $\phi_1, \cdots, \phi_k$ are linearly dependent on $(a,b)$. There exists constants $c_1, \cdots, c_k$, not all zero, such that, for all $x \in (a,b)$,

$$c_1\phi_1(x) + \cdots + c_k\phi_k(x) = 0.$$

Differentiating this equality successively we find that

$$\begin{aligned} &c_1\phi_1'(x) + \cdots + c_k\phi_k'(x) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &c_1\phi_1^{(n-1)}(x) + \cdots + c_k\phi_k^{(n-1)}(x) = 0. \end{aligned}$$

Thus

$$c_1\begin{pmatrix} \phi_1(x) \\ \phi_1'(x) \\ \cdots \\ \phi_1^{(n-1)}(x) \end{pmatrix} + \cdots + c_k\begin{pmatrix} \phi_k(x) \\ \phi_k'(x) \\ \cdots \\ \phi_k^{(n-1)}(x) \end{pmatrix} = \mathbf{0}$$

for all $x \in (a,b)$. Hence the $k$ vector-valued functions are linearly dependent on $(a,b)$. □

Recall that, $n$ vectors in $\mathbb{R}^n$ are linearly dependent if and only if the determinant of matrix formed by these vectors is zero.

**Definition.** The Wronskian of $n$ functions $\phi_1(x), \cdots, \phi_n(x)$ is defined by

$$W(\phi_1, \cdots, \phi_n)(x) = \begin{vmatrix} \phi_1(x) & \cdots & \phi_n(x) \\ \cdots & \cdots & \cdots \\ \phi_1^{(n-1)}(x) & \cdots & \phi_n^{(n-1)}(x) \end{vmatrix}. \tag{2.1.7}$$

Note that Wronskian of $\phi_1,\ \cdots,\ \phi_n$ is the determinant of the matrix formed by the vector-valued functions given in (2.1.6).

**Theorem 2.5** *Let $y_1(x),\ \cdots,\ y_n(x)$ be $n$ solutions of (2.1.2) on $(a,b)$ and let $W(x)$ be their Wronskian.*
*(1) $y_1(x),\ \cdots,\ y_n(x)$ are linearly dependent on $(a,b)$ if and only if $W(x)\equiv 0$ on $(a,b)$.*
*(2) $y_1(x),\ \cdots,\ y_n(x)$ are linearly independent on $(a,b)$ if and only if $W(x)$ does not vanish on $(a,b)$.*

**Corollary 2.6** *(1) The Wronskian of $n$ solutions of (2.1.2) is either identically zero, or nowhere zero.*
*(2) $n$ solutions $y_1,\cdots,y_n$ of (2.1.2) are linearly independent on $(a,b)$ if and only if the set of vectors*

$$\begin{pmatrix} y_1(x_0)\\ y_1'(x_0)\\ \cdots\\ y_1^{(n-1)}(x_0)\end{pmatrix},\ \cdots,\ \begin{pmatrix} y_n(x_0)\\ y_n'(x_0)\\ \cdots\\ y_n^{(n-1)}(x_0)\end{pmatrix}$$

*are linearly independent for some $x_0\in(a,b)$.*

**Proof of Theorem 2.5.** Let $y_1,\ \cdots,\ y_n$ be solutions of (2.1.2) on $(a,b)$, and let $W(x)$ be their Wronskian.

*Step 1.* We first show that, if $y_1,\cdots,y_n$ are linearly dependent on $(a,b)$, then $W(x)\equiv 0$.
Since these solutions are linearly dependent, from Lemma 2.4, $n$ vector-valued functions

$$\begin{pmatrix} y_1(x)\\ y_1'(x)\\ \cdots\\ y_1^{(n-1)}(x)\end{pmatrix},\ \cdots,\ \begin{pmatrix} y_n(x)\\ y_n'(x)\\ \cdots\\ y_n^{(n-1)}(x)\end{pmatrix}$$

are linearly dependent on $(a,b)$. Thus for all $x\in(a,b)$, the determinant of the matrix formed by these vectors, namely, the Wronskian of $y_1,\cdots,y_n$, is zero.

*Step 2.* Now, assume that the Wronskian $W(x)$ of $n$ solutions $y_1,\cdots,y_n$ vanishes at $x_0\in(a,b)$. We shall show that $y_1,\cdots,y_n$ are linearly dependent on $(a,b)$.
Since $W(x_0)=0$, the $n$ vectors

$$\begin{pmatrix} y_1(x_0)\\ y_1'(x_0)\\ \cdots\\ y_1^{(n-1)}(x_0)\end{pmatrix},\ \cdots,\ \begin{pmatrix} y_n(x_0)\\ y_n'(x_0)\\ \cdots\\ y_n^{(n-1)}(x_0)\end{pmatrix}$$

are linearly dependent. Thus there exist $n$ constants $c_1,\cdots,c_n$, not all zero, such that

$$c_1\begin{pmatrix} y_1(x_0)\\ y_1'(x_0)\\ \cdots\\ y_1^{(n-1)}(x_0)\end{pmatrix}+\cdots+c_n\begin{pmatrix} y_n(x_0)\\ y_n'(x_0)\\ \cdots\\ y_n^{(n-1)}(x_0)\end{pmatrix}=\mathbf{0} \tag{2.1.8}$$

Define

$$y_0(x) = c_1 y_1(x) + \cdots + c_n y_n(x).$$

From Theorem 2.3, $y_0$ is a solution of (2.1.2). From (2.1.8), $y_0$ satisfies the initial conditions

$$y(x_0) = 0,\ y'(x_0) = 0, \cdots, y^{(n-1)}(x_0) = 0. \tag{2.1.9}$$

From Corollary 2.2, we have $y_0 \equiv 0$, namely,

$$c_1 y_1(x) + \cdots + c_n y_n(x) = 0$$

for all $x \in (a, b)$. Thus $y_1, \cdots, y_n$ are linearly dependent on $(a, b)$. □

**Example 2.1** Consider the differential equation $y'' - \frac{1}{x}y' = 0$ on the interval $(0, \infty)$. Both $\phi_1(x) = 1$ and $\phi_2(x) = x^2$ are solutions of the differential equation. $W(\phi_1, \phi_2)(x) = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x \neq 0$ for $x > 0$. Thus $\phi_1$ and $\phi_2$ are linearly independent solutions.

**Theorem 2.7** *(1) Let $a_1(x)$, $\cdots$, $a_n(x)$ and $f(x)$ be continuous on the interval $(a, b)$. The homogeneous equation (2.1.2) has $n$ linearly independent solutions on $(a, b)$.*
*(2) Let $y_1$, $\cdots$, $y_n$ be $n$ linearly independent solutions of (2.1.2) defined on $(a, b)$. The general solution of (2.1.2) is given by*

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x), \tag{2.1.10}$$

*where $c_1$, $\cdots$, $c_n$ are arbitrary constants.*

**Proof.** (1) Fix $x_0 \in (a, b)$. For $k = 1, 2, \cdots, n$, let $y_k$ be the solution of (2.1.2) satisfying the initial conditions

$$y_k^{(j)}(x_0) = \begin{cases} 0 & \text{if} \quad j \neq k-1, \\ 1 & \text{if} \quad j = k-1. \end{cases}$$

The $n$ vectors

$$\begin{pmatrix} y_1(x_0) \\ y_1'(x_0) \\ \cdots \\ y_1^{(n-1)}(x_0) \end{pmatrix}, \cdots, \begin{pmatrix} y_n(x_0) \\ y_n'(x_0) \\ \cdots \\ y_n^{(n-1)}(x_0) \end{pmatrix}$$

are lineally independent since they form the identity matrix. From Corollary 2.6, $y_1, \cdots, y_n$ are linearly independent on $(a, b)$. From Theorem 2.3, for any constants $c_1, \cdots, c_n$, $y = c_1 y_1 + \cdots + c_n y_n$ is a solution of (2.1.2).

(2) Now let $y_1, \cdots, y_n$ be $n$ linearly independent solutions of (2.1.2) on $(a, b)$. We shall show that the general solution of (2.1.2) is given by

$$y = c_1 y_1 + \cdots + c_n y_n. \tag{2.1.11}$$

Given a solution $\tilde{y}$ of (2.1.2), and fix $x_0 \in (a, b)$. Since $y_1, \cdots, y_n$ are linearly independent on $(a, b)$, the vectors

$$\begin{pmatrix} y_1(x_0) \\ y_1'(x_0) \\ \cdots \\ y_1^{(n-1)}(x_0) \end{pmatrix}, \cdots, \begin{pmatrix} y_n(x_0) \\ y_n'(x_0) \\ \cdots \\ y_n^{(n-1)}(x_0) \end{pmatrix}$$

are linearly independent vectors. They form a basis for $\mathbb{R}^n$. Thus the vector

$$\begin{pmatrix} \tilde{y}(x_0) \\ \tilde{y}'(x_0) \\ \cdots \\ \tilde{y}^{(n-1)}(x_0) \end{pmatrix}$$

can be represented as a linear combination of the $n$ vectors, namely, there exist $n$ constants $\tilde{c}_1, \cdots, \tilde{c}_n$ such that

$$\begin{pmatrix} \tilde{y}(x_0) \\ \tilde{y}'(x_0) \\ \cdots \\ \tilde{y}^{(n-1)}(x_0) \end{pmatrix} = \tilde{c}_1 \begin{pmatrix} y_1(x_0) \\ y_1'(x_0) \\ \cdots \\ y_1^{(n-1)}(x_0) \end{pmatrix} + \cdots + \tilde{c}_n \begin{pmatrix} y_n(x_0) \\ y_n'(x_0) \\ \cdots \\ y_n^{(n-1)}(x_0) \end{pmatrix}.$$

Let

$$\phi(x) = \tilde{y}(x) - [\tilde{c}_1 y_1(x) + \cdots + \tilde{c}_n y_n(x)].$$

$\phi(x)$ is a solution of (2.1.2) and satisfies the initial conditions (2.1.4) at $x = x_0$. By Corollary 2.2, $\phi(x) \equiv 0$ on $(a, b)$. Thus

$$\tilde{y}(x) = \tilde{c}_1 y_1(x) + \cdots + \tilde{c}_n y_n(x).$$

So (2.1.11) gives a general solution of (2.1.2). □

Any set of $n$ linearly independent solutions is called a *fundamental set of solutions*.

Now we consider the non-homogeneous equation (2.1.1). We have

**Theorem 2.8** *Let $y_p$ be a particular solution of (2.1.1), and $y_1, \cdots, y_n$ be a fundamental set of solutions for the associated homogeneous equation (2.1.2). The general solution of (2.1.1) is given by*

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) + y_p(x). \tag{2.1.12}$$

**Proof.** Let $y$ be a solution of the non-homogeneous equation. Then $y - y_p$ is a solution of the homogeneous equation. Thus $y(x) - y_p(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$. □

## 2.2 Linear Equations with Constant Coefficients

Let us begin with second order linear equation with constant coefficients

$$y'' + ay' + by = 0, \tag{2.2.1}$$

where $a$ and $b$ are constants. We look for a solution of the form $y = e^{\lambda x}$. Plugging into (2.2.1) we find that, $e^{\lambda x}$ is a solution of (2.2.1) if and only if

$$\lambda^2 + a\lambda + b = 0. \tag{2.2.2}$$

(2.2.2) is called the *auxiliary equation* or *characteristic equation* of (2.2.1). The roots of (2.2.2) are called *characteristic values* (or eigenvalues):

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}),$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}).$$

1. If $a^2 - 4b > 0$, (2.2.2) has two distinct real roots $\lambda_1$, $\lambda_2$, and the general solutions of (2.2.1) is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

2. If $a^2 - 4b = 0$, (2.2.2) has one real root $\lambda$ (we may say that (2.2.2) has two equal roots $\lambda_1 = \lambda_2$). The general solution of (2.2.1) is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}.$$

3. If $a^2 - 4b < 0$, (2.2.2) has a pair of complex conjugate roots

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta.$$

The general solution of (2.2.1) is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x).$$

**Example 2.2** Solve $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$.

Ans: $\lambda_1 = 1$, $\lambda_2 = -2$, $y = e^x + 3e^{-2x}$.

**Example 2.3** Solve $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

Ans: $\lambda_1 = \lambda_2 = 2$, $y = (3 - 5x)e^{2x}$.

**Example 2.4** Solve $y'' - 2y' + 10y = 0$.

Ans: $\lambda_1 = 1 + 3i$, $\lambda_2 = 1 - 3i$, $y = e^x(c_1 \cos 3x + c_2 \sin 3x)$.

Now we consider $n$-th order homogeneous linear equations with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \tag{2.2.3}$$

where $a_1, \cdots, a_n$ are real constants.

$y = e^{\lambda x}$ is a solution of (2.2.3) if and only if $\lambda$ satisfies

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0. \tag{2.2.4}$$

The solutions of (2.2.4) are called *characteristic values* or eigenvalues for the equation (2.2.3).

Let $\lambda_1, \cdots, \lambda_s$ be the distinct eigenvalues for (2.2.3). Then we can write

$$\begin{aligned} &\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \\ &= (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s}, \end{aligned} \tag{2.2.5}$$

where $m_1, \cdots, m_s$ are positive integers and

$$m_1 + \cdots + m_s = n.$$

We call them the *multiplicity* of the eigenvalues $\lambda_1, \cdots, \lambda_s$ respectively.

**Lemma 2.9** *Assume $\lambda$ is an eigenvalue of (2.2.3) of multiplicity $m$.*
*(i) $e^{\lambda x}$ is a solution of (2.2.3).*
*(ii) If $m > 1$, then for any positive integer $1 \le k \le m-1$, $x^k e^{\lambda x}$ is a solution of (2.2.3).*
*(iii) If $\lambda = \alpha + i\beta$, then*

$$x^k e^{\alpha x}\cos(\beta x), \quad x^k e^{\alpha x}\sin(\beta x)$$

*are solutions of (2.2.3), where $0 \le k \le m-1$.*

**Theorem 2.10** *Let $\lambda_1, \cdots, \lambda_s$ be the distinct eigenvalues for (2.2.3), with multiplicity $m_1, \cdots, m_s$ respectively. Then (2.2.3) has a fundamental set of solutions*

$$\begin{aligned} &e^{\lambda_1 x},\ xe^{\lambda_1 x},\ \cdots,\ x^{m_1-1}e^{\lambda_1 x}; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots; \\ &e^{\lambda_s x},\ xe^{\lambda_s x},\ \cdots,\ x^{m_s-1}e^{\lambda_s x}. \end{aligned} \tag{2.2.6}$$

**Proof of Lemma 2.9 and 2.10**

Consider the $n$-th order linear equation with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0, \tag{A1}$$

where $y^{(k)} = \frac{d^k y}{dx^k}$. Let $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y$ and $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$. Note that $p$ is a polynomial in $z$ of degree $n$. Then we have

$$L(e^{zx}) = p(z)e^{zx} \tag{A2}$$

Before we begin the proof, let's observe that $\frac{\partial^2}{\partial z\partial x}e^{zx} = \frac{\partial^2}{\partial x\partial z}e^{zx}$ by Clairaut's theorem because $e^{zx}$ is differentiable in $(x,z)$ as a function of two variables and all the higher order partial derivatives exist and continuous. That means we can interchange the order of differentiation with respect to $x$ and $z$ as we wish. Therefore $\frac{d}{dz}L(e^{zx}) = L(\frac{d}{dz}e^{zx})$. For instance, one may verify directly that

$$\frac{d}{dz}\frac{d^k}{dx^k}(e^{zx}) = xz^k e^{zx} + kz^{k-1}e^{zx} = \frac{d^k}{dx^k}(\frac{d}{dz}e^{zx}).$$

Here one may need to use Leibniz's rule of taking the $k$-th derivative of a product of two functions:

$$(u \cdot v)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} u^{(i)} v^{(k-i)}. \tag{A3}$$

More generally, $\frac{d^k}{dz^k} L(e^{zx}) = L(\frac{d^k}{dz^k} e^{zx})$. (Strictly speaking, partial derivative notations should be used.) Now let's prove our results.

(1) If $\lambda$ is a root of $p$, then $L(e^{\lambda x}) = 0$ by (A2) so that $e^{\lambda x}$ is a solution of (A1).
(2) If $\lambda$ is a root of $p$ of multiplicity $m$, then $p(\lambda) = 0, p'(\lambda) = 0, p''(\lambda) = 0, \ldots, p^{(m-1)}(\lambda) = 0$. Now for $k = 1, \ldots, m-1$, differentiating (A2) $k$ times with respect to $z$, we have

$$L(x^k e^{zx}) = L(\frac{d^k}{dz^k} e^{zx}) = \frac{d^k}{dz^k} L(e^{zx}) = \frac{d^k}{dz^k}(p(z)e^{zx}) = \sum_{i=0}^{k} \binom{k}{i} p^{(i)}(z) x^{k-i} e^{zx}.$$

Thus $L(x^k e^{\lambda x}) = 0$ and $x^k e^{\lambda x}$ is solution of (A1).
(3) Let $\lambda_1, \cdots, \lambda_s$ be the distinct roots of $p$, with multiplicity $m_1, \cdots, m_s$ respectively. Then we wish to prove that

$$\begin{aligned} &e^{\lambda_1 x},\ xe^{\lambda_1 x},\ \cdots,\ x^{m_1-1}e^{\lambda_1 x}; \\ &\cdots\cdots\cdots\cdots\cdots\cdots; \\ &e^{\lambda_s x},\ xe^{\lambda_s x},\ \cdots,\ x^{m_s-1}e^{\lambda_s x}. \end{aligned} \tag{A4}$$

are linearly independent over $\mathbb{R}$. To prove this, suppose for all $x$ in $\mathbb{R}$

$$\begin{aligned} &c_{11}e^{\lambda_1 x} + c_{12}xe^{\lambda_1 x} + \cdots + c_{1m_1}x^{m_1-1}e^{\lambda_1 x} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \\ &c_{s1}e^{\lambda_s x} + c_{s2}xe^{\lambda_s x} + \cdots + c_{sm_s}x^{m_s-1}e^{\lambda_s x} = 0. \end{aligned}$$

Let's write this as

$$P_1(x)e^{\lambda_1 x} + P_2(x)e^{\lambda_2 x} + \cdots + P_s(x)e^{\lambda_s x} = 0,$$

for all $x$ in $\mathbb{R}$, where $P_i(x) = c_{i1} + c_{i2}x + \cdots + c_{im_i}x^{m_i-1}$. We need to prove $P_i(x) \equiv 0$ for all $i$. By discarding those $P_i$'s which are identically zero, we may assume all $P_i$'s are not identically zero. Dividing the above equation by $e^{\lambda_1 x}$, we have

$$P_1(x) + P_2(x)e^{(\lambda_2-\lambda_1)x} + \cdots + P_s(x)e^{(\lambda_s-\lambda_1)x} = 0,$$

for all $x$ in $\mathbb{R}$. Upon differentiating this equation sufficiently many times (at most $m_1$ times since $P_1(x)$ is a polynomial of degree $m_1 - 1$), we can reduce $P_1(x)$ to 0. Note that in this process, the degree of the resulting polynomial multiplied by $e^{(\lambda_i-\lambda_1)x}$ remains unchanged. Therefore, we get

$$Q_2(x)e^{(\lambda_2-\lambda_1)x} + \cdots + Q_s(x)e^{(\lambda_s-\lambda_1)x} = 0,$$

where deg $Q_i =$ deg $P_i$. Canceling the term $e^{\lambda_1}x$ we have

$$Q_2(x)e^{\lambda_2 x} + \cdots + Q_s(x)e^{\lambda_s x} = 0.$$

For instance, if deg $P_i = 0$, then $P_i$ is a nonzero constant. The resulting $Q_i$ is equal to $(\lambda_i - \lambda_1)^\alpha P_i$ for some positive integer $\alpha$. Since $\lambda_i \neq \lambda_1$, $Q_i$ is also a nonzero constant. Thus deg $Q_i = 0$.

Repeating this procedure, we arrive at

$$R_s(x)e^{\lambda_s x} = 0,$$

where deg $R_s$ = deg $P_s$. Hence $R_s(x) \equiv 0$ which is a contradiction. Thus all the $P_i(x)$'s are identically zero. That means all $c_{ij}$'s are zero and the functions in (A4) are linearly independent.

**Remark.** If (2.2.3) has a complex eigenvalue $\lambda = \alpha + i\beta$, then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue. Thus both $x^k e^{(\alpha+i\beta)x}$ and $x^k e^{(\alpha-i\beta)x}$ appear in (2.2.6), where $0 \le k \le m-1$. In order to obtain a fundamental set of real solutions, the pair of solutions $x^k e^{(\alpha+i\beta)x}$ and $x^k e^{(\alpha-i\beta)x}$ in (2.2.6) should be replaced by $x^k e^{\alpha x}\cos(\beta x)$ and $x^k e^{\alpha x}\sin(\beta x)$.

In the following we discuss solution of the non-homogeneous equation.

$$y'' + P(x)y' + Q(x)y = f(x), \tag{2.2.7}$$

The associated homogeneous equation is

$$y'' + P(x)y' + Q(x)y = 0.$$

The method applies to higher order equations.

**1. Methods of variation of parameters.**

Let $y_1$ and $y_2$ be two linearly independent solutions of the associated homogeneous equation $y'' + P(x)y' + Q(x)y = 0$ and $W(x)$ be their Wronskian. We look for a particular solution of (2.2.7) in the form

$$y_p = u_1 y_1 + u_2 y_2,$$

where $u_1$ and $u_2$ are functions to be determined. Suppose

$$u_1' y_1 + u_2' y_2 = 0.$$

Differentiating this equation once, we get $u_1'' y_1 + u_2'' y_2 = -u_1' y_1' - u_2' y_2'$. Plugging $y_p$ into (2.2.7) we get

$$u_1' y_1' + u_2' y_2' = f.$$

Hence $u_1'$ and $u_2'$ satisfy

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0, \\ u_1' y_1' + u_2' y_2' = f. \end{cases} \tag{2.2.8}$$

Solving it, we find that

$$u_1' = -\frac{y_2}{W} f, \qquad u_2' = \frac{y_1}{W} f.$$

Integrating yields

$$\begin{aligned} u_1(x) &= -\int_{x_0}^{x} \frac{y_2(t)}{W(t)} f(t)dt, \\ u_2(x) &= \int_{x_0}^{x} \frac{y_1(t)}{W(t)} f(t)dt. \end{aligned} \tag{2.2.9}$$

**Example 2.5** Solve the differential equation $y'' + y = \sec x$.

**Solution**. A basis for the solutions of the homogeneous equation consists of $y_1 = \cos x$ and $y_2 = \sin x$. Now $W(y_1, y_2) = \cos x \cos x - (-\sin x)\sin x = 1$. Thus $u_1 = -\int \sin x \sec x\, dx = \ln|\cos x| + c_1$ and $u_2 = \int \cos x \sec x\, dx = x + c_2$. From this, a particular solution is given by $y_p = \cos x \ln|\cos x| + x \sin x$. Therefore, the general solution is $y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$.

The method of variation of parameters can also be used to find another solution of a second order homogeneous linear differential equation when one solution is given. Suppose $z$ is a known solution of the equation

$$y'' + P(x)y' + Q(x)y = 0.$$

We assume $y = vz$ is a solution so that

$$\begin{aligned} 0 &= (vz)'' + P(vz)' + Q(vz) = (v''z + 2v'z' + vz'') + P(v'z + vz') + Qvz \\ &= (v''z + 2v'z' + Pv'z) + v(z'' + Pz' + Qz) = v''z + v'(2z' + Pz). \end{aligned}$$

That is

$$\frac{v''}{v'} = -2\frac{z'}{z} - P.$$

An integration gives $v' = z^{-2}e^{-\int P\,dx}$ and $v = \int z^{-2}e^{-\int P\,dx}\, dx$. We leave it as an exercise to show that $z$ and $vz$ are linearly independent solutions by computing their Wronskian.

**Example 2.6** Given $y_1 = x$ is a solution of $x^2y'' + xy' - y = 0$, find another solution.

**Solution.** Let's write the DE in the form $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$. Then $P(x) = 1/x$. Thus a second linearly independent solution is given $y = vx$, where

$$v = \int x^{-2}e^{-\int 1/x\,dx}\, dx = \int x^{-2}x^{-1}\, dx = -\frac{1}{2x^2}.$$

Therefore the second solution is $y = -\frac{1}{2}x^{-1}$ and the general solution is $y = c_1x + c_2x^{-1}$.

**2. Method of undetermined coefficients.**
Consider the equation $y'' + ay' + by = f(x)$, where $a$ and $b$ are real constants.
**Case 1.** $f(x) = P_n(x)e^{\alpha x}$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.
We look for a particular solution in the form

$$y = Q(x)e^{\alpha x},$$

where $Q(x)$ is a polynomial. Plugging it into $y'' + ay' + by = f(x)$ we find

$$Q'' + (2\alpha + a)Q' + (\alpha^2 + a\alpha + b)Q = P_n(x). \tag{2.2.10}$$

**Subcase 1.1.** If $\alpha^2 + a\alpha + b \neq 0$, namely, $\alpha$ is not a root of the characteristic equation, we choose

$Q = R_n$, a polynomial of degree $n$, and

$$y = R_n(x)e^{\alpha x}.$$

The coefficients of $R_n$ can be determined by comparing the terms of same power in the two sides of (2.2.10). Note that in this case both sides of (2.2.10) are polynomials of degree $n$.

**Subcase 1.2.** If $\alpha^2 + a\alpha + b = 0$ but $2\alpha + a \neq 0$, namely, $\alpha$ is a simple root of the characteristic equation, then (2.2.10) is reduced to

$$Q'' + (2\alpha + a)Q' = P_n. \tag{2.2.11}$$

We choose $Q$ to be a polynomial of degree $n+1$. Since the constant term of $Q$ does not appear in (2.2.11), we may choose $Q(x) = xR_n(x)$, where $R_n(x)$ is a polynomial of degree $n$.

$$y = xR_n(x)e^{\alpha x}.$$

**Subcase 1.3** If $\alpha^2 + a\alpha + b = 0$ and $2\alpha + a = 0$, namely, $\alpha$ is a root of the characteristic equation with multiplicity 2, then (2.2.10) is reduced to

$$Q'' = P_n. \tag{2.2.12}$$

We choose $Q(x) = x^2R_n(x)$, where $R_n(x)$ is a polynomial of degree $n$.

$$y = x^2R_n(x)e^{\alpha x}.$$

**Example 2.7** Find the general solution of $y'' - y' - 2y = 4x^2$.

Ans: $y = c_1e^{2x} + c_2e^{-x} - 3 + 2x - 2x^2$.

**Example 2.8** Find a particular solution of $y''' + 2y'' - y' = 3x^2 - 2x + 1$.

Ans: $y = -27x - 5x^2 - x^3$.

**Example 2.9** Solve $y'' - 2y' + y = xe^x$.

Ans: $y = c_1e^x + c_2xe^x + \frac{1}{6}x^3e^x$.

**Case 2.** $f(x) = P_n(x)e^{\alpha x}\cos(\beta x)$ or $f(x) = P_n(x)e^{\alpha x}\sin(\beta x)$, where $P_n(x)$ is a polynomial of degree $n \geq 0$.
We first look for a solution of

$$y'' + ay' + by = P_n(x)e^{(\alpha+i\beta)x}. \tag{2.2.13}$$

Using the method in Case 1 we obtain a complex-valued solution

$$z(x) = u(x) + iv(x),$$

where $u(x) = \Re(z(x))$, $v(x) = \Im(z(x))$. Substituting $z(x) = u(x)+iv(x)$ into (2.2.13) and taking the real and imaginary parts, we can show that $u(x) = \Re(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\cos(\beta x), \tag{2.2.14}$$

and $v(x) = \Im(z(x))$ is a solution of

$$y'' + ay' + by = P_n(x)e^{\alpha x}\sin(\beta x). \tag{2.2.15}$$

**Example 2.10** Solve $y'' - 2y' + 2y = e^x \cos x$.

Ans: $y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{2} x e^x \sin x$.

Alternatively to solve (2.2.14) or (2.2.15), one can try a solution of the form

$$Q_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is not a root of $\lambda^2 + a\lambda + b = 0$, and

$$xQ_n(x)e^{\alpha x}\cos(\beta x) + xR_n(x)e^{\alpha x}\sin(\beta x)$$

if $\alpha + i\beta$ is a root of $\lambda^2 + a\lambda + b = 0$, where $Q_n$ and $R_n$ are polynomials of degree $n$

The following conclusions will be useful.

**Theorem 2.11** *Let $y_1$ and $y_2$ be particular solutions of the equations*

$$y'' + ay' + by = f_1(x)$$

*and*

$$y'' + ay' + by = f_2(x)$$

*respectively, then $y_p = y_1 + y_2$ is a particular solution of*

$$y'' + ay' + by = f_1(x) + f_2(x).$$

**Proof.** Exercise.

**Example 2.11** Solve $y'' - y = e^x + \sin x$.

**Solution.** A particular solution for $y'' - y = e^x$ is given by $y_1 = \frac{1}{2} x e^x$. Also a particular solution for $y'' - y = \sin x$ is given by $y_2 = -\frac{1}{2}\sin x$. Thus $\frac{1}{2}(xe^x - \sin x)$ is a particular solution of the given differential equation. The general solution of the corresponding homogeneous differential equation is given by $c_1 e^{-x} + c_2 e^x$. Hence the general solution of the given differential equation is $c_1 e^{-x} + c_2 e^x + \frac{1}{2}(xe^x - \sin x)$.

## 2.3 Operator Methods

Let $x$ denote independent variable, and $y$ dependent variable. Introduce

$$Dy = \frac{d}{dx}y, \quad D^n y = \frac{d^n}{dx^n}y = y^{(n)}.$$

We define $D^0 y = y$. Given a polynomial $L(x) = \sum_{j=0}^{n} a_j x^j$, where $a_j$'s are constants, we define a differential operator $L(D)$ by

$$L(D)y = \sum_{j=0}^{n} a_j D^j y.$$

Then the equation

$$\sum_{j=0}^{n} a_j y^{(j)} = f(x) \tag{2.3.1}$$

can be written as

$$L(D)y = f(x). \tag{2.3.2}$$

Let $L(D)^{-1}f$ denote any solution of (2.3.2). We have

$$\begin{aligned} &D^{-1}D = DD^{-1} = D^0, \\ &L(D)^{-1}L(D) = L(D)L(D)^{-1} = D^0. \end{aligned}$$

However, $L(D)^{-1}f$ is not unique.

To see the above properties, first recall that $D^{-1}f$ means a solution of $y' = f$. Thus $D^{-1}f = \int f$. Hence it follows that $D^{-1}D = DD^{-1}$= identity operator $D^0$.
For the second equality, note that a solution of $L(D)y = L(D)f$ is simply $f$. Thus by definition of $L(D)^{-1}$, we have $L(D)^{-1}(L(D)f) = f$. This means $L(D)^{-1}L(D) = D^0$. Lastly, since $L(D)^{-1}f$ is a solution of $L(D)y = f(x)$, it is clear that $L(D)(L(D)^{-1}f) = f$. In other words, $L(D)L(D)^{-1} = D^0$.

More generally, we have:

$$\begin{aligned} &1. \quad D^{-1}f(x) = \int f(x)dx + C, \\ &2. \quad (D-a)^{-1}f(x) = Ce^{ax} + e^{ax}\int e^{-ax}f(x)dx, \\ &3. \quad L(D)(e^{ax}f(x)) = e^{ax}L(D+a)f(x), \\ &4. \quad L(D)^{-1}(e^{ax}f(x)) = e^{ax}L(D+a)^{-1}f(x). \end{aligned} \tag{2.3.3}$$

**Proof.** Property 2 is just the solution of the first order linear ODE. To prove Property 3, first observe that $(D-r)(e^{ax}f(x)) = e^{ax}D(f(x)) + ae^{ax}f(x) - re^{ax}f(x) = e^{ax}(D+a-r)(f(x))$. Thus $(D-s)(D-r)(e^{ax}f(x)) = (D-s)[e^{ax}(D+a-r)(f(x))] = e^{ax}(D+a-s)(D+a-r)(f(x))$. Now we may write $L(D) = (D-r_1)\cdots(D-r_n)$. Then $L(D)(e^{ax}f(x)) = e^{ax}L(D+a)f(x)$.

This says that we can move the factor $e^{ax}$ to the left of the operator $L(D)$ if we replace $L(D)$ by $L(D+a)$.

To prove Property 4, apply $L(D)$ to the right hand side. We have

$$L(D)[e^{ax}L(D+a)^{-1}f(x)] = e^{ax}[L(D+a)(L(D+a)^{-1}f(x))] = e^{ax}f(x).$$

Thus $L(D)^{-1}(e^{ax}f(x)) = e^{ax}L(D+a)^{-1}f(x)$. □

Let $L(x) = (x-r_1)\cdots(x-r_n)$. The solution of (2.3.2) is given by

$$y = L(D)^{-1}f(x) = (D-r_1)^{-1}\cdots(D-r_n)^{-1}f(x). \tag{2.3.4}$$

Then we obtain the solution by successive integration. Moreover, if $r_j's$ are distinct, we can write

$$\frac{1}{L(x)} = \frac{A_1}{x-r_1} + \cdots + \frac{A_n}{x-r_n},$$

where $A_j's$ can be found by the method of partial fractions. Then the solution is given by

$$y = [A_1(D-r_1)^{-1} + \cdots + A_n(D-r_n)^{-1}]f(x). \tag{2.3.5}$$

Next consider the case of repeated roots. Let the multiple root be equal to $m$ and the equation to be solved is

$$(D-m)^n y = f(x) \tag{2.3.6}$$

To solve this equation, let us assume a solution of the form $y = e^{mx}v(x)$, where $v(x)$ is a function of $x$ to be determined. One can easily verify that $(D-m)^n e^{mx}v = e^{mx}D^n v$. Thus equation (2.3.6) reduces to

$$D^n v = e^{-mx}f(x) \tag{2.3.7}$$

If we integrate (2.3.7) $n$ times, we obtain

$$v = \int\int\cdots\int\int e^{-mx}f(x)\,dx\cdots dx + c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \tag{2.3.8}$$

Thus we see that

$$(D-m)^{-n}f(x) = e^{mx}\left[\iint\cdots\iint e^{-mx}f(x)dx\cdots dx + c_0 + c_1x + \cdots + c_{n-1}x^{n-1}\right] \tag{2.3.9}$$

**Example 2.12** Solve $(D^2-3D+2)y = xe^x$.

**Solution**. First $\frac{1}{D^2-3D+2} = \frac{1}{D-2} - \frac{1}{D-1}$. Therefore

$$\begin{aligned} y &= (D^2-3D+2)^{-1}(xe^x) \\ &= (D-2)^{-1}(xe^x) - (D-1)^{-1}(xe^x) \\ &= e^{2x}D^{-1}(e^{-2x}xe^x) - e^xD^{-1}(e^{-x}xe^x) \\ &= e^{2x}D^{-1}(e^{-x}x) - e^xD^{-1}(x) \\ &= e^{2x}(-xe^{-x} - e^{-x} + c_1) - e^x(\tfrac{1}{2}x^2 + c_2) \\ &= -e^x(\tfrac{1}{2}x^2 + x + 1) + c_1e^{2x} + c_2e^x. \end{aligned}$$

**Example 2.13** Solve $(D^3 - 3D^2 + 3D - 1)y = e^x$.

**Solution.** The DE is equivalent to $(D-1)^3 y = e^x$. Therefore,

$$y = (D-1)^{-3} e^x = e^x \left[ \iiint e^{-x} e^x \, dx + c_0 + c_1 x + c_2 x^2 \right] = e^x \left[ \frac{1}{6} x^3 + c_0 + c_1 x + c_2 x^2 \right].$$

If $f(x)$ is a polynomial in $x$, then $(1-D)(1 + D + D^2 + D^3 + \cdots) f = f$. Thus $(1-D)^{-1}(f) = (1 + D + D^2 + D^3 + \cdots) f$. Therefore, if $f$ is a polynomial, we may formally expand $(D-r)^{-1}$ into power series in $D$ and apply it to $f$. If the degree of $f$ is $n$, then it is only necessary to expand $(D-r)^{-1}$ up to $D^n$.

