

A BRIEF HISTORY OF GREEK MATHEMATICS

At the dawn of civilization, man discovered two mathematical concepts: “multiplicity” and “space”. The first notion involved counting (of animals, days, *etc.*) and the second involved areas and volumes (of land, water, crop yield, *etc.*). These evolved into two major branches of mathematics—arithmetic and geometry. (The word “geometry” is derived from the greek roots “geo” meaning “earth” and “metrein” meaning “measure”.) The mathematics of the Egyptians and the Babylonians was essentially empirical in nature. It has been traditional to state that demonstrative mathematics first appeared in the sixth century B.C. The Greek geometer, Thales of Miletus, is credited for giving some logical reasoning (rather than by intuition and experimentation) for several elementary results involving circles and angles of triangles. The next major Greek mathematician is Pythagoras (born *ca.* 572 B.C.) of Samos. He founded a scholarly society called the Pythagorean brotherhood (it was an academy for the study of philosophy, mathematics and natural science; it was also a society with secret rites). The Pythagoreans believed in the special role of “whole number” as the foundation of all natural phenomena. The Pythagoreans gave us 2 important results: Pythagorean theorem, and more importantly (albeit reluctantly), the irrational quantities (which struck a blow against the supremacy of the whole numbers). The Pythagorean, Hippasus, is credited with the discovery that the side of a square and its diagonal are incommensurable (*i.e.* a square of length 1 has a diagonal with irrational length). This showed that the whole numbers are inadequate to represent the ratios of all geometric lengths. This discovery established the supremacy of geometry over arithmetic in all subsequent greek mathematics. For all the trouble that Hippasus caused, the Pythagoreans supposedly took him far out into the Mediterranean and tossed him overboard to his death—thereby indicating the dangers inherent in free thinking, even in the relatively austere discipline of mathematics! Hippocrates of Chios (born *ca.* 440 B.C.) is credited with two significant contributions to geometry. The first was his composition of the first *Elements*: the first exposition developing the theorems of geometry precisely and logically from a few given axioms and postulates. This treatise (which has been lost to history) was rendered obsolete by that of Euclid. His other contribution was the quadrature of the lune.

Plato (427–347 B.C.) studied philosophy in Athens under Socrates. He then set out on his travels, studying mathematics under Theodorus of Cyrene in North Africa. On his return to Athens in 387 B.C. he founded the Academy. Plato’s influence on mathematics was not due to any mathematical discoveries, but rather to his conviction that the study of mathematics provides the best training for the mind, and was hence essential for the cultivation of philosophers. The renowned motto over the door of his Academy states:

Let no one ignorant of geometry enter here.

Aristotle, a pupil of Plato, was primarily a philosopher. His contribution to mathematics is his analysis of the roles of definitions and hypotheses in mathematics. Plato’s Academy did however produce some great mathematicians, one of whom is Eudoxus (*ca.* 408–355 B.C.). With the discovery of the incommensurables certain proofs of the Pythagoreans in geometric theorems (such as those on similar triangles) were rendered false. Eudoxus developed the theory of proportions which circumvented these problems. This theory led directly to the work of Dedekind (Dedekind cuts) in the nineteenth century. His other great contribution, the method of exhaustion, has applications in the determination of areas and volumes of sophisticated geometric figures. This process was used by Archimedes to determine the area of the circle. The method of exhaustion can be considered the geometric forerunner of the modern notion of “limit” in integral calculus. Menaechmus, a pupil of Eudoxus, discovered the conic sections. There is a legendary story told about Alexander the Great (356–323 B.C.) who is said to have asked his tutor, Menaechmus, to teach him geometry concisely, to which the latter replied,

O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all.

