THE BELT TRICK

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1. Introduction

With some practice, it is not too hard to see that it is possible to untie two full twists of a belt with one end attached to a ball and the other end attached to the ceiling by simply isotoping the belt in $\mathbb{R}^3$ but not moving the ball. To do so, one simply loops the belt underneath the ball. This is the so-called belt trick. The belt trick unfolds some of the interesting results in both topology and algebra. It explains why the fundamental group of the unit tangent bundle of the 2-sphere, or equivalently $\pi_1(SO_3)$ is $\mathbb{Z}_2$. It also provides a geometric interpretation for the group extension of the symmetry group of a geometric figure by $\mathbb{Z}_2$. In this note, we introduce Kauffman's Belt Theorem in [1]. Related results can be found in [2] and [3]

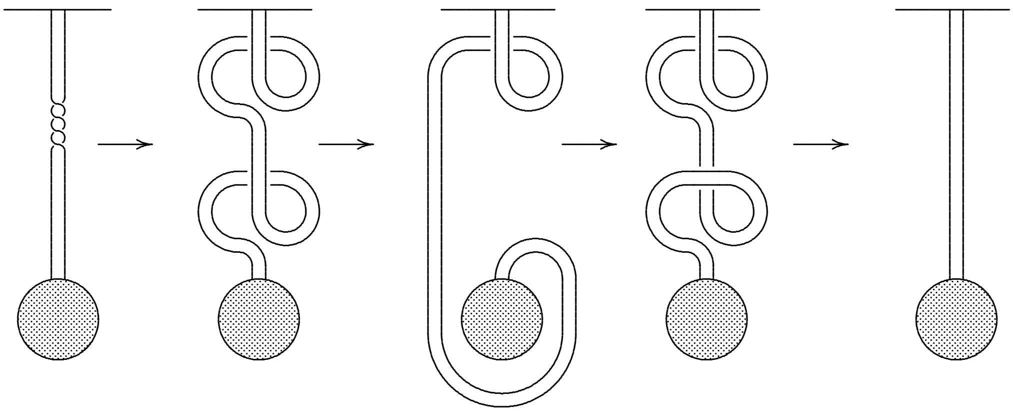
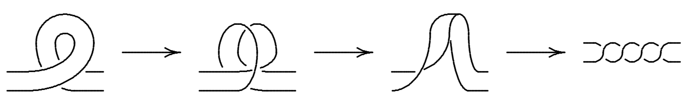


Figure 1. The belt trick

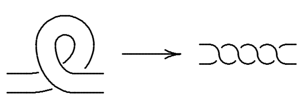
2. Curls and Twists

A curl consists of a band that loops over or under itself once. Up to isotopy fixing the ends, a curl is the same as a twisted band.

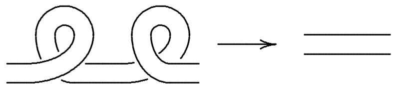
(I) Left-handed curl



(II) Right-handed curl



In figure 2 (I), the left-handed curl is shown to be the same as a band with a left-handed twist. Similarly, the right-handed curl is simply a band with a right-handed twist. Obviously, two curls of opposite sense put together cancel each other.

Cancellation of curls 

To see how the belt trick works, it is easier to regard two full left-handed twists as two consecutive curls as in figure 1. The lower curl is brought down from the front and looped underneath the ball. Bringing it back to the belt, it changes to a right-handed curl and thus cancels with the remaining left-handed curl.

3. The Klein 4-group

In this section, we recall the familiar Klein 4-group as the symmetry group of a rectangle in $\mathbb{R}^3$. We shall see in the next section how its double group which is the quaternion group can be described as the symmetry group in $\mathbb{R}^3$ of a rectangle with a belt attached to it to the ceiling.

The Klein 4-group $\mathbb{K} = \{1, I, J, K\}$ has four elements and its multiplication table is shown in table 1.

$\bullet$	1	I	J	K
1	1	I	J	K
I	I	1	K	J
J	J	K	1	I
K	K	J	I	1

Table 1.

$+$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$

Table 2.

By comparing the multiplication tables of $\mathbb{K}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, we see that $\mathbb{K}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Also $\mathbb{K}$ is the symmetry group of a rectangle in $\mathbb{R}^3$.

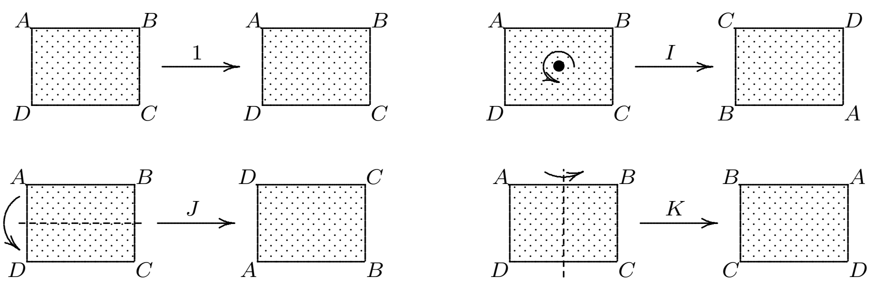


Figure 2. $\mathbb{K}$ is the symmetry group of a rectangle in $\mathbb{R}^3$

Here 1 is the identity transformation, I is the rotation through an angle $180°$ about the axis perpendicular to the plane of the rectangle and passing through the center of the rectangle, J is the rotation through an angle $180°$ about the axis passing through the midpoints of AD and BC and K is the rotation through an angle $180°$ about the axis passing through the midpoints of AB and DC. Check that for instance $IJ = K$. One may also consider $\mathbb{K}$ as the set of configurations of the images of $1, I, J, K$. Note that the symmetry group of a square is the dihedral group D_4 of order 8.