**Example 2.14** Solve $(D^4 - 2D^3 + D^2)y = x^3$.

**Solution.** We have

$$\begin{aligned} y &= (D^4 - 2D^3 + D^2)^{-1} f = \tfrac{1}{D^2(1-D)^2} x^3 \\ &= D^{-2}(1 + 2D + 3D^2 + 4D^3 + 5D^4 + 6D^5) x^3 \\ &= D^{-2}(x^3 + 6x^2 + 18x + 24) \\ &= D^{-1}(\tfrac{x^4}{4} + 2x^3 + 9x^2 + 24x) \\ &= \tfrac{x^5}{20} + \tfrac{x^4}{2} + 3x^3 + 12x^2. \end{aligned}$$

Therefore, the general solution is $y = (c_1 + c_2 x)e^x + (c_3 + c_4 x) + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$.

**Exercise 2.1** For all real $x$, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.

(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain.

**Exercise 2.2** Find the general solution of $y'' - 2y' - 3y = x(e^x + e^{-x})$.

Ans: $y = c_1 e^{-x} + c_2 e^{3x} - \frac{x}{4} e^x - \frac{x^2}{8} e^{-x} - \frac{x}{16} e^{-x}$.

# Chapter 3

# Second Order Linear Differential Equations

## 3.1 Exact 2nd Order Equations

The general 2nd order linear differential equation is of the form

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = f(x) \tag{3.1.1}$$

The equation can be written as

$$(p_0y' - p_0'y)' + (p_1y)' + (p_0'' - p_1' + p_2)y = f(x) \tag{3.1.2}$$

It is said to be exact if

$$p_0'' - p_1' + p_2 \equiv 0. \tag{3.1.3}$$

In the event that the equation is exact, a first integral to (3.1.1) is

$$p_0(x)y' - p_0'(x)y + p_1(x)y = \int f(x)\,dx + C_1.$$

**Example 3.1** Find the general solution of the DE

$$\frac{1}{x}y'' + (\frac{1}{x} - \frac{2}{x^2})y' - (\frac{1}{x^2} - \frac{2}{x^3})y = e^x.$$

**Solution.** Condition (3.1.3) is fulfilled. The first integral is

$$\frac{1}{x}y' + \frac{1}{x^2}y + (\frac{1}{x} - \frac{2}{x^2})y = e^x + C_1.$$

That is

$$y' + (1 - \frac{1}{x})y = xe^x + C_1x.$$

From the last equation, the general solution is found to be

$$y = \frac{1}{2}xe^x + C_1x + C_2xe^{-x}.$$

## 3.2 The Adjoint Differential Equation and Integrating Factor

If (3.1.1) is multiplied by a function $v(x)$ so that the resulting equation is exact, then $v(x)$ is called an integrating factor of (3.1.1). That is

$$(p_0 v)'' - (p_1 v)' + p_2 v = 0. \tag{3.2.4}$$

This is a differential equation for $v$, which is, more explicitly,

$$p_0(x)v'' + (2p_0'(x) - p_1(x))v' + (p_0''(x) - p_1'(x) + p_2(x))v = 0. \tag{3.2.5}$$

Equation (3.2.5) is called the *adjoint* of the given differential equation (3.1.1). A function $v(x)$ is thus an integrating factor for a given differential equation, if and only if it is a solution of the adjoint equation. Note that the adjoint of (3.2.5) is in turn found to be the associated homogeneous equation of (3.1.1), thus each is the adjoint of the other.
In this case, a first integral to (3.1.1) is

$$v(x)p_0(x)y' - (v(x)p_0(x))'y + v(x)p_1(x)y = \int v(x)f(x)\,dx + C_1.$$

**Example 3.2** Find the general solution of the DE

$$(x^2 - x)y'' + (2x^2 + 4x - 3)y' + 8xy = 1.$$

**Solution.** The adjoint of this equation is

$$(x^2 - x)v'' - (2x^2 - 1)v' + (4x - 2)v = 0.$$

By the trial of $x^m$, this equation is found to have $x^2$ as a solution. Thus $x^2$ is an integrating factor of the given differential equation. Multiplying the original equation by $x^2$, we obtain

$$(x^4 - x^3)y'' + (2x^4 + 4x^3 - 3x^2)y' + 8x^3 y = x^2.$$

Thus a first integral to it is

$$(x^4 - x^3)y' - (4x^3 - 3x^2)y + (2x^4 + 4x^3 - 3x^2)y = \int x^2\,dx + C.$$

After simplification, we have

$$y' + \frac{2x}{x-1}y = \frac{1}{3(x-1)} + \frac{C}{x^3(x-1)}.$$

An integrating factor for this first order linear equation is $e^{2x}(x-1)^2$. Thus the above equation becomes

$$e^{2x}(x-1)^2 y = \frac{1}{3}\int (x-1)e^{2x}\,dx + C\int \frac{e^{2x}(x-1)}{x^3}\,dx + C_2.$$

That is

$$e^{2x}(x-1)^2 y = \frac{1}{3}\left[\frac{x}{2} - \frac{3}{4}\right]e^{2x} + C\frac{e^{2x}}{2x^2} + C_2.$$

Thus the general solution is

$$y = \frac{1}{(x-1)^2}\left(\frac{x}{6} - \frac{1}{4} + \frac{C_1}{x^2} + C_2 e^{-2x}\right).$$

**Exercise 3.1** Solve the following differential equation by finding an integrating factor of it.

$$y'' + \frac{4x}{2x-1}y' + \frac{8x-8}{(2x-1)^2}y = 0.$$

Ans: $y = \frac{C_1}{2x-1} + \frac{C_2 x}{2x-1}e^{-2x}$.

**Exercise 3.2** The equation

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0 \tag{3.2.6}$$

is called a Sturm-Liouville equation, where $\lambda$ is a real parameter. Show that the adjoint of a Sturm-Liouville equation is itself.

## 3.3 Lagrange's Identity and Green's Formula

Let $L$ be the differential operator given by the left hand side of (3.1.1), that is $L[y] = p_0(x)y'' + p_1(x)y' + p_2(x)y$. The *formal adjoint* of $L$ is the differential operator defined by $L^+[y] = (p_0(x)y)'' - (p_1(x)y)' + p_2(x)y$, where $p_0'', p_1'$ and $p_2$ are continuous on an interval $[a, b]$. Let $u$ and $v$ be functions having continuous second derivatives on the interval $[a, b]$. Direct simplification gives the following identity relating $L$ and $L^+$.

**Theorem 3.1 (Lagrange's identity)**

$$L[u]v - uL^+[v] = \frac{d}{dx}[P(u, v)],$$

*where* $P(u, v) = up_1v - u(p_0v)' + u'p_0v$.

**Proof**. Expanding both sides, we get $p_0u''v + p_1u'v - p_0uv'' - 2p_0'uv' - p_0''uv + p_1'uv + p_1uv'$.□

Integrating Lagrange's identity leads to Green's formula.

**Corollary 3.2 (Green's formula)**

$$\int_a^b \left(L[u]v - uL^+[v]\right)\, dx = P(u, v)(x)\Big|_a^b .$$

Let's define an inner product for continuous real-valued functions on $[a, b]$ by

$$(f, g) = \int_a^b f(x)g(x)\, dx.$$

With this notation, Green's formula becomes

$$(L[u], v) = (u, L^+[v]) + P(u, v)(x)\Big|_a^b . \tag{3.3.7}$$

Though the operators $L$ and $L^+$ are meaningful when applied to any function $y$ having a continuous second derivative on $[a, b]$, it is usually more advantageous to further restrict the domains they act. In general, it is evidently more useful to restrict $L$ and $L^+$ so that

$$(L[u], v) = (u, L^+[v]) \tag{3.3.8}$$

holds for all $u$ in the domain of $L$ and all $v$ in the domain of $L^+$. This is the case when we try to solve boundary value problems in which $L$ and $L^+$ are restricted to those functions that satisfy the given boundary conditions. When (3.3.8) holds, $L^+$ is called the *adjoint operator* of $L$ (i.e. without the adjective "formal"). For the special case when $L^+ = L$ and (3.3.8) holds, we say that $L$ is self-adjoint.

As an example, let $L$ be the differential operator given by the first two terms on the left hand side of the Sturm-Liouville equation (3.2.6), that is $L[y] = (p(x)y')' + q(x)y$. By the exercise for the equation in (3.2.6), it follows that $L^+ = L$. Then Lagrange's identity and Green's formula reduce to the followings.

**Theorem 3.3 (Lagrange's identity for $L[y] = (p(x)y')' + q(x)y$)**

$$L[u]v - uL[v] = -\frac{d}{dx}[pW(u,v)],$$

*where $W(u,v) = uv' - vu'$ is the Wronskian of $u$ and $v$.*

**Corollary 3.4 (Green's formula for $L[y] = (p(x)y')' + q(x)y$)**

$$(L[u],v) - (u,L[v]) = \int_a^b (L[u]v - uL[v])\,dx = -pW(u,v)(x)\Big|_a^b. \tag{3.3.9}$$

**Exercise 3.3** Let $L[y] = p_0(x)y'' + p_1(x)y' + p_2(x)y$. Prove that $L^+ = L$ if and only if $p_1(x) = p_0'(x)$.

**Exercise 3.4** Prove theorem 3.3.

**Exercise 3.5** Prove corollary 3.4.

**Exercise 3.6** Prove that if $u$ and $v$ are solutions of $(p(x)y')' + q(x)y = 0$, then $pW(u,v)(x)$ is a constant for all $x \in [a,b]$

## 3.4 Regular Boundary Value Problem

The problem of finding a solution to a second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x), \;\; a < x < b; \tag{3.4.10}$$

satisfying the boundary conditions

$$\begin{aligned} a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = d_1, \\ a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = d_2, \end{aligned} \tag{3.4.11}$$

is called a *two-point boundary value problem.*

When $d_1 = d_2 = 0$, the boundary conditions are said to be homogeneous; otherwise we refer them as nonhomogeneous.

For the homogeneous equation with homogeneous boundary condition

$$\begin{aligned} & y'' + p(x)y' + q(x)y = 0, \;\; a < x < b; \\ & a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = 0, \\ & a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = 0, \end{aligned} \tag{3.4.12}$$

one can verify the following properties:

1. If $\phi(x)$ is a nontrivial solution to (3.4.12), then so is $c\phi(x)$ for any constant $c$. Thus (3.4.12) has a one-parameter family of solutions.
2. If (3.4.12) has two linearly independent solutions $\phi_1$ and $\phi_2$, then $c_1\phi_1+c_2\phi_2$ is also a solution of (3.4.12) for any constants $c_1$ and $c_2$. Thus (3.4.12) has a two-parameter family of solutions.
3. The remaining possibility is that $\phi(x) \equiv 0$ is the unique solution to (3.4.12).

For nonhomogeneous equation (3.4.10), there is an additional possibility that it has no solution. The following examples illustrate some of these cases.

**Example 3.3** Find all solutions to the boundary value problem

$$y'' + 2y' + 5y = 0;\ y(0) = 2, y(\pi/4) = e^{-\pi/4}.$$

**Solution**. The general solution to the equation $y''+2y'+5y = 0$ is $y = c_1e^{-x}\cos 2x+c_2e^{-x}\sin 2x$. Substituting the boundary conditions, we have $y(0) = c_1 = 2$ and $y(\pi/4) = c_2e^{-\pi/4} = e^{-\pi/4}$. Thus $c_1 = 2, c_2 = 1$, and so the boundary value problem has the unique solution $y = 2e^{-x}\cos 2x + e^{-x}\sin 2x$.

**Example 3.4** Find all solutions to the boundary value problem

$$y'' + y = \cos 2x;\ y'(0) = 0, y'(\pi) = 0.$$

**Solution**. The general solution to the equation $y'' + y = \cos 2x$ is $y = c_1\cos x + c_2\sin x - (1/3)\cos 2x$. Thus $y' = -c_1\sin x + c_2\cos x + (2/3)\sin 2x$. Substituting the boundary conditions, we have $y'(0) = c_2 = 0$ and $y'(\pi) = -c_2 = 0$. Thus the boundary value problem has a one-parameter family of solutions $y = c_1\cos x - (1/3)\cos 2x$, where $c_1$ is any real number.

**Example 3.5** Find all solutions to the boundary value problem

$$y'' + 4y = 0;\ y(-\pi) = y(\pi), y'(-\pi) = y'(\pi).$$

**Solution**. The general solution to the equation $y''+4y = 0$ is $y = c_1\cos 2x+c_2\sin 2x$. Since $\cos 2x$ and $\sin 2x$ are periodic functions of period $\pi$, $y = c_1\cos 2x + c_2\sin 2x$ and $y' = -2c_1\sin 2x + 2c_2\cos 2x$ automatically satisfy the boundary conditions $y(-\pi) = y(\pi), y'(-\pi) = y'(\pi)$. Hence the boundary value problem has a two-parameter family of solutions $y = c_1\cos 2x + c_2\sin 2x$, where $c_1$ and $c_2$ are any real numbers.

**Example 3.6** Find all solutions to the boundary value problem

$$y'' + 4y = 4x;\ y(-\pi) = y(\pi), y'(-\pi) = y'(\pi).$$

**Solution**. The general solution to the equation $y'' + 4y = 4x$ is $y = x + c_1\cos 2x + c_2\sin 2x$. Since $y(-\pi) = -\pi + c_1$ and $y(\pi) = \pi + c_1$, there are no solutions satisfying $y(-\pi) = y(\pi)$. Hence the boundary value problem has no solution.

**Exercise 3.7** Find all solutions to the boundary value problem

$$y'' + y = e^x;\ y(0) = 0, y(\pi) + y'(\pi) = 0.$$

**Ans**. $y = (\frac{1}{2} + e^{\pi})\sin x - \frac{1}{2}\cos x + \frac{1}{2}e^x$.

## 3.5 Regular Sturm-Liouville Boundary Value Problem

Let $L[y] = (p(x)y')' + q(x)y$. Consider the *regular* Sturm-Liouville boundary value problem:

$$\begin{aligned} &L[y] + \lambda r(x)y = 0, \quad a < x < b; \\ &a_1 y(a) + a_2 y'(a) = 0, \\ &b_1 y(b) + b_2 y'(b) = 0, \end{aligned} \tag{3.5.13}$$

where $p(x), p'(x), q(x)$ and $r(x)$ are continuous functions on $[a, b]$ and $p(x) > 0$ and $r(x) > 0$ on $[a, b]$. We only consider *separated* boundary conditions. Also we exclude the cases when $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$.

Let $u$ and $v$ be functions that have continuous second derivatives on $[a, b]$ and satisfy the boundary conditions in (3.5.13). The boundary conditions in (3.5.13) imply that $W(u, v)(b) = W(u, v)(a) = 0$. This is because the system of equations $a_1 u(a) + a_2 u'(a) = 0, a_1 v(a) + a_2 v'(a) = 0$ has nontrivial solutions in $a_1$ and $a_2$ since $a_1$ and $a_2$ are not both zero. Therefore the determinant of the system $W(u, v)(a)$ must be zero. Similarly $W(u, v)(b) = 0$.

Thus by Green's formula (3.3.9),

$$(L[u], v) = (u, L[v]).$$

Therefore $L$ is a self-adjoint operator with domain equal to the set of functions that have continuous second derivatives on $[a, b]$ and satisfy the boundary conditions in (3.5.13). Self-adjoint operators are like symmetric matrices in that their eigenvalues are always real.

**Remark**. Here we only require $u$ and $v$ satisfy the boundary conditions in (3.5.13) but not necessarily the differential equation $L[y] + \lambda r(x)y = 0$. However, if $u$ and $v$ satisfy the differential equation, then $L[u]v = uL[v] = -\lambda ruv$ and hence $(L[u], v) = (u, L[v])$ too.

The regular Sturm-Liouville boundary value problem in (3.5.13) involves a parameter $\lambda$. The objective is to determine for which values of $\lambda$, the equation in (3.5.13) has nontrivial solutions satisfying the given boundary conditions. Such problems are called *eigenvalue problems*. The nontrivial solutions are called *eigenfunctions*, and the corresponding number $\lambda$ an *eigenvalue*. If all the eigenfunctions associated with a particular eigenvalue are just scalar multiple of each other, then the eigenvalue is called *simple*.

**Theorem 3.5** *All the eigenvalues of the regular Sturm-Liouville boundary value problem (3.5.13) are (i) real, (ii) have real-valued eigenfunctions and (iii) simple.*

**Proof**. (i) Let $\lambda$ be a possibly complex eigenvalue with eigenfunction $u$ for (3.5.13). Also $u$ may be complex-valued. Thus $u \not\equiv 0$, $L[u] = -\lambda ru$, and $u$ satisfies the boundary conditions in (3.5.13).

Then $L[\overline{u}] = \overline{L[u]} = \overline{-\lambda ru} = -\overline{\lambda} r\overline{u}$. Also $\overline{u}$ satisfies the boundary conditions in (3.5.13) as $a_1, a_2, b_1, b_2$ are real. Thus $\overline{\lambda}$ is an eigenvalue with eigenfunction $\overline{u}$ for (3.5.13). We have

$$(L[u], \overline{u}) = (-\lambda ru, \overline{u}) = -\lambda(ru, \overline{u}) = -\lambda \int_a^b r|u|^2 \, dx.$$

By Green's formula,

$$(L[u], \overline{u}) = (u, L[\overline{u}]) = (u, -\overline{\lambda} r\overline{u}) = -\overline{\lambda}(u, r\overline{u}) = -\overline{\lambda} \int_a^b r|u|^2 \, dx.$$

Since $r > 0$, $u \not\equiv 0$ and $|u| \geq 0$, we have $\int_a^b r|u|^2\,dx > 0$. Therefore $\lambda = \overline{\lambda}$ and $\lambda$ is real.

(ii) If $u(x) = u_1(x) + iu_2(x)$ is a complex-valued eigenfunction with eigenvalue $\lambda$. By (i), $\lambda$ is real. Then $-\lambda r(u_1 + iu_2) = L[u_1 + iu_2] = L[u_1] + iL[u_2]$. Equating real and imaginary parts, we have $L[u_1] = -\lambda r u_1$ and $L[u_2] = -\lambda r u_2$. Furthermore both $u_1$ and $u_2$ satisfy the boundary conditions in (3.5.13). Therefore we can take either $u_1$ or $u_2$ as the real-valued eigenfunction associated with $\lambda$.

(iii) Suppose $u_1$ and $u_2$ are 2 eigenfunctions corresponding to the eigenvalue $\lambda$ and both are solutions to (3.5.13). Thus

$$a_1u_1(a) + a_2u_1'(a) = 0, \text{ and } a_1u_2(a) + a_2u_2'(a) = 0.$$

Since $a_1$ and $a_2$ are not both zero, this system of linear equations (in $a_1$ and $a_2$) has zero determinant. That is $W(u_1, u_2)(a) = 0$. Similarly $W(u_1, u_2)(b) = 0$.

Suppose $u_1$ and $u_2$ are linearly independent on $(a, b)$. Then $W(u_1, u_2)(x) \neq 0$ for all $x \in (a, b)$. Since $u_1$ and $u_2$ are solutions of (3.5.13), we have by exercise 3.6 $p(x)W(u_1, u_2)(x) = C$, for all $x \in [a, b]$, where $C$ is a nonzero constant as $p(x) > 0$ for all $x \in [a, b]$. But this contradicts $C = p(a)W(u_1, u_2)(a) = 0$. Consequently $u_1$ and $u_2$ are linearly dependent. Therefore one must be a multiple of the other. That is $\lambda$ is a simple eigenvalue.

Two real-valued functions $f$ and $g$ defined on $[a, b]$ are said to be *orthogonal* with respect to a positive weight function $r(x)$ on the interval $[a, b]$ if

$$\int_a^b f(x)g(x)r(x)\,dx = 0.$$

**Theorem 3.6** *Eigenfunctions that correspond to distinct eigenvalues of the regular Sturm-Liouville boundary value problem in (3.5.13) are orthogonal with respect to the weight function* $r(x)$ *on* $[a, b]$.

**Proof**. Let $\lambda$ and $\mu$ be distinct eigenvalues with corresponding real-valued eigenfunctions $u$ and $v$ for (3.5.13) respectively. That is $L[u] = -\lambda r u$ and $L[v] = -\mu r v$. Then $-\mu(u, rv) = (u, -\mu r v) = (u, L[v]) = (L[u], v) = (-\lambda r u, v) = -\lambda(ru, v)$. Since $(u, rv) = (ru, v)$ and $\mu \neq \lambda$, we have $(u, rv) = (ru, v) = 0$. That is $\int_a^b ruv\,dx = 0$ which means $u$ and $v$ are orthogonal with respect to the weight function $r(x)$ on $[a, b]$.

**Theorem 3.7** *The eigenvalues of the regular Sturm-Liouville boundary value problem in (3.5.13) form a countable, increasing sequence*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

*with* $\lim_{n\to\infty} \lambda_n = +\infty$.

For a proof of this result, see [4], page 333.

**Example 3.7** Consider the regular Sturm-Liouville boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0. \tag{3.5.14}$$

When $\lambda \leq 0$, the boundary value problem (3.5.14) has only the trivial solution $y = 0$. Thus for $\lambda \leq 0$, it is not an eigenvalue. Let's consider $\lambda > 0$. The general solution to the equation $y'' + \lambda y = 0$ is given by $y = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$. Now $y(0) = 0$ implies that $A = 0$, and $y(\pi) = 0$ implies that $B\sin(\sqrt{\lambda}\, \pi) = 0$. Since we are looking for nontrivial solutions, we must have $B \neq 0$ so that $\sin(\sqrt{\lambda}\, \pi) = 0$. Therefore $\lambda = n^2$, $n = 1, 2, 3, \ldots$ with corresponding eigenfunctions $\phi_n(x) = B_n \sin(nx)$. One can easily verify that $(\phi_m, \phi_n) = 0$ for $m \neq n$.

Thus, associated to a regular Sturm-Liouville boundary value problem, there is a sequence of orthogonal eigenfunctions $\{\phi_n\}$ defined on $[a, b]$. We can use these eigenfunctions to form an *orthonormal* system with respect to $r(x)$ simply by normalizing each eigenfunction $\phi_n$ so that

$$\int_a^b \phi_n^2(x) r(x)\, dx = 1.$$

Now suppose $\{\phi_n\}$ is orthonormal with respect to a positive weight function $r(x)$ on $[a, b]$, that is

$$\int_a^b \phi_n(x)\phi_m(x) r(x)\, dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

Then with any *piecewise continuous*[1] function $f$ on $[a, b]$, we can identify an orthogonal expansion

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \int_a^b f(x)\phi_n(x) r(x)\, dx.$$

For instance, the eigenfunctions $\phi_n(x) = \sin nx$ for (3.5.14) gives rise to the Fourier sine series expansion.

**Exercise 3.8** Consider the regular Sturm-Liouville boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, y'(\pi) = 0. \tag{3.5.15}$$

Show that the eigenvalues of (3.5.15) are $\lambda_n = (2n-1)^2/4$, $n = 1, 2, 3, \ldots$, with corresponding eigenfunctions

$$\phi_n(x) = a_n \sin\left(\frac{2n-1}{2} x\right).$$

**Example 3.8** Construct an orthonormal system of eigenfunctions corresponding to the regular Sturm-Liouville boundary value problem (3.5.15).

**Solution**. By Exercise 3.8 and Theorem 3.6, the eigenfunctions $\phi_n(x) = a_n \sin\left(\frac{2n-1}{2} x\right)$ are orthogonal with respect to $r(x) = 1$ on $[0, \pi]$. Direct computation gives

$$\int_0^\pi a_n^2 \sin^2\left(\frac{2n-1}{2} x\right)\, dx = a_n^2 \pi/2.$$

[1]A function $f(x)$ is piecewise continuous on a finite interval $[a, b]$ if $f(x)$ is continuous at every point in $[a, b]$, except possibly for a finite number of points at which $f(x)$ has a jump discontinuity.

Thus, if we take $a_n = \sqrt{2/\pi}$, then

$$\left\{\sqrt{2/\pi}\sin\left(\frac{2n-1}{2}x\right)\right\}$$

is an orthonormal system of eigenfunctions.

Like the theory of Fourier series, the eigenfunction expansion of $f$ converges uniformly to $f$ on $[a, b]$ under suitable conditions.

**Theorem 3.8** *Let $\{\phi_n\}$ be an orthonormal system of eigenfunctions for the regular Sturm-Liouville boundary value problem in (3.5.13). Let $f$ be a continuous function on $[a, b]$ such that $f'$ is piecewise continuous on $[a, b]$ and $f$ satisfies the boundary conditions in (3.5.13). Then*

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \qquad a \leq x \leq b, \tag{3.5.16}$$

*where*

$$c_n = \int_a^b f(x)\phi_n(x)r(x)\,dx.$$

*Furthermore, the eigenfunction expansion in (3.5.16) converges uniformly on $[a, b]$.*

For a proof of this result, see [2].

**Example 3.9** Express

$$f(x) = \begin{cases} 2x/\pi, & 0 \leq x \leq \pi/2, \\ 1, & \pi/2 \leq x \leq \pi, \end{cases}$$

in an eigenfunction expansion using the orthonormal system of eigenfunctions

$$\left\{\sqrt{2/\pi}\sin\left(\frac{2n-1}{2}x\right)\right\}.$$

**Solution**. Referring to the problem (3.5.15), we compute

$$c_n = \int_0^{\pi} f(x)\sqrt{2/\pi}\sin\left(\frac{2n-1}{2}x\right)\,dx = \frac{2^{7/2}}{\pi^{3/2}(2n-1)^2}\sin\left(\frac{n\pi}{2} - \frac{\pi}{4}\right).$$

Thus

$$f(x) = \sum_{n=1}^{\infty} c_n\sqrt{2/\pi}\sin\left(\frac{2n-1}{2}x\right). \tag{3.5.17}$$

Since $f(0) = 0$, $f'(\pi) = 0$, the function $f$ satisfies the boundary conditions in (3.5.15). Moreover, $f$ is continuous and $f'$ is piecewise continuous on $[0, \pi]$. Thus by Theorem 3.8, the series in (3.5.17) converges uniformly to $f$ on $[0, \pi]$.

The Sturm-Liouville boundary value problem arises from solving partial differential equations such as the heat equation in mathematical physics. The eigenfunction expansion can be used to solve the nonhomogeneous regular Sturm-Liouville boundary value problem. For further discussion on the Sturm-Liouville boundary value problem, see [4].

## 3.6 Nonhomogeneous Boundary Value Problem

We only consider the nonhomogeneous regular Sturm-Liouville boundary value problem with homogeneous boundary conditions. Let

$$\begin{aligned} &L[y] = f(x), \quad a < x < b; \\ &a_1 y(a) + a_2 y'(a) = 0, \\ &b_1 y(b) + b_2 y'(b) = 0, \end{aligned} \tag{3.6.18}$$

where $L[y] = (p(x)y')' + q(x)y$, $p(x), p'(x)$ and $q(x)$ are continuous functions on $[a, b]$ and $p(x) > 0$ on $[a, b]$, and $f(x)$ is continuous on $[a, b]$.

Let

$$\begin{aligned} &L[y] = 0, \quad a < x < b; \\ &a_1 y(a) + a_2 y'(a) = 0, \\ &b_1 y(b) + b_2 y'(b) = 0, \end{aligned} \tag{3.6.19}$$

be the corresponding homogeneous regular Sturm-Liouville boundary value problem.

**Theorem 3.9** *The nonhomogeneous problem (3.6.18) has a unique solution if and only if the homogeneous problem (3.6.19) has only the trivial solution.*

**Proof**. To prove the necessity, let $z$ be the unique solution to (3.6.18). Suppose (3.6.19) has a nontrivial solution $u$. Thus $u \not\equiv 0$ on $[a, b]$. Then $z + u$ is also a solution to (3.6.18). But $z + u \not\equiv z$ on $[a, b]$, contradicting the fact that $z$ is the unique solution to (3.6.18). Thus (3.6.19) has only the trivial solution.

To prove the sufficiency, we have to prove both the existence and uniqueness of the solution for (3.6.18). The uniqueness is easy. Suppose $z_1$ and $z_2$ are two solutions to (3.6.18). Then $z_1 - z_2$ is a solution of (3.6.19). Since we assume (3.6.19) has only the trivial solution, we must have $z_1 - z_2 \equiv 0$ so that $z_1 \equiv z_2$ on $[a, b]$.

To prove the existence of solution to (3.6.18), we use the method of variation of parameters. First let $y_1$ and $y_2$ be nontrivial solutions to the equation $L[y] = 0$ satisfying respectively

$$a_1 y_1(a) + a_2 y_1'(a) = 0, \tag{3.6.20}$$

and

$$b_1 y_2(b) + b_2 y_2'(b) = 0. \tag{3.6.21}$$

The existence of $y_1$ and $y_2$ follows from the existence and uniqueness theorem for initial value problems.

We need to prove $y_1$ and $y_2$ are linearly independent. Suppose not, we may assume $y_1 = cy_2$ for some nonzero constant $c$. Then $y_1$ also satisfies (3.6.21). Thus $y_1$ is a solution of the homogeneous problem (3.6.19). Since (3.6.19) has only the trivial solution, we have $y_1 \equiv 0$ which is a contradiction. Therefore, $y_1$ and $y_2$ are linearly independent.

Let's write $L[y] = f$ in the standard form

$$y'' + \frac{p'}{p} y' + \frac{q}{p} y = \frac{f}{p}.$$

By the method of variation of parameters, we have

$$y = u_1 y_1 + u_2 y_2,$$

where

$$u_1 = -\int_a^x \frac{y_2(t)}{W(t)}\frac{f(t)}{p(t)}\,dt, \quad \text{and} \quad u_2 = \int_a^x \frac{y_1(t)}{W(t)}\frac{f(t)}{p(t)}\,dt.$$

Here $W(t)$ is the Wronskian of $y_1$ and $y_2$. Since we are free to pick the antiderivative for $u_1$ and $u_2$, it turns out to be convenient to choose

$$u_1 = \int_x^b \frac{y_2(t)}{W(t)}\frac{f(t)}{p(t)}\,dt, \tag{3.6.22}$$

and

$$u_2 = \int_a^x \frac{y_1(t)}{W(t)}\frac{f(t)}{p(t)}\,dt. \tag{3.6.23}$$

Thus we obtain the following solution to $L[y] = f$.

$$\begin{aligned} y(x) &= y_1(x)\int_x^b \frac{y_2(t)}{W(t)}\frac{f(t)}{p(t)}\,dt + y_2(x)\int_a^x \frac{y_1(t)}{W(t)}\frac{f(t)}{p(t)}\,dt. \\ &= \int_a^b G(x,t)f(t)\,dt, \end{aligned} \tag{3.6.24}$$

where

$$G(x,t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{W(t)p(t)}, & a \le t \le x \\ \dfrac{y_1(x)y_2(t)}{W(t)p(t)}, & x \le t \le b. \end{cases} \tag{3.6.25}$$

Using the fact that $y_1$ and $y_2$ satisfy the equation $L[y] = 0$, it follows from Lagrange's identity that $W(x)p(x) = C$, where $C$ is a constant. Note that $C \neq 0$ since $p(x) > 0$ and $W(x) \neq 0$ for all $x \in [a, b]$. Thus $G(x,t)$ has the simpler form

$$G(x,t) = \begin{cases} y_1(t)y_2(x)/C, & a \le t \le x \\ y_1(x)y_2(t)/C, & x \le t \le b. \end{cases} \tag{3.6.26}$$

The function $G(x,t)$ is called the *Green's function* for the problem (3.6.18). Next we verify that $y(x) = \displaystyle\int_a^b G(x,t)f(t)\,dt$, is a solution to (3.6.18).

$$\begin{aligned} y'(x) &= \frac{d}{dx}\left(y_1(x)\int_x^b \frac{y_2(t)f(t)}{C}\,dt\right) + \frac{d}{dx}\left(y_2(x)\int_a^x \frac{y_1(t)f(t)}{C}\,dt\right) \\ &= y_1'(x)\int_x^b \frac{y_2(t)f(t)}{C}\,dt - \frac{y_1(x)y_2(x)f(x)}{C} + y_2'(x)\int_a^x \frac{y_1(t)f(t)}{C}\,dt + \frac{y_1(x)y_2(x)f(x)}{C} \\ &= y_1'(x)\int_x^b \frac{y_2(t)f(t)}{C}\,dt + y_2'(x)\int_a^x \frac{y_1(t)f(t)}{C}\,dt \end{aligned} \tag{3.6.27}$$

Thus

$$\begin{aligned} a_1 y(a) + a_2 y'(a) &= a_1 y_1(a)\int_a^b \frac{y_2(t)f(t)}{C}\,dt + a_2 y_1'(a)\int_a^b \frac{y_2(t)f(t)}{C}\,dt \\ &= (a_1 y_1(a) + a_2 y_1'(a))\int_a^b \frac{y_2(t)f(t)}{C}\,dt = 0, \end{aligned} \tag{3.6.28}$$

since $y_1$ satisfies (3.6.20). Similarly using (3.6.21), one can verify that

$$b_1y(b) + b_2y'(b) = (b_1y_2(b) + b_2y_2'(b)) \int_a^b \frac{y_1(t)f(t)}{C}\,dt = 0. \tag{3.6.29}$$

**Example 3.10** Find the Green's function $G(x,t)$ for the boundary value problem

$$y'' = f,\ y(0) = 0, y(\pi) = 0.$$

Use the Green's function to obtain the solution when $f(x) = -6x$.