Alexander entered Egypt and established the city of Alexandria at the mouth of the Nile in 332 B.C. This city grew rapidly and reached a population of half a million within three decades. Alexander’s empire fell apart after his death. In 306 B.C. one of his generals, Ptolemy, son of Lagos, declared himself King Ptolemy I (thereby establishing the Ptolemaic dynasty). The Museum and Library of Alexandria were built under Ptolemy I. Alexandria soon supplanted the Academy as the foremost center of scholarship in the world. At one point, the Library had over 600,000 papyrus rolls. Alexandria remained the intellectual metropolis of the greek race until its destruction in A.D. 641 at the hands of the Arabs. Among the scholars attracted to Alexandria around 300 B.C. was Euclid, who set up a school

of mathematics. He wrote the *Elements*. This had a profound influence on western thought as it was studied and analyzed for centuries. It was divided into thirteen books and contained 465 propositions from plane and solid geometry and from number theory. His genius was not so much in creating new mathematics but rather in the presentation of old mathematics in a clear, logical and organized manner. He provided us with an *axiomatic* development of the subject. The *Elements* begins with 23 definitions, 5 postulates and 5 common notions or general axioms. From these he proved his first proposition. All subsequent results were obtained from a blend of his definitions, postulates, axioms and previously proven propositions. He thus avoided circular arguments. There is a story about Euclid (reminiscent of the one about Menaechmus and Alexander): Ptolemy I once asked Euclid if there was in geometry any shorter way than that of the *Elements*, to which Euclid replied, “There is no royal road to geometry”.

The greatest mathematician of antiquity is Archimedes (287–212 B.C.) of Syracuse, Sicily. He made great contributions in applied mechanics (especially during the second Punic War against the Romans—missile weapons, crane-like beaks and iron claws for seizing ships, spinning them around, sinking or shattering them against cliffs, ...), astronomy and hydrostatics. He devised methods for computing areas of curvilinear plane figures, volumes bounded by curved surfaces and methods of approximating π . Using the method of exhaustion (of Eudoxes), he anticipated the integral calculus of Newton and Leibniz by more than 2000 years; in one of his problems he also anticipated their invention of differential calculus. The above-mentioned problem involved constructing a tangent at any point of his spiral. There are numerous stories told about Archimedes. According to Plutarch, Archimedes would

... forget his food and neglect his person, to that degree that when he was occasionally carried by absolute violence to bathe or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil of his body, being in a state of entire preoccupation, and, in the truest sense, divine possession with his love and delight in science.

The death of Archimedes as told by Plutarch:

... as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, he never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declined to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through.

The next major mathematician of the third century B.C. was Apollonius (262–190 B.C.) of Perga, Asia Minor. His claim to fame rests on his work *Conic Sections* in eight books. It contains 400 propositions and supercedes the work in that subject of Menaechmus, Aristaeus and Euclid. The names “ellipse”, “parabola” and “hyperbola” were supplied by Apollonius. His methods are similar to modern methods and he is said to have anticipated the analytic geometry of Descartes by 1800 years.

The end of the third century B.C. saw the end of the Golden Age of Greek Mathematics. In the next three centuries only one mathematician made a significant contribution, Hipparchus of Nicaea (180–125 B.C.), who founded trigonometry.

SOME QUOTATIONS

Carl Friedrich Gauss: “... that the sum of the angles cannot be less than 180° : this is the critical point, the reef on which all the wrecks occur.”

Janos Bolyai: “Out of nothing I have created a strange new universe.”

Bernhard Riemann: “The unboundedness of space possesses a greater empirical certainty than any external experience. But its infinite extent by no means follows from this.”

Bertrand Russell: “... what matters in mathematics ... is not the intrinsic nature of our terms but the logical nature of their interrelations.”

EXTRACT FROM BOOK I OF EUCLID'S ELEMENTS

To illustrate the systematic approach that Euclid used in his elements, we include below an extract from Book 1 of the Elements.

