

## A STRONGER NOTION OF CONNECTEDNESS

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In the discussion of connectedness of topological spaces, we often come across the result that a space  $X$  is connected if and only if every continuous map from  $X$  to the discrete space of two points must be constant. One would wonder what sort of notion of connectedness would come up if the discrete space of two points is replaced by some other spaces. It turns out if one replaces the discrete space of two points by any finite space, it will not give rise to any interesting concept of connectedness other than the usual one. However, if one considers the space of all integers  $\mathcal{Z}$  equipped with the finite complement topology, then a slightly stronger notion of connectedness is generated. Let's call a space  $X$  **strongly connected** if  $X$  is not a disjoint union of countably many but more than one closed sets. Equivalently we have the following characterization.

**Theorem 1.**  $X$  is strongly connected if and only if every continuous map from  $X$  to  $\mathcal{Z}$  is constant.

**Proof.** Let  $f : X \rightarrow \mathcal{Z}$  be continuous. Since a continuous image of a strongly connected space is strongly connected,  $f[X]$  is strongly connected. The only strongly connected subspaces of  $\mathcal{Z}$  are the one-point spaces. Hence  $f$  is constant. Conversely, suppose that  $X$  is a disjoint union  $\cup_{i=1}^{\infty} E_i$  of closed sets  $E_i$ , where at least two of them are nonempty. Define  $f : X \rightarrow \mathcal{Z}$  by taking  $f(x) = i$  whenever  $x \in E_i$ . Then  $f$  is continuous and not constant. ■

Clearly if a space is strongly connected, then it is connected. On the other hand, the space  $\mathcal{Z}$  is connected but not strongly connected. In [1], Bing produces a countable Hausdorff connected space which is not strongly connected. In [4], there is an example of a connected subset of  $\mathbb{R}^2$  which is not strongly connected. Strong connectedness have been widely discussed in the context of continuum in the literature such as [2] and [3]. The following result is proved in [3].

**Theorem 2.** Let  $X$  be a compact Hausdorff space. Then  $X$  is connected if and only if  $X$  is strongly connected.

However this result cannot be applied directly to noncompact spaces. In this note, we would like to introduce a couple of results which enable us to deduce that  $\mathbb{R}^n$  is strongly connected. For this, we need a local concept of strong connectedness. We say that a space  $X$  is **locally strongly connected** if it has a basis consisting of strongly connected open sets.

**Theorem 3.** Let  $X$  be a locally compact Hausdorff space. Suppose  $X$  is locally connected. Then  $X$  is locally strongly connected.

**Proof.** Let  $O$  be an open neighborhood of a point  $x \in X$ . Then there exists a compact neighborhood  $V$  of  $x$  lying inside  $O$ . Let  $C$  be a connected component of  $V$  containing  $x$ . Since  $V$  is a neighborhood of  $x$  and  $X$  is locally connected,  $C$  is a neighborhood of  $x$ . Since  $C$  is closed in  $V$  and  $V$  is compact,  $C$  is also compact. This shows that  $C$  is a compact connected neighborhood of  $x$  lying inside  $O$ . By Theorem 2,  $C$  is strongly connected. ■

**Corollary 4.** Let  $X$  be a locally compact Hausdorff space. Suppose  $X$  is locally connected and connected. Then  $X$  is strongly connected.

**Proof.** The strongly connected components of a space  $X$  are the maximal strongly connected subsets of  $X$ . They are clearly closed in  $X$ . If the space is locally strongly connected, then they are also open in  $X$ . Hence if  $X$  is connected and locally strongly connected, then there is only one strongly connected component of  $X$ . In other words,  $X$  is strongly connected. ■

Lastly, the completeness of a metric space is another property which can promote connectedness to strong connectedness.

**Theorem 5.** A connected locally connected complete metric space is strongly connected.

**Proof.** Let  $X$  be a connected locally connected complete metric space. Suppose that  $X$  is a countable disjoint union  $\cup_{i=1}^{\infty} E_i$ , where each  $E_i$  is a closed subset of  $X$ . Since  $X$  is connected, we may assume that each  $E_i$  is nonempty.

Let  $k$  be a positive integer. Since  $X$  is a locally connected metric space, the collection  $\mathcal{C}_k$  of all connected open sets of diameter less than  $\frac{1}{k}$  is a basis for the topology of  $X$ .

Let  $\mathcal{D}_1 = \{O \in \mathcal{C}_1 \mid O \subset X \setminus E_1\}$ . Then  $X \setminus E_1 = \cup_{i=2}^{\infty} E_i = \cup_{O \in \mathcal{D}_1} O$ . We assert that one of the member in  $\mathcal{D}_1$  must intersect infinitely many  $E_i$ 's for  $i \geq 2$ . If not, then each of these connected open sets in  $\mathcal{D}_1$  must lie in some  $E_i$  for some  $i \geq 2$ . Thus each  $E_i$  with  $i \geq 2$  is a union of some of these open sets and hence is open. This contradicts the fact that  $X$  is connected. Let  $O_1$  be a member in  $\mathcal{D}_1$  which intersects infinitely many  $E_i$ 's for  $i \geq 2$ . In particular, this implies that  $O_1 \setminus E_2$  is nonempty.

Next let  $\mathcal{D}_2 = \{O \in \mathcal{C}_2 \mid O \subset O_1 \setminus E_2\}$ . Then  $O_1 \setminus E_2 = \cup_{O \in \mathcal{D}_2} O$ . As before the connectedness of  $O_1$  implies that there exists a member  $O_2$  in  $\mathcal{D}_2$  which intersects infinitely many  $E_i$ 's for  $i \geq 3$ . Therefore  $O_2 \setminus E_3$  is nonempty.

Continuing in this way produces a decreasing sequence  $O_1 \supset O_2 \supset O_3 \supset \dots$  of nonempty connected open subsets of  $X$  such that for each positive integer  $k$ ,  $O_k \in \mathcal{C}_k$  and  $O_k \subset O_{k-1} \setminus E_k$ . Here  $O_0$  is taken to be  $X$ .

Now, for each positive integer  $k$ , pick an element  $x_k \in O_k$ . Since the diameter of  $O_k$  is less than  $\frac{1}{k}$ ,  $(x_k)$  is a Cauchy sequence. By the completeness of  $X$ ,  $(x_k)$  converges to an element  $x \in X$ . Hence  $x$  lies in some  $E_n$ . But  $x$  also lies in  $O_n$ . This contradicts the fact that  $O_n$  is disjoint from  $E_n$ . Consequently,  $X$  cannot be expressed as a disjoint union of countably many but more than one nonempty closed subsets. This shows that  $X$  is strongly connected. ■

**Corollary 6.**  $\mathbb{R}^n$  is strongly connected and locally strongly connected.

**Corollary 7.** A subset in  $\mathbb{R}$  is strongly connected if and only if it is an interval.

## REFERENCES

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