**Solution**. A general solution to $y'' = 0$ is $y = Ax+B$. Thus $y_1 = x$ is a solution satisfying $y(0) = 0$, and $y = \pi - x$ a solution satisfying $y(\pi) = 0$. Furthermore $y_1$ and $y_2$ are linearly independent with $W(x) = -x - (\pi - x) = -\pi$. We now compute $C = p(x)W(x) = (1)(-\pi) = -\pi$. Therefore the Green's function is

$$G(x,t) = \begin{cases} y_1(t)y_2(x)/C, & 0 \le t \le x \\ y_1(x)y_2(t)/C, & x \le t \le \pi \end{cases} = \begin{cases} t(x-\pi)/\pi, & 0 \le t \le x \\ x(t-\pi)/\pi, & x \le t \le \pi. \end{cases} \tag{3.6.30}$$

Hence for $f(x) = -6x$, the solution is given by

$$\begin{aligned} y(x) &= \int_0^\pi G(x,t)f(t)\,dt = \int_0^x (t(x-\pi)/\pi)\,(-6t)\,dt + \int_x^\pi (x(t-\pi)/\pi)\,(-6t)\,dt \\ &= x(\pi^2 - x^2). \end{aligned}$$

Since the homogeneous problem $y'' = 0,\ y(0) = 0, y(\pi) = 0$ has only the trivial solution, the above solution to the nonhomogeneous problem $y'' = -6x,\ y(0) = 0, y(\pi) = 0$ is unique.

Theorem 3.9 can be strengthened to include other cases.

**Theorem 3.10 (Fredholm)** *The nonhomogeneous problem (3.6.18) has a solution if and only if for every solution $y(x)$ of the homogeneous problem (3.6.19),*

$$\int_a^b f(t)y(t)\,dt = 0.$$

**Proof**. To prove the necessity, suppose $z$ is a solution of (3.6.18) so that $L[z] = f$. Let $y$ be any solution of (3.6.19). Thus $L[y] = 0$. Since $L$ is self-adjoint, we have

$$\int_a^b f(t)y(t)\,dt = (f,y) = (L[z],y) = (z, L[y]) = (z,0) = 0.$$

To prove the sufficiency, let's consider 2 cases.

(Case 1) The problem (3.6.19) has only the trivial solution. By Theorem 3.9, the problem (3.6.18) has a unique solution.

(Case 2) The problem (3.6.19) has a nontrivial solution $y_1$. By hypothesis, $\int_a^b y_1(t)f(t)\,dt = 0$. Also $y_1$ satisfies both boundary conditions (3.6.20) and (3.6.21). Let $y_2$ be any solution of $L(y) = 0$ such that $y_1$ and $y_2$ are linearly independent. Following the proof of Theorem 3.9, we can verify that $y(x) = \displaystyle\int_a^b G(x,t)f(t)\,dt$ is a solution to (3.6.18). For the boundary conditions, we have

$$a_1y(a) + a_2y'(a) \quad = \quad (a_1y_1(a) + a_2y_1'(a)) \int_a^b \frac{y_2(t)f(t)}{C}\, dt = 0, \tag{3.6.31}$$

because $y_1$ satisfies (3.6.20), and

$$b_1y(b) + b_2y'(b) \quad = \quad (b_1y_2(b) + b_2y_2'(b)) \int_a^b \frac{y_1(t)f(t)}{C}\, dt = 0, \tag{3.6.32}$$

because $\int_a^b y_1(t)f(t)\, dt = 0$.

**Remark**. In case 2, if $z$ is another solution of (3.6.19), then both $y_1$ and $z$ satisfy the boundary conditions of (3.6.19). In particular at the point $a$, we have $a_1y_1(a) + a_2y_1'(a) = 0$ and $a_1z(a) + a_2z'(a) = 0$. Since $a_1 \neq 0$ and $a_2 \neq 0$, we have $W(y_1, z)(a) = 0$. This implies that $y_1$ and $z$ are linearly dependent. Thus $z$ is just a multiple of $y_1$, and the problem (3.6.19) has a one-parameter family of solutions.

**Example 3.11** Show that the boundary value problem

$$y'' + y' + \frac{5}{2}y = f; \;\; y(0) = 0, y(\frac{2\pi}{3}) = 0$$

has a solution if and only if

$$\int_0^{\frac{2\pi}{3}} e^{\frac{x}{2}} \sin(\frac{3x}{2}) f(x)\, dx = 0.$$

**Solution**. Consider the homogeneous boundary value problem $y''+y'+\frac{5}{2}y = 0;\;\; y(0) = 0, y(\frac{2\pi}{3}) = 0$. The general solution of the equation $y'' + y' + \frac{5}{2}y = 0$ is $y = e^{-\frac{1}{2}x}(A\cos(\frac{3x}{2}) + B\sin(\frac{3x}{2}))$. As $y(0) = 0$, we deduce that $A = 0$. Thus $y = Be^{-\frac{1}{2}x}\sin(\frac{3x}{2})$. This also satisfies $y(\frac{2\pi}{3}) = 0$. Thus the homogeneous boundary value problem has a one parameter family of solutions. Note that the given equation is not self-adjoint. However it can be converted into a self-adjoint equation by multiplying throughout by a factor $e^x$. Thus we may write the problem as

$$(e^xy')' + \frac{5}{2}e^xy = e^xf(x); \;\; y(0) = 0, y(\frac{2\pi}{3}) = 0.$$

By Theorem 3.10, this problem has a solution if and only if

$$\int_0^{\frac{2\pi}{3}} (Be^{-\frac{1}{2}x}\sin(\frac{3x}{2}))(e^xf(x))\, dx = 0.$$

This is equivalent to the condition

$$\int_0^{\frac{2\pi}{3}} e^{\frac{x}{2}} \sin(\frac{3x}{2}) f(x)\, dx = 0.$$

**Exercise 3.9** Using Green's function, solve the boundary value problem

$$(xy')' - (4/x)y = -x; \;\; y(1) = 0, y(2) = 0.$$

**Ans**: $y = \frac{1}{60}(16x^2\ln 2 - 16x^{-2}\ln 2 - 15x^2\ln x)$.

**Exercise 3.10** Show that any equation $p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$ can be made self-adjoint by multiplying throughout by $\frac{1}{p_0}e^{\int p_1/p_0\,dx}$.

**Exercise 3.11** Consider the regular Sturm-Liouville boundary value problem

$$y'' - xy + \lambda y = 0, \quad y(0) = 0,\ y(\pi) = 0.$$

Let $\lambda$ be an eigenvalue and $\phi$ a corresponding eigenfunction. Show that $\lambda > 0$.

## 3.7 Oscillations and the Sturm Separation Theorem

Consider the homogeneous second order linear differential equation.

$$y'' + P(x)y' + Q(x)y = 0 \tag{3.7.33}$$

It is rarely possible to solve this equation in general. However by studying the properties of the coefficient functions, it is sometimes possible to describe the behavior of the solutions. One of the essential characteristics that is of primary interest to mathematicians is the number of zeros of a solution to (3.7.33). If a function has an infinite number of zeros in an interval $[a, \infty)$, we say that the function is *oscillatory*. Therefore, studying the oscillatory behavior of a function means investigating the number and locations of its zeros.

The familiar equation $y'' + y = 0$ has two linearly independent solutions $s(x) = \sin(x)$ satisfying $y(0) = 0, y'(0) = 1$, and $c(x) = \cos(x)$ satisfying $y(0) = 1, y'(0) = 0$ respectively. The positive zeros of $s(x)$ and $c(x)$ are $\pi, 2\pi, 3\pi, \ldots$, and $\pi/2, 3\pi/2, 5\pi/2, \ldots$, respectively. Note that the zeros of $s(x)$ and $c(x)$ interlace each other in the sense that between two successive zeros of $s(x)$, there is a zero of $c(x)$, and vice versa. This property is described by the Sturm separation theorem.

**Theorem 3.11 (Sturm Separation Theorem)** *If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of*

$$y'' + P(x)y' + Q(x)y = 0,$$

*then the zeros of these functions are distinct and occur alternatively in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$, and conversely.*

**Proof**. Let $x_1$ and $x_2$ be two consecutive zeros of $y_1$ with $x_1 < x_2$. Since $y_1$ and $y_2$ are linearly independent, their Wronskian $W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ never vanish. It follows that $y_2(x_1) \neq 0$ and $y_2(x_2) \neq 0$, otherwise their Wronskian is zero at these points.

Assume that the conclusion is false, that is $y_2(x) \neq 0$ for all $x \in [x_1, x_2]$. Then the function $f(x) = y_1(x)/y_2(x)$ is well-defined and continuous on $[x_1, x_2]$, and continuously differentiable on $(x_1, x_2)$. Since $y_1(x_1) = y_1(x_2) = 0$, we have $f(x_1) = f(x_2) = 0$. By Rolle's theorem, there exists $z \in (x_1, x_2)$ such that $f'(z) = 0$. Computing $f'(z)$, we find that

$$0 = f'(z) = \frac{y_1'(z)y_2(z) - y_1(z)y_2'(z)}{[y_2(z)]^2} = -\frac{W(y_1, y_2)(z)}{[y_2(z)]^2}.$$

But this means $W(y_1, y_2)(z) = 0$, which contradicts the fact that $y_1$ and $y_2$ are linearly independent. Therefore $y_2$ must have a zero in $(x_1, x_2)$.

By interchanging the roles of $y_1$ and $y_2$, we see that between any two consecutive zeros of $y_2$, there is a zero of $y_1$. Consequently, there cannot be more than one zero of $y_2$ between two consecutive zeros of $y_1$. □

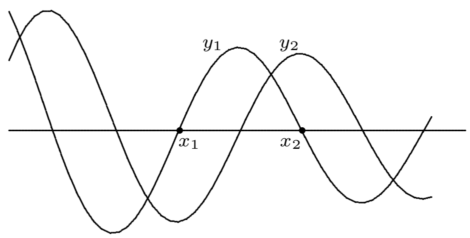



**Figure 3.1**: Interlacing of zeros of two linearly independent solutions

**Corollary 3.12** *Suppose one nontrivial solution to $y'' + P(x)y' + Q(x)y = 0$ is oscillatory on $[a, \infty)$. Then all solutions are oscillatory.*

**Proof**. Let $y_1$ be a nontrivial solution that has an infinite number of zeros on $[a, \infty)$. By the Sturm separation theorem, if $y_2$ is any other solution such that $y_1$ and $y_2$ are linearly independent, then $y_2$ must have a zero between each consecutive pair of zeros of $y_1$, and hence it must also have an infinite number of zeros on $[a, \infty)$. If $y_1$ and $y_2$ are linearly dependent, then $y_2 = \lambda y_1$ since $y_1$ is nontrivial. Thus $y_2$ also has an infinite number of zeros on $[a, \infty)$. □

**Theorem 3.13** *Let $y$ be a nontrivial solution of $y'' + P(x)y' + Q(x)y = 0$ on a closed interval $[a, b]$. Then $y$ has at most a finite number of zeros in this interval.*

**Proof**. We assume the contrary that $y$ has an infinite number of zeros in $[a, b]$. By the Bolzano-Weierstrass theorem, there exist in $[a, b]$ a point $x_0$, and a sequence of zeros $x_n$ not equal to $x_0$ such that $\lim_{n\to\infty} x_n = x_0$. Since $y$ is continuous and differentiable at $x_0$, we have

$$y(x_0) = \lim_{n\to\infty} y(x_n) = 0$$

and

$$y'(x_0) = \lim_{n\to\infty} \frac{y(x_n) - y(x_0)}{x_n - x_0} = 0.$$

By the existence and uniqueness theorem (Theorem 2.1), $y$ must be the trivial solution which is a contradiction. Therefore, $y$ has at most a finite number of zeros in $[a, b]$. □

## 3.8 Sturm Comparison Theorem

The equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{3.8.34}$$

can be written in the form

$$u'' + q(x)u = 0 \tag{3.8.35}$$

by putting $y = uv$, where $v = e^{-\frac{1}{2}\int P\,dx}$ and $q(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$. It is customary to refer to (3.8.34) as the *standard form*, and to (3.8.35) as the *normal form*, of a homogeneous second order linear equation. Since $v(x) > 0$ for all $x$, the above transformation of (3.8.34) into (3.8.35) has no effect on the zeros of the solutions, and therefore leaves unaltered the oscillation behavior of the solutions. Next we shall restrict our discussion on (3.8.35).

**Exercise 3.12** Using the substitution $y = e^{-\frac{x^3}{3}}u$ , find the general solution of the equation

$$y'' + 2x^2y' + (x^4 + 2x + 1)y = 0.$$

Show that the distance between two consecutive zeros of any nontrivial solution is $\pi$.

Ans: $y = Ae^{-\frac{x^3}{3}}\sin(x - \theta)$.

The Sturm Separation Theorem compares the zeros of two solutions to the same equation. For solutions of two different equations, it may still be possible to relate their zeros. For example, consider the equations

$$y'' + m^2y = 0 \quad \text{and} \quad y'' + n^2y = 0.$$

The first has a general solution of the form $y_1(x) = A_1\sin(m(x - \theta_1))$; and the second, $y_2(x) = A_2\sin(n(x - \theta_2))$. The distance between consecutive zeros of $y_1$ is $\pi/m$, and of $y_2$ is $\pi/n$. Therefore, for $n > m$, the distance between two zeros of $y_1$ is greater than the distance between two zeros of $y_2$. Thus for $n^2 > m^2$ (equivalent to $n > m$), between two consecutive zeros of $y_1$, there is a zero of $y_2$. A similar result holds when the constants $m^2$ and $n^2$ are replaced by functions of $x$.

**Theorem 3.14 (Sturm comparison theorem)** *Let $y_1$ be a nontrivial solution to*

$$y'' + q_1(x)y = 0, \quad a < x < b, \tag{3.8.36}$$

*and let $y_2$ be a nontrivial solution to*

$$y'' + q_2(x)y = 0, \quad a < x < b. \tag{3.8.37}$$

*Assume $q_2(x) \ge q_1(x)$ for all $x \in (a,b)$. If $x_1$ and $x_2$ are two consecutive zeros of $y_1$ on $(a,b)$ with $x_1 < x_2$, then there exists a zero of $y_2$ in $(x_1,x_2)$, unless $q_2(x) = q_1(x)$ on $[x_1,x_2]$ in which case $y_1$ and $y_2$ are linearly dependent on $[x_1,x_2]$.*

**Proof**. Suppose $y_2(x) \neq 0$ in $(x_1,x_2)$. We wish to show that $q_1(x) = q_2(x)$ and $y_1, y_2$ are linearly dependent on $[x_1,x_2]$. Without loss of generality, we may assume $y_1(x) > 0$ and $y_2(x) > 0$ in $(x_1,x_2)$. First we have

$$\begin{aligned}\frac{d}{dx}\left(W(y_2,y_1)\right) &= \frac{d}{dx}(y_2y_1' - y_2'y_1) = y_2'y_1' + y_2y_1'' - y_2''y_1 - y_2'y_1' \\ &= y_2y_1'' - y_2''y_1 = y_2(-q_1y_1) - (-q_2y_2)y_1 \\ &= y_1y_2(q_2 - q_1) \geq 0,\end{aligned}$$

for all $x$ in $(x_1, x_2)$. Hence, $W(y_2, y_1)$ is non-decreasing on $(x_1, x_2)$. However, since $y_1(x_1) = y_1(x_2) = 0$ and $y_1$ is positive on $(x_1, x_2)$, we must have $y_1'(x_1) \geq 0$ and $y_1'(x_2) \leq 0$. Therefore, $W(y_2, y_1)(x_1) = y_2(x_1)y_1'(x_1) \geq 0$ and $W(y_2, y_1)(x_2) = y_2(x_2)y_1'(x_2) \leq 0$.

Since $W(y_2, y_1)(x)$ is non-decreasing, the only way for it to be nonnegative at $x_1$ and nonpositive at $x_2$ is $W(y_2, y_1)(x) = 0$ for all $x$ in $[x_1, x_2]$. This also implies $\frac{d}{dx}W(y_2, y_1)(x) = 0$ in $[x_1, x_2]$. That means $q_1(x) = q_2(x)$ in $[x_1, x_2]$. Then $y_1$ and $y_2$ satisfy the same equation on $[x_1, x_2]$, and their Wronskian vanishes on this interval. Hence, $y_1$ and $y_2$ are linearly dependent. □

**Remark**. The Sturm comparison theorem 3.14 asserts that either $y_2$ has a zero between $x_1$ and $x_2$ or $y_2(x_1) = y_2(x_2) = 0$ (since $y_1$ and $y_2$ are linearly dependent in the later case).

**Corollary 3.15** *Suppose $q(x) \leq 0$ for all $x \in [a, b]$. If $y$ is a nontrivial solution of $y'' + q(x)y = 0$ on $[a, b]$, then $y$ has at most one zero on $[a, b]$.*

**Proof**. Since $u(x) \equiv 1$ is a solution to the equation $y'' + 0 \cdot y = 0$ and $q(x) \leq 0$, it follows from 3.14 that if $y$ has two or more zeros in $[a, b]$, then $u$ must have a zero between them. Since $u$ is never zero, $y$ can have at most 1 zero in $[a, b]$. □

**Example 3.12** The equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \;\; x > 0 \tag{3.8.38}$$

is called Bessel's equation of order $p$. For $a > 0$, let's discuss the number of zeros in the interval $[a, a + \pi)$. The substitution $y = ux^{-\frac{1}{2}}$ transforms equation (3.8.38) into the following form.

$$\frac{d^2u}{dx^2} + \left(1 - \frac{p^2 - \frac{1}{4}}{x^2}\right)u = 0, \quad x > 0. \tag{3.8.39}$$

Since $y = ux^{-\frac{1}{2}}$, the distribution of zeros of a solution $y$ to (3.8.39) is the same as the corresponding solution $u$ to (3.8.39). Let's compare the solutions of (3.8.39) with those of $u'' + u = 0$. Observe that $u(x) = A\sin(x - a)$ is a solution of $u'' + u = 0$ and has zeros at $a$ and $a + \pi$.

**Case 1**. ($p > 1/2$). In this case, $4p^2 - 1 > 0$ so that $1 - \frac{p^2 - \frac{1}{4}}{x^2} < 1$ for $x$ in $[a, a + \pi)$. By the Sturm comparison theorem 3.14, a solution to (3.8.39) cannot have more than one zero in $[a, a + \pi)$ because $u(x) = A\sin(x - a)$ does not have a zero in $(a, a + \pi)$.

**Case 2**. ($0 \leq p < 1/2$). In this case, $4p^2 - 1 < 0$ so that $1 - \frac{p^2 - \frac{1}{4}}{x^2} > 1$ for $x$ in $[a, a + \pi)$. By the Sturm comparison theorem 3.14, a solution to (3.8.39) must have a zero in $(a, a + \pi)$, since $a$ and $a + \pi$ are consecutive zeros of $u(x) = A\sin(x - a)$.

**Case 3**. ($p = 1/2$). In this case, (3.8.39) reduces to $u'' + u = 0$, which has the general solution $u(x) = A\sin(x - a)$. Consequently, it has exactly one zero in $[a, a + \pi)$.

For further discussion of Bessel's equation, see section 5.5.

**Exercise 3.13** Show that any nontrivial solution to $y'' + (x^2 - 1)y = 0$ has at most one zero in $[-1, 1]$ but has infinitely many zeros in $(-\infty, -1)$ and in $(1, \infty)$.

**Exercise 3.14** Let $y(x)$ be a nontrivial solution of $y'' + q(x)y = 0$, $x > 0$. Suppose $q(x) > 1/x^2$ for $x > 0$. Show that $y(x)$ has an infinite number of positive zeros.

**Exercise 3.15** Let $q(x)$ be a continuous increasing function on $[0, \infty)$, and $\phi(x)$ a nontrivial solution to

$$y'' + q(x)y = 0, \quad 0 < x < \infty.$$

Prove that if $x_{t-1}, x_t$ and $x_{t+1}$ are three consecutive positive zeros of $\phi$ with $x_{t-1} < x_t < x_{t+1}$, then $x_{t+1} - x_t < x_t - x_{t-1}$.

**Exercise 3.16** Show that if $w(x)$ and $q(x)$ are continuous on $[a, b]$ and $q(x) < 0$ on $[a, b]$, then a nontrivial solution $\phi$ of the equation

$$y'' + w(x)y' + q(x)y = 0$$

can have at most one zero in $[a, b]$.

**Exercise 3.17** Show that the solution to the boundary value problem

$$y'' + y' + \frac{5}{2}y = f; \;\; y(0) = 0, \; y(\frac{2\pi}{3}) = 0,$$

where $f(x) = e^{-\frac{1}{2}x}\sin 3x$, is $y = Be^{-\frac{1}{2}x}\sin(\frac{3x}{2}) - \frac{4}{27}e^{-\frac{1}{2}x}\sin 3x$.

# Chapter 4

# Linear Differential Systems

## 4.1 Linear Systems

The following system is called a linear differential system of first order in *normal form*:

$$\begin{cases} x_1' = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + g_1(t), \\ \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + g_n(t), \end{cases}$$

where $a_{ij}(t)$ and $g_j(t)$ are continuous functions of $t$ and $'$ denotes differentiation with respect to $t$. Denote

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \cdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \cdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \cdots & \cdots & \cdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}.$$

We call $\mathbf{g}(t)$ a continuous vector field, or a continuous vector-valued function, if all its components are continuous functions. We call $\mathbf{A}(t)$ a continuous matrix, or a continuous matrix-valued function, if all its entries are continuous functions. Define

$$\mathbf{x}' = \begin{pmatrix} x_1'(t) \\ \cdots \\ x_n'(t) \end{pmatrix}, \quad \int \mathbf{x}(t)dt = \begin{pmatrix} \int x_1(t)dt \\ \cdots \\ \int x_n(t)dt \end{pmatrix}.$$

Then the linear system can be written in the *matrix form*:

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t). \tag{4.1.1}$$

When $\mathbf{g} \equiv 0$, (4.1.1) is reduced to

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}. \tag{4.1.2}$$

(4.1.2) is called a homogeneous differential system, and (4.1.1) is called a non-homogeneous system if $\mathbf{g}(t) \not\equiv \mathbf{0}$. We shall also call (4.1.2) the *homogeneous system associated with* (4.1.1), or the *associated homogeneous system*.

**Example 4.1**

$$\begin{aligned} x_1' &= 2x_1 + 3x_2 + 3t, \\ x_2' &= -x_1 + x_2 - 7\sin t \end{aligned}$$

is equivalent to

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3t \\ -7\sin t \end{pmatrix}.$$

**Example 4.2** Given a second order system

$$\begin{cases} \dfrac{d^2x}{dt^2} = x + 2y + 3t, \\ \dfrac{d^2y}{dt^2} = 4x + 5y + 6t, \end{cases}$$

it can be expressed into an equivalent first order differential system by introducing more variables. For this example, let $u = x'$ and $v = y'$. Then we have

$$\begin{aligned} x' &= u \\ u' &= x + 2y + 3t \\ y' &= v \\ v' &= 4x + 5y + 6t \end{aligned} .$$

Next, we begin with the initial value problem

$$\begin{cases} \mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t), \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases} \tag{4.1.3}$$

where $\mathbf{x}_0$ is a constant vector. Similar to Theorem 2.1 we can show the following theorem.

**Theorem 4.1** *Assume that $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $a < t < b$ containing $t_0$. Then, for any constant vector $\mathbf{x}_0$, (4.1.3) has a solution $\mathbf{x}(t)$ defined on this interval. This solution is unique.*
*Especially, if $\mathbf{A}(t)$ and $\mathbf{g}(t)$ are continuous on $\mathbb{R}$, then for any $t_0 \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, (4.1.3) has a unique solution $\mathbf{x}(t)$ defined on $\mathbb{R}$.*

## 4.2 Homogeneous Linear Systems

In this section we assume $\mathbf{A} = \Big(a_{ij}(t)\Big)$ is a continuous $n$ by $n$ matrix-valued function defined on the interval $(a, b)$. We shall discuss the structure of the set of all solutions of (4.1.2).

**Lemma 4.2** *Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two solutions of (4.1.2) on $(a, b)$. Then for any numbers $c_1$, $c_2$, $\mathbf{z}(t) = c_1\mathbf{x}(t) + c_2\mathbf{y}(t)$ is also a solution of (4.1.2) on $(a, b)$.*

**Definition 4.1** $\mathbf{x}_1(t), \cdots, \mathbf{x}_r(t)$ are *linearly dependent* in $(a, b)$, if there exist numbers $c_1, \cdots, c_r$, not all zero, such that

$$c_1\mathbf{x}_1(t) + \cdots + c_r\mathbf{x}_r(t) = \mathbf{0} \qquad \text{for all} \quad t \in (a, b).$$

$\mathbf{x}_1(t), \cdots, \mathbf{x}_r(t)$ are *linearly independent* on $(a, b)$ if they are not linearly dependent.

**Lemma 4.3** *A set of solutions $\mathbf{x}_1(t), \cdots, \mathbf{x}_r(t)$ of (4.1.2) are linearly dependent on $(a, b)$ if and only if $\mathbf{x}_1(t_0), \cdots, \mathbf{x}_r(t_0)$ are linearly dependent vectors for any fixed $t_0 \in (a, b)$.*

**Proof.** Obviously "$\Longrightarrow$" is true. We show "$\Longleftarrow$". Suppose that, for some $t_0 \in (a, b)$, $\mathbf{x}_1(t_0), \cdots$, $\mathbf{x}_r(t_0)$ are linearly dependent. Then there exist constants $c_1, \cdots, c_r$, not all zero, such that

$$c_1\mathbf{x}_1(t_0) + \cdots + c_r\mathbf{x}_r(t_0) = \mathbf{0}.$$

Let $\mathbf{y}(t) = c_1\mathbf{x}_1(t) + \cdots + c_r\mathbf{x}_r(t)$. Then $\mathbf{y}(t)$ is the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{0}.$$

Since $\mathbf{x}(t) = \mathbf{0}$ is also a solution of the initial value problem, by the uniqueness we have $\mathbf{y}(t) \equiv \mathbf{0}$ on $(a, b)$, i.e.

$$c_1\mathbf{x}_1(t) + \cdots + c_r\mathbf{x}_r(t) \equiv \mathbf{0}$$

on $(a, b)$. Since $c_j$'s are not all zero, $\mathbf{x}_1(t), \cdots, \mathbf{x}_r(t)$ are linearly dependent on $(a, b)$. □

**Theorem 4.4** *(i) The differential system in (4.1.2) has $n$ linearly independent solutions.*
*(ii) Let $\mathbf{x}_1, \cdots, \mathbf{x}_n$ be any set of $n$ linearly independent solutions of (4.1.2) on $(a, b)$. Then the general solution of (4.1.2) is given by*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t), \tag{4.2.1}$$

*where $c_j$'s are arbitrary constants.*

**Proof.** (i) Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be a set of linearly independent vectors in $\mathbb{R}^n$. Fix $t_0 \in (a, b)$. For each $j$ from 1 to $n$, consider the initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{e}_j.$$

From Theorem 4.1, there exists a unique solution $\mathbf{x}_j(t)$ defined on $(a, b)$. From Lemma 4.3, $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ are linearly independent on $(a, b)$.

(ii) Now let $\mathbf{x}_1(t), \cdots, \mathbf{x}_n(t)$ be any set of $n$ linearly independent solutions of (4.1.2) on $(a, b)$. Fix $t_0 \in (a, b)$. From Lemma 4.3, $\mathbf{x}_1(t_0), \cdots, \mathbf{x}_n(t_0)$ are linearly independent vectors. Let $\tilde{\mathbf{x}}(t)$ be any solution of (4.1.2). Then $\tilde{\mathbf{x}}(t_0)$ can be represented by a linear combination of $\mathbf{x}_1(t_0), \cdots$, $\mathbf{x}_n(t_0)$, namely, there exist $n$ constants $\tilde{c}_1, \cdots, \tilde{c}_n$ such that

$$\tilde{\mathbf{x}}(t_0) = \tilde{c}_1\mathbf{x}_1(t_0) + \cdots + \tilde{c}_n\mathbf{x}_n(t_0).$$

As in the proof of Lemma 4.3, we can show that

$$\tilde{\mathbf{x}}(t) = \tilde{c}_1\mathbf{x}_1(t) + \cdots + \tilde{c}_n\mathbf{x}_n(t).$$

Thus $c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ is the general solution of (4.1.2). □

Recall that, $n$ vectors

$$\mathbf{a}_1 = \begin{pmatrix} a_{11} \\ \cdots \\ a_{n1} \end{pmatrix}, \quad \cdots\cdots, \quad \mathbf{a}_n = \begin{pmatrix} a_{1n} \\ \cdots \\ a_{nn} \end{pmatrix}$$

are linearly dependent if and only if the determinant

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = 0.$$

In order to check whether $n$ solutions are linearly independent, we need the following notation.

**Definition 4.2** The Wronskian of $n$ vector-valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ \cdots \\ x_{n1}(t) \end{pmatrix}, \quad \cdots, \quad \mathbf{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \cdots \\ x_{nn}(t) \end{pmatrix}$$

is the determinant

$$W(t) \equiv W(\mathbf{x}_1, \cdots, \mathbf{x}_n)(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix}.$$

Using Lemma 4.3 we can show that

**Theorem 4.5** *(i) The Wronskian of $n$ solutions of (4.1.2) is either identically zero or nowhere zero in $(a, b)$.*
*(ii) $n$ solutions of (4.1.2) are linearly dependent in $(a, b)$ if and only if their Wronskian is identically zero in $(a, b)$.*

**Definition 4.3** A set of $n$ linearly independent solutions of (4.1.2) is called a *fundamental set of solutions*, or a *basis of solutions*. Let

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ \cdots \\ x_{n1}(t) \end{pmatrix}, \quad \cdots, \quad \mathbf{x}_n(t) = \begin{pmatrix} x_{1n}(t) \\ \cdots \\ x_{nn}(t) \end{pmatrix}$$

be a fundamental set of solutions of (4.1.2) on $(a, b)$. The matrix-valued function

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

is called a *fundamental matrix* of (4.1.2) on $(a, b)$.

**Remark.** (i) From Theorem 4.5, a fundamental matrix is non-singular for all $t \in (a, b)$.
(ii) A fundamental matrix $\mathbf{\Phi}(t)$ satisfies the following *matrix equation*:

$$\mathbf{\Phi}' = \mathbf{A}(t)\mathbf{\Phi}. \tag{4.2.2}$$

(iii) Let $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ are two fundamental matrices defined on $(a, b)$. Then there exists a constant, non-singular matrix $\mathbf{C}$ such that

$$\Psi(t) = \Phi(t)\mathbf{C}.$$

**Theorem 4.6** *Let $\mathbf{\Phi}(t)$ be a fundamental matrix of (4.1.2) on $(a, b)$. Then the general solution of (4.1.2) is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}, \tag{4.2.3}$$

*where $\mathbf{c} = \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix}$ is an arbitrary constant vector.*

## 4.3 Non-Homogeneous Linear Systems

In this section we consider the solutions of the non-homogeneous system (4.1.1), where $\mathbf{A} = \Big(a_{ij}(t)\Big)$ is a continuous $n$ by $n$ matrix-valued function and $\mathbf{g}(t)$ is a continuous vector-valued function, both defined on the interval $(a, b)$.

**Theorem 4.7** *Let $\mathbf{x}_p(t)$ be a particular solution of (4.1.1), and $\mathbf{\Phi}(t)$ be a fundamental matrix of the associated homogeneous system (4.1.2). Then the general solution of (4.1.1) is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{x}_p(t), \tag{4.3.1}$$

*where $\mathbf{c}$ is an arbitrary constant vector.*

**Proof.** For any constant vector $\mathbf{c}$, $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{x}_p(t)$ is a solution of (4.1.1). On the other hand, let $\mathbf{x}(t)$ be a solution of (4.1.1) and set $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}_p(t)$. Then $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$. From (4.2.3), there exists a constant vector $\tilde{\mathbf{c}}$ such that $\mathbf{y}(t) = \mathbf{\Phi}(t)\tilde{\mathbf{c}}$. So $\mathbf{x}(t) = \mathbf{\Phi}(t)\tilde{\mathbf{c}} + \mathbf{x}_p(t)$. Thus (4.3.1) gives a general solution of (4.1.1). □

**Method of variation of parameters.**

Let $\mathbf{\Phi}$ be a fundamental matrix of (4.1.2). We look for a particular solution of (4.1.1) in the form

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{u}(t), \qquad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ \cdots \\ u_n(t) \end{pmatrix}.$$

Plugging into (4.1.1) we get

$$\mathbf{\Phi}'\mathbf{u} + \mathbf{\Phi}\mathbf{u}' = \mathbf{A}\mathbf{\Phi}\mathbf{u} + \mathbf{g}.$$

From (4.2.2), $\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}$. So $\mathbf{\Phi}\mathbf{u}' = \mathbf{g}$, and thus $\mathbf{u}' = \mathbf{\Phi}^{-1}\mathbf{g}$.

$$\mathbf{u}(t) = \int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds. \tag{4.3.2}$$

So we obtain the following:

**Theorem 4.8** *The general solution of (4.1.1) is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}^{-1}(s)\mathbf{g}(s)ds, \tag{4.3.3}$$

*where $\Phi(t)$ is a fundamental matrix of the associated homogeneous system (4.1.2).*

**Exercise 4.1** Solve $\begin{cases} x_1' &= 3x_1 - x_2 + t \\ x_2' &= 2x_1 + t \end{cases}.$

Ans: $x_1 = c_1e^t + c_2e^{2t} - \frac{t}{2} - \frac{1}{4},\ x_2 = 2c_1e^t + c_2e^{2t} - \frac{t}{2} - \frac{1}{4}$.