DEFINITIONS

- (1) A *point* is that which has no part.
- (2) A *line* is breadthless length.
- (3) The extremities of a line are points.
- (4) A *straight line* is a line which lies evenly with the points on itself.
- (5) A *surface* is that which has length and breadth only.
- (6) The extremities of a surface are lines.
- (7) A *plane surface* is a surface which lies evenly with the straight lines on itself.
- (8) A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
- (9) And where the lines containing the angles are straight, the angle is called *rectilineal*.
- (10) When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other side is called a *perpendicular* to that on which it stands.
- (11) An *obtuse angle* is an angle greater than a right angle.
- (12) An *acute angle* is an angle less than a right angle.
- (13) A *boundary* is that which is an extremity of anything.
- (14) A *figure* is that which is contained by any boundary or boundaries.
- (15) A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
- (16) And the point is called the *centre* of the circle.
- (17) A *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
- (18) A *semicircle* is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.
- (19) *Rectilineal figures* are those which are contained by straight lines, *trilateral* figures being those contained by three, *quadrilateral* those contained by four, and *multilateral* those contained by more than four straight lines.
- (20) Of trilateral figures, an *equilateral triangle* is that which has its three sides equal, an *isosceles triangle* that which has two of its sides alone equal, and a *scalene triangle* that which has its three sides unequal.
- (21) Further, of trilateral figures, a *right-angled triangle* is that which has a right angle, an *obtuse-angled triangle* that which has an obtuse angle, and an *acute-angled triangle* that which has its three angles acute.
- (22) Of quadrilateral figures, a *square* is that which is both equilateral and right-angled; an *oblong* that which is right-angled but not equilateral; a *rhombus* that which is equilateral but not right-angled; and a *rhomboid* that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let *quadrilaterals* other than these be called *trapezia*.
- (23) *Parallel* straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

COMMON NOTIONS

- (1) Things which are equal to the same thing are also equal to one another.
- (2) If equals be added to equals, the wholes are equal.
- (3) If equals be subtracted from equals, the remainders are equal.
- (4) Things which coincide with one another are equal to one another.
- (5) The whole is greater than the part.

POSTULATES

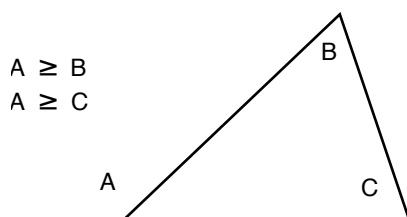
Let the following be postulated:

- (1) To draw a straight line from any point to any point.
- (2) To produce a finite straight line continuously in a straight line.
- (3) To describe a circle with any centre and distance.
- (4) That all right angles are equal to one another.
- (5) That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

There is much that can be said about Euclid's "definitions". However, we shall refrain from doing so. We shall instead examine more closely the controversial fifth postulate. Over a hundred years ago, a postulate was supposed to be a "self-evident truth". Observe that the statements of the first four postulates are short and "self-evident", and thus, readily acceptable by most people. However, the statement of the fifth postulate is rather long and sounds complicated. Nevertheless, it was deemed to be necessarily true, and hence one should be able to derive it from the other four postulates and the definitions. The problem then is to prove the fifth postulate as a theorem.

Let us first take another look at Postulate 2. It pertains to extensions of a finite straight line. Euclid implicitly assumed that Postulate 2 implies that straight lines must be "unbounded in extent", or are "infinitely long". As it turns out, this is a hidden postulate (*i.e.* it should be considered a separate postulate). Euclid made tacit use of this assumption in proving Proposition 16 of Book I of "Elements":

Proposition I.16. *In any triangle, if any one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.*

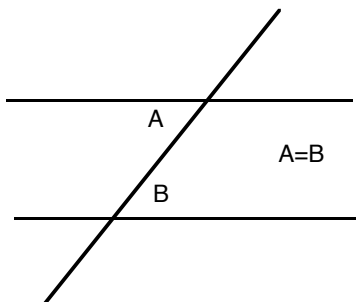


This proposition was then used to prove:

Proposition I.17. *In any triangle, two angles taken together in any manner are less than two right angles.*

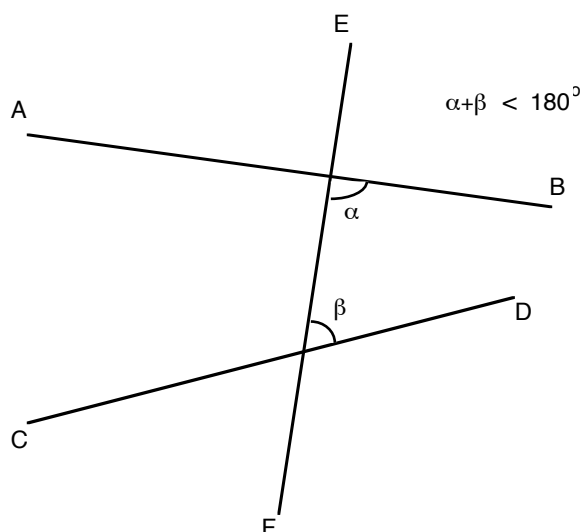
The following proposition also uses Proposition I.16 and hence requires that straight lines be infinitely long:

Proposition I.27. *If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.*



Let us state Postulate 5 in modern language and notation.