4. Quaternion Group

The quaternion group $\mathbb{Q} = \{1, i, j, k, -1, -i, -j, -k\}$ has 8 elements. Its multiplication table is shown in table 3.

$\bullet$	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

Table 3. The multiplication table for $\mathbb{Q}$

Alternatively, the group multiplication can be determined by the rules: $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$ and 1=idenity.

Let $p : \mathbb{Q} \longrightarrow \mathbb{K}$ be given by $p(\pm 1) = 1$, $p(\pm i) = I$, $p(\pm j) = J$ and $p(\pm k) = K$. Then p is a group homomorphism and is a two-to-one map. Now can we still interpret elements of $\mathbb{Q}$ as motions of geometric objects as those in the Klein 4-group? The answer is to attach a belt to the rectangle and realize $-1, i, j, k$ as the motions described in figure 4. Note that all rotations are in the anticlockwise sense and along the directions of the positive right-handed xyz-coordinate axes. The resulting half twist is left-handed. Alternatively, regard each of the elements in $\mathbb{Q}$ as the corresponding configuration of the rectangle in $\mathbb{R}^3$ together with the number of twists in the belt. Here the number of twists in the belt is measured up to isotopy of the belt fixing its ends. By the belt trick, we know that it can have at most up to two half twists in either sense. Under these interpretations, check that, $i^2 = j^2 = k^2 = -1$. Also $ij = k, jk = i$ and $ki = j$. Now the map $p : \mathbb{Q} \longrightarrow \mathbb{K}$ is given by forgetting the belt or the twists. Moreover $\operatorname{Ker} p = \{-1, 1\} \cong \mathbb{Z}_2$. However, $\mathbb{Q}$ does not contain a copy of $\mathbb{K}$ so that $\mathbb{Q} \ncong \mathbb{K} \rtimes \mathbb{Z}_2$.

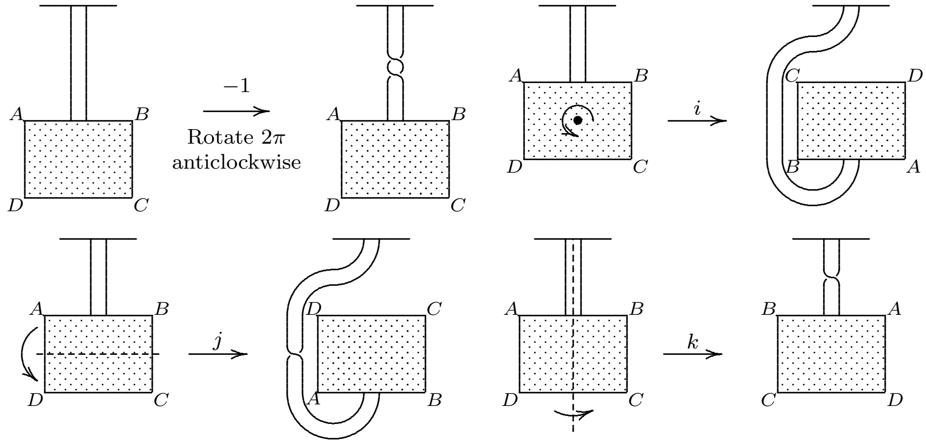


Figure 4. The quaternions as configurations of a rectangle in $\mathbb{R}^3$ with a belt attached to it.

We have seen that the quaternion group arises from the Klein 4-group, by attaching a belt to the rectangle whose symmetries generate $\mathbb{K}$. With this attachment, the rotations all double in value (2π rotation is nontrivial and is of order 2), and the resulting group of rotations coupled with the sign indicated on the belt is isomorphic to $\mathbb{Q}$. One may also consider $\mathbb{Q}$ as the set of configurations of the rectangle together with the state of the belt, which we shall call a Λ-configuration.

5. The double group of a subgroup of $SO(3)$

Let G be a subgroup of $SO(3)$. Regard the 3-sphere S^3 as the group of unit quaternions. That is $S^3 = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\}$. Let $p : S^3 \longrightarrow SO(3)$ be the homomorphism given by $p(g)(v) = gvg^{-1}$, where $v \in \mathbb{R}^3 =$ Set of quaternions of the form $ai + bj + ck,\ a, b, c \in \mathbb{R}$. The group homomorphism p is a two-to-one and the preimage $\overline{G} = p^{-1}[G]$ is a subgroup of S^3, called the double group of G.
Let A be an object in $\mathbb{R}^3$, and let $G(A)$ denote the subgroup of $SO(3)$ of rotations that preserve A. That is $G(A) = \{s \in SO(3) \mid s(A) = A\}$. Now attach one end of a belt to A and the other end to the ceiling. As in [1], we say that two rotations of A are Λ-equivalent if they are identical on A, and they produce the same state on the belt (i.e. its isotopy class in $\mathbb{R}^3$). We shall call a Λ-equivalence class of rotations of A a Λ-configuration of A.