## 4.4 Homogeneous Linear Systems with Constant Coefficients

Consider a homogeneous linear system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{4.4.1}$$

where $\mathbf{A} = (a_{ij})$ is a constant $n$ by $n$ matrix.

Let us try to find a solution of (4.4.1) in the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{k}$, where $\mathbf{k}$ is a constant vector, $\mathbf{k} \neq \mathbf{0}$. Plugging it into (4.4.1) we find

$$\mathbf{A}\mathbf{k} = \lambda\mathbf{k}. \tag{4.4.2}$$

**Definition 4.4** Assume that a number $\lambda$ and a vector $\mathbf{k} \neq \mathbf{0}$ satisfy (4.4.2), then we call $\lambda$ an *eigenvalue* of $\mathbf{A}$, and $\mathbf{k}$ an *eigenvector* associated with $\lambda$.

**Lemma 4.9** *$\lambda$ is an eigenvalue of $\mathbf{A}$ if and only if*

$$\det\,(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.4.3}$$

*(where $\mathbf{I}$ is the $n \times n$ unit matrix), namely,*

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

**Remark.** Let $\mathbf{A}$ be an $n$ by $n$ matrix and $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the distinct roots of (4.4.3). Then there exist positive integers $m_1, m_2, \cdots, m_k$, such that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

and

$$m_1 + m_2 + \cdots + m_k = n.$$

$m_j$ is called the *algebraic multiplicity* (or simply *multiplicity*) of the eigenvalue $\lambda_j$. The number of linearly independent eigenvectors of $\mathbf{A}$ associated with $\lambda_j$ is called the *geometric multiplicity* of the eigenvalue $\lambda_j$ and is denoted by $\mu(\lambda_j)$. We always have

$$\mu(\lambda_j) \le m_j.$$

If $\mu(\lambda_j) = m_j$ then we say that the eigenvalue $\lambda_j$ is *quasi-simple.* Especially if $m_j = 1$ we say that $\lambda_j$ is a *simple* eigenvalue. Note that in this case $\lambda_j$ is a simple root of (4.4.3).

**Theorem 4.10** *If $\lambda$ is an eigenvalue of $\mathbf{A}$ and $\mathbf{k}$ is an associated eigenvector, then*

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{k}$$

*is a solution of (4.4.1).*

*Let $\mathbf{A}$ be a real matrix. If $\lambda$ is a complex eigenvalue of $\mathbf{A}$, and $\mathbf{k}$ is an eigenvector associated with $\lambda$, then*

$$\mathbf{x}_1 = \Re(e^{\lambda t}\mathbf{k}), \quad \mathbf{x}_2 = \Im(e^{\lambda t}\mathbf{k})$$

*are two linearly independent real solutions of (4.4.1).*

In the following we always assume that $\mathbf{A}$ is a real matrix.

**Theorem 4.11** *If $\mathbf{A}$ has $n$ linearly independent eigenvectors $\mathbf{k}_1, \cdots, \mathbf{k}_n$ associated with eigenvalues $\lambda_1, \cdots, \lambda_n$ respectively, then*

$$\Phi(t) = (e^{\lambda_1 t}\mathbf{k}_1, \cdots, e^{\lambda_n t}\mathbf{k}_n)$$

*is a fundamental matrix of (4.4.1), and the general solution is given by*

$$\mathbf{x}(t) = \Phi(t)\mathbf{c} = c_1 e^{\lambda_1 t}\mathbf{k}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{k}_n, \tag{4.4.4}$$

*where* $\mathbf{c} = \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix}$ *is an arbitrary constant vector.*

**Proof.** We only need to show $\det \Phi(t) \neq 0$. Since $\mathbf{k}_1, \cdots, \mathbf{k}_n$ are linearly independent, so $\det \Phi(0) \neq 0$. From Theorem 4.5 we see that $\det \Phi(t) \neq 0$ for any $t$. Hence $\Phi(t)$ is a fundamental matrix. If one of the $\lambda$'s is complex, then replace $e^{\lambda t}\mathbf{k}$ and $e^{\overline{\lambda} t}\overline{\mathbf{k}}$ by $\Re e^{\lambda t}\mathbf{k}$ and $\Im e^{\lambda t}\mathbf{k}$. The resulting set of solutions is a linearly independent set over $\mathbb{R}$. □

**Remark.** Under the conditions of Theorem 4.11, the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $\mathbf{A}$ need not to be distinct. In fact we only assume that all the eigenvalues of $\mathbf{A}$ are quasi-simple.
If $\mathbf{A}$ has $n$ distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, and let $\mathbf{k}_1, \cdots, \mathbf{k}_n$ be the associated eigenvectors, then they are linearly independent. Hence the general solution is given by (4.4.4).

**Example 4.3** Solve the system $\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{x}$.

**Solution**. The matrix $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$ has eigenvalues $\lambda_1 = -2$, and $\lambda_2 = -4$.
For $\lambda_1 = -2$ we find an eigenvector $\mathbf{k}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
For $\lambda_2 = -4$ we find an eigenvector $\mathbf{k}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The general solution is given by

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

**Example 4.4** Solve the system

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{x} + e^{-2t}\begin{pmatrix} -6 \\ 2 \end{pmatrix}.$$

**Solution**. We first solve the associated homogeneous system

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{x}$$

and find two linearly independent solutions $\mathbf{x}_1(t) = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^{-4t} \\ -e^{-4t} \end{pmatrix}$. The fundamental matrix is

$$\mathbf{\Phi} = (\mathbf{x}_1(t), \mathbf{x}_2(t)) = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix}.$$

$$\mathbf{\Phi}^{-1} = \frac{1}{-2e^{-6t}} \begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix},$$

Let $\mathbf{g}(t) = e^{-2t} \begin{pmatrix} -6 \\ 2 \end{pmatrix}$. Then

$$\mathbf{\Phi}^{-1}(t)\mathbf{g}(t) = \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix},$$

$$\mathbf{u}(t) = \int_0^t \Phi^{-1}(s)\mathbf{g}(s)ds = \begin{pmatrix} -2t \\ -2e^{2t} + 2 \end{pmatrix},$$

$$\Phi(t)\mathbf{u}(t) = 2e^{-2t} \begin{pmatrix} -t-1 \\ -t+1 \end{pmatrix} + 2e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x} = c_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2te^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2e^{-2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**Example 4.5** Solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{x}$.

**Solution**. The matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ has eigenvalues $\pm 2i$.

For $\lambda = 2i$, we find an eigenvector $\mathbf{k} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$.

$$e^{2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix} = (\cos 2t + i \sin 2t) \begin{pmatrix} 1 \\ 2i \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + i \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

The general solution is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

**Example 4.6** Solve

$$\begin{cases} x' = -3x + 4y - 2z, \\ y' = x + z, \\ z' = 6x - 6y + 5z. \end{cases}$$

**Solution**. The matrix $\mathbf{A} = \begin{pmatrix} -3 & 4 & -2 \\ 1 & 0 & 1 \\ 6 & -6 & 5 \end{pmatrix}$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$.

For $\lambda_1 = 2$ we find an eigenvector $\mathbf{k}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

For $\lambda_2 = 1$ we find an eigenvector $\mathbf{k}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -1$ we find an eigenvector $\mathbf{k}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

The general solution is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

namely

$$\begin{aligned} x(t) &= c_2 e^{t} + c_3 e^{-t}, \\ y(t) &= c_1 e^{2t} + c_2 e^{t}, \\ z(t) &= 2c_1 e^{2t} - c_3 e^{-t}. \end{aligned}$$

**Example 4.7** Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}.$$

**Solution**. The matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 3 \pm i$.

For $\lambda_1 = 2$, we find an eigenvector $\mathbf{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 3 + i$, we find an eigenvector $\mathbf{k}_2 = \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}$.

We have

$$\begin{aligned} e^{(3+i)t}\mathbf{k}_2 &= e^{3t} \begin{pmatrix} \cos t + i \sin t \\ \cos t - \sin t + i(\cos t + \sin t) \\ 2\cos t + \sin t + i(2\sin t - \cos t) \end{pmatrix}, \\ \Re(e^{(3+i)t}\mathbf{k}_2) &= e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} \\ \Im(e^{(3+i)t}\mathbf{k}_2) &= e^{3t} \begin{pmatrix} \sin t \\ \cos t + \sin t \\ 2\sin t - \cos t \end{pmatrix}. \end{aligned}$$

The general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} \sin t \\ \cos t + \sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

**Example 4.8** Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

**Solution**. We have

$$\det(\mathbf{A} - \lambda \mathbf{I}) = -(\lambda - 3)^2(\lambda + 3).$$

$\mathbf{A}$ has eigenvalues $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = -3$ (We may say that, $\lambda = 3$ is an eigenvalue of algebraic multiplicity 2, and $\lambda = -3$ is a simple eigenvalue).
For $\lambda = 3$ we solve the equation $\mathbf{A}\mathbf{k} = 3\mathbf{k}$, namely

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \mathbf{0}.$$

The solution is $\mathbf{k} = \begin{pmatrix} v - u \\ u \\ v \end{pmatrix}$. So we find two eigenvectors $\mathbf{k}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{k}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -3$ we find an eigenvector $\mathbf{k}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

The general solution is given by

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Now we consider the solutions of (4.4.1) associated with a multiple eigenvalue $\lambda$, with geometric multiplicity $\mu(\lambda)$ less than the algebraic multiplicity. The following lemma is proved in appendix 1.

**Lemma 4.12** *Assume $\lambda$ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$. Then the following system*

$$(\mathbf{A} - \lambda \mathbf{I})^m \mathbf{v} = \mathbf{0} \tag{4.4.5}$$

*has exactly $m$ linearly independent solutions.*

By direct computations we can prove the following theorem.

**Theorem 4.13** *Assume that $\lambda$ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m > 1$. Let $\mathbf{v}_0 \neq \mathbf{0}$ be a solution of (4.4.5). Define*

$$\mathbf{v}_l = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{l-1}, \quad l = 1, 2, \cdots, m-1, \tag{4.4.6}$$

*and let*

$$\mathbf{x}(t) = e^{\lambda t}\left[\mathbf{v}_0 + t\mathbf{v}_1 + \frac{t^2}{2}\mathbf{v}_2 + \cdots + \frac{t^{m-1}}{(m-1)!}\mathbf{v}_{m-1}\right]. \tag{4.4.7}$$

*Then $\mathbf{x}(t)$ is a solution of (4.4.1).*
*Let $\mathbf{v}_0^{(1)}$, $\cdots$, $\mathbf{v}_0^{(m)}$ be $m$ linearly independent solutions of (4.4.5). They generate $m$ linearly independent solutions of (4.4.1) via (4.4.6) and (4.4.7).*

**Remark.** In (4.4.6), we always have

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{m-1} = \mathbf{0}.$$

If $\mathbf{v}_{m-1} \neq \mathbf{0}$ then $\mathbf{v}_{m-1}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue $\lambda$.

In practice, to find the solutions of (4.4.1) associated with an eigenvalue $\lambda$ of multiplicity $m$, we first solve (4.4.5) and find $m$ linearly independent solutions

$$\mathbf{v}_0^{(1)}, \quad \mathbf{v}_0^{(2)}, \quad \cdots, \quad \mathbf{v}_0^{(m)}.$$

For each of these vectors, say $\mathbf{v}_0^{(k)}$, we compute the iteration sequence

$$\mathbf{v}_l^{(k)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{l-1}^{(k)}, \quad l = 1, 2, \cdots$$

There is an integer $0 \leq j \leq m-1$ ($j$ depends on the choice of $\mathbf{v}_0^{(k)}$) such that

$$\mathbf{v}_j^{(k)} \neq \mathbf{0}, \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_j^{(k)} = \mathbf{0}.$$

Thus $\mathbf{v}_j$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue $\lambda$. Then the iteration stops and yields a solution

$$\mathbf{x}^{(k)}(t) = e^{\lambda t}\left[\mathbf{v}_0^{(k)} + t\mathbf{v}_1^{(k)} + \frac{t^2}{2}\mathbf{v}_2^{(k)} + \cdots + \frac{t^j}{j!}\mathbf{v}_j^{(k)}\right]. \tag{4.4.8}$$

**Example 4.9** Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -4 \end{pmatrix}.$$

**Solution**. From $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(\lambda+3)^2 = 0$, we find eigenvalues $\lambda_1 = -3$ with multiplicity 2, and $\lambda_2 = 0$ simple.
For the double eigenvalue $\lambda_1 = -3$, we solve

$$(\mathbf{A} + 3\mathbf{I})^2\mathbf{v} = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0},$$

and find two linearly independent solutions $\mathbf{v}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. Plugging $\mathbf{v}_0^{(1)}$, $\mathbf{v}_0^{(2)}$ into (4.4.6), (4.4.7) we get

$$\mathbf{v}_1^{(1)} = (\mathbf{A} + 3\mathbf{I})\mathbf{v}_0^{(1)} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix},$$

$$\mathbf{x}^{(1)} = e^{-3t}(\mathbf{v}_0^{(1)} + t\mathbf{v}_1^{(1)}) = e^{-3t}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}\right],$$

$$\mathbf{v}_1^{(2)} = (\mathbf{A} + 3\mathbf{I})\mathbf{v}_0^{(2)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

$$\mathbf{x}^{(2)} = e^{-3t}(\mathbf{v}_0^{(2)} + t\mathbf{v}_1^{(2)}) = e^{-3t}\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right].$$

For the simple eigenvalue $\lambda_2 = 0$ we find an eigenvector $\mathbf{k}_3 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$.

So the general solution is

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{k}_3 \\ &= c_1 e^{-3t}\left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}\right] + c_2 e^{-3t}\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] + c_3\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

**Example 4.10** Solve the system

$$\begin{cases} x' = 2x + y + 2z, \\ y' = -x + 4y + 2z, \\ z' = 3z. \end{cases}$$

**Solution**. Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

The eigenvalue of $\mathbf{A}$ is 3 with multiplicity 3. Solving the linear system:

$$(\mathbf{A} - 3\mathbf{I})^3\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0},$$

we obtain 3 obvious linearly independent solutions

$$\mathbf{v}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{v}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_0^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Plugging $\mathbf{v}_0^{(j)}$ into (4.4.6), (4.4.7) we get

$$\begin{aligned}
\mathbf{v}_1^{(1)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_0^{(1)} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \\
\mathbf{v}_2^{(1)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{x}^{(1)} &= e^{3t}(\mathbf{v}_0^{(1)} + t\mathbf{v}_1^{(1)}) = e^{3t}\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right];
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_1^{(2)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_0^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \\
\mathbf{v}_2^{(2)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{x}^{(2)} &= e^{3t}(\mathbf{v}_0^{(2)} + t\mathbf{v}_1^{(2)}) = e^{3t}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right];
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_1^{(3)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_0^{(3)} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \\
\mathbf{v}_2^{(3)} &= (\mathbf{A} - 3\mathbf{I})\mathbf{v}_1^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{x}^{(3)} &= e^{3t}(\mathbf{v}_0^{(3)} + t\mathbf{v}_1^{(3)}) = e^{3t}\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\right].
\end{aligned}$$

The general solution is

$$\begin{aligned}\mathbf{x}(t) &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)}\\ &= c_1e^{3t}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\-1\\0\end{pmatrix}\right] + c_2e^{3t}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\0\end{pmatrix}\right]\\ &+ c_3e^{3t}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} + t\begin{pmatrix}2\\2\\0\end{pmatrix}\right].\end{aligned}$$

**Remark**. It is possible to reduce the number of constant vectors in the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ by using a basis for the Jordan canonical form of $\mathbf{A}$. For details of the Jordan canonical form, see appendix 1. However the following algorithm usually works well if the size of $\mathbf{A}$ is small.

Consider an eigenvalue $\lambda$ of $A$ with algebraic multiplicity $m$.
Start with $r = m$. Let $\mathbf{v}_0$ be a vector such that $(\mathbf{A} - \lambda\mathbf{I})^r\mathbf{v}_0 = 0$ but $(\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v}_0 \neq 0$. $\mathbf{v}_0$ is called a generalized eigenvector of rank $r$ associated with the eigenvalue $\lambda$. If no such $\mathbf{v}_0$ exists, reduce $r$ by 1. Then

$$\mathbf{v}_0, \mathbf{v}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_0, \cdots, \mathbf{v}_{r-1} = (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{v}_0,$$

form a chain of linearly independent solutions of (4.4.5) with $\mathbf{v}_{r-1}$ being the base eigenvector associated with the eigenvalue $\lambda$. This gives $r$ independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda t}(\mathbf{v}_0 + t\mathbf{v}_1 + \cdots + \tfrac{t^{r-1}}{(r-1)!}\mathbf{v}_{r-1}).\\ &\vdots\\ \mathbf{x}_{r-1}(t) &= e^{\lambda t}(\mathbf{v}_{r-2} + t\mathbf{v}_{r-1}),\\ \mathbf{x}_r(t) &= e^{\lambda t}\mathbf{v}_{r-1}.\end{aligned}$$

Repeat this procedure by finding another $\mathbf{v}$ which is not in the span of the previous chains of vectors and the resulting base eigenvectors are linearly independent. Do this for each eigenvalue of $\mathbf{A}$.

Results of linear algebra shows that
(1) Any chain of generalized eigenvectors constitutes a linearly independent set of vectors.
(2) If two chains of generalized eigenvectors are based on linearly independent eigenvectors, then the union of these vectors is a linearly independent set of vectors (whether the two base eigenvectors are associated with different eigenvalues or with the same eigenvalue).

**Example 4.11** Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix}3&0&0&0\\0&3&0&0\\0&1&3&0\\0&0&1&3\end{pmatrix}$.

**Solution**. $\mathbf{A}$ has an eigenvalue $\lambda = 3$ of multiplicity 4. Direct calculation gives $(\mathbf{A} - 3\mathbf{I}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $(\mathbf{A} - 3\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $(\mathbf{A} - 3\mathbf{I})^3 = \mathbf{0}$, and $(\mathbf{A} - 3\mathbf{I})^4 = \mathbf{0}$.

It can be seen that $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent eigenvectors of $\mathbf{A}$.

Together with $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, they form a basis of $\{\mathbf{v} \mid (\mathbf{A} - 3\mathbf{I})^4\mathbf{v} = \mathbf{0}\} = \mathbb{R}^4$.

Note that $(\mathbf{A}-3\mathbf{I})\mathbf{v_2} = \mathbf{v_3}$, and $(\mathbf{A}-3\mathbf{I})\mathbf{v_3} = \mathbf{v_4}$. Hence $\{\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ forms a chain of generalized eigenvectors associated with the eigenvalue 3. $\{\mathbf{v_1}\}$ alone is another chain. Therefore the general solution is

$$\mathbf{x}(t) = e^{3t}\left(c_1\mathbf{v_1} + c_2(\mathbf{v_2} + t\mathbf{v_3} + \frac{t^2}{2}\mathbf{v_4}) + c_3(\mathbf{v_3} + t\mathbf{v_4}) + c_4\mathbf{v_4}\right).$$

That is

$$\mathbf{x}(t) = \begin{pmatrix} c_1e^{3t} \\ c_2e^{3t} \\ (c_2t + c_3)e^{3t} \\ (\frac{c_2t^2}{2} + c_3t + c_4)e^{3t} \end{pmatrix}.$$

**Exercise 4.2** Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 7 & 5 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 12 & 10 & -5 & 4 \\ -4 & -4 & 2 & -1 \end{pmatrix}.$$

Ans: $\mathbf{x}(t) = c_1e^{-t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + c_2e^{t}\begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix} + c_3e^{t}\begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + c_4e^{t}\begin{pmatrix} -t-1 \\ 1 \\ -2t \\ 0 \end{pmatrix}.$

## 4.5 Higher Order Linear Equations

Consider the $n$-th order linear equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = f(t), \tag{4.5.1}$$

where $y^{(k)} = \frac{d^k y}{dt^k}$. We assume that $a_j(t)$'s and $f(t)$ are continuous functions defined on the interval $(a, b)$. The associated homogeneous equation is

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0. \tag{4.5.2}$$

The general theory of solutions of (4.5.1) and (4.5.2) can be established by applying the results in the previous sections to the equivalent system.
Recall the initial value problem

$$\begin{cases} y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_1, \\ \cdots\cdots\cdots \\ y^{(n-1)}(t_0) = y_{n-1}. \end{cases} \tag{4.5.3}$$

If we let $x_1 = y$, $x_2 = y'$, $x_3 = y''$,..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)}$, then (4.5.3) is equivalent to the following linear system:

$$\begin{array}{rcl} x_1' &=& x_2 \\ x_2' &=& x_3 \\ x_3' &=& x_4 \\ \vdots & \vdots & \vdots \\ x_{n-1}' &=& x_n \\ x_n' &=& -a_n(t)x_1 - a_{n-1}(t)x_2 \cdots - a_1(t)x_n + f(t) \end{array}$$

That is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & \cdots & -a_2(t) & -a_1(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f(t) \end{pmatrix}.$$

The initial values can be expressed as

$$\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Thus the first component $x_1$ of each solution of this system corresponds to a solution of the original $n$th order linear ODE (4.5.1). In fact a solution of this system gives rise to a vector-valued function where the $i$th component is the $(i-1)$th derivative of a solution of (4.5.1). That is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

**Remarks.**

1. The existence and uniqueness theorem for first order differential systems implies the existence and uniqueness theorem for higher order linear ODEs.
2. The linear structure of the solution space of the homogeneous linear system translates to the linear structure of the solution space of the homogeneous higher order linear ODE.
3. If $\mathbf{x}_1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \\ \vdots \\ y_1^{(n-1)}(t) \end{pmatrix}, \ldots, \mathbf{x}_n(t) = \begin{pmatrix} y_n(t) \\ y_n'(t) \\ \vdots \\ y_n^{(n-1)}(t) \end{pmatrix}$ are $n$ linearly independent solutions of the equivalent linear system of (4.5.2), then the first components $y_1(t), \ldots, y_n(t)$ of these vector-valued functions form $n$ linearly independent solutions of (4.5.2).
4. In particular the Wronskian of $n$ solutions of (4.5.2) is the same as the Wronskian of the corresponding $n$ solutions of the equivalent linear system.

That is

$$W(y_1, \ldots, y_n) = W(\mathbf{x}_1, \ldots, \mathbf{x}_n).$$

**Variation of parameters**.

From (4.3.3) we can derive the variation of parameter formula for higher order equations. Consider a second order equation

$$y'' + p(t)y' + q(t)y = f(t). \tag{4.5.7}$$

Let $x_1 = y$, $x_2 = y'$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then (4.5.7) is written as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{4.5.8}$$

Assume $y_1(t)$ and $y_2(t)$ are two linearly independent solutions of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Eventually, we look for a solution of (4.5.7) in the form

$$y = u_1 y_1 + u_2 y_2.$$

Choose a fundamental matrix $\Phi(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$. By the method of variation of parameters (the matrix version), the corresponding solution of (4.5.8) is in the form $\mathbf{x} = \Phi\mathbf{u}$. That is

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y_1 u_1 + y_2 u_2 \\ y_1' u_1 + y_2' u_2 \end{pmatrix} \tag{4.5.9}$$

Now

$$\Phi(t)^{-1} = \frac{1}{W(t)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix},$$

where $W(t)$ is the Wronskian of $y_1(t)$ and $y_2(t)$. Using (4.3.3), we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int \frac{1}{W(t)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} dt.$$

Thus

$$u_1(t) = -\int \frac{y_2(t)}{W(t)} f(t)dt, \qquad u_2(t) = \int \frac{y_1(t)}{W(t)} f(t)dt. \tag{4.5.10}$$

Note that, (4.5.9) implies

$$y_1'u_1 + y_2'u_2 = y' = y_1'u_1 + y_2'u_2 + y_1u_1' + y_2u_2'.$$

Hence $y_1u_1' + y_2u_2' = 0$. Plugging $y' = y_1'u_1 + y_2'u_2$ into (4.5.7) we find $y_1'u_1' + y_2'u_2' = f$. Solving these two equations, we find that

$$u_1' = -\frac{y_2}{W} f, \quad u_2' = \frac{y_1}{W} f.$$

Again we get (4.5.10).

**Linear equations with constant coefficients**.
Now we consider linear equations with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t), \tag{4.5.11}$$

and the associated homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0, \tag{4.5.12}$$

where $a_1, \cdots, a_n$ are real constants. Recall that (4.5.12) is equivalent to a system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are roots of the equation

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{4.5.13}$$

which is the same as the characteristic equation of (4.5.12).

Thus the eigenvalues of $\mathbf{A}$ are the same as the characteristic values of (4.5.12).
Then the following result can be deduced using the theory of linear differential system with constant coefficients.

**Theorem 4.14** *Let $\lambda_1, \cdots, \lambda_s$ be the distinct eigenvalues for (4.5.12), with multiplicity $m_1, \cdots, m_s$ respectively. Then (4.5.12) has a fundamental set of solutions*

$$\begin{aligned} &e^{\lambda_1 t},\ te^{\lambda_1 t},\ \cdots,\ t^{m_1-1}e^{\lambda_1 t};\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;\\ &e^{\lambda_s t},\ te^{\lambda_s t},\ \cdots,\ t^{m_s-1}e^{\lambda_s t}. \end{aligned} \tag{4.5.14}$$

## 4.6 Introduction to the Phase Plane

Consider a system of two first order equations

$$\begin{aligned} x'(t) &= f(x,y)\\ y'(t) &= g(x,y). \end{aligned} \tag{4.6.1}$$

It is called an *autonomous system* since $f$ and $g$ are independent of $t$. This property implies that if the pair $(x(t), y(t))$ is a solution of (4.6.1), then the pair $(x(t+c), y(t+c))$ is also a solution of (4.6.1) for any constant $c$.

Let the pair $(x(t), y(t))$ be a solution to (4.6.1). If we plot the points $(x(t), y(t))$ on the $xy$-plane, the resulting curve is called an *integral curve* of the system (4.6.1), and the $xy$-plane is called the *phase plane*.

Let $\mathbf{F}(x,y) = \langle f(x,y), g(x,y)\rangle$ be the vector field on the $xy$-plane defined by $f$ and $g$. If we plot the unit vectors defined by those nonzero $\mathbf{F}(x,y)$ at various points of the phase plane, we get the so called *direction field* of (4.6.1). For any point $\mathbf{p}(t) = \langle x(t), y(t)\rangle$ on an integral curve of (4.6.1), we have

$$\mathbf{p}'(t) = \langle x'(t), y'(t)\rangle = \langle f(x,y), g(x,y)\rangle = \mathbf{F}(x,y).$$

Thus $\mathbf{F}(x,y)$ is everywhere tangent to the integral curve $\mathbf{p}(t)$. See figure 4.1.

By eliminating $t$ in (4.6.1), it leads to the equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}.$$

It is called the *phase plane equation*. Thus any integral curve of (4.6.1) satisfies the phase plane equation.

**Exercise 4.3** Solve the system $x'(t) = 1, y'(t) = 3x^2 + x - 1$. Sketch a few integral curves of the system.

## 4.7 Linear Autonomous System in the Plane

A linear autonomous system in the plane has the form

$$\begin{aligned} x'(t) &= a_{11}x + a_{12}y + b_1\\ y'(t) &= a_{21}x + a_{22}y + b_2 \end{aligned}$$

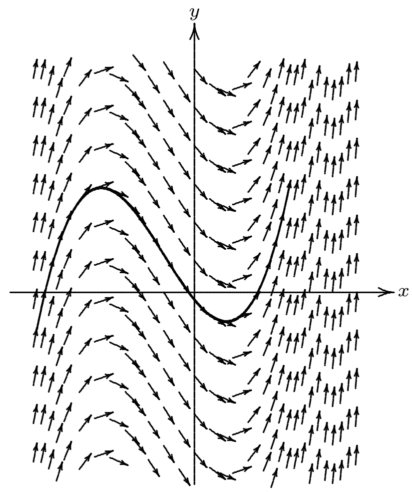



Figure 4.1: Direction field and an integral curve of $\frac{dx}{dt} = f(x,y), \frac{dy}{dt} = g(x,y)$

where $a_{ij}$'s and $b_j$'s are constants. We assume $a_{11}a_{22} - a_{21}a_{12} \neq 0$. By a simple translation the system can always be put in form

$$\begin{aligned} x'(t) &= ax + by \\ y'(t) &= cx + dy, \end{aligned} \tag{4.7.2}$$

where $ad - bc \neq 0$. The point $(0,0)$ is called a *critical point* of (4.7.2) since both $ax + by$ and $cx + dy$ are zero at $x = 0, y = 0$. (In general a *critical point* of (4.6.1) is a solution of the system of equations $f(x,y) = 0, g(x,y) = 0$.) Corresponding to the critical point $(0,0)$ of (4.7.2), we have $x(t) = 0$, $y(t) = 0$ for all $t$ is a constant solution of (4.7.2). Our interest is to investigate the behavior of the integral curves near the critical point $(0,0)$. The behavior of the solutions to (4.7.2) is linked to the nature of the roots $r_1$ and $r_2$ of the characteristic equation of (4.7.2). That is $r_1$ and $r_2$ are the roots of the quadratic equation

$$X^2 - (a+d)X + (ad - bc) = 0. \tag{4.7.3}$$

**Case 1**. If $r_1, r_2$ are real, distinct and of the same sign, then the critical point $(0,0)$ is a node. Suppose $r_1 < r_2 < 0$. The general solution to (4.7.2) has the form

$$\begin{aligned} x(t) &= c_1A_1e^{r_1t} + c_2A_2e^{r_2t} \\ y(t) &= c_1B_1e^{r_1t} + c_2B_2e^{r_2t}, \end{aligned} \tag{4.7.4}$$

where $A$'s and $B$'s are fixed constants such that $B_1/A_1 \neq B_2/A_2$ and $c_1, c_2$ are arbitrary constants. When $c_2 = 0$, we have the solution $x(t) = c_1A_1e^{r_1t}, y(t) = c_1B_1e^{r_1t}$. For any $c_1 > 0$, it gives the parametric equation of a half-line $A_1y = B_1x$ with slope $B_1/A_1$. As $t \to \infty$, the point on this half-line approaches the origin $(0,0)$. For any $c_1 < 0$, it represents the other half of the line $A_1y = B_1x$. As $t \to \infty$, the point on this half-line also approaches the origin $(0,0)$.
When $c_1 = 0$, we have the solution $x(t) = c_2A_2e^{r_2t}, y(t) = c_2B_2e^{r_2t}$. In exactly the same way, it represents two half-lines lying on the line $A_2y = B_2x$ with slope $B_2/A_2$. The two half-lines approach $(0,0)$ as $t \to \infty$.

The two lines $A_1y = B_1x$ and $A_2y = B_2x$ are called the *transformed axes*, usually denoted by $\hat{x}$ and $\hat{y}$ on the phase plane.

If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4.7.4) are parametric equations of some curves. Since $r_1 < 0$ and $r_2 < 0$, these curves approach $(0,0)$ as $t \to \infty$. Furthermore,

$$\frac{y}{x} = \frac{c_1B_1e^{r_1t} + c_2B_2e^{r_2t}}{c_1A_1e^{r_1t} + c_2A_2e^{r_2t}} = \frac{(c_1B_1/c_2)e^{(r_1-r_2)t} + B_2}{(c_1A_1/c_2)e^{(r_1-r_2)t} + A_2}.$$

As $r_1 - r_2 < 0$, we see that $\frac{y}{x} \to \frac{B_2}{A_2}$ as $t \to \infty$ so that all the curves enter $(0,0)$ with slope $\frac{B_2}{A_2}$.

A critical point is called a *node* if it is approached and entered (with a well-defined tangent line) by each integral curve as $t \to \infty$ or $t \to -\infty$. A critical point is said to be *stable* if for each $R > 0$, there exists a positive $r \leq R$ such that every integral curve which is inside the circle $x^2 + y^2 = r^2$ for some $t = t_0$ remains inside the circle $x^2 + y^2 = R^2$ for all $t > t_0$. Roughly speaking, a critical point is stable if all integral curves that get sufficiently close to the point stay close to the point. If the critical point is not stable, it is called *unstable*. A critical point is said to be *asymptotically stable* if it is stable and there exists a circle $x^2 + y^2 = r_0^2$ such that every integral curve which is inside this circle for some $t = t_0$ approaches the critical point as $t \to \infty$. A node is said to be *proper* if every direction through the node defines an integral curve, otherwise it is said to be *improper*.

In our situation, we have $(0,0)$ is an asymptotically stable improper node. See figure 4.2(i).

If $r_1 > r_2 > 0$, then the situation is exactly the same except that all curves now approach and enter $(0,0)$ as $t \to -\infty$. So all the arrow showing the directions are reversed. See figure 4.2(ii). The point $(0,0)$ is a unstable improper node.

**Case 2**. If $r_1, r_2$ are real, distinct and of opposite sign, then the critical point $(0,0)$ is a saddle point. Let's suppose $r_1 < 0 < r_2$. The general solution is still represented by (4.7.4). The two half-line solutions $x(t) = c_1A_1e^{r_1t}, y(t) = c_1B_1e^{r_1t}$ (for $c_1 > 0$ and $c_1 < 0$) still approach and enter $(0,0)$ as $t \to \infty$, but the other two half-line solutions $x(t) = c_2A_2e^{r_2t}, y(t) = c_2B_2e^{r_2t}$ approach and enter $(0,0)$ as $t \to -\infty$. If $c_1 \neq 0$ and $c_2 \neq 0$, the general solution (4.7.4) defines integral curves of (4.7.2), but since $r_1 < 0 < r_2$, none of these curves approaches $(0,0)$ as $t \to \infty$ or $t \to -\infty$. So $(0,0)$ is not a node. Instead, as $t \to \infty$, each of these curves is asymptotic to one of the half-lines of the line $A_2y = B_2x$; whereas as $t \to -\infty$, each of these curves is asymptotic to one of the half-lines of the line $A_1y = B_1x$. In this case, the critical point is called a *saddle point* and is certainly unstable. See figure 4.2(iii).