Postulate 5. *If AB and CD are cut by EF so that $\alpha + \beta < 180^\circ$, then AB and CD meet in the direction of B and D .*



Euclid did not make use of Postulate 5 until Proposition 29 of Book I:

Proposition I.29. *A straight line falling on parallel straight lines makes the alternate angles equal to one another.*

He used this to prove the following (stated here in modern language and notation):

Proposition I.32. *The exterior angle of a triangle is equal to the sum of the two interior and opposite angles; the sum of the interior angles of a triangle is 180° .*

Mathematicians, as early as Proclus (410–485 A.D.), have tried to prove Postulate 5 from Postulates 1–4 *directly*. However, they make *implicit* use of “unstated axioms” in their “proofs”. These unstated axioms were later found to be *logically equivalent* to the fifth postulate. This means that

- (1) Postulates 1–4 + “unstated axiom” implies Postulate 5; and conversely,
- (2) Postulates 1–5 implies “unstated axiom”.

Thus, the “proofs” involved circular reasoning.

Proclus’ proof involved the use of the following unstated axiom:

Proclus’ Axiom. *If a straight line cuts one of 2 parallels, it must cut the other side also.*

Other famous axioms logically equivalent to Postulate 5 include:

Playfair’s Axiom. *If p is a point not on a line ℓ , then there is exactly one line through p that is parallel to ℓ .*

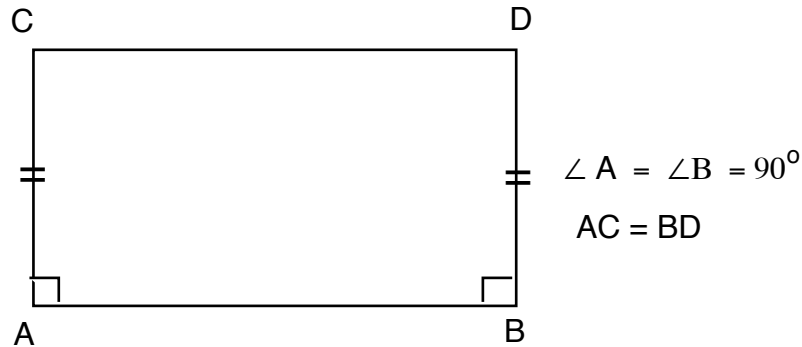
(Playfair’s Axiom is actually a restatement of Euclid’s Proposition I.31 in modern language and notation.)

Equidistance Axiom. *Parallel lines are everywhere equidistant.*

It turns out the the second part of Proposition I.32 is also logically equivalent to Postulate 5:

Angle sum of triangle Axiom. *The sum of the interior angles of a triangle is 180° .*

Gerolamo Saccheri (1667–1733) was the first mathematician who attempted to prove the fifth postulate via an *indirect* method—*reductio ad absurdum*. This means that he assumed that Postulate 5 was not true and he attempted to derive a contradiction. In his book *Euclides ad omni naevo vindicatus* (Euclid vindicated of all flaws), he introduced what is now called the Saccheri quadrilateral. It is a quadrilateral $ABCD$ such that AB forms the base, AC and BD the sides such that $AC = BD$, and the angles at A and B are right angles. We shall refer to the angles $\angle C$ and $\angle D$ as *summit angles*.



Saccheri first proved that $\angle C = \angle D$. He then considered three mutually exclusive hypotheses regarding the summit angles:

- HRA** (Hypothesis of the Right Angle.) The summit angles are right angles. ($\angle C = \angle D = 90^\circ$.)
- HAA** (Hypothesis of the Acute Angle.) The summit angles are acute angles. ($\angle C = \angle D < 90^\circ$.)
- HOA** (Hypothesis of the Obtuse Angle.) The summit angles are obtuse angles. ($\angle C = \angle D > 90^\circ$.)