The Belt theorem. The set of Λ-configurations of A is isomorphic to the double group $\overline{G}(A)$.

Proof. Each position or configuration of A in space curls the belt, but by the belt trick we may compare 2π and 4π rotations of A and find that they diff by a 2π twist in the belt, say away from the curl. That is, one can make the Λ-configurations of A the same as the usual configurations of A together possibly with an extra twist in the belt. As a result, we get a doubling of the set of configurations of A to the set of Λ-configurations.

6. The unit tangent bundle of the sphere and the real projective space

Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the 2-dimensional sphere with radius 1. The unit tangent bundle is the space $TS^2 = \{(x, v) \in S^2 \times \mathbb{R}^3 \mid v \text{ is a unit tangent vector to } S^2 \text{ at } x\}$. For example $\mathbf{u} = ((0,0,1), \langle 1,0,0\rangle)$ and $\mathbf{v} = ((0,0,1), \langle 0,1,0\rangle)$ are elements of TS^2. The set $\{\mathbf{u}, \mathbf{v}\}$ spans the tangent plane to S^2 at the point $(0,0,1)$. Consider the path $\alpha : [0,1] \longrightarrow TS^2$ given by $\alpha(t) = ((0,0,1), \langle \cos(4\pi t), \sin(4\pi t), 0\rangle)$. It is a loop inside TS^2 joining $\mathbf{u}$ to $\mathbf{u}$. It may also be represented as a belt attached to the north pole $(0,0,1)$ with a 4π twist. In other words, a twisted belt attached to the north pole corresponds to a loop in TS^2 based at the north pole. The belt trick demonstrates the fact that the loop α can be deformed to the trivial constant loop inside TS^2. This essentially proves that $\pi_1(TS^2) = \mathbb{Z}_2$.

$$\begin{aligned}\text{Let } SO(3) &= \{A \mid A \text{ is a } 3\times 3 \text{ orthogoanl matrix with determinant equals to } 1\} \\ &= \{\langle v_1, v_2, v_3\rangle \mid v_1, v_2, v_3 \text{ form a right-handed orthonormal frame in } \mathbb{R}^3\} \\ &= \{R \mid R \text{ is a rotation in } \mathbb{R}^3\}.\end{aligned}$$

Each element $\mathbf{w} = (x, v)$ in TS^2 corresponds to an element $(x, v, x \times v)$ in $SO(3)$, where $x \times v$ means the cross product of the two vectors x and v in $\mathbb{R}^3$. Thus $TS^2 \cong SO(3)$.

The 3-dimensional projective real space RP^3 is defined as the quotient space of the 3-dimensional ball B^3 with radius π under the equivalence relation that identifies antipodal points. Alternatively, $RP^3 = \{\ell \mid \ell \text{ is a line through the origin in } \mathbb{R}^3\}$. Now for each element $x \in RP^3$, we can associate a rotation about the axis in the direction from the origin to the point x through an angle $\|x\|$ in the anticlockwise sense. Thus $RP^3 \cong SO(3) \cong TS^2$. Combining all, the belt trick shows that $\pi_1(TS^2) = \pi_1(SO(3)) = \pi_1(RP^3) = \mathbb{Z}_2$.

Lastly we include the following Maple statement which animates the belt trick.

```
animate3d([-s*sin(2*Pi*u)*cos(Pi*t)+(9*u+1)*sin(Pi*u)*sin(Pi*u)*sin(2*Pi*t),
s*cos(2*Pi*u)-(9*u+1)*sin(2*Pi*u)*sin(Pi*t),s*sin(2*Pi*u)*sin(Pi*t)
+(9*u+1)*(sin(Pi*u)*sin(Pi*u)*cos(2*Pi*t)+cos(Pi*u)*cos(Pi*u))],
[3*sin(Pi*u)*cos(Pi*(s/2+1)),3*sin(Pi*u)*sin(Pi*(s/2+1)),3*cos(Pi*u)-2],
s=-2.5..2.5,u=0..1,t=0..1,numpoints=200,frames=15,scaling=CONSTRAINED,
shading=ZHUE);
```

References

[1] Kauffman, L.H., Knots and Physics, 2nd edition, *World Scientific* (1993), 419-436.

[2] Jurišić, A. The Mercedes Knot Problem, *The American Mathematical Monthly*, Vol. 103, No.**9**, November, (1996), 756-770.

[3] Stojanoska, V., Stoytchev, O., Touching the $\mathbb{Z}_2$ in Three-Dimensional Rotations, *Mathematics Magazine*, Vol. 81, No., 5, December (2008), 345-357.

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