**Case 3**. If $r_1, r_2$ are real and equal, then the critical point $(0,0)$ is a node.

Let $r_1 = r_2 = r$. In this case, the general solution may or may not involve a factor of $t$ times $e^{rt}$. Let's consider first the subcase where $t$ is not present. Then the general solution is

$$x(t) = c_1e^{rt},\ y(t) = c_2e^{rt},$$

where $c_1$ and $c_2$ are arbitrary constants. The integral curves lie on the lines $c_1y = c_2x$. When $r > 0$, the integral curves move away from $(0,0)$, so $(0,0)$ is unstable. See figure 4.3(i). When $r < 0$, the integral curves approach $(0,0)$ and $(0,0)$ is asymptotically stable. See figure 4.3(ii). In either situation all integral curves lie on lines passing through $(0,0)$. Because every direction through $(0,0)$ defines an integral curve, the point $(0,0)$ is a proper node.

Now let's discuss the case where a factor of $t$ times $e^{rt}$ is present. Suppose $r < 0$. The general solution can be written in the form

$$\begin{aligned} x(t) &= c_1 A e^{rt} + c_2(A_1 + At)e^{rt} \\ y(t) &= c_1 B e^{rt} + c_2(B_1 + Bt)e^{rt}, \end{aligned} \tag{4.7.5}$$

where $A$'s and $B$'s are definite constants and $c_1, c_2$ are arbitrary constants. When $c_2 = 0$, we obtain the solutions $x(t) = c_1 A e^{rt}, y(t) = c_1 B e^{rt}$. These are solutions representing the two half-lines (for $c_1 > 0$ and $c_1 < 0$) lying on the line $\hat{y}$ with equation $Ay = Bx$ and slope $B/A$; and since $r < 0$, both approach $(0,0)$ as $t \to \infty$. Also since $y/x = B/A$, it is clear that both of these half lines enter $(0,0)$ with slope $B/A$.

If $c_2 \neq 0$, the solutions (4.7.5) are curves. As $r < 0$, these curves approach $(0,0)$ as $t \to \infty$. Furthermore,

$$\frac{y}{x} = \frac{c_1 B e^{rt} + c_2(B_1 + Bt)e^{rt}}{c_1 A e^{rt} + c_2(A_1 + At)e^{rt}} = \frac{c_1 B/c_2 + B_1 + Bt}{c_1 A/c_2 + A_1 + At}$$

approaches $B/A$ as $t \to \infty$; so these curves all enter $(0,0)$ with slope $B/A$. We also have $y/x \to B/A$ as $t \to -\infty$. Therefore each of these curves is tangent to $\hat{y}$ as $t \to \pm\infty$. Consequently $(0,0)$ is a node that is asymptotically stable. See figure 4.2(iv).

If $r > 0$, the situation is unchanged except that the directions of the curves are reversed and the critical point is unstable.

**Exercise 4.4** Find and classify the critical point of the system $x' = 2x + y + 3, y' = -3x - 2y - 4$.

**Case 4**. If $r_1, r_2$ are conjugate complex but not purely imaginary, then the critical point $(0,0)$ is a spiral.

Let $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$. First observe that the discriminant of (4.7.3) is negative. That is

$$(a + d)^2 - 4(ad - bc) = (a - d)^2 + 4bc < 0. \tag{4.7.6}$$

The general solution of (4.7.2) is given by

$$\begin{aligned} x(t) &= e^{\alpha t}[c_1(A_1 \cos\beta t - A_2 \sin\beta t) + c_2(A_1 \sin\beta t + A_2 \cos\beta t)], \\ y(t) &= e^{\alpha t}[c_1(B_1 \cos\beta t - B_2 \sin\beta t) + c_2(B_1 \sin\beta t + B_2 \cos\beta t)], \end{aligned} \tag{4.7.7}$$

where $A$'s and $B$'s are definite constants and $c$'s are arbitrary constants.

Suppose $\alpha < 0$. Then from (4.7.7) we see that $x \to 0$ and $y \to 0$ as $t \to \infty$. That means all integral curves approach $(0,0)$ as $t \to \infty$. Next we shall show that the integral curves do not enter $(0,0)$ as $t \to \infty$. Instead they wind around like a spiral towards $(0,0)$. To do so, we shall show that the angular coordinate $\theta = \tan^{-1}(y/x)$ is always strictly increasing or always strictly decreasing. That is $d\theta/dt > 0$ for all $t > 0$, or $d\theta/dt < 0$ for all $t > 0$. Differentiating $\theta = \tan^{-1}(y/x)$, we have

$$\frac{d\theta}{dt} = \frac{x dy/dt - y dx/dt}{x^2 + y^2}.$$

Using (4.7.2), we get

$$\frac{d\theta}{dt} = \frac{cx^2 + (d - a)xy - by^2}{x^2 + y^2}. \tag{4.7.8}$$

Since we are interested in solutions that represent integral curves, we assume $x^2 + y^2 \neq 0$. Now (4.7.6) implies that $b$ and $c$ have opposite signs. Let's consider the case $b < 0$ and $c > 0$. When $y = 0$, (4.7.8) gives $d\theta/dt = c > 0$. If $y \neq 0$, $d\theta/dt$ cannot be 0. If it were, then by (4.7.8) we have $cx^2 + (d-a)xy - by^2 = 0$ or $c(x/y)^2 + (d-a)(x/y) - b = 0$, for some real number $x/y$. But this contradicts the fact that its discriminant also given by (4.7.6) is negative. Thus we conclude that $d\theta/dt > 0$ for all $t > 0$ when $c > 0$. Similarly in case $b > 0$ and $c < 0$, $d\theta/dt < 0$ for all $t > 0$. Since by (4.7.7), $x$ and $y$ change sign infinitely often as $t \to \infty$, all integral curves must spiral in to $(0,0)$ (counterclockwise or clockwise according to $c > 0$ or $c < 0$). The critical point in this case is a spiral, and is asymptotically stable. See figure 4.3(iii).
If $\alpha > 0$, the situation is the same except that the integral curves approach $(0,0)$ as $t \to -\infty$ and the critical point is unstable. See figure 4.3(iv).

**Exercise 4.5** Classify the critical point at the origin and sketch the phase plane diagram for the system $x' = x - 4y$, $y' = 4x + y$. See figure 4.3(v).

**Case 5**. If $r_1, r_2$ are purely imaginary, then the critical point $(0,0)$ is a centre.
The general solution is given by (4.7.7) without the exponential term. Thus $x(t)$ and $y(t)$ are periodic functions of period $2\pi$ so that each integral curve is a closed path surrounding the origin. These curves can be shown to be ellipses by solving the phase plane equation

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

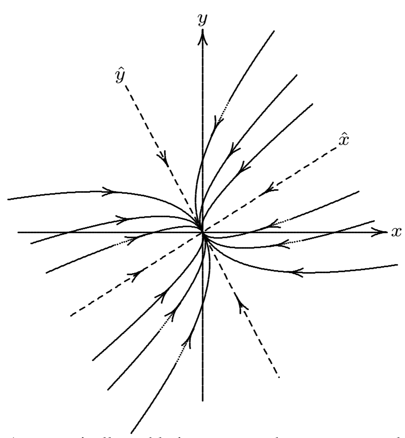
See figure 4.3(vi).

**Exercise 4.6** Show that integral curves of the system $x' = -\beta y, y' = \beta x$ are circles.

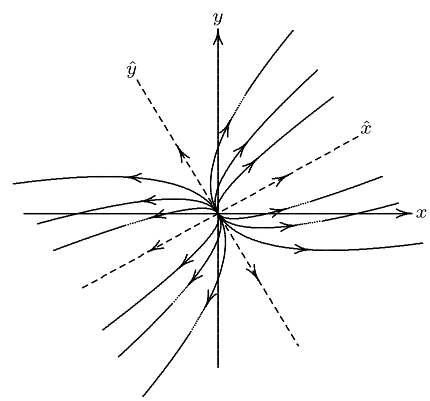
The following result summarizes the above discussion.

**Theorem 4.15** *Assume $(0,0)$ is an isolated critical point of the linear system $x' = ax + by, y' = cx + dy$, where $a, b, c, d$ are real and $ad - bc \neq 0$. Let $r_1, r_2$ be the roots of the characteristic equation $X^2 - (a+d)X + (ad - bc) = 0$. The stability of the origin and the classification of the origin as a critical point depends on the roots $r_1, r_2$ as follows.*

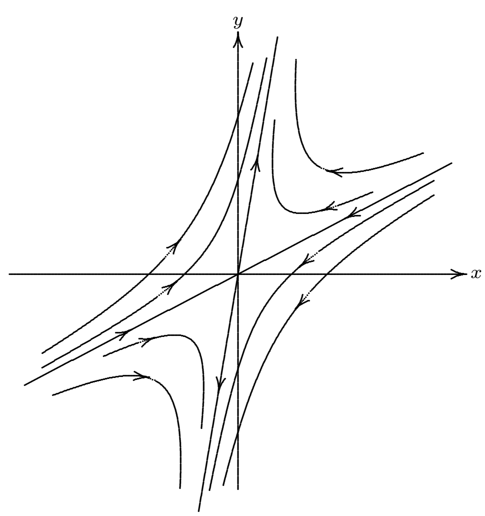
| Roots | Type of critical point | Stability |
|---|---|---|
| *distinct, positive* | *improper node* | *unstable* |
| *distinct, negative* | *improper node* | *asymptotically stable* |
| *opposite signs* | *saddle point* | *unstable* |
| *equal, positive* | *proper or improper node* | *unstable* |
| *equal, negative* | *proper or improper node* | *asymptotically stable* |
| *complex value with positive real part* | *spiral point* | *unstable* |
| *complex value with negative real part* | *spiral point* | *asymptotically stable* |
| *purely imaginary* | *centre* | *stable* |



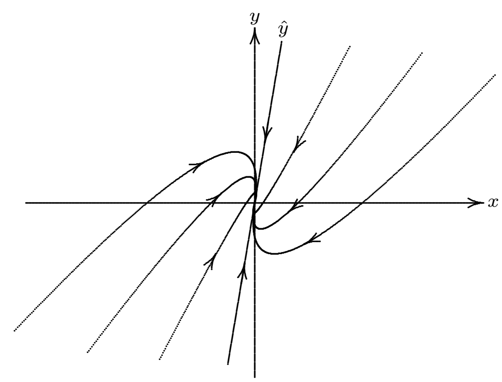

(i) Asymptotically stable improper node: roots are real, distinct and negative




(ii) Unstable improper node: roots are real, distinct and positive

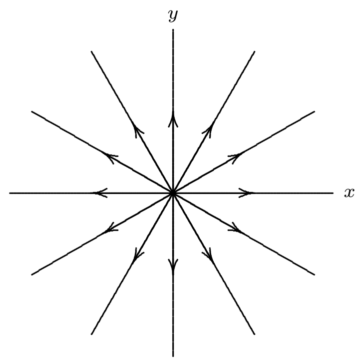



(iii) Unstable saddle point: roots are real and of opposite sign

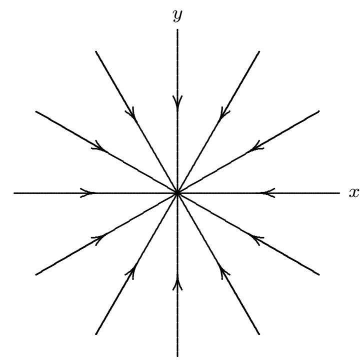



(iv) Equal roots $r$ with a factor of $t$, $r < 0$ stable. Trajectories are tangent to $\hat{y}$ at the origin, and parallel to $\hat{y}$ when $t$ approaches $\pm\infty$.

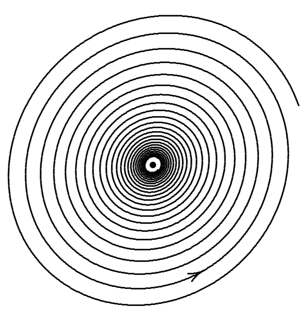
Figure 4.2: Improper node and saddle point



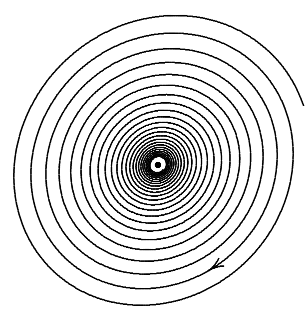

(i) Proper node: equal positive roots with no factor of $t$, unstable



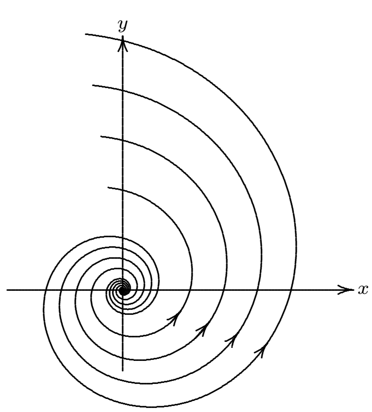

(ii) Proper node: equal negative roots with no factor of $t$, stable



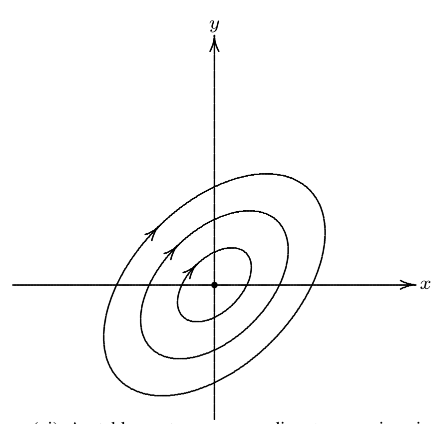
(iii) Spiral curve corresponding to complex roots, real part of the root $\alpha < 0$, asymptotically stable, $c > 0$



(iv) Spiral curve corresponding to complex roots, real part of the root $\alpha > 0$, unstable, $c < 0$




(v) Integral curves for $x' = x - 4y, y' = 4x + y$




(vi) A stable center corresponding to pure imaginary roots

Figure 4.3: Stable and unstable nodes

## 4.8 Appendix 1: Proof of Lemma 4.12

**Lemma 4.12** Let $\mathbf{A}$ be an $n \times n$ complex matrix and $\lambda$ an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $m$. Then

$$\dim \{\mathbf{x} \in \mathbb{C}^n \mid (\lambda \mathbf{I} - \mathbf{A})^m \mathbf{x} = \mathbf{0}\} = m.$$

**Proof**. The proof consists of several steps. Let $T = \{\mathbf{x} \in \mathbb{C}^n \mid (\lambda \mathbf{I} - \mathbf{A})^m \mathbf{x} = \mathbf{0}\}$. The space $T$ is called the generalized eigenspace corresponding to the eigenvalue $\lambda$.
**Step 1**. $T$ is a subspace of $\mathbb{C}^n$. This is just direct verification.
**Step 2**. $T$ is invariant under $\mathbf{A}$ meaning $\mathbf{A}[T] \subseteq T$. This is because if we take a vector $\mathbf{x}$ in $T$, then $(\lambda \mathbf{I} - \mathbf{A})^m \mathbf{x} = \mathbf{0}$ so that $\mathbf{A}(\lambda \mathbf{I} - \mathbf{A})^m \mathbf{x} = \mathbf{0}$, which is the same as $(\lambda \mathbf{I} - \mathbf{A})^m (\mathbf{A}\mathbf{x}) = \mathbf{0}$. Therefore, $\mathbf{A}\mathbf{x}$ is also in $T$.
**Step 3**. By Step 2 which says that $\mathbf{A}[T] \subseteq T$, we may consider $\mathbf{A}$ as a linear transformation on the subspace $T$. In other words, $\mathbf{A} : T \longrightarrow T$. Let $\mu$ be an eigenvalue of $\mathbf{A} : T \longrightarrow T$. That is $\mathbf{A}\mathbf{v} = \mu \mathbf{v}$. Then $\mathbf{v} \in T$ implies that $(\lambda \mathbf{I} - \mathbf{A})^m \mathbf{v} = \mathbf{0}$. Since $\mathbf{A}\mathbf{v} = \mu \mathbf{v}$, this simplifies to $(\lambda - \mu)^m \mathbf{v} = \mathbf{0}$. Being an eigenvector, $\mathbf{v} \neq \mathbf{0}$ so that $\mu = \lambda$. Therefore, all eigenvalues of $\mathbf{A} : T \longrightarrow T$ are equal to $\lambda$.
**Step 4**. Let $\dim T = r$. Certainly $r \leq n$. Then by Step 3, the characteristic polynomial of $\mathbf{A} : T \longrightarrow T$ is $(\lambda - z)^r$. Since $T$ is an invariant subspace of $\mathbf{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, one can choose a basis of $T$ and then extend it to a basis of $\mathbb{C}^n$ so that with respect to this new basis of $\mathbb{C}^n$, the matrix $\mathbf{A}$ is similar to a matrix where the upper left hand $r \times r$ submatrix represents $\mathbf{A}$ on $T$ and the lower left hand $(n - r) \times r$ submatrix is the zero matrix. From this, we see that $(\lambda - z)^r$ is a factor of the characteristic polynomial of $\mathbf{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Hence $r \leq m$.

We also need the Cayley-Hamilton Theorem in the last step of the proof.
**Cayley-Hamilton Theorem** Let $p(z)$ be the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$. Then $p(\mathbf{A}) = \mathbf{0}$.
**Step 5**. Let $p(z) = (\lambda_1 - z)^{m_1} \cdots (\lambda_k - z)^{m_k}$ be the characteristic polynomial of $\mathbf{A}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of $\mathbf{A} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. For each $i$ from 1 to $k$, let

$$T_i = \{\mathbf{x} \in \mathbb{C}^n \mid (\lambda_i \mathbf{I} - \mathbf{A})^{m_i} \mathbf{x} = \mathbf{0}\}.$$

By Step 4, we have $\dim T_i \leq m_i$.
Let $f_i(z) = (\lambda_i - z)^{m_i}$ and $g_i(z) = f_1(z) \cdots \hat{f}_i(z) \cdots f_k(z)$, where $\hat{f}_i(z)$ means the polynomial $f_i(z)$ is omitted. Note that $f_i(z) g_i(z) = p(z)$ for all $i$.
Resolving $\frac{1}{(\lambda_1 - z)^{m_1} \cdots (\lambda_k - z)^{m_k}}$ into partial fractions, we have the identity

$$\frac{1}{(\lambda_1 - z)^{m_1} \cdots (\lambda_k - z)^{m_k}} \equiv \frac{h_1(z)}{(\lambda_1 - z)^{m_1}} + \frac{h_2(z)}{(\lambda_2 - z)^{m_2}} + \cdots + \frac{h_k(z)}{(\lambda_k - z)^{m_k}},$$

where $h_1(z), \ldots, h_k(z)$ are polynomials in $z$. Finding the common denominator of the right hand side and equate the numerators on both sides, we have $1 \equiv g_1(z) h_1(z) + \cdots + g_k(z) h_k(z)$. Substituting the matrix $A$ into this polynomial identity, we have

$$g_1(\mathbf{A}) h_1(\mathbf{A}) + \cdots + g_k(\mathbf{A}) h_k(\mathbf{A}) = \mathbf{I},$$

where $\mathbf{I}$ is the identity $n \times n$ matrix.

Now for any $\mathbf{x} \in \mathbb{C}^n$, we have

$$g_1(\mathbf{A})h_1(\mathbf{A})\mathbf{x} + \cdots + g_k(\mathbf{A})h_k(\mathbf{A})\mathbf{x} = \mathbf{x}.$$

Note that each $g_i(\mathbf{A})h_i(\mathbf{A})\mathbf{x}$ is in $T_i$ because $f_i(\mathbf{A})[g_i(\mathbf{A})h_i(\mathbf{A})\mathbf{x}] = p(\mathbf{A})h_i(\mathbf{A})\mathbf{x} = \mathbf{0}$ by the Cayley-Hamilton Theorem. This shows that any vector in $\mathbb{C}^n$ can be expressed as a sum of vectors where the $i$-summand is in $T_i$. In other words,

$$\mathbb{C}^n = T_1 + T_2 + \cdots + T_k.$$

Consequently, $m_1 + \cdots m_k = n \leq \text{dim}T_1 + \cdots + \text{dim}T_k \leq m_1 + \cdots + m_k$ so that $\text{dim}T_i = m_i$.

**Remarks**.

1. In fact

$$\mathbb{C}^n = T_1 \oplus \cdots \oplus T_k,$$

which is the primary decomposition theorem. To prove this, we know from the dimension theorem that $\text{dim}T_1 + \text{dim}(T_2 + \cdots + T_k) = \text{dim}\mathbb{C}^n + \text{dim}T_1 \cap (T_2 + \cdots + T_k)$. Thus, $n = m_1 + (m_2 + \cdots + m_k) \geq m_1 + \text{dim}(T_2 + \cdots + T_k) = n + \text{dim}T_1 \cap (T_2 + \cdots + T_k)$ so that $\text{dim}T_1 \cap (T_2 + \cdots + T_k) = 0$. That is $T_1 \cap (T_2 + \cdots + T_k) = \{\mathbf{0}\}$. Similarly, $T_i \cap (T_1 + \cdots + \hat{T}_i + \cdots + T_k) = \{\mathbf{0}\}$, where the summand $\hat{T}_i$ is omitted for $i = 1, \ldots, k$. This proves that $T_1 + \cdots + T_k$ is a direct sum.

2. If $\mathbf{A}$ is a real matrix and $\lambda$ is a real eigenvalue of $\mathbf{A}$ of algebraic multiplicity $m$, then $T$ is a real vector space and the real dimension of $T$ is $m$. This is because for any set of real vectors in $\mathbb{R}^n$, it is linearly independent over $\mathbb{R}$ if and only if it is linearly independent over $\mathbb{C}$.

3. If $\mathbf{A}$ is a real matrix and $\lambda$ is a complex eigenvalue of $\mathbf{A}$ of algebraic multiplicity $m$, then $\overline{\lambda}$ is also an eigenvalue of $A$ of algebraic multiplicity $m$. In this case, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis over $\mathbb{C}$ of $T_\lambda$, where $T_\lambda$ is the generalized eigenspace corresponding to $\lambda$, then $\{\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_m\}$ is a basis over $\mathbb{C}$ of $T_{\overline{\lambda}}$. It can be shown that the $2m$ real vectors $\Re\mathbf{v}_1, \ldots, \Re\mathbf{v}_m, \Im\mathbf{v}_1, \ldots, \Im\mathbf{v}_m$ are linearly independent over $\mathbb{R}$ and form a basis of $(T_\lambda \oplus T_{\overline{\lambda}}) \cap \mathbb{R}^n$.

**Jordan canonical form**.

Consider an eigenvalue $\lambda$ of $\mathbf{A}$ with algebraic multiplicity $m$. Let $T = \{\mathbf{x} \in \mathbb{C}^n \mid (\lambda\mathbf{I} - \mathbf{A})^m\mathbf{x} = \mathbf{0}\}$. We give an algorithm for finding a Jordan basis for $T$ such that the matrix representing $\mathbf{A}$ on $T$ is in Jordan form. There are many possible choices of the Jordan basis, though the Jordan form will always consist of the same number of blocks of each size. For instance, the matrix $I$ is in Jordan form in any basis.

**Definition**. Let $W$ be a linear subspace of $V$. We say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ in $V$ is linearly independent mod $W$ if $\sum_i \alpha_i\mathbf{v}_i \in W$ implies $\alpha_i = 0$ for all $i$. We say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ is a basis of $V$ mod $W$ if it is linearly independent mod $W$ and $V = W + \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$.

To find a basis of $V$ mod $W$, we can simply choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $W$ and extend it to a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_1, \ldots, \mathbf{v}_l\}$ for $V$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ is a basis of $V$ mod $W$.

We begin by finding a "cyclic vector" for each Jordan block which generates the Jordan basis for that block. Let $t_i = \text{dimKer}(\lambda\mathbf{I} - \mathbf{A})^i$, $i = 1, \ldots, m$. Note that $t_{i+1} - t_i$ is the number of Jordan blocks of size greater $i$, and thus the number of Jordan blocks of size exactly $i$ is $s_i = (t_i - t_{i-1}) - (t_{i+1} - t_i)$ for $0 < i < k$. Here $t_0$ is taken as 0. Start with the largest $k$ such that $t_{k-1} < t_k$. Note that $1 \leq k \leq m$.

1. Choose a basis $\{\mathbf{v}_k^1, \ldots, \mathbf{v}_k^{s_k}\}$ of $T = \text{Ker}(\lambda\mathbf{I} - \mathbf{A})^k$ mod $\text{Ker}(\lambda\mathbf{I} - \mathbf{A})^{k-1}$. These are the cyclic vectors for the Jordan blocks of size $k$. There are $s_k = t_k - t_{k-1}$ of them.

2. If $k = 1$, stop. Otherwise, apply $\lambda\mathbf{I} - \mathbf{A}$ to the cyclic vectors from the previous step to obtain $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_k^i$ in $\text{Ker}(\lambda\mathbf{I} - \mathbf{A})^{k-1}$. The important point is that

$$\{(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_k^1, \ldots, (\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_k^{s_k}\}$$

   is linearly independent mod $\text{Ker}(\lambda\mathbf{I} - \mathbf{A})^{k-2}$.

   If $k = 2$, stop. Otherwise, extend to a basis of $\text{Ker}(\lambda\mathbf{I} - \mathbf{A})^{k-1}$ mod $\text{Ker}(\lambda\mathbf{I} - \mathbf{A})^{k-2}$ by adding $\{\mathbf{v}_{k-1}^1, \ldots, \mathbf{v}_{k-1}^{s_{k-1}}\}$, which are then the cyclic vectors for the Jordan blocks of size $k - 1$. Since we have extended a list of size $s_k$ to reach a length $t_{k-1} - t_{k-2}$, there are $s_{k-1} = (t_{k-1} - t_{k-2}) - (t_k - t_{k-1})$ of these new cyclic vectors.

3. Repeat step 2, with $k$ replaced by $k - 1$.

Once the algorithm terminates, we may arrange the basis vectors as follows:

$$\begin{array}{l}
\{\ \mathbf{v}_k^1, \ldots, (\lambda\mathbf{I} - \mathbf{A})^{k-1}\mathbf{v}_k^1, \ldots, \mathbf{v}_k^{s_k}, \ldots, (\lambda\mathbf{I} - \mathbf{A})^{k-1}\mathbf{v}_k^{s_k}, \\
\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\quad \mathbf{v}_i^1, \ldots, (\lambda\mathbf{I} - \mathbf{A})^{i-1}\mathbf{v}_i^1, \ldots, \mathbf{v}_i^{s_i}, \ldots, (\lambda\mathbf{I} - \mathbf{A})^{i-1}\mathbf{v}_i^{s_i}, \\
\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\quad \mathbf{v}_2^1, (\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2^1, \ldots, \mathbf{v}_2^{s_2}, (\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2^{s_2}, \\
\quad \mathbf{v}_1^1, \ldots, \mathbf{v}_1^{s_1}\}
\end{array}$$

This puts the Jordan form of $\mathbf{A}$ on $T$ into blocks of decreasing size down the diagonal.

**Example 4.12** Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. Find a Jordan basis for $\mathbf{A}$.

**Solution**. $2\mathbf{I}-\mathbf{A}=\begin{bmatrix} 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$ has rank 5, so that $t_1=2$.

$(2\mathbf{I}-\mathbf{A})^2=\begin{bmatrix} 0&0&0&0&1&0&1\\ 0&0&0&0&1&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$ has rank 3, so that $t_2=4$.

$(2\mathbf{I}-\mathbf{A})^3=\begin{bmatrix} 0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$ has rank 1, so that $t_3=6$.

$(2\mathbf{I}-\mathbf{A})^4$ is the zero matrix so that $t_4=7$, and so $t_5=t_6=t_7=7$. Thus $k=4$.

As $(t_4-t_3)-(t_5-t_4)=1$, there is 1 Jordan block of size 4.
As $(t_3-t_2)-(t_4-t_3)=1$, there is 1 Jordan block of size 3.
As $(t_2-t_1)-(t_3-t_2)=0$, there is no Jordan block of size 2.
As $(t_1-t_0)-(t_2-t_1)=0$, there is no Jordan block of size 1.

Let $\mathbf{e}_i$, $i=1,\dots,7$ be the canonical basis vectors. From the matrix $2\mathbf{I}-\mathbf{A}$ , we know that $\mathbf{e}_1,\mathbf{e}_2$ are eigenvectors. Let's follow the algorithm to find a Jordan basis for $A$. Start with $k=4$.

($k=4$) A basis for $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})^4$ mod $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})^3$ is $\{\mathbf{e}_6\}$. Then $(2\mathbf{I}-\mathbf{A})(\mathbf{e}_6)=\mathbf{e}_3+\mathbf{e}_5$, which is in $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})^3$. Extending it to a basis mod $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})^2$, we get $\{\mathbf{e}_3+\mathbf{e}_5,\mathbf{e}_7\}$. Here $\mathbf{e}_7$ is the added vector.

($k=3$) $(2\mathbf{I}-\mathbf{A})(\mathbf{e}_3+\mathbf{e}_5)=\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4$ and $(2\mathbf{I}-\mathbf{A})(\mathbf{e}_7)=\mathbf{e}_4$, which are in $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})^2$. Extending it to a basis mod $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})$, we get itself $\{\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\mathbf{e}_4\}$. That is no more extension is necessary.

($k=2$) $(2\mathbf{I}-\mathbf{A})(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)=\mathbf{e}_1+\mathbf{e}_2$, $(2\mathbf{I}-\mathbf{A})(\mathbf{e}_4)=\mathbf{e}_1$. So we choose $\{\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1\}$ to be the basis for the eigenspace $\mathrm{Ker}(2\mathbf{I}-\mathbf{A})$.

($k=1$) Stop.

Therefore, a Jordan basis is

$$\{\mathbf{e}_6, \mathbf{e}_3+\mathbf{e}_5, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_7, \mathbf{e}_4, \mathbf{e}_1\}.$$

Consequently, the Jordan form of $\mathbf{A}$ is $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$.

**Example 4.13** Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. Solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

**Solution**. $\mathbf{A}$ has an eigenvalue 2 of algebraic multiplicity 7. We use the Jordan basis $\{\mathbf{e}_6, \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_7, \mathbf{e}_4, \mathbf{e}_1\}$ for $\mathbf{A}$. Thus

$$\begin{aligned}\mathbf{x}(t) = e^{2t} [\ & c_1(\mathbf{e}_6 + t(\mathbf{e}_3+\mathbf{e}_5) + \tfrac{t^2}{2}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4) + \tfrac{t^3}{6}(\mathbf{e}_1+\mathbf{e}_2)) \\ & +c_2((\mathbf{e}_3+\mathbf{e}_5) + t(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4) + \tfrac{t^2}{2}(\mathbf{e}_1+\mathbf{e}_2)) \\ & +c_3((\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4) + t(\mathbf{e}_1+\mathbf{e}_2)) \\ & +c_4(\mathbf{e}_1+\mathbf{e}_2) \\ & +c_5(\mathbf{e}_7 + t\mathbf{e}_4 + \tfrac{t^2}{2}\mathbf{e}_1) \\ & +c_6(\mathbf{e}_4 + t\mathbf{e}_1) \\ & +c_7\mathbf{e}_1]\,.\end{aligned}$$

**Exercise 4.7** Find a Jordan basis of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 & 0 & -1 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Hence solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Ans: There is 1 block of size 3 and 1 block of size 2. A Jordan basis is $\{\mathbf{e}_5, \mathbf{e}_1+\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_2\}$.
$\mathbf{x}(t) = e^{3t}\Big(c_1(\mathbf{e}_5 + t(\mathbf{e}_1+\mathbf{e}_3) + \tfrac{t^2}{2}\mathbf{e}_1) + c_2((\mathbf{e}_1+\mathbf{e}_3) + t\mathbf{e}_1) + c_3\mathbf{e}_1 + c_4(\mathbf{e}_4 + t\mathbf{e}_2) + c_5\mathbf{e}_2\Big)$.

# Chapter 5

# Power Series Solutions

## 5.1 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{5.1.1}$$

is a power series in $x - x_0$. In what follows, we will be focusing mostly at the point $x_0 = 0$. That is

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{5.1.2}$$

(5.1.2) is said to *converge* at a point $x$ if the limit $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists, and in this case the *sum* of the series is the value of this limit. It is obvious that (5.1.2) always converges at $x = 0$. It can be showed that each power series like (5.1.2) corresponds to a positive real number $R$, called the *radius of convergence*, with the property that the series converges if $|x| < R$ and diverges if $|x| > R$. It is customary to put $R$ equal to 0 when the series converges only at $x = 0$, and equal to $\infty$ when it converges for all $x$. In many important cases, $R$ can be found by the ratio test as follow.
If each $a_n \neq 0$ in (5.1.2), and if for a fixed point $x \neq 0$ we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L,$$

then (5.1.2) converges for $L < 1$ and diverges if $L > 1$. It follows from this that

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

if this limit exists (we put $R = \infty$, if $|a_n/a_{n+1}| \longrightarrow \infty$)
The interval $(-R, R)$ is called the *interval of convergence* in the sense that inside the interval the series converges and outside the interval the series diverges.

Consider the following power series

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots \tag{5.1.3}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{5.1.4}$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{5.1.5}$$

It is easy to verify that (5.1.3) converges only at $x = 0$. Thus $R = 0$. For (5.1.4), it converges for all $x$ so that $R = \infty$. For (5.1.5), the power series converges for $|x| < 1$ and $R = 1$.

Suppose that (5.1.2) converges for $|x| < R$ with $R > 0$, and denote its sum by $f(x)$. That is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{5.1.6}$$

Then one can prove that $f$ is continuous and has derivatives of all orders for $|x| < R$. Also the series can be differentiated termwise in the sense that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots ,$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + \cdots ,$$

and so on. Furthermore, the resulting series are still convergent for $|x| < R$. These successive differentiated series yield the following basic formula relating $a_n$ to with $f(x)$ and its derivatives.

$$a_n = \frac{f^n(0)}{n!} \tag{5.1.7}$$

Moreover, (5.1.6) can be integrated termwise provided the limits of integration lie inside the interval of convergence.

If

$$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots \tag{5.1.8}$$

is another power series with interval of convergence $|x| < R$, then (5.1.6) and (5.1.8) can be added or subtracted termwise:

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n = (a_0 \pm b_0) + (a_1 \pm b_1) x + (a_2 \pm b_2) x^2 + \cdots \tag{5.1.9}$$

They can also be multiplied like polynomials, in the sense that

$$f(x) g(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$.