HRA turns out to be logically equivalent to Postulate 5. To use the method of *reductio ad absurdum* Saccheri assumed that HRA is false. First, he would assume that HOA is true and show that this leads to a contradiction. Next, he would assume that HAA is true and show that this also leads to a contradiction.

When he adopted HOA, Saccheri indeed reached a contradiction. But in doing so, he made use of Proposition I.16, hence the hidden assumption that straight lines are infinitely long. It is precisely this hidden postulate that the HOA is contradicting. Saccheri thus proved (without knowing it) that Postulates 1–4, together with the additional postulate that straight lines are infinitely long, implies that the sum of the interior angles of a quadrilateral is equal to or less than 360° .

When Saccheri attempted to eliminate the possibility of HAA, he experienced greater difficulties. Assuming HAA, he derived the fact that the sum of the interior angles of a triangle is less than 180° , and other “strange results”; however, he never reached a contradiction. Nevertheless, he rejected HAA because “it is repugnant to the nature of the straight line”. He thus missed his chance of being a founder of the first non-euclidean geometry. Instead, the honors belong to three men—who discovered it independently.

Carl Friedrich Gauss (1777–1855) developed the geometry implied by HAA between the years 1810–1820, but he did not publish his results as he was too “timid”. We know this only because of his correspondence and his private papers which became available after his death.

Two contemporaries of Gauss, Janos Bolyai (1775–1856) and Nicolai Ivanovitch Lobachevsky (1793–1856), who worked independently of each other, are officially credited with the discovery of the first non-euclidean geometry. In 1829 Lobachevsky published his results (in russian) on the new geometry in the journal “Kazan Messenger”. Because this is a rather obscure journal, his results went unnoticed by the scientific community. In 1832 Janos Bolyai’s results were published in an appendix to a work called “Tentamen”, written by his father Wolfgang Bolyai, a lifelong friend of Gauss. Gauss received the manuscript and instantly recognized it as a work of a genius. However, Gauss was determined to avoid all controversy regarding non-euclidean geometry, and he not only suppressed his own work on the subject, but he also remained silent about the work of others on that subject.

In 1837 Lobachevsky published a paper on that subject in Crelle’s journal and in 1840 he wrote a small book in german on the same subject. Again Gauss recognized genius and in 1842 he proposed Lobachevsky for membership in the Royal Society of Göttingen. But Gauss made no statement regarding

Lobachevsky's work on non-euclidean geometry. He did not tell Lobachevsky about Janos Bolyai. Bolyai learned about Lobachevsky around 1848, but Lobachevsky died without knowing that he had a co-discoverer. Just before he died, when he was blind, Lobachevsky dictated an account of his revolutionary ideas about geometry. This was translated into french by Jules Houel in 1866, who also translated the works of Bolyai the following year. Thus began the widespread circulation of the ideas on the new non-euclidean geometry. Euclid was finally vindicated—Postulate 5 is indeed an independent postulate.

Consequence of HAA include:

- the sum of the angles of any triangle is less than 180° ,
- the existence of many parallel lines through a point not on a given line.

We now return to the hidden postulate on infinitely long lines. Bernhard Riemann (1826–1866) diluted Euclid's implicit assumption that lines are *infinitely long* and replaced it with *endlessness* or *unboundedness*. As a result, he discovered that

- (1) Proposition I.16 no longer holds;
- (2) Proposition I.17 no longer holds;
- (3) Proposition I.27 no longer holds.

Now Proposition I.27 allows us to construct at least one parallel through a point not on a given line. A consequence of the non-validity of Proposition I.27 is that under HOA, there are no parallel lines! In this new geometry, the sum of the angles of a triangle is greater than 180° .

By 1873 Felix Klein classified, unified and named the three geometries:

	Parabolic Geometry	Hyperbolic Geometry	Elliptic Geometry
Saccheri	HRA	HAA	HOA
Angle sum of \triangle	$= 180^\circ$	$< 180^\circ$	$> 180^\circ$
Playfair's	exactly 1 parallel	more than 1 parallel	no parallels
curvature	zero	negative	positive
founder	Euclid	Gauss, Bolyai, Lobachevski	Riemann