Suppose two power series (5.1.6) and (5.1.8) converge to the same function so that $f(x) = g(x)$ for $|x| < R$, then (5.1.7) implies that they have the same coefficients, $a_n = b_n$ for all $n$. In particular, if $f(x) = 0$ for all $|x| < R$, then $a_n = 0$, for all $n$.

Let $f(x)$ be a continuous function that has derivatives of all orders for $|x| < R$. Can it be represented by a power series? If we use (5.1.7), it is natural to expect

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots \tag{5.1.10}$$

to hold for all $|x| < R$. Unfortunately, this is not always true. Instead, one can use Taylor's expansion for $f(x)$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \; + \; R_n(x),$$

where the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\overline{x})}{(n+1)!} x^{n+1}$$

for some point $\overline{x}$ between $0$ and $x$. To verify (5.1.6), it suffices to show that $R_n(x) \longrightarrow 0$ as $n \longrightarrow \infty$.

**Example 5.1** The following familiar expansions are valid for all $x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

A function $f(x)$ with the property that a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{5.1.11}$$

is valid in some interval containing the point $x_0$ is said to be *analytic* at $x_0$. In this case, $a_n$ is necessarily given by

$$a_n = \frac{f^{(n)}(x_0)}{n!},$$

and (5.1.11) is called the Taylor series of $f(x)$ at $x_0$.
Thus $e^x, \sin x, \cos x$ are analytic at all points. Concerning analytic functions, we have the following basic results.

1. Polynomials, $e^x, \sin x, \cos x$ are analytic at all points.
2. If $f(x)$ and $g(x)$ are analytic at $x_0$, then $f(x) \pm g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ [provided $g(x_0) \neq 0$] are also analytic at $x_0$.
3. If $f(x)$ is analytic at $x_0$, and $f^{-1}(x)$ is a continuous inverse, then $f^{-1}(x)$ is analytic at $f(x_0)$ if $f'(x_0) \neq 0$.
4. If $g(x)$ is analytic at $x_0$ and $f(x)$ is analytic at $g(x_0)$, then $f(g(x))$ is analytic at $x_0$.
5. The sum of a power series is analytic at all points inside the interval of convergence.

## 5.2 Series Solutions of First Order Equations

A first order differential equation $y' = f(x, y)$ can be solved by assuming that it has a power series solution. Let's illustrate this with two familiar examples.

**Example 5.2** Consider the differential equation $y' = y$. We assume it has a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{5.2.1}$$

that converges for $|x| < R$. That is the equation $y' = y$ has a solution which is analytic at the origin. Then

$$y' = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots \tag{5.2.2}$$

has the same interval of convergence. Since $y' = y$, the series (5.2.1) and (5.2.2) have the same coefficients. That is

$$(n+1)a_{n+1} = a_n, \qquad \text{all for } n = 0, 1, 2, \ldots$$

Thus $a_n = \frac{1}{n} a_{n-1} = \frac{1}{n(n-1)} a_{n-2} = \cdots = \frac{1}{n!} a_0$. Therefore

$$y = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \right),$$

where $a_0$ is an arbitrary constant. In this case, we recognize this as the power series of $e^x$. Thus the general solution is $y = a_0 e^x$.

**Example 5.3** The function $y = (1+x)^p$, where $p$ is a real constant satisfies the differential equation

$$(1+x)y' = py, \quad y(0) = 1. \tag{5.2.3}$$

As before, we assume it has a power series solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

with positive radius of convergence. Then

$$\begin{aligned} y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n+1)a_{n+1} x^n + \cdots, \\ xy' &= a_1 x + 2a_2 x^2 + \cdots + n a_n x^n + \cdots, \\ py &= pa_0 + pa_1 x + pa_2 x^2 + \cdots + pa_n x^n + \cdots, \end{aligned}$$

Using (5.2.3) and equating coefficients, we have

$$(n+1)a_{n+1} + n a_n = p a_n, \qquad \text{for all } n = 0, 1, 2, \ldots$$

That is

$$a_{n+1} = \frac{p-n}{n+1} a_n,$$

so that

$$a_1 = p, a_2 = \frac{p(p-1)}{2}, a_3 = \frac{p(p-1)(p-2)}{2 \cdot 3}, \ldots, a_n = \frac{p(p-1)\cdots(p-n+1)}{n!}.$$

In other words,

$$y = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{2\cdot 3}x^3 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots .$$

By ratio test, this series converges for $|x| < 1$. Since (5.2.3) has a unique solution, we conclude that

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{2\cdot 3}x^3 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots ,$$

for $|x| < 1$, This is just the binomial series of $(1+x)^p$.

## 5.3 Second Order Linear Equations and Ordinary Points

Consider the homogeneous second order linear differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{5.3.1}$$

**Definition 5.1** The point $x_0$ is said to be an *ordinary point* of (5.3.1) if $P(x)$ and $Q(x)$ are analytic at $x_0$. If at $x = x_0$, $P(x)$ and/or $Q(x)$ are not analytic, then $x_0$ is said to be a *singular point* of (5.3.1). A singular point $x_0$ at which the functions $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic is called a *regular singular point* of (5.3.1). If a singular point $x_0$ is not a regular singular point, then it is called an *irregular singular point*.

**Example 5.4** If $P(x)$ and $Q(x)$ are constant, then every point is an ordinary point of (5.3.1).

**Example 5.5** Consider the equation $y'' + xy = 0$. Since the function $Q(x) = x$ is analytic at every point, every point is an ordinary point.

**Example 5.6** In the Cauchy-Euler equation $y'' + \frac{a_1}{x}y' + \frac{a_2}{x^2}y = 0$, where $a_1$ and $a_2$ are constants, the point $x = 0$ is a singular point, but every other point is an ordinary point.

**Example 5.7** Consider the differential equation

$$y'' + \frac{1}{(x-1)^2}y' + \frac{8}{x(x-1)}y = 0.$$

The singular points are 0 and 1. At the point 0, $xP(x) = x(1-x)^{-2}$ and $x^2Q(x) = -8x(1-x)^{-1}$, which are analytic at $x = 0$, and hence the point 0 is a regular singular point. At the point 1, we have $(x-1)P(x) = 1/(x-1)$ which is not analytic at $x = 1$, and hence the point 1 is an irregular singular point.

To discuss the behavior of the singularities at infinity, we use the transformation $x = 1/t$, which converts the problem to the behavior of the transformed equation near the origin. Using the substitution $x = 1/t$, (5.3.1) becomes

$$\frac{d^2y}{dt^2} + \left(\frac{2}{t} - \frac{1}{t^2}P(\frac{1}{t})\right)\frac{dy}{dt} + \frac{1}{t^4}Q(\frac{1}{t})y = 0 \tag{5.3.2}$$

We define the point at infinity to be an ordinary point, a regular singular point, or an irregular singular point of (5.3.1) according as the origin of (5.3.2) is an ordinary point, a regular singular point, or an irregular singular point.

**Example 5.8** Consider the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{x}\right)\frac{dy}{dx} + \frac{1}{2x^3}y = 0.$$

The substitution $x = 1/t$ transforms the equation into

$$\frac{d^2y}{dt^2} + \left(\frac{3-t}{2t}\right)\frac{dy}{dt} + \frac{1}{2t}y = 0.$$

Hence the point at infinity is a regular singular point of the original differential equation.

**Exercise 5.1** Show that the hypergeometric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0,$$

where $a, b, c$ are constants has precisely 3 regular singular points at $0, 1, \infty$.

**Theorem 5.1** *Let $x_0$ be an ordinary point of the differential equation*

$$y'' + P(x)y' + Q(x)y = 0,$$

*and let $a_0$ and $a_1$ be arbitrary constants. Then there exists a unique function $y(x)$ that is analytic at $x_0$, is a solution of the differential equation in an interval containing $x_0$, and satisfies the initial conditions $y(x_0) = a_0, y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on an interval $|x - x_0| < R$, $R > 0$, then the power series expansion of this solution is also valid on the same interval.*

**Proof**. Let $x_0$ be an ordinary point of (5.3.1). That is $P(x)$ and $Q(x)$ are analytic at $x_0$. For simplicity, we take $x_0 = 0$. Thus $P(x) = \sum_{n=0}^{\infty} p_n x^n$ and $Q(x) = \sum_{n=0}^{\infty} q_n x^n$, for $|x| < R$, with $R > 0$. We look for a solution of (5.3.1) in form of a power series: $y = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence at least $R$. Here $a_0$ and $a_1$ are the two given numbers for the initial value problem.

$$y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + 2a_2x + 3a_3x^2 + \cdots$$
$$y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n = 2a_2 + (2)(3)a_3x + (3)(4)a_4x^2 + \cdots$$

Thus $P(x)y' = \left(\sum_{n=0}^{\infty} p_n x^n\right)\left[\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n\right]$
$$= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} p_{n-k}(k+1)a_{k+1}\right]x^n.$$

$$Q(x)y = \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} q_{n-k}a_k\right]x^n.$$

Substituting these into (5.3.1), we have

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2}+\sum_{k=0}^{n}p_{n-k}(k+1)a_{k+1}+\sum_{k=0}^{n}q_{n-k}a_k\right]x^n=0.$$

Equating the coefficient of $x^n$ to 0, we get the following recursion formula for $a_n$.

$$(n+1)(n+2)a_{n+2}=-\sum_{k=0}^{n}\left[(k+1)p_{n-k}a_{k+1}+q_{n-k}a_k\right] \tag{5.3.12}$$

For example,

$$\begin{aligned}2a_2 &= -(p_0a_1+q_0a_0),\\ (2)(3)a_3 &= -(p_1a_1+2p_0a_2+q_1a_0+q_0a_1),\\ (3)(4)a_4 &= -(p_2a_1+2p_1a_2+3p_0a_3+q_2a_0+q_1a_1+q_0a_2),\end{aligned}$$

and so on ...

The formula (5.3.12) determines all $a_2, a_3, \ldots$ in terms of $a_0$ and $a_1$ so that the power series $y=\sum_{n=0}^{\infty}a_nx^n$ formally satisfies the DE and the given initial conditions. It remains to prove the series $\sum_{n=0}^{\infty}a_nx^n$, with $a_n$ defined by (5.3.12) converges for $|x|<R$. Recall $R$ is the radius of convergence for

$$P(x)=\sum_{n=0}^{\infty}p_nx^n \quad \text{and} \quad Q(x)=\sum_{n=0}^{\infty}q_nx^n. \tag{5.3.13}$$

Let $r$ be a positive number such that $r<R$. The two series (5.3.13) converge at $x=r$ so that the terms approach zero and are therefore bounded. Thus there exists a constant $M>0$ such that

$$|p_n|r^n\le M \quad \text{and} \quad |q_n|r^n\le M \text{ for all } n.$$

Therefore

$$\begin{aligned}(n+1)(n+2)|a_{n+2}| &\le \frac{M}{r^n}\sum_{k=0}^{n}\left[(k+1)|a_{k+1}|+|a_k|\right]r^k\\ &\le \frac{M}{r^n}\sum_{k=0}^{n}\left[(k+1)|a_{k+1}|+|a_k|\right]r^k+M|a_{n+1}|r.\end{aligned}$$

Let's consider a sequence $\{b_n\}_{n=0}^{\infty}$ defined as follow: $b_0=|a_0|, b_1=|a_1|$, and $b_{n+2}$ given by

$$(n+1)(n+2)b_{n+2}=\frac{M}{r^n}\sum_{k=0}^{n}\left[(k+1)b_{k+1}+b_k\right]r^k+Mb_{n+1}r \tag{5.3.14}$$

Observe that $0\le|a_n|\le b_n$ for all $n$. Now we try to prove that $\sum_{n=0}^{\infty}b_nx^n$ converges for $|x|<r$. Then by comparison test, $\sum_{n=0}^{\infty}a_nx^n$ also converges for $|x|<r$. Since $r$ is an arbitrary positive

number less than $R$, we conclude that $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. We shall apply ratio test to show $\sum_{n=0}^{\infty} b_n x^n$ converges. From (5.3.14), replacing $n$ by $n-1$ and then by $n-2$, we obtain

$$n(n+1)b_{n+1} = \frac{M}{r^{n-1}} \sum_{k=0}^{n-1} [(k+1)b_{k+1} + b_k] r^k + Mb_n r, \text{ and}$$
$$(n-1)nb_n = \frac{M}{r^{n-2}} \sum_{k=0}^{n-2} [(k+1)b_{k+1} + b_k] r^k + Mb_{n-1} r.$$

Now multiplying the first equation by $r$ and by the second equation, we obtain

$$\begin{aligned} rn(n+1)b_{n+1} &= \frac{M}{r^{n-2}} \sum_{k=0}^{n-1} [(k+1)b_{k+1} + b_k] r^k + Mb_n r^2 \\ &= \frac{M}{r^{n-2}} \sum_{k=0}^{n-2} [(k+1)b_{k+1} + b_k] r^k + rM(nb_n + b_{n-1}) \\ &\qquad + Mb_n r^2 \\ &= (n-1)nb_n - Mb_{n-1}r + rM(nb_n + b_{n-1}) + Mb_n r^2 \\ &= [(n-1)n + rMn + Mr^2]b_n. \end{aligned}$$

Therefore

$$\frac{b_{n+1}}{b_n} = \frac{(n-1)n + rMn + Mr^2}{rn(n+1)}.$$

Thus $\lim_{n\to\infty} \left| \frac{b_{n+1}x^{n+1}}{b_n x^n} \right| = \lim_{n\to\infty} \frac{b_{n+1}}{b_n}|x| = \frac{|x|}{r}$, and the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < r$. Consequently, by comparison test the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < r$. □

**Exercise 5.2** Find two linearly independent solutions of $y'' - xy' - x^2 y = 0$

Ans: $y_1(x) = 1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \frac{3}{1120}x^8 + \cdots$ and $y_2(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{13}{1008}x^7 + \cdots$.

**Exercise 5.3** Using power series method, solve the initial value problem $(1+x^2)y'' + 2xy' - 2y = 0$, $y(0) = 0, y'(0) = 1$.

Ans: $y = x$.

**Legendre's equation**

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0,$$

where $p$ is a constant called the order of Legendre's equation.
That is $P(x) = -\frac{2x}{1-x^2}$ and $Q(x) = \frac{p(p+1)}{1-x^2}$. The origin is an ordinary point, and we expect a solution of the form $y = \sum a_n x^n$. Thus the left hand side of the equation becomes

$$(1-x^2) \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - 2x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + p(p+1) \sum_{n=0}^{\infty} a_n x^n,$$

or

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=2}^{\infty}(n-1)na_nx^n - \sum_{n=1}^{\infty}2na_nx^n + \sum_{n=0}^{\infty}p(p+1)a_nx^n.$$

The sum of these series is required to be zero, so the coefficient of $x^n$ must be zero for every $n$. This gives

$$(n+1)(n+2)a_{n+2} - (n-1)na_n - 2na_n + p(p+1)a_n = 0,$$

for $n = 2, 3, \ldots$ In other words,

$$a_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}a_n.$$

This recursion formula enables us to express $a_n$ in terms of $a_0$ or $a_1$ according as $n$ is even or odd. In fact, for $m > 0$, we have

$$a_{2m} = (-1)^m \frac{p(p-2)(p-4)\cdots(p-2m+2)(p+1)(p+3)\cdots(p+2m-1)}{(2m)!}a_0,$$

$$a_{2m+1} = (-1)^m \frac{(p-1)(p-3)\cdots(p-2m+1)(p+2)(p+4)\cdots(p+2m)}{(2m+1)!}a_1.$$

With that, we get two linearly independent solutions

$$y_1(x) = \sum_{m=0}^{\infty} a_{2m}x^{2m} \quad \text{and} \quad y_2(x) = \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1},$$

and the general solution is given by

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - \frac{p(p-2)(p-4)(p+1)(p+3)(p+5)}{6!}x^6 + \cdots\right]$$
$$+a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - \frac{(p-1)(p-3)(p-5)(p+2)(p+4)(p+6)}{7!}x^7 + \cdots\right].$$

When $p$ is not an integer, the series representing $y_1$ and $y_2$ have radius of convergence $R = 1$. For example,

$$\left|\frac{a_{2n+2}x^{2n+2}}{a_{2n}x^{2n}}\right| = \left|-\frac{(p-2n)(p+2n+1)}{(2n+1)(2n+2)}\right| |x^2| \longrightarrow |x|^2$$

as $n \longrightarrow \infty$, and similarly for the second series. In fact, by Theorem 4.1, and the familiar expansion

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots, \quad |x| < 1,$$

shows that $R = 1$ for both $P(x)$ and $Q(x)$. Thus, we know any solution of the form $y = \sum a_n x^n$ must be valid at least for $|x| < 1$.

The functions defined in the series solution of Legendre's equation are called Legendre functions. When $p$ is a nonnegative integer, one of these series terminates and becomes a polynomial in $x$. For instance, if $p = n$ is an even positive integer, the series representing $y_1$ terminates and $y_1$ is a polynomial of degree $n$. If $p = n$ is odd, $y_2$ again is a polynomial of degree $n$. These are called Legendre polynomials $P_n(x)$ and they give particular solutions to Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

where $n$ is a nonnegative integer. It is customary to choose the arbitrary constants $a_0$ or $a_1$ so that the coefficient of $x^n$ in $P_n(x)$ is $(2n)!/[2^n(n!)^2]$. This implies $P_n(1) = 1$. Then

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

The six Legendre polynomials are

$$\begin{array}{ll} P_0 = 1, & P_1(x) = x \\ P_2(x) = \frac{1}{2}(3x^2-1), & P_3(x) = \frac{1}{2}(5x^3-3x) \\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{array}$$

There is also a Rodrigues' formula for the Legendre polynomial given by

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}(x^2-1)^n.$$

**Hermite's equation**

$$y'' - 2xy' + 2py = 0,$$

where $p$ is a constant. The general solution of Hermite's equation is $y(x) = a_0 y_1(x) + a_1 y_2(x)$, where

$$y_1(x) = 1 - \frac{2p}{2!}x^2 + \frac{2^2 p(p-2)}{4!}x^4 - \frac{2^3 p(p-2)(p-4)}{6!}x^6 + \cdots,$$

$$y_2(x) = x - \frac{2(p-1)}{3!}x^3 + \frac{2^2(p-1)(p-3)}{5!}x^5 - \frac{2^3(p-1)(p-3)(p-5)}{7!}x^7 + \cdots.$$

By Theorem 4.1, both series for $y_1$ and $y_2$ converge for all $x$. Note that $y_1$ is a polynomial if $p$ is an even integer, whereas $y_2$ is a polynomial if $p$ is an odd integer.

The Hermite polynomial of degree $n$ denoted by $H_n(x)$ is the $n$th-degree polynomial solution of Hermite's equation, multiplied by a suitable constant so that the coefficient of $x^n$ is $2^n$. The first six Hermite's polynomials are

$$\begin{array}{ll} H_0(x) = 1, & H_1(x) = 2x, \\ H_2(x) = 4x^2 - 2, & H_3(x) = 8x^3 - 12x, \\ H_4(x) = 16x^4 - 48x^2 + 12, & H_5(x) = 32x^5 - 160x^3 + 120x \end{array}$$

A general formula for the Hermite polynomials is

$$H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

## 5.4 Regular Singular Points and the Method of Frobenius

Consider the second order linear homogeneous differential equation

$$x^2y'' + xp(x)y' + q(x)y = 0, \tag{5.4.1}$$

where $p(x)$ and $q(x)$ are analytic at $x = 0$. In other words, 0 is a regular singular point of (5.4.1). Let $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots$, and $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots$. Suppose (5.4.1) has a series solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{5.4.2}$$

An infinite series of the form (5.4.2) is called a Frobenius series, and the method that we are going to describe is called the method of Frobenius. We may assume $a_0 \neq 0$ because the series must have a first nonzero term. Termwise differentiation gives

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \tag{5.4.3}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}. \tag{5.4.4}$$

Substituting the series of $y$, $y'$ and $y''$ into (5.4.1) yields

$$\begin{aligned}[r(r-1)a_0x^r + (r+1)ra_1x^{r+1} + \cdots] + [p_0x + p_1x^2 + \cdots] \cdot [ra_0x^{r-1} + (r+1)a_1x^r + \cdots] \\ +[q_0 + q_1x + \cdots] \cdot [a_0x^r + a_1x^{r+1} + \cdots] = 0.\end{aligned} \tag{5.4.5}$$

The lowest power of $x$ in (5.4.5) is $x^r$. If (5.4.5) is to be satisfied identically, the coefficient $r(r-1)a_0 + p_0ra_0 + q_0a_0$ of $x^r$ must vanish. As $a_0 \neq 0$, it follows that $r$ satisfies the quadratic equation

$$r(r-1) + p_0r + q_0 = 0. \tag{5.4.6}$$

This is the same equation obtained with the Cauchy-Euler equation. Equation (5.4.6) is called the *indicial equation* of (5.4.1) and its two roots (possibly equal) are the *exponents* of the differential equation at the regular singular point $x = 0$.

Let $r_1$ and $r_2$ be the roots of the indicial equation. If $r_1 \neq r_2$, then there are two possible Frobenius solutions and they are linearly independent. Whereas $r_1 = r_2$, there is only one possible Frobenius series solution. The second one cannot be a Frobenius series and can only be found by other means.

**Example 5.9** Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3y' - (1-x^2)y = 0.$$

**Solution.** Clearly $x = 0$ is a regular singular point since $p(x) = \frac{3}{2}(1+x)^2$ and $q(x) = -\frac{1}{2}(1-x)$ are polynomials. Rewrite the equation in the standard form:

$$y'' + \frac{\frac{3}{2}(1+2x+x^2)}{x}y' + \frac{-\frac{1}{2}(1-x)}{x^2}y = 0.$$

We see that $p_0 = \frac{3}{2}$ and $q_0 = -\frac{1}{2}$. Hence the indicial equation is

$$r(r-1) + \frac{3}{2}r - \frac{1}{2} = r^2 + \frac{1}{2}r - \frac{1}{2} = (r+1)(r - \frac{1}{2}) = 0,$$

with roots $r_1 = \frac{1}{2}$ and $r_2 = -1$. The two possible Frobenius series solutions are of the forms

$$y_1(x) = x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n.$$

Once the exponents $r_1$ and $r_2$ are known, the coefficients in a Frobenius series solution can be found by substituting the series (5.4.2),(5.4.3) and (5.4.4) into the differential equation (5.4.1). If $r_1$ and $r_2$ are complex conjugates, we always get two linearly independent solutions. We shall restrict our attention for *real* solutions of the indicial equation and seek solutions only for $x > 0$. The solutions on the interval $x < 0$ can be studied by changing the variable to $t = -x$ and solving the resulting equation for $t > 0$.

Let's work out the recursion relations for the coefficients. By (5.4.3), we have

$$\begin{aligned}
\tfrac{1}{x}p(x)y' &= \frac{1}{x}\left(\sum_{n=0}^{\infty} p_n x^n\right)\left[\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}\right] \\
&= x^{r-2}\left(\sum_{n=0}^{\infty} p_n x^n\right)\left[\sum_{n=0}^{\infty} a_n(n+r)x^{n}\right] \\
&= x^{r-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} p_{n-k}a_k(r+k)\right]x^n \\
&= x^{r-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1} p_{n-k}a_k(r+k) + p_0 a_n(r+n)\right]x^n.
\end{aligned}$$

Also we have

$$\begin{aligned}
\tfrac{1}{x^2}q(x)y &= \frac{1}{x^2}\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{r+n}\right) \\
&= x^{r-2}\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^{n}\right) \\
&= x^{r-2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}a_k\right)x^n \\
&= x^{r-2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1} q_{n-k}a_k + q_0 a_n\right)x^n.
\end{aligned}$$

Substituting these into the differential equation (5.4.1) and canceling the term $x^{r-2}$, we have

$$\sum_{n=0}^{\infty}\left\{a_n[(r+n)(r+n-1)+(r+n)p_0+q_0] + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k}+q_{n-k}]\right\}x^n = 0.$$

Thus, equating the coefficients to zero, we have for $n \geq 0$,

$$a_n[(r+n)(r+n-1)+(r+n)p_0+q_0]+\sum_{k=0}^{n-1} a_k[(r+k)p_{n-k}+q_{n-k}]=0. \qquad (5.4.7)$$

When $n=0$, we get $r(r-1)+rp_0+q_0=0$, which is true because $r$ is a root of the indicial equation. Then $a_n$ can be determined by (5.4.7) recursively provided

$$(r+n)(r+n-1)+(r+n)p_0+q_0 \neq 0.$$

This would be the case if the two roots of the indicial equation do not differ by an integer. Suppose $r_1 > r_2$ are the two roots of the indicial equation with $r_1 = r_2 + N$ for some positive integer $N$. If we start with the Frobenius series with the smaller exponent $r_2$, then at the $N$-th step the process breaks off because the coefficient $a_N$ in (5.4.7) is zero. In this case, only the Frobenius series solution with the larger exponent is guaranteed to exist. The other solution may not be a Frobenius series or does not exist. The proof of the following theorem is similar to theorem 4.1, see [4].

**Theorem 5.2** *Assume that $x=0$ is a regular singular point of the differential equation (5.4.1) and that the power series expansions of $p(x)$ and $q(x)$ are valid on an interval $|x| < R$ with $R > 0$. Let the indicial equation (5.4.6) have real roots $r_1$ and $r_2$ with $r_1 \geq r_2$. Then (5.4.1) has at least one solution*

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n, \quad (a_0 \neq 0) \qquad (5.4.8)$$

*on the interval $0 < x < R$, where $a_n$ are determined in terms of $a_0$ by the recursion formula (5.4.7) with $r$ replaced by $r_1$, and the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. Furthermore, if $r_1 - r_2$ is not zero or a positive integer, then equation (5.4.1) has a second independent solution*

$$y_1 = x^{r_2}\sum_{n=0}^{\infty} a_n x^n, \quad (a_0 \neq 0) \qquad (5.4.9)$$

*on the same interval, where $a_n$ are determined in terms of $a_0$ by the recursion formula (5.4.7) with $r$ replaced by $r_2$, and again the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$.*

**Remark.** (1) If $r_1 = r_2$, then there cannot be a second Frobenius series solution. (2) If $r_1 - r_2 = n$ is a positive integer and the summation of (5.4.7) is nonzero, then there cannot be a second Frobenius series solution. (3) If $r_1 - r_2 = n$ is a positive integer and the summation of (5.4.7) is zero, then $a_n$ is unrestricted and can be assigned any value whatever. In particular, we can put $a_n = 0$ and continue to compute the coefficients without difficulties. Hence, in this case, there does exist a second Frobenius series solution. In many cases of (1) and (2), it is possible to determine a second solution by the method of variation of parameters. For instance a second solution for the Cauchy-Euler equation for the case where its indicial equation has equal roots is given by $x^r \ln x$.

**Exercise 5.4** Find two linearly independent Frobenius series solutions of the differential equation $2x^2y'' + x(2x+1)y' - y = 0$.

Ans: $y_1 = x(1 - \frac{2}{5}x + \frac{4}{35}x^2 + \cdots)$, $y_2 = x^{-\frac{1}{2}}(1 - x + \frac{1}{2}x^2 + \cdots)$.
More precisely, $y_1 = x\sum_{n=0}^{\infty}\frac{(-1)^n 6\cdot 4^n (n+1)!}{(2n+3)!}x^n$ and $y_2 = x^{-\frac{1}{2}}e^{-x}$.

**Example 5.10** Find the Frobenius series solutions of $xy'' + 2y' + xy = 0$.

**Solution.** Rewrite the equation in the standard form $x^2y'' + 2xy' + x^2y = 0$. We see that $p(x) = 2$ and $q(x) = x^2$. Thus $p_0 = 2$ and $q_0 = 0$ and the indicial equation is $r(r-1) + 2r = r(r+1) = 0$ so that the exponents of the equation are $r_1 = 0$ and $r_2 = -1$. In this case, $r_1 - r_2$ is an integer and we may not have two Frobenius series solutions. We know there is a Frobenius series solution corresponding to $r_1 = 0$. Let's consider the possibility of the solution corresponding to the smaller exponent $r_2 = -1$. Let's begin with $y = x^{-1}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n-1}$. Substituting this into the given equation, we obtain

$$\sum_{n=0}^{\infty}(n-1)(n-2)c_n x^{n-2} + 2\sum_{n=0}^{\infty}(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

or equivalently

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

or

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} c_{n-2} x^{n-2} = 0.$$

The cases $n = 0$ and $n = 1$ reduce to $0 \cdot c_0 = 0$ and $0 \cdot c_1 = 0$. Thus $c_0$ and $c_1$ are arbitrary and we can expect to get two linearly independent Frobenius series solutions. Equating coefficients, we obtain the recurrence relation

$$c_n = -\frac{c_{n-2}}{n(n-1)}, \quad \text{for } n \geq 2.$$

It follows from this that for $n \geq 1$.

$$c_{2n} = \frac{(-1)^n c_0}{(2n)!} \quad \text{and} \quad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}.$$

Therefore, we have

$$y = x^{-1}\sum_{n=0}^{\infty} c_n x^n = \frac{c_0}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + \frac{c_1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

We recognize this general solution as

$$y = \frac{1}{x}(c_0\cos x + c_1\sin x).$$

If we begin with the larger exponent, we will get the solution $(\sin x)/x$.

**Exercise 5.5** The equation $xy'' + (1-x)y' + \alpha y = 0$, where $\alpha$ is a constant is called the Laguerre equation. Find a Frobenius series solution of the Laguerre equation. Show that if $\alpha = k$ is a nonnegative integer, the Laguerre equation has the monic polynomial solution

$$L_k(x) = (k!)^2\sum_{n=0}^{k}\frac{(-1)^n}{n![(k-n)!]^2}x^{k-n}.$$

Ans: $y = 1 + \sum_{n=1}^{\infty}\frac{(0-\alpha)(1-\alpha)\cdots(n-1-\alpha)}{(n!)^2}x^n$.

## 5.5 Bessel's Equation

The second order linear homogeneous differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \tag{5.5.1}$$

where $p$ is a constant is called Bessel's equation. Its general solution is of the form

$$y = c_1 J_p(x) + c_2 Y_p(x). \tag{5.5.2}$$

The function $J_p(x)$ is called the Bessel function of order $p$ of the first kind and the $Y_p(x)$ is the Bessel function of order $p$ of the second kind. These functions have been tabulated and behave somewhat like the trigonometric functions of damped amplitude. If we let $y = u/\sqrt{x}$, we obtain

$$\frac{d^2u}{dx^2} + \left(1 - \frac{p^2 - \frac{1}{4}}{x^2}\right)u = 0. \tag{5.5.3}$$

In the special case in which $p = \pm\frac{1}{2}$, this equation becomes

$$\frac{d^2u}{dx^2} + u = 0.$$

Hence $u = c_1 \sin x + c_2 \cos x$ and

$$y = c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}. \tag{5.5.4}$$

Also we see that as $x \to \infty$ in (5.5.3), and $p$ is finite, we would expect the solution of (5.5.1) to behave as (5.5.4). For the distribution of zeros of the solutions of Bessel'e equation, see section 3.8.

It is easy to see that $x = 0$ is a regular singular point of Bessel's equation. Here $p(x) = 1$ and $q(x) = -p^2 + x^2$. Thus the indicial equation is $r(r-1) + r - p^2 = r^2 - p^2 = 0$. Therefore, the exponents are $\pm p$. Let $r$ be either $-p$ or $p$. If we substitute $y = \sum_{m=0}^{\infty} c_m x^{m+r}$ into Bessel's equation, we find in the usual manner that $c_1 = 0$ and that for $m \geq 2$,

$$[(m+r)^2 - p^2]c_m + c_{m-2} = 0 \tag{5.5.5}$$

**The case** $r = p \geq 0$**.** If we use $r = p$ and write $a_m$ in place of $c_m$, then (5.5.5) yields the recursion formula

$$a_m = -\frac{a_{m-2}}{m(2p+m)} \tag{5.5.6}$$

As $a_1 = 0$, it follows that $a_m = 0$ for all odd values of $m$. The first few coefficients $a_m$ for $m$ even are

$$a_2 = -\frac{a_0}{2(2p+2)} = -\frac{a_0}{2^2(p+1)},$$

$$a_4 = -\frac{a_2}{4(2p+4)} = \frac{a_0}{2^4 \cdot 2(p+1)(p+2)},$$

$$a_6 = -\frac{a_4}{6(2p+6)} = -\frac{a_0}{2^6 \cdot 2 \cdot 3(p+1)(p+2)(p+3)}.$$

In general, one can show that

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(p+1)(p+2)\cdots(p+m)}.$$

Thus we have a solution associated with the larger exponent $p$

$$y_1 = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m!(p+1)(p+2)\cdots(p+m)} x^{2m+p}.$$

If $p = 0$, this is the only Frobenius series solution. In this case, if we choose $a_0 = 1$, we get a solution of Bessel's equation of order 0 given by

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots.$$

This special function $J_0(x)$ is called the Bessel function of order zero of the first kind. A second linearly independent solution can be obtained by other means, but it is not a Frobenius series.

**The case** $r = -p < 0$**.** Our theorem does not guarantee the existence of a Frobenius solution associated with the smaller exponent. However, as we shall see, it does have a second Frobenius series solution so long as $p$ is not an integer. Let's write $b_m$ in place of $c_m$ in (5.5.5). Thus we have $b_1 = 0$ and for $m \geq 2$,

$$m(m-2p)b_m + b_{m-2} = 0 \tag{5.5.7}$$

Note that there is a potential problem if it happens that $2p$ is a positive integer, or equivalently if $p$ is a positive integer or an odd integral multiple of $\frac{1}{2}$. Suppose $p = k/2$ where $k$ is an odd positive integer. Then for $m \geq 2$, (5.5.7) becomes

$$m(m-k)b_m = -b_{m-2} \tag{5.5.8}$$

Recall $b_1 = 0$ so that $b_3 = 0$, $b_5 = 0$, $\cdots$, $b_{k-2} = 0$ by (5.5.8). Now in order to satisfy (5.5.8) for $m = k$, we can simply choose $b_k = 0$. Subsequently all $b_m = 0$ for all odd values of $m$. [If we let $b_k$ to be arbitrary and non-zero, the subsequent solution so obtained is just $b_k y_1(x)$ if we also take $b_0 = 0$. Thus no new solution arises in this situation.]

So we only have to work out $b_m$ in terms of $b_0$ for even values of $m$. In view of (5.5.8), it is possible to solve $b_m$ in terms of $b_{m-2}$ since $m(m-k) \neq 0$ as $m$ is always even while $k$ is odd. The result is the same as before except we should replace $p$ by $-p$. Thus in this case, we have a second solution

$$y_2 = b_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m!(-p+1)(-p+2)\cdots(-p+m)} x^{2m-p}.$$

Since $p(x) = 1$ and $q(x) = x^2 - p^2$ are just polynomials. The series representing $y_1$ and $y_2$ converge for all $x > 0$. If $p > 0$, then the first term in $y_1$ is $a_0 x^p$, whereas the first term in $y_2$ is $b_0 x^{-p}$. Hence $y_1(0) = 0$, but $y_2(x) \longrightarrow \pm\infty$ as $x \longrightarrow 0$, so that $y_1$ and $y_2$ are linearly independent. So we have two linearly independent solutions as long as $p$ is not an integer.

If $p = n$ is an nonnegative integer and we take $a_0 = \frac{1}{2^n n!}$, the solution $y_1$ becomes

$$J_n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}.$$

$J_n$ is called the Bessel function of the first kind of integral order $n$.

**Remarks**.

**1.** For Bessel's equation of order $p$, if $p$ is not an integer, the factorials in $J_p$ can be replaced by the so called Gamma functions and the general solution is $Y = c_1 J_p + c_2 J_{-p}$. Here

$$J_{\pm p} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m \pm n + 1)} \left(\frac{x}{2}\right)^{2m \pm p}.$$

If $p$ is an integer, (5.5.7) can still be used to get a solution $J_{-p}$, but it turns out it is just $(-1)^p J_p$, so there is only one Frobenius series solution. A second solution can be obtained by considering the function

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}.$$

If $p$ is not an integer, $Y_p$ is a solution of Bessel's equation of order $p$ as it is a linear combination of $J_p$ and $J_{-p}$. If $p$ approaches to an integer, the expression of $Y_p$ gives an indeterminate form as both the numerator and denominator approach zero. To get a second solution when $p = n$ is an integer, we take limit as $p$ tends to $n$ to get a solution $Y_n$.

$$Y_n(x) = \lim_{p \to n} \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi} = \lim_{p \to n} \frac{1}{\pi} \left[\frac{\partial J_p}{\partial p} - (-1)^n \frac{\partial J_{-p}}{\partial p}\right].$$

This limit was first evaluated using L'Hôpital's rule by Hankel in 1868. $Y_n$ is called a Bessel function of the second kind, and it follows that $y = c_1 J_p + c_2 Y_p$ is the general solution of Bessel's equation in all cases, whether $p$ is an integer or not. For a general form of $Y_p$, see [2].

**2. The case** $r_1 = r_2$. Let $L(y) = x^2 y'' + xp(x)y' + q(x)y$. We are solving $L(y) = 0$ by taking a series solution of the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$. If we treat $r$ as a variable, then $a_n$'s are functions of $r$. That is $y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n$. Substituting this into $L(y)$ and requires it to be a solution, we get (5.4.7), which can be used to determine $a_n(r)$ recursively provided

$$(r+n)(r+n-1) + (r+n)p_0 + q_0 \neq 0.$$

When $r$ is near the double root $r_1 = r_2$, this expression is nonzero so that all $a_n$ can be determined from (5.4.7). This means

$$L(y(x, r)) = a_0 (r - r_1)^2 x^r.$$

So if $a_0 \neq 0$, we take $r = r_1$, we get one Frobenius series solution $y_1(x)$. Now let's differentiate the above equation with respect to $r$. We get

$$L(\frac{\partial y}{\partial r}) = \frac{\partial}{\partial r} L(y) = a_0[(r - r_1)^2 x^r \ln x + 2(r - r_1)x^r].$$

Evaluating at $r = r_1$, we obtain

$$L(\frac{\partial y}{\partial r}\bigg|_{r=r_1}) = \frac{\partial}{\partial r} L(y)\bigg|_{r=r_1} = 0.$$

Consequently, we have the second solution

$$y_2(x) = \frac{\partial y}{\partial r}(x, r_1) = x^{r_1} \ln x \sum_{n=0}^{\infty} a_n(r_1)x^n + x^{r_1} \sum_{n=0}^{\infty} a'_n(r_1)x^n = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1)x^n.$$

Note that the sum in the last expression starts at $n = 1$ because $a_0$ is a constant and $a'_0 = 0$.
If we apply this method to Bessel's equation of order $p = 0$, we get by choosing $a_0 = 1$ the solutions

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}, \quad \text{and}$$

$$y_2(x) = y_1(x) \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n H(n)}{(n!)^2} \left(\frac{x}{2}\right)^{2n},$$

where $H(n) = \sum_{k=1}^{n} \frac{1}{k}$.

**3. The case $r_1 - r_2$ is a nonnegative integer**
Consider

$$x^2 y'' + xp(x)y' + q(x)y = 0, \quad x > 0,$$

where $p(x) = \sum_{n=0}^{\infty} p_n x^n$ and $q(x) = \sum_{n=0}^{\infty} q_n x^n$. Let $r_1$ and $r_2$ be the roots (exponents) with $r_1 \geq r_2$ of the indicial equation $r(r-1) + p_0 r + q_0 = 0$.
Write $r_1 = r_2 + m$, where $m$ is a nonnegative integer. Let $y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$ be a Frobenius series solution corresponding to the larger exponent $r_1$. For simplicity, we take $a_0 = 1$.
Let $u = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$ and make a change of variable:

$$y(x) = u(x) - b_m y_1(x) \ln x.$$

We get

$$x^2 u'' + xp(x)u' + q(x)u = b_m[2xy'_1 + (p(x) - 1)y_1].$$

Now let's substitute $u = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$ to see if we can determine the $b_n$'s. Note that the first term in the power series expansion of $b_m[2xy'_1 + (p(x) - 1)y_1]$ is $mb_m$, with $m \geq 0$.
Hence after substituting the power series of $u$ into the above equation, we have

$$(r_2(r_2 - 1) + p_0 r_2 + q_0)b_0 x^{r_2} + A_1 x^{r_2+1} + \cdots + A_m x^{r_2+m} + \cdots = mb_m x^{r_1} + \cdots. \quad (5.5.9)$$

The first term on the left hand side is 0 as $r_2$ is a root of the indicial equation. This means $b_0$ can be arbitrary. The coefficients $A_1, A_2, \ldots$ are given by the main recurrence relation (5.4.7). Thus by equating $A_1, \ldots, A_{m-1}$ to 0, one can determine $b_1, \ldots b_{m-1}$. The next term on the left hand side of (5.5.9) is the coefficient $A_m$ of $x^{r_1}$. In the expression of $A_m$ given by (5.4.7), the coefficient of $b_m$ is 0. Previously, this forbids the determination of $b_m$ and possibly runs into a contradiction. Now on the right hand side of (5.5.9), if $m > 0$, then one can determine $b_m$ by equating the coefficients of $x^{r_1}$ on both sides. From then on, all the subsequent $b_n$'s can be determined and we get a solution of the form $y(x) = u(x) - b_m y_1(x) \ln x$. Note that if $b_m = 0$ in this determination, then a second Frobenius series solution in fact can be obtained with the smaller exponent $r_2$.

**Example 5.11** Consider $x^2y'' + xy = 0$. Here $p(x) = 0, q(x) = x$. The exponents are 0 and 1. Hence $m = 1$. Corresponding to the exponent 1, the recurrence relation is $n(n+1)a_n + a_{n-1} = 0$ for $n \geq 0$.

We have the solution

$$y_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{(n!)^2} x^n = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \cdots .$$

Now $b_1[2xy_1' + (p(x) - 1)y_1] = b_1(2x(1 - x + \frac{1}{4}x^2 - \cdots) - (x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \cdots)] = b_1[x - \frac{3}{2}x^2 + \frac{5}{12}x^3 - \cdots]$.

Substituting $u = x^0 \sum_{n=0}^{\infty} b_n x^n$ into $x^2u'' + xu = b_1[2xy_1' + (p(x) - 1)y_1]$, we get

$$0\cdot(0-1)b_0 + [(1)(0)b_1 + b_0]x + [(2)(1)b_2 + b_1]x^2 + [(3)(2)b_3 + b_2]x^3 + \cdots = b_1[x - \frac{3}{2}x^2 + \frac{5}{12}x^3 - \cdots].$$

Comparing coefficients, we have $b_0 = b_1$, $2b_2 + b_1 = -\frac{3}{2}b_1$ and $6b_3 + b_2 = \frac{5}{12}b_1, \cdots$. Thus $b_1 = b_0$, $b_2 = -\frac{5}{4}b_0$, $b_3 = \frac{5}{18}b_0, \ldots$. Therefore $u = b_0(1 + x - \frac{5}{4}x^2 + \frac{5}{18}x^3 - \cdots)$. By taking $b_0 = 1$, we get the solution $y = (1 + x - \frac{5}{4}x^2 + \frac{5}{18}x^3 - \cdots) - y_1(x)\ln x$.

If $m = 0$, then $r_1 = r_2$ and the first terms on both sides of (5.5.9) are 0. Thus we can continue to determine the rest of $b_n$'s. In this case, the $\ln$ term is definitely present.

**Exercise 5.6** Find the general solution of the differential equation

$$x^2(1 + x^2)y'' - x(1 + 2x + 3x^2)y' + (x + 5x^2)y = 0.$$

Ans: $y_1 = x^2(1 + x + \frac{1}{2}x^2 + \cdots),\ y_2 = (1 + x + 2x^2 + \frac{8}{3}x^3 + \cdots) - 2y_1 \ln x$. The general solution is a linear combination of $y_1$ and $y_2$.

## 5.6 Appendix 2: Some Properties of the Legendre Polynomials

The Legendre polynomial $P_n(x)$ is a polynomial of degree $n$ satisfying Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

where $n$ is a nonnegative integer. It is normalized so that the coefficient of $x^n$ is $(2n)!/[2^n(n!)^2]$. Explicitly it is given by

$$P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$

There is also a Rodrigues' formula for the Legendre polynomial given by

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n.$$

Note that in Rodrigues' formula, the coefficient of $x^n$ is $(2n)!/[2^n(n!)^2]$. We can use Rodrigues' formula to show that $P_n(1) = 1$. By this formula, we have $2^n P_n(1)$ is the coefficient of $(x-1)^n$ in the Taylor polynomial expansion of $(x^2-1)^n$ at $x = 1$. As $(x^2-1)^n = (x-1)^n(x-1+2)^n = (x-1)^n[(x-1)^n + n(x-1)^{n-1}2 + \cdots + 2^n]$, it is clear that the coefficient of $(x-1)^n$ is $2^n$. Thus $P_n(1) = 1$.

The Legendre polynomial $P_n(x)$ has the generating function $\phi(Z) = (1-2xZ+Z^2)^{-\frac{1}{2}} = (1 + Z^2 - 2xZ)^{-\frac{1}{2}}$. That is $P_n(x)$ is the coefficient of $Z^n$ in the expansion of $\phi$. To see this, let's expand $\phi$ as a power series in $Z$. It can be shown under the condition that $|x| \leq 1$, this power series converges to $\phi(Z)$ for $|Z| < 1$. That is

$$\phi(Z) = \sum_{n=0}^{\infty} A_n Z^n, \quad \text{for } |Z| < 1, |x| \leq 1. \tag{A.1}$$

Using Binomial expansion,

$$(1+Z^2-2xZ)^{-\frac{1}{2}} = 1 - \frac{1}{2}(Z^2-2xZ) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(Z^2-2xZ)^2 + \cdots,$$

we see that $A_n$ is a polynomial in $x$ of degree $n$. If we let $x = 1$, we obtain

$$\phi(Z)|_{x=1} = (1-2Z+Z^2)^{-\frac{1}{2}} = (1-Z)^{-1} = 1 + Z + Z^2 + Z^3 + \cdots, \quad |Z| < 1.$$

Hence $A_n(1) = 1$ for all $n$. Now, if we can show that $A_n$ satisfies Legendre's equation, it will be identical with $P_n(x)$ as the $A_n$'s are the only polynomials of degree $n$ that satisfy the equation and have the value 1 when $x = 1$. Differentiating $\phi$ with respect to $Z$ and $x$, we obtain

$$(1-2Zx+Z^2)\frac{\partial \phi}{\partial Z} = (x-Z)\phi, \tag{A.2}$$

$$Z\frac{\partial \phi}{\partial Z} = (x-Z)\frac{\partial \phi}{\partial x}. \tag{A.3}$$

Substituting (A.1) into (A.2) and equating the coefficients of $Z^{n-1}$, we obtain

$$nA_n - (2n-1)xA_{n-1} + (n-1)A_{n-2} = 0 \tag{A.4}$$

Also substituting (A.1) into (A.3) and equating the coefficients of $Z^{n-1}$, we obtain

$$x\frac{dA_{n-1}}{dx} - \frac{dA_{n-2}}{dx} = (n-1)A_{n-1} \tag{A.5}$$

In (A.5), replace $n$ by $n+1$ to get

$$x\frac{dA_n}{dx} - \frac{dA_{n-1}}{dx} = nA_n \tag{A.6}$$

Now differentiate (A.4) with respect to $x$ and eliminate $dA_{n-2}/dx$ by (A.5), we have

$$\frac{dA_n}{dx} - x\frac{dA_{n-1}}{dx} = nA_{n-1} \tag{A.7}$$

We now multiply (A.6) by $-x$ and add it to (A.7) and obtain

$$(1-x^2)\frac{dA_n}{dx} = n(A_{n-1} - xA_n) \tag{A.8}$$

Differentiating (A.8) with respect to $x$ and simplifying the result by (A.6), we finally obtain

$$(1-x^2)\frac{d^2A_n}{dx^2} - 2x\frac{dA_n}{dx} + n(n+1)A_n = 0 \tag{A.9}$$

This shows that $A_n$ is a solution of Legendre's equation. Using this generating function and Legendre's equation, it can be shown that $P_n(x)$ satisfy the following orthogonal relations.

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{2}{2n+1} & \text{if } m = n \end{cases} . \tag{A.10}$$

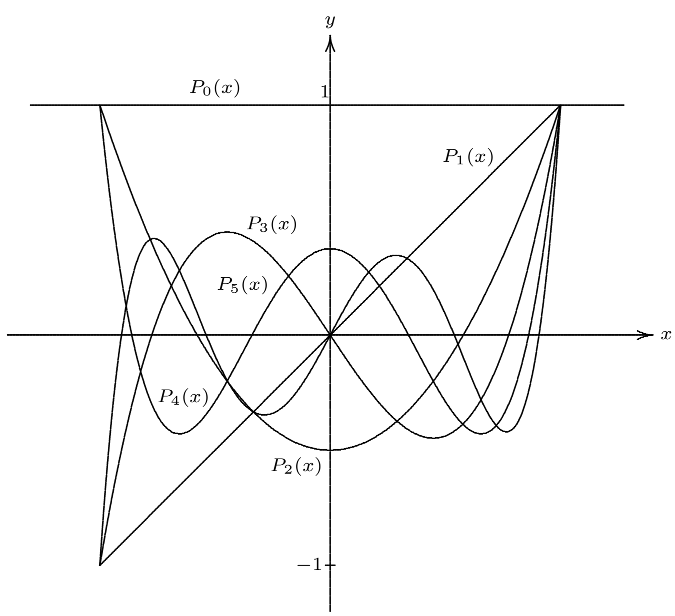



Figure 5.1: The Legendre polynomials

# Chapter 6

# Fundamental Theory of ODEs

## 6.1 Existence-Uniqueness Theorem

We consider the initial value problem

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0, \tag{6.1.1}$$

**Definition 6.1** Let $G$ be a subset in $\mathbb{R}^2$. $f(t,x) : G \to \mathbb{R}$ is said to satisfy a Lipschitz condition with respect to $x$ in $G$ if there exists a constant $L > 0$ such that, for any $(t,x_1), (t,x_2) \in G$,

$$|f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|.$$

$L$ is called a Lipschitz constant.

**Theorem 6.1 (Picard)** *Let $f(t,x)$ be continuous on the rectangle*

$$R : |t - t_0| \le a, \ |x - x_0| \le b \quad (a, b > 0),$$

*and let*

$$|f(t,x)| \le M$$

*for all $(t,x) \in R$. Furthermore, assume $f$ satisfies a Lipschitz condition with constant $L$ in $R$. Then there is a unique solution to the initial value problem*

$$\frac{dx}{dt} = f(t,x), \ \ x(t_0) = x_0$$

*on the interval $I = [t_0 - \alpha, t_0 + \alpha]$, where $\alpha = \min\{a, b/M\}$.*

Proof of the existence of solution will be given in section 6.2 and 6.3. The uniqueness of solution will be proved in section 6.5.

**Example 6.1** Let $f(t,x) = x^2 e^{-t^2} \sin t$ be defined on

$$G = \{(t,x) \in \mathbb{R}^2 : 0 \le x \le 2\}.$$

Let $(t,x_1), (t,x_2) \in G$.

$$
\begin{aligned}
& |f(t,x_1) - f(t,x_2)| \\
= \; & |x_1^2 e^{-t^2} \sin t - x_2^2 e^{-t^2} \sin t| \\
= \; & |e^{-t^2} \sin t||x_1 + x_2||x_1 - x_2| \\
\leq \; & (1)(4)|x_1 - x_2|
\end{aligned}
$$

Thus we may take $L = 4$ and $f$ satisfies a Lipschitz condition in $G$ with Lipschitz constant 4.

**Example 6.2** Let $f(t,x) = t\sqrt{x}$ be defined on

$$G = \{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq 1, 0 \leq x \leq 1\}.$$

Consider the two points $(1,x), (1,0) \in G$. We have $|f(1,x) - f(1,0)| = \sqrt{x} = \frac{1}{\sqrt{x}}|x - 0|$. However, as $x \to 0^+$, $\frac{1}{\sqrt{x}} \to +\infty$, so that $f$ cannot satisfy a Lipschitz condition with any finite constant $L > 0$ on $G$.

**Proposition 6.1.1** *Suppose $f(t,x)$ has a continuous partial derivative $f_x(t,x)$ on a rectangle $R = \{(t,x) \in \mathbb{R}^2 : a_1 \leq t \leq a_2, b_1 \leq x \leq b_2\}$ in the $tx$-plane. Then $f$ satisfies a Lipschitz condition on $R$.*

**Proof**. Since $|f_x(t,x)|$ is continuous on $R$, it attains its maximum value in $R$ by the extreme value theorem. Let $K$ be the maximum value of $|f_x(t,x)|$ on $R$. By mean value theorem, we have

$$|f(t,x_1) - f(t,x_2)| = |f_x(t,c)||x_1 - x_2|,$$

for some $c$ between $x_1$ and $x_2$.
Therefore,

$$|f(t,x_1) - f(t,x_2)| \leq K|x_1 - x_2|$$

for all $(t,x_1), (t,x_2) \in R$. Thus, $f$ satisfies a Lipschitz condition in $R$ with Lipschitz constant $K$.□

**Example 6.3** Let $f(t,x) = x^2$ be defined on

$$G = \{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq 1\}.$$

First

$$|f(t,x_1) - f(t,x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2||x_1 - x_2|.$$

Since $x_1$ and $x_2$ can be arbitrarily large, $f$ cannot satisfy a Lipschitz condition on $G$. If we replace $G$ by any closed and bounded region, then $f$ will satisfy a Lipschitz condition.

## 6.2 The Method of Successive Approximations

We will give the proof of Theorem 6.1 in several steps. Let's fix $f(t,x)$ to be a continuous function defined on the rectangle

$$R : |t - t_0| \leq a, \; |x - x_0| \leq b \; (a, b > 0).$$

The objective is to show that on some interval $I$ containing $t_0$, there is a solution $\phi$ to (6.1.1). The first step will be to show that the initial value problem (6.1.1) is equivalent to an integral equation, namely

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{6.2.1}$$

By a solution of this equation on $I$ is meant a continuous function $\phi$ on $I$ such that $(t, \phi(t))$ is in $R$ for all $t \in I$, and

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

**Theorem 6.2** *A function $\phi$ is a solution of the initial value problem (6.1.1) on an interval $I$ if and only if it is a solution of the integral equation (6.1.2) on $I$.*

**Proof**. Suppose $\phi$ is a solution of the initial value problem on $I$. Then

$$\phi'(t) = f(t, \phi(t)) \tag{6.2.2}$$

on $I$. Since $\phi$ is continuous on $I$, and $f$ is continuous on $R$, the function $f(t, \phi(t))$ is continuous on $I$. Integrating (6.2.2) from $t_0$ to $t$ we obtain

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

Since $\phi(t_0) = x_0$, we see that $\phi$ is a solution of (6.2.1).
Conversely, suppose $\phi$ satisfies (6.2.1). Differentiating we find, using the fundamental theorem of Calculus, that $\phi'(t) = f(t, \phi(t))$ for all $t \in I$. Moreover, from (6.2.1), it is clear that $\phi(t_0) = x_0$ and thus $\phi$ is a solution of (6.1.1). $\square$

As a first approximation to the solution of (6.2.1), we consider $\phi_0$ defined by $\phi_0(t) = x_0$. This function satisfies the initial condition $\phi_0(t_0) = x_0$, but does not in general satisfy (6.2.1). However, if we compute

$$\phi_1(t) = x_0 + \int_{t_0}^{t} f(s, \phi_0(s))\, ds = x_0 + \int_{t_0}^{t} f(s, x_0)\, ds,$$

we might expect $\phi_1$ is a closer approximation to a solution than $\phi_0$. In fact, if we continue the process and define successively

$$\phi_0(t) = x_0, \qquad \phi_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds, \quad k = 0, 1, 2, \ldots \tag{6.2.3}$$

we might expect, on taking the limit as $k \to \infty$, that we would obtain $\phi_k(t) \to \phi(t)$, where $\phi$ would satisfy

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

Thus $\phi$ would be our desired solution.

We call the functions $\phi_0, \phi_1, \phi_2, \cdots$ defined by (6.2.3) *successive approximations* to a solution of the integral equation (6.2.1), or the initial value problem (6.1.1).

**Example 6.4** Consider the initial value problem $x' = tx, \;\; x(0) = 1$.

The integral equation corresponding to this problem is

$$x(t) = 1 + \int_0^t s \cdot x(s)\, ds,$$

and the successive approximations are given by

$$\phi_0(t) = 1, \qquad \phi_{k+1}(t) = 1 + \int_0^t s\phi_k(s)\, ds, \quad k = 0, 1, 2, \ldots.$$

Thus $\phi_1(t) = 1 + \int_0^t s\, ds = 1 + \frac{t^2}{2}$, $\phi_2(t) = 1 + \int_0^t s(1 + \frac{s^2}{2})\, ds = 1 + \frac{t^2}{2} + \frac{t^4}{2 \cdot 4}$, and it may be established by induction that

$$\phi_k(t) = 1 + \left(\frac{t^2}{2}\right) + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \cdots + \frac{1}{k!}\left(\frac{t^2}{2}\right)^k.$$

We recognize $\phi_k(x)$ as a partial sum for the series expansion of the function $\phi(t) = e^{t^2/2}$. We know that this series converges for all $t$ and this means that $\phi_k(t) \to \phi(t)$ as $k \to \infty$, for all $x \in \mathbb{R}$. Indeed $\phi$ is a solution of this initial value problem.

**Theorem 6.3** *Suppose $|f(t, x)| \leq M$ for all $(t, x) \in R$. Then the successive approximations $\phi_k$, defined by (6.2.3), exist as continuous functions on*

$$I : \ |t - t_0| \leq \alpha = \min\{a, b/M\},$$

*and $(t, \phi_k(t))$ is in $R$ for $t \in I$. Indeed, the $\phi_k$'s satisfy*

$$|\phi_k(t) - x_0| \leq M|t - t_0| \tag{6.2.4}$$

*for all $t \in I$.*

Note: Since for $t \in I$, $|t - t_0| \leq b/M$, the inequality (6.2.4) implies that $|\phi_k(t) - x_0| \leq b$ for all $t \in I$, which shows that the points $(t, \phi_k(t))$ are in $R$ for $t \in I$.
The geometric interpretation of the inequality (6.2.4) is that the graph of each $\phi_k$ lies in the region $T$ in $R$ bounded by the two lines $x - x_0 = M(t - t_0)$, $x - x_0 = -M(t - t_0)$, and the lines $t - t_0 = \alpha$, $t - t_0 = -\alpha$.

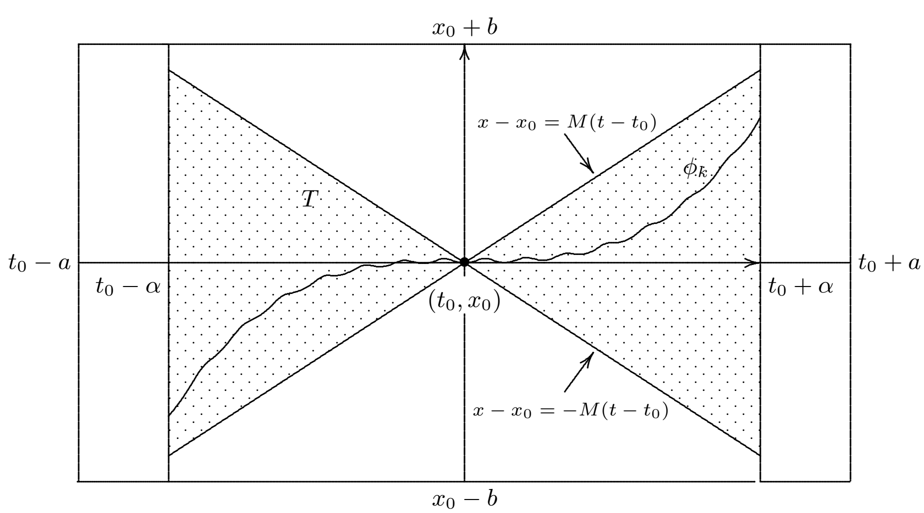



**Figure 6.1**: The graph of each $\phi_k$ lies in the region $T$

**Proof**. We prove it by induction on $k$. Clearly $\phi_0$ exists on $I$ as a continuous function, and satisfies (6.2.4) with $k = 0$. Now suppose the theorem has been proved for the functions $\phi_0, \phi_1, \dots, \phi_k$, with $k \geq 0$. We shall prove that it is valid for $\phi_{k+1}$. By induction hypothesis, the point $(t, \phi_k(t))$ is in $R$ for $t \in I$. Thus the function $f(t, \phi_k(t))$ exists for $t \in I$ and is continuous on $I$. Therefore, $\phi_{k+1}$, which is given by

$$\phi_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds,$$

exists as a continuous function on $I$. Moreover,

$$|\phi_{k+1}(t) - x_0| \leq \left| \int_{t_0}^{t} |f(s, \phi_k(s))|\, ds \right| \leq M|t - t_0|,$$

which shows that $\phi_{k+1}$ satisfies (6.2.4). □

## 6.3 Convergence of the Successive Approximations

We now prove the main existence theorem

**Theorem 6.4** *Let $f(t, x)$ be continuous on the rectangle*

$$R : |t - t_0| \leq a,\ |x - x_0| \leq b \quad (a, b > 0),$$

*and let*

$$|f(t, x)| \leq M$$

*for all $(t, x) \in R$. Furthermore, assume $f$ satisfies a Lipschitz condition with constant $L$ in $R$. Then the successive approximations*

$$\phi_0(t) = x_0, \qquad \phi_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds, \quad k = 0, 1, 2, \dots$$

*converges uniformly on the interval $I = [t_0 - \alpha, t_0 + \alpha]$ with $\alpha = \min\{a, b/M\}$, to a solution of the initial value problem $\dfrac{dx}{dt} = f(t, x),\ \ x(t_0) = x_0$ on $I$.*

**Proof**. (a) *Convergence of* $\{\phi_k(t)\}$. The key to the proof is the observation that $\phi_k$ may be written as

$$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \dots + (\phi_k - \phi_{k-1}),$$

and hence $\phi_k(t)$ is a partial sum for the series

$$\phi_0(t) + \sum_{p=1}^{\infty} [\phi_p(t) - \phi_{p-1}(t)]. \tag{6.3.1}$$

Therefore to show that the sequence $\{\phi_k(t)\}$ converges uniformly is equivalent to show that the series (6.3.1) converges uniformly.

By Theorem 6.3, the functions $\phi_k$ all exist as continuous functions on $I$, and $(t, \phi_p(t))$ is in $R$ for $t \in I$. Moreover,

$$|\phi_1(t) - \phi_0(t)| \leq M|t - t_0|, \tag{6.3.2}$$

for $t \in I$. Next consider the difference of $\phi_2$ and $\phi_1$. We have

$$\phi_2(t) - \phi_1(t) = \int_{t_0}^{t} [f(s, \phi_1(s)) - f(s, \phi_0(s))]\, ds.$$

Therefore

$$|\phi_2(t) - \phi_1(t)| \leq \left| \int_{t_0}^{t} |f(s, \phi_1(s)) - f(s, \phi_0(s))|\, ds \right|,$$

and since $f$ satisfies a Lipschitz condition

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|,$$

we have

$$|\phi_2(t) - \phi_1(t)| \leq L \left| \int_{t_0}^{t} |\phi_1(s) - \phi_0(s)|\, ds \right|.$$

Using (6.3.2), we obtain

$$|\phi_2(t) - \phi_1(t)| \leq ML \left| \int_{t_0}^{t} |s - t_0|\, ds \right|.$$

Thus if $t \geq t_0$,

$$|\phi_2(t) - \phi_1(t)| \leq ML \int_{t_0}^{t} (s - t_0)\, ds = \frac{ML(t - t_0)^2}{2}.$$

The same result is valid in case $t \leq t_0$.

We shall prove by induction that

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \frac{ML^{p-1}|t - t_0|^p}{p!} \tag{6.3.3}$$

for all $t \in I$.

We have proved this for $p = 1$ and $p = 2$. Let's assume $t \geq t_0$. The proof is similar for $t \leq t_0$. Assume (6.3.3) is true for $p = m$. Using the definition of $\phi_{m+1}$ and $\phi_m$, we have

$$\phi_{m+1}(t) - \phi_m(t) = \int_{t_0}^{t} [f(s, \phi_m(s)) - f(s, \phi_{m-1}(s))]\, ds,$$

and thus

$$|\phi_{m+1}(t) - \phi_m(t)| \leq \left| \int_{t_0}^{t} |f(s, \phi_m(s)) - f(s, \phi_{m-1}(s))|\, ds \right|.$$

Using a Lipschitz condition, we get

$$|\phi_{m+1}(t) - \phi_m(t)| \leq L \left| \int_{t_0}^{t} |\phi_m(s) - \phi_{m-1}(s)|\, ds \right|.$$

By induction hypothesis, we obtain

$$|\phi_{m+1}(t) - \phi_m(t)| \leq \frac{ML^m}{m!} \left| \int_{t_0}^{t} |s - t_0|^m\, ds \right| = \frac{ML^m|t - t_0|^{m+1}}{(m+1)!}.$$

Thus, (6.3.3) is true for all positive integer $p$.

Since $|t - t_0| \leq \alpha$ for all $t \in I$, we can further deduce from (6.3.3) that

$$|\phi_p(t) - \phi_{p-1}(t)| \leq \frac{ML^{p-1}\alpha^p}{p!} = \frac{M}{L}\frac{(L\alpha)^p}{p!}. \tag{6.3.4}$$

Since the series $\sum_{p=1}^{\infty} \frac{M}{L}\frac{(L\alpha)^p}{p!}$ converges to $\frac{M}{L}(e^{L\alpha}-1)$, we have by Weierstrass M-test that the series

$$\phi_0(t) + \sum_{p=1}^{\infty}[\phi_p(t) - \phi_{p-1}(t)]$$

converges absolutely and uniformly on $I$. Thus the sequence of partial sum which is $\phi_k(t)$ converges uniformly on $I$ to a limit $\phi(t)$. Next we shall show that this limit $\phi$ is a solution of the integral equation (6.2.1).

(b) *Properties of the limit* $\phi$. Since each $\phi_k$ is continuous on $I$ and the sequence converges uniformly to $\phi$, the function $\phi$ is also continuous on $I$. Now if $t_1$ and $t_2$ are in $I$, we have

$$|\phi_{k+1}(t_1) - \phi_{k+1}(t_2)| = \left| \int_{t_2}^{t_1} f(s, \phi_k(s))\, ds \right| \leq M|t_1 - t_2|,$$

which implies, by letting $k \to \infty$,

$$|\phi(t_1) - \phi(t_2)| \leq M|t_1 - t_2|. \tag{6.3.5}$$

It also follows from (6.3.5) that the function $\phi$ is continuous on $I$. In fact $\phi$ is uniformly continuous on $I$. Letting $t_1 = t, t_2 = t_0$ in (6.3.5), we see that

$$|\phi(t) - \phi(t_0)| \leq M|t - t_0|$$

which implies that the points $(t, \phi(t))$ are in $R$ for all $t \in I$.

(c) *Estimate for* $|\phi(t) - \phi_k(t)|$. We have

$$\phi(t) = \phi_0(t) + \sum_{p=1}^{\infty}[\phi_p(t) - \phi_{p-1}(t)],$$

and

$$\phi_k(t) = \phi_0(t) + \sum_{p=1}^{k}[\phi_p(t) - \phi_{p-1}(t)].$$

Using (6.3.4), we have

$$\begin{aligned}
|\phi(t) - \phi_k(t)| &= \left| \sum_{p=k+1}^{\infty} [\phi_p(t) - \phi_{p-1}(t)] \right| \\
&\leq \sum_{p=k+1}^{\infty} |\phi_p(t) - \phi_{p-1}(t)| \\
&\leq \sum_{p=k+1}^{\infty} \frac{M}{L}\frac{(L\alpha)^p}{p!} \\
&\leq \frac{M}{L}\frac{(L\alpha)^{k+1}}{(k+1)!} \sum_{p=0}^{\infty} \frac{(L\alpha)^p}{p!} \\
&\leq \frac{M}{L}\frac{(L\alpha)^{k+1}}{(k+1)!} e^{L\alpha}.
\end{aligned}$$

Letting $\epsilon_k = \frac{(L\alpha)^{k+1}}{(k+1)!}$, we see that $\epsilon_k \to 0$ as $k \to \infty$ as $\epsilon_k$ is a general term for the series $e^{L\alpha}$. In terms of $\epsilon_k$, we may rewrite the above inequality as

$$|\phi(t) - \phi_k(t)| \leq \frac{M}{L} e^{L\alpha} \epsilon_k, \quad \text{and } \epsilon_k \to 0 \text{ as } k \to \infty \tag{6.3.6}$$

(d) *The limit $\phi$ is a solution*. To complete the proof we must show that

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds,$$

for all $t \in I$. Note that since $\phi$ is continuous, the integrand $f(s, \phi(s))$ of the right hand side is continuous on $I$. Since

$$\phi_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, \phi_k(s))\, ds,$$

we get the result by taking limit on both sides as $k \to \infty$ provided we can show

$$\int_{t_0}^{t} f(s, \phi_k(s))\, ds \to \int_{t_0}^{t} f(s, \phi(s))\, ds, \;\; \text{as } k \to \infty.$$

Now
$$\begin{aligned}
\left| \int_{t_0}^{t} f(s, \phi(s))\, ds - \int_{t_0}^{t} f(s, \phi_k(s))\, ds \right| &\leq \left| \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \phi_k(s))|\, ds \right| \\
&\leq L \left| \int_{t_0}^{t} |\phi(s) - \phi_k(s)|\, ds \right| \\
&\leq M e^{L\alpha} \epsilon_k |t - t_0| \qquad \text{by (6.3.6)} \\
&\leq M\alpha e^{L\alpha} \epsilon_k \;\to\; 0 \text{ as } k \to \infty.
\end{aligned}$$

This completes the proof of the Theorem 6.4. □

**Example 6.5** Consider the initial value problem $x' = (\sin t)x^2$, $x(0) = \frac{1}{2}$. Let $f(t, x) = (\sin t)x^2$ be defined on

$$R = \{(t, x) : |t| \leq 1, |x - \frac{1}{2}| \leq \frac{1}{2}\}.$$

$|f(t, x)| = |(\sin t)x^2| \leq 1$. Thus we may take $M = 1$. Therefore by Theorem 6.4 a solution exists on $[-\alpha, \alpha]$ where $\alpha = \min\{1, \frac{1}{2}\} = \frac{1}{2}$. In fact $x(t) = (1 + \cos t)^{-1}$ is a solution defined on the maximal domain $(-\pi, \pi)$.

**Exercise 6.1** Consider the initial value problem $\frac{dx}{dt} = tx + x^{10}$, $x(0) = \frac{1}{10}$. Show that there exists a unique solution on $[-\frac{1}{2}, \frac{1}{2}]$.

## 6.4 Non-local Existence of Solutions

**Theorem 6.5** *Let $f(t, x)$ be a continuous function on the strip $S = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \leq a\}$, where $a$ is a given positive number, and $f$ satisfies a Lipschitz condition with respect to $S$. Then the initial value problem*

$$x'(t) = f(t, x), \;\; x(t_0) = x_0,$$

*where $(t_0, x_0) \in S$ has a unique solution on the entire interval $[-a + t_0, a + t_0]$.*

**Remark**. If $f$ is bounded on $S$, the result can be deduced from Picard's Theorem by taking $b > Ma$. If $f$ is not necessarily bounded, the proof is slightly different.

**Proof of Theorem 6.5.** First note that the given region $S$ is not bounded above or below. Hence $f(t,x)$ needs not be bounded in $S$. However, as in Theorem 5.4, we shall consider the series

$$\phi_0(t) + \sum_{p=1}^{\infty}(\phi_p(t) - \phi_{p-1}(t))$$

whose $n$-th partial sum is $\phi_n(t)$ and $\phi_n(t) \to \phi(t)$ giving the solution of the initial value problem. Since $f(t,x)$ is not bounded in $S$, we adopt a different method of estimating different terms of the series. Let $M_0 = |x_0|$ and $M_1 = \max|\phi_1(t)|$. The fact that $M_1$ exists can be seen as follows. Since $f(t,x)$ is continuous in $S$, for a fixed $x_0$, $f(t,x_0)$ is a continuous function on $|t - t_0| \le a$. Thus $\phi_1(t) = x_0 + \int_{t_0}^{t} f(s,x_0)\,ds$ is a continuous function in this interval so that $|\phi_1(t)|$ attains its maximum in this interval. We take it to be $M_1$ and let $M = M_0 + M_1$.

Thus, $|\phi_0(t)| = |x_0| \le M$ and $|\phi_1(t) - \phi_0(t)| \le M$. If $t_0 \le t \le t_0 + a$, then we have

$$|\phi_2(t) - \phi_1(t)| = \left|\int_{t_0}^{t}[f(s,\phi_1(s)) - f(s,\phi_0(s))]\,ds\right| \le \int_{t_0}^{t}|f(s,\phi_1(s)) - f(s,\phi_0(s))|\,ds$$
$$\le L\int_{t_0}^{t}|\phi_1(s) - \phi_0(s)|\,ds \le LM(t - t_0),$$ where $L$ is the Lipschitz constant.

Now

$$|\phi_3(t) - \phi_2(t)| = \left|\int_{t_0}^{t}[f(s,\phi_2(s)) - f(s,\phi_1(s))]\,ds\right| \le \int_{t_0}^{t}|f(s,\phi_2(s)) - f(s,\phi_1(s))|\,ds$$
$$\le L\int_{t_0}^{t}|\phi_2(s) - \phi_1(s)|\,ds \le L^2M\int_{t_0}^{t}|(s - t_0)|\,ds = \frac{L^2M}{2}(t - t_0)^2.$$

Hence, in general, we can prove by induction that

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{L^{n-1}M(t - t_0)^{n-1}}{(n-1)!}.$$

Similar argument is true for the interval $t_0 - a \le t \le t_0$. Hence for every $t$ with $|t - t_0| \le a$,

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{L^{n-1}M|t - t_0|^{n-1}}{(n-1)!} \le \frac{L^{n-1}M}{(n-1)!}a^{n-1}.$$

Thus

$$|\phi_0(t)| + \sum_{n=1}^{\infty}|\phi_n(t) - \phi_{n-1}(t)| \le M + M\sum_{n=1}^{\infty}\frac{(La)^{n-1}}{(n-1)!}.$$

Hence each term on the left hand side of the above equation is less than the corresponding term of the convergent series of positive constants. Hence, by Weierstrass $M$-test, the series on the left converges uniformly on the whole interval $|t - t_0| \le a$ and let's denote its limit by $\phi(t)$.

Next we show that $\phi(t)$ is a solution of the initial value problem. We need to show that $\phi(t)$ satisfies the integral equation

$$\phi(t) - x_0 - \int_{t_0}^{t} f(s,\phi(s))\,ds = 0. \tag{6.4.1}$$

We know that

$$\phi_n(t) - x_0 - \int_{t_0}^{t} f(s, \phi_{n-1}(s))\, ds = 0. \tag{6.4.2}$$

Substituting the value of $x_0$ in (6.4.2) into the left hand side of (6.4.1), we get

$$\phi(t) - x_0 - \int_{t_0}^{t} f(s, \phi(s))\, ds = \phi(t) - \phi_n(t) - \int_{t_0}^{t} \left[f(s, \phi(s)) - f(s, \phi_{n-1}(s))\right]\, ds.$$

Thus we obtain

$$\begin{aligned} \left|\phi(t) - x_0 - \textstyle\int_{t_0}^{t} f(s, \phi(s))\, ds\right| &\leq |\phi(t) - \phi_n(t)| + \left|\textstyle\int_{t_0}^{t} |f(s, \phi(s)) - f(s, \phi_{n-1}(s))|\, ds\right| \\ &\leq |\phi(t) - \phi_n(t)| + L\left|\textstyle\int_{t_0}^{t} |\phi(s) - \phi_{n-1}(s)|\, ds\right| \end{aligned} \tag{6.4.3}$$

Since $\phi_n(t) \to \phi(t)$ uniformly for $t \in [t_0 - a, t_0 + a]$, the right hand side of (6.4.3) tends to zero as $n \to \infty$. Hence

$$\phi(t) - x_0 - \int_{t_0}^{t} f(s, \phi(s))\, ds = 0.$$

The uniqueness of solution will be proved in section 6.5. □

**Corollary 6.6** *Let $f(t, x)$ be a continuous function defined on $\mathbb{R}^2$. Let $(t_0, x_0) \in \mathbb{R}^2$. Suppose that for any $a > 0$, $f$ satisfies a Lipschitz condition with respect to $S = \{(t, x) \in \mathbb{R}^2 : |t| \leq a\}$. Then the initial value problem*

$$x'(t) = f(t, x), \;\; x(t_0) = x_0$$

*has a unique solution on the entire $\mathbb{R}$.*

**Proof**. If $t$ is any real number, there is an $a > 0$ such that $t$ is contained in $[t_0 - a, t_0 + a]$. For this $a$, the function $f$ satisfies the condition of Theorem 5.5 on the strip

$$\{(t, x) \in \mathbb{R}^2 : |t - t_0| \leq a\},$$

since this strip is contained in the strip

$$\{(t, x) \in \mathbb{R}^2 : |t| \leq a + |t_0|\}.$$

Thus there is a unique solution $\phi(t)$ to the initial value problem for all $t \in \mathbb{R}$. □

**Example 6.6** Consider the initial value problem $x' = \sin(tx), \; x(0) = 1$.

Let $f(t, x) = \sin(tx)$. Let $a > 0$. Using the mean value theorem, we have for any $t \in [-a, a]$, $|f(t, x_1) - f(t, x_2)| = |\sin(tx_1) - \sin(tx_2)| = |t\cos(t\zeta)(x_1 - x_2)| \leq |t||x_1 - x_2| \leq a|x_1 - x_2|$. Thus $f$ satisfies a Lipschitz condition on the strip $S = \{(t, x) \in \mathbb{R}^2 : |t| \leq a\}$ for any $a > 0$, and by corollary 6.6 there exists a unique solution on the entire $\mathbb{R}$.

**Exercise 6.2** Let $g(t)$ and $h(t)$ be continuous on the interval $I = [t_0 - a, t_0 + a]$, where $a > 0$. Prove that the initial value problem $x' + g(t)x = h(t), \; x(t_0) = x_0$ has a solution defined on $I$.

**Exercise 6.3** Show that the initial value problem $x' = \dfrac{x^3 e^t}{1 + x^2} + t^2 \cos x, \; x(0) = 1$ has a solution on $\mathbb{R}$.

## 6.5 Gronwall's Inequality and Uniqueness of Solution

**Theorem 6.7** *Let $f$, $g$, and $h$ be continuous nonnegative functions defined for $t \geq t_0$. If*

$$f(t) \leq h(t) + \int_{t_0}^{t} g(s)f(s)\,ds, \;\; t \geq t_0,$$

*then*

$$f(t) \leq h(t) + \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)\,du}\,ds, \;\; t \geq t_0.$$

**Proof.** First we are given

$$f(t) \leq h(t) + \int_{t_0}^{t} g(s)f(s)\,ds \tag{6.5.1}$$

Let $z(t) = \int_{t_0}^{t} g(s)f(s)\,ds$. Then for $t \geq t_0$,

$$z'(t) = g(t)f(t) \tag{6.5.2}$$

Since $g(t) \geq 0$, multiplying both sides of (6.5.1) by $g(t)$ and using (6.5.2), we get

$$z'(t) \leq g(t)[h(t) + z(t)]$$

which gives

$$z'(t) - g(t)z(t) \leq g(t)h(t).$$

This is a first order differential inequality which can be solved by finding an integrating factor $e^{-\int_{t_0}^{t} g(u)\,du}$. Hence the solution is

$$z(t)e^{-\int_{t_0}^{t} g(u)\,du} \leq \int_{t_0}^{t} g(s)h(s)e^{-\int_{t_0}^{s} g(u)\,du}\,ds$$

Or equivalently,

$$z(t) \leq \int_{t_0}^{t} g(s)h(s)e^{-\int_{t_0}^{s} g(u)\,du}e^{\int_{t_0}^{t} g(u)\,du}\,ds = \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)\,du}\,ds \tag{6.5.3}$$

Substituting for $z(t)$ in (6.5.3), we get

$$\int_{t_0}^{t} g(s)f(s)\,ds \leq \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)\,du}\,ds \tag{6.5.4}$$

From (6.5.1), we can replace the left side of (6.5.4) by the lesser inequality to obtain

$$f(t) - h(t) \leq \int_{t_0}^{t} g(s)h(s)e^{\int_s^t g(u)\,du}\,ds.$$

□

**Theorem 6.8** *(Gronwall's Inequality) Let $f$ and $g$ be continuous nonnegative functions for $t \geq t_0$. Let $k$ be any nonnegative constant. If*

$$f(t) \leq k + \int_{t_0}^{t} g(s)f(s)\,ds, \quad \textit{for } \ t \geq t_0,$$

*then*

$$f(t) \leq k e^{\int_{t_0}^{t} g(s)\,ds}, \quad \textit{for } \ t \geq t_0.$$

**Proof.** The result follows by letting $h(t) = k$ for all $t \geq t_0$ in 6.7. □

**Corollary 6.9** *Let $f$ be a continuous nonnegative function for $t \geq t_0$ and $k$ a nonnegative constant. If*

$$f(t) \leq k \int_{t_0}^{t} f(s)\,ds$$

*for all $t \geq t_0$, then $f(t) \equiv 0$ for all $t \geq t_0$.*

**Proof.** For any $\epsilon > 0$, we can rewrite the given hypothesis as

$$f(t) \leq \epsilon + k \int_{t_0}^{t} f(s)\,ds,$$

for all $t \geq t_0$. Hence applying Gronwall's inequality, we have

$$f(t) \leq \epsilon e^{\int_{t_0}^{t} k\,ds},$$

for all $t \geq t_0$, which gives $f(t) \leq \epsilon e^{k(t-t_0)}$, for all $t \geq t_0$. Since $\epsilon$ is arbitrary, we get $f(t) \equiv 0$ by taking limit as $\epsilon \to 0^+$. □

**Remark**. Similar results hold for $t \leq t_0$ when all the integrals are integrated from $t$ to $t_0$. For example, in Corollary 6.9, if

$$f(t) \leq k \int_{t}^{t_0} f(s)\,ds$$

for all $t \leq t_0$, then $f(t) \equiv 0$ for all $t \leq t_0$.

**Corollary 6.10** *Let $f(t, x)$ be a continuous function which satisfies a Lipschitz condition on $R$ with a Lipschitz constant $L$, where $R$ is either a rectangle or a strip. If $\phi$ and $\varphi$ are two solutions of*

$$x' = f(t, x), \quad x(t_0) = x_0,$$

*on an interval $I$ containing $t_0$, then $\phi(t) = \varphi(t)$ for all $t \in I$.*

**Proof**. Let $I = [t_0 - \alpha, t_0 + \alpha]$. For $t \in [t_0, t_0 + \alpha]$, we have

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\,ds,$$

and

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds.$$

Thus

$$|\phi(t) - \varphi(t)| \leq \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \varphi(s))|\, ds \leq L \int_{t_0}^{t} |\phi(s) - \varphi(s)|\, ds.$$

By Corollary 6.9, $|\phi(t) - \varphi(t)| \equiv 0$ for $t \in [t_0, t_0 + \alpha]$. Thus $\phi(t) = \varphi(t)$ for $t \in [t_0, t_0 + \alpha]$. Similarly, $\phi(t) = \varphi(t)$ for $t \in [t_0 - \alpha, t_0]$. □

**Remark.** If we only assume that $f(t, x)$ is a continuous function, we can still show that (6.1.1) has at least one solution, but the solution may not be unique.

**Theorem 6.11** *(Peano) Assume $G$ is an open subset of $\mathbb{R}^2$ containing $(t_0, x_0)$ and $f(t, x)$ is continuous in $G$. Then there exists $a > 0$ such that (6.1.1) has at least one solution on the interval $[t_0 - a, t_0 + a]$.*

The proof of Peano's theorem uses the Arzela-Ascoli theorem. See [1] or [2].

**Example 6.7** Consider the initial value problem $x' = x^{2/3}$, $x(0) = 0$. We find that $x(t) = 0$ and $x(t) = \frac{1}{27}t^3$ are both solutions.

**Example 6.8** Suppose $\phi(t)$ is a solution to the initial value problem

$$x' = \frac{x^3 - x}{1 + t^2 x^2}, \qquad x(0) = \frac{1}{2}.$$

Show that $0 < \phi(t) < 1$ for all $t \in J$, where $\phi(t)$ is defined on the open interval $J$ containing 0.

**Solution**. Let $\phi(t)$ be a solution defined on the open interval $J$ to the initial value problem, where $0 \in J$. Suppose there exists $s \in J$ such that $\phi(s) \geq 1$. Without loss of generality, we may assume $s > 0$. Since $\phi(t)$ is continuous and $\phi(0) = 1/2$, we have by the intermediate value theorem, that $\phi(s_0) = 1$ for some $s_0 \in (0, s)$. We may take $s_0$ to be the least value in $(0, s)$ such that $\phi(s_0) = 1$. In other words, $\phi(t) < 1$ for all $t \in (0, s_0)$ and $\phi(s_0) = 1$.

Now consider the initial value problem

$$x' = \frac{x^3 - x}{1 + t^2 x^2}, \qquad x(s_0) = 1.$$

The function $f(t, x) = \frac{x^3 - x}{1 + t^2 x^2}$ satisfies the conditions of the existence and uniqueness theorem. Thus there is a unique solution defined on an interval $I = [s_0 - \alpha, s_0 + \alpha]$ for some $\alpha > 0$. The above function $\phi(t)$ defined on $J$ is a solution to this initial value problem, and it has the property that $\phi(t) < 1$ for all $t < s_0$. However, $\varphi(t) \equiv 1$ is clearly a solution to this initial value problem on $I$. But $\varphi$ and $\phi$ are different solutions to the initial value problem contradicting the uniqueness of the solution. Consequently, $\phi(t) < 1$ for all $t \in J$. Similarly, $\phi(t) > 0$ for all $t \in J$.

**Corollary 6.12** *Let $f(t, x)$ be a continuous function which is defined either on a strip*

$$R = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \leq a\},$$

*or a rectangle*

$$R = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \leq a, |x - x_0| \leq b\}.$$

*Assume $f$ satisfies a Lipschitz condition on $R$ with a Lipschitz constant $L$. Let $\phi$ and $\varphi$ be solutions defined on $I = [-a + t_0, t_0 + a]$ of $x' = f(t, x)$ satisfying the initial condition $x(t_0) = x_0$ and $x(t_0) = x_1$ respectively on $I$, then*

$$|\phi(t) - \varphi(t)| \leq |x_0 - x_1| e^{L|t - t_0|}$$

*for all $t \in I$.*

**Remark**. In particular

$$|\phi(t) - \varphi(t)| \leq |x_0 - x_1| e^{La},$$

for all $t \in I$. Thus if the initial values $x_0$ and $x_1$ are close, the resulting solutions $\phi$ and $\varphi$ are also close. This gives the continuous dependence on initial value.

The proof of this corollary is by Gronwall's inequality and is left as an exercise.

**Corollary 6.13** *Let $f(t, x)$ be a continuous function which is defined either on a strip*

$$R = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \leq a\},$$

*or a rectangle*

$$R = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \leq a, |x - x_0| \leq b\}.$$

*Assume $f$ satisfies a Lipschitz condition on $R$ with a Lipschitz constant $L$. Let $g(t, x)$ be continuous on $R$ and*

$$|g(t, x) - f(t, x)| \leq \epsilon, \text{ for all } (t, x) \in R. \tag{6.5.5}$$

*Let $\phi$ and $\varphi$ be solutions of the initial value problems $x' = f(t, x)$, $x(t_0) = x_0$, and $x' = g(t, x)$, $x(t_0) = x_0$ respectively. Assume both $\phi$ and $\varphi$ are defined on $I = [-a + t_0, t_0 + a]$. Then for all $t \in I$,*

$$|\phi(t) - \varphi(t)| \leq \epsilon a e^{La}.$$

*In particular, as $g(t, x)$ approaches $f(t, x)$ uniformly on $R$, that is, as $\epsilon \to 0^+$ in (6.5.5), the solution $\phi(t)$ approaches $\varphi(t)$ uniformly on $I$.*

**Proof.** By the integral representations for $\phi(t)$ and $\varphi(t)$, we have

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds, \quad \varphi(t) = x_0 + \int_{t_0}^{t} g(s, \varphi(s))\, ds.$$

Subtracting these equations gives

$$\phi(t) - \varphi(t) = \int_{t_0}^{t} [f(s, \phi(s)) - g(s, \varphi(s))]\, ds.$$

Inserting the term $0 = -f(s, \varphi(s)) + f(s, \varphi(s))$ and using triangle inequality, we have for $t_0 \leq t \leq t_0 + a$,

$$
\begin{aligned}
|\phi(t)-\varphi(t)| &\le \int_{t_0}^{t} |f(s,\phi(s))-f(s,\varphi(s))|\ ds + \int_{t_0}^{t} |f(s,\varphi(s))-g(s,\varphi(s))|\ ds \\
&\le L\int_{t_0}^{t} |\phi(s)-\varphi(s)|\, ds + \int_{t_0}^{t} \epsilon\, ds \\
&\le L\int_{t_0}^{t} |\phi(s)-\varphi(s)|\, ds + \epsilon a.
\end{aligned}
$$

By Gronwall's inequality, we obtain

$$|\phi(t)-\varphi(t)| \le \epsilon a e^{\int_{t_0}^{t} L\, ds} \le \epsilon a e^{La}, \ \text{ for } \ t_0 \le t \le t_0 + a.$$

The last inequality is also valid for $t_0 - a \le t \le t_0$. Thus

$$|\phi(t)-\varphi(t)| \le \epsilon a e^{La} \ \text{ for } \ t \in I.$$

□

**Exercise 6.4** Prove corollary 6.12.

## 6.6 Existence and Uniqueness of Solutions to Systems

Consider a system of differential equations

$$
\begin{cases}
x_1' = f_1(t, x_1, \cdots, x_n), \\
x_2' = f_2(t, x_1, \cdots, x_n), \\
\quad \cdots\cdots\cdots\cdots\cdots \\
x_n' = f_n(t, x_1, \cdots, x_n),
\end{cases}
$$

where $x_j' = \frac{dx_j}{dt}$. Let us introduce notations

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}, \ \mathbf{x}' = \begin{pmatrix} x_1' \\ \cdots \\ x_n' \end{pmatrix}, \ \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} f_1(t,\mathbf{x}) \\ \cdots \\ f_n(t,\mathbf{x}) \end{pmatrix}.$$

Then the system can be written in a vector form:

$$\mathbf{x}' = \mathbf{f}(t,\mathbf{x}). \tag{6.6.1}$$

Differential equations of higher order can be reduced to equivalent systems. Let us consider

$$\frac{d^n y}{dt^n} + F(t, y, y', \cdots, \frac{d^{n-1}y}{dt^{n-1}}) = 0. \tag{6.6.2}$$

Let

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad \cdots, \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}.$$

Then (6.6.2) is equivalent to the following system

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ \cdots\cdots\cdots \\ x_n' = -F(t, x_1, x_2, \cdots, x_n). \end{cases}$$

It can be written in the form of (6.6.1) if we let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2, \\ x_3, \\ \cdots \\ -F(t, x_1, \cdots, x_n) \end{pmatrix}.$$

Recall that for a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, its magnitude $|\mathbf{x}|$ is defined to be

$$|\mathbf{x}| = |x_1| + |x_2| + \cdots + |x_n|.$$

The following 2 properties of the magnitude can be proved easily.
1. Triangle Inequality $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.
2. If $A = (a_{ij})$ is an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$, then $|A\mathbf{x}| \leq |A||\mathbf{x}|$ where $|A| = \sum_{i,j}^n |a_{ij}|$.

**Definition 6.2** Let $G$ be a subset in $\mathbb{R}^{1+n}$. $\mathbf{f}(t, \mathbf{x}) : G \to \mathbb{R}^n$ is said to satisfy a Lipschitz condition with respect to $\mathbf{x}$ in $G$ if there exists a constant $L > 0$ such that, for all $(t, \mathbf{x}), (t, \mathbf{y}) \in G$,

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

**Example 6.9** Let $\mathbf{f} : \mathbb{R}^{1+2} \longrightarrow \mathbb{R}^2$ be given by

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} 2x_2 \cos t \\ x_1 \sin t \end{pmatrix}, \qquad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\begin{aligned} \text{Then } |\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| &= \left| \begin{pmatrix} 2x_2 \cos t \\ x_1 \sin t \end{pmatrix} - \begin{pmatrix} 2y_2 \cos t \\ y_1 \sin t \end{pmatrix} \right| \\ &= |2 \cos t(x_2 - y_2)| + |\sin t(x_1 - y_1)| \\ &\leq 2|x_2 - y_2| + |x_1 - y_1| \\ &\leq 2(|x_2 - y_2| + |x_1 - y_1|) \\ &= 2|\mathbf{x} - \mathbf{y}|. \end{aligned}$$

Thus $\mathbf{f}$ satisfies a Lipschitz condition with respect to $\mathbf{x}$ in $\mathbb{R}^3$ with Lipschitz constant 2.

**Theorem 6.14** *Suppose* $\mathbf{f}$ *is defined on a set* $G \subset \mathbb{R}^{1+n}$ *of the form*

$$|t - t_0| \leq a, \ |\mathbf{x} - \mathbf{x}_0| \leq b, \quad (a, b > 0)$$

*or of the form*

$$|t-t_0| \leq a,\ |\mathbf{x}| < \infty, \quad (a>0).$$

*If* $\partial \mathbf{f}/\partial x_k$ *$(k=1,\ldots,n)$ exists, is continuous on $G$, and there is a constant $L>0$ such that*

$$\left|\frac{\partial \mathbf{f}}{\partial x_k}\right| \leq L, \quad (k=1,\ldots,n),$$

*for all $(t,\mathbf{x}) \in G$, then $\mathbf{f}$ satisfies a Lipschitz condition on $G$ with Lipschitz constant $L$.*

**Proof**. Let $\mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} f_1(t,\mathbf{x}) \\ f_2(t,\mathbf{x}) \\ \vdots \\ f_n(t,\mathbf{x}) \end{pmatrix}$, where each $f_i(t,\mathbf{x}) : \mathbb{R}^{1+n} \longrightarrow \mathbb{R}$.

Thus

$$\frac{\partial \mathbf{f}}{\partial x_k} = \begin{pmatrix} \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_k} \\ \vdots \\ \frac{\partial f_n}{\partial x_k} \end{pmatrix}.$$

Let $(t,\mathbf{z}), (t,\mathbf{y}) \in G \subseteq \mathbb{R}^{1+n}$. Define $\mathbf{F} : [0,1] \longrightarrow \mathbb{R}^n$ by

$$\mathbf{F}(s) = \mathbf{f}(t, s\mathbf{z} + (1-s)\mathbf{y}) = \mathbf{f}(t, \mathbf{y} + s(\mathbf{z}-\mathbf{y})).$$

The point $s\mathbf{z} + (1-s)\mathbf{y}$ lies on the segment joining $\mathbf{z}$ and $\mathbf{y}$, hence the point $(t, s\mathbf{z} + (1-s)\mathbf{y})$ is in $G$.

Now $\quad \mathbf{F}'(s) = \displaystyle\sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial x_k}\frac{dx_k}{ds} = \sum_{k=1}^{n} \begin{pmatrix} \frac{\partial f_1}{\partial x_k} \\ \frac{\partial f_2}{\partial x_k} \\ \vdots \\ \frac{\partial f_n}{\partial x_k} \end{pmatrix} (z_k - y_k).$

Therefore,

$$|\mathbf{F}'(s)| \leq \sum_{k=1}^{n} \left|\frac{\partial \mathbf{f}}{\partial x_k}\right| |z_k - y_k| \leq L \sum_{k=1}^{n} |z_k - y_k| = L|\mathbf{z}-\mathbf{y}|,$$

for $s \in [0,1]$.
Since

$$\mathbf{f}(t,\mathbf{z}) - \mathbf{f}(t,\mathbf{y}) = \mathbf{F}(1) - \mathbf{F}(0) = \int_0^1 \mathbf{F}'(s)\,ds$$

we have $|\mathbf{f}(t,\mathbf{z}) - \mathbf{f}(t,\mathbf{y})| \leq \int_0^1 |\mathbf{F}'(s)|\,ds \leq L|\mathbf{z}-\mathbf{y}|$. □

**Theorem 6.15 (Picard)** *Let $\mathbf{f}(t,x)$ be continuous on the set*

$$R : |t-t_0| \leq a,\ |\mathbf{x}-\mathbf{x}_0| \leq b \quad (a,b>0),$$

*and let*

$$|\mathbf{f}(t,x)| \leq M$$

*for all* $(t, \mathbf{x}) \in R$. *Furthermore, assume* $\mathbf{f}$ *satisfies a Lipschitz condition with constant* $L$ *in* $R$. *Then there is a unique solution to the initial value problem*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0$$

*on the interval* $I = [t_0 - \alpha, t_0 + \alpha]$, *where* $\alpha = \min\{a, b/M\}$.

**Theorem 6.16** *Let* $\mathbf{f}(t, x)$ *be a continuous function on the strip* $S = \{(t, \mathbf{x}) \in \mathbb{R}^{n+1} : |t - t_0| \le a\}$, *where* $a$ *is a given positive number, and* $\mathbf{f}$ *satisfies a Lipschitz condition with respect to* $S$. *Then the initial value problem*

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0,$$

*where* $(t_0, \mathbf{x}_0) \in S$ *has a unique solution on the entire interval* $[-a + t_0, a + t_0]$.

**Corollary 6.17** *Let* $\mathbf{f}(t, \mathbf{x})$ *be a continuous function defined on* $\mathbb{R}^{n+1}$. *Let* $(t_0, \mathbf{x}_0) \in \mathbb{R}^{n+1}$. *Suppose that for any* $a > 0$, $\mathbf{f}$ *satisfies a Lipschitz condition with respect to*

$$S = \{(t, \mathbf{x}) \in \mathbb{R}^{n+1} : |t| \le a\}.$$

*Then the initial value problem*

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0$$

*has a unique solution on the entire* $\mathbb{R}$.

The proofs carry over directly from those for Theorem 5.1 and 5.5 and Corollary 5.6 using the method of successive approximations. That is the successive approximations

$$\phi_0(t) = \mathbf{x}_0, \qquad \phi_{k+1}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \phi_k(s))\, ds, \quad k = 0, 1, 2, \ldots$$

converge uniformly on the interval $I = [t_0 - \alpha, t_0 + \alpha]$ with $\alpha = \min\{a, b/M\}$, to a solution of the initial value problem $\dfrac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0$ on $I$. Uniqueness is proved by Gronwall's inequality as before.

**Example 6.10** Find the first 5 successive approximations to the initial value problem

$$x'' = -e^t x, \; x(0) = 1, x'(0) = 0.$$

**Solution**. The initial value problem is equivalent to the following initial value problem of differential system.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} y(t) \\ -e^t x(t) \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We start with

$$\begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Then

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ -e^s \times 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 1 - e^t \end{pmatrix}.$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1-e^s \\ -e^s \end{pmatrix} ds = \begin{pmatrix} 2+t-e^{-t} \\ 1-e^t \end{pmatrix}.$$

$$\begin{pmatrix} x_3(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 1-e^s \\ -e^s(2+s-e^s) \end{pmatrix} ds = \begin{pmatrix} 2+t-e^{-t} \\ \frac{1}{2}-e^t-te^t+\frac{1}{2}e^{2t} \end{pmatrix}.$$

$$\begin{pmatrix} x_4(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{2}-e^s-se^s+\frac{1}{2}e^{2s} \\ -e^s(2+s-e^s) \end{pmatrix} ds = \begin{pmatrix} \frac{3}{4}+\frac{t}{2}-te^t+\frac{1}{4}e^{2t} \\ 1+t-e^t-te^t \end{pmatrix}.$$

**Example 6.11** Consider the linear differential system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = (a_{ij})$ is an $n \times n$ constant matrix. Let $\mathbf{f}(t, \mathbf{x}) = \mathbf{A}\mathbf{x}$. For any $a > 0$ and for all $|t| < a$, we have $|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)| = |\mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2)| \leq |\mathbf{A}||\mathbf{x}_1 - \mathbf{x}_2|$, where $|\mathbf{A}| = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|$, so that $\mathbf{f}$ satisfies a Lipschitz condition on the strip $S = \{(t, \mathbf{x}) \in \mathbb{R}^{n+1} : |t| \leq a\}$. Therefore the system has a unique solution for any initial value and is defined on the entire $\mathbb{R}$.

**Example 6.12** Let $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t) = (a_{ij}(t))$ is an $n \times n$ matrix of continuous functions defined on a closed interval $I$. Let $|a_{ij}(t)| \leq K$ for all $t \in I$ and all $i, j = 1, \ldots n$.
Thus if $\mathbf{f}(t, \mathbf{x}) = \mathbf{A}(t)\mathbf{x}$, then

$$\frac{\partial \mathbf{f}}{\partial x_k} = \begin{pmatrix} a_{1k}(t) \\ a_{2k}(t) \\ \vdots \\ a_{nk}(t) \end{pmatrix},$$

which is independent of $\mathbf{x}$.
Therefore,

$$\left| \frac{\partial \mathbf{f}}{\partial x_k} \right| = \sum_{i=1}^n |a_{ik}(t)| \leq nK \equiv L, \qquad \text{for all } t \in I \text{ and } k = 1, \ldots, n.$$

By theorem 6.14 the function $\mathbf{f}$ satisfies a Lipschitz condition on the strip

$$S = \{(t, \mathbf{x}) \in \mathbb{R}^{1+n} \mid t \in I\}$$

with Lipschitz constant $L$. Thus by corollary 6.17, the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ has a unique solution for any initial value in $S$ and is defined on all of $I$. Thus if $\mathbf{A}(t) = (a_{ij}(t))$ is continuous on an open interval $I$ or $\mathbb{R}$, then $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ has a unique solution for any initial value $\mathbf{x}(t_0) = \mathbf{x}_0$ with $t_0 \in I$ or $\mathbb{R}$, and is defined on $I$ or $\mathbb{R}$. (See corollary 6.6.) In particular this together with corollary 6.17 prove theorem 2.1 and theorem 4.1.

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