Point Set Topology

A. Topological Spaces and Continuous Maps

Definition 1.1 A topology on a set X is a collection $\mathcal{T}$ of subsets of X satisfying the following axioms:

$\mathcal{T}$1. $\emptyset, X \in \mathcal{T}$.

$\mathcal{T}$2. $\{O_\alpha \mid \alpha \in I\} \subseteq \mathcal{T} \implies \bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$.

$\mathcal{T}$3. $O, O' \in \mathcal{T} \implies O \cap O' \in \mathcal{T}$.

A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a topology on X. When the topology $\mathcal{T}$ on X under discussion is clear, we simply denote $(X, \mathcal{T})$ by X. Let $(X, \mathcal{T})$ be a topological space. Members of $\mathcal{T}$ are called open sets. ($\mathcal{T}$3) implies that a finite intersection of open sets is open.

Examples

1. Let X be a set. The power set 2^X of X is a topology on X and is called the discrete topology on X. The collection $\mathcal{I} = \{\emptyset, X\}$ is also a topology on X and is called the indiscrete topology on X.

2. Let (X, d) be a metric space. Define $O \subseteq X$ to be open if for any $x \in O$, there exists an open ball $B(x, r)$ lying inside O. Then, $\mathcal{T}_d = \{O \subseteq X \mid O \text{ is open}\} \cup \{\emptyset\}$ is a topology on X. $\mathcal{T}_d$ is called the topology induced by the metric d.

3. Since $\mathbb{R}^n$ is a metric space with the usual metric:

$$d((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

$\mathbb{R}^n$ has a topology $\mathcal{U}$ induced by d. This topology on $\mathbb{R}^n$ is called the usual topology.

4. Let X be an infinite set. Then, $\mathcal{T} = \{\emptyset, X\} \cup \{O \subseteq X \mid X \setminus O \text{ is finite}\}$ is a topology on X. $\mathcal{T}$ is called the complement finite topology on X.

5. Define $O \subseteq \mathbb{R}$ to be open if for any $x \in O$, there exists $\delta > 0$ such that $[x, x + \delta) \subseteq O$. Then, the collection $\mathcal{T}'$ of all such open sets is a topology on $\mathbb{R}$. $\mathcal{T}'$ is called the lower limit topology on $\mathbb{R}$. Note that $\mathcal{U} \subset \mathcal{T}'$ but $\mathcal{U} \neq \mathcal{T}'$. We shall denote this topological space by $\mathbb{R}_l$.

Proposition 1.2 Let $(X, \mathcal{T})$ be a topological space and A a subset of X. Then, the collection $\mathcal{T}_A = \{O \cap A \mid O \in \mathcal{T}\}$ is a topology on A. $\mathcal{T}_A$ is called the subspace topology on A and $(A, \mathcal{T}_A)$ is called a subspace of $(X, \mathcal{T})$.

Proof Exercise.

Let $(X, \mathcal{T})$ be a topological space.
Definition 1.3 $E \subseteq X$ is said to be closed if $X \setminus E$ is open. (i.e. $X \setminus E \in \mathcal{T}$.)

Proposition 1.4
(1) $\emptyset$ and X are closed in X.

(2) If E, F are closed in X, then $E \cup F$ is closed in X.

(3) If $\{E_\alpha \mid \alpha \in I\}$ is a collection of closed sets in X, then $\bigcap_{\alpha \in I} E_\alpha$ is closed in X.

Proof Exercise.

Definition 1.5 A basis for the topology $\mathcal{T}$ on X is a subcollection $\mathcal{B}$ of $\mathcal{T}$ such that any open set in X is a union of members of $\mathcal{B}$.

Definition 1.6 A subbasis for the topology $\mathcal{T}$ on X is subcollection $\mathcal{S}$ of $\mathcal{T}$ such that the collection of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$. Hence, any open subset of X is a union of finite intersections of members in $\mathcal{S}$.

Examples

1. $\{(a, b) \mid a < b\}$ is a basis for the usual topology on $\mathbb{R}$. $\{(a, \infty), (-\infty, b) \mid a, b \in \mathbb{R}\}$ is a subbasis for the usual topology on R.

2. Let (X, d) be a metric space and $\mathcal{T}_d$ the topology on X induced by d. The collection $\mathcal{B}$ of all open balls of X is a basis for $\mathcal{T}_d$.

3. $\{[a, b) \mid a < b\}$ is a basis for the lower limit topology on $\mathbb{R}_l$. What would be a subbasis for the lower limit topology on $\mathbb{R}_l$?

4. If $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ itself is a basis for $\mathcal{T}$.

Proposition 1.7 Let X be a set and $\mathcal{B}$ a collection of subsets of X such that

$\mathcal{B}$**1.** $\bigcup_{U \in \mathcal{B}} U = X$,

$\mathcal{B}$**2.** for any $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$, there exists $U \in \mathcal{B}$ such that $x \in U \subseteq U_1 \cap U_2$.

Then the collection $\mathcal{T}_\mathcal{B}$ of all unions of members of $\mathcal{B}$ is a topology on X and $\mathcal{B}$ is a basis for $\mathcal{T}_\mathcal{B}$.

Proof $\mathcal{T}1$ and $\mathcal{T}2$ are clearly satisfied. $\mathcal{B}2$ implies that if $U_\alpha, U_\beta \in \mathcal{B}$, then $U_\alpha \cap U_\beta \in \mathcal{T}_\mathcal{B}$. Let $O_1 = \cup U_\alpha$ and let $O_2 = \cup U_\beta$ be in $\mathcal{T}_\mathcal{B}$, where $U_\alpha, U_\beta \in \mathcal{B}$. Then

$O_1 \cap O_2 = \bigcup(U_\alpha \cap U_\beta)$ is in $\mathcal{T}$. That $\mathcal{B}$ is a basis for $\mathcal{T}_\mathcal{B}$ follows from the definition of $\mathcal{T}_\mathcal{B}$.

Proposition 1.8 Let X be a set and $\mathcal{S}$ be a collection of subsets of X such that $X = \bigcup_{V\in\mathcal{S}} V$. Then the collection $\mathcal{T}_\mathcal{S}$ of all unions of finite intersections of members of $\mathcal{S}$ is a topology on X and $\mathcal{S}$ is a subbasis for $\mathcal{T}_\mathcal{S}$.

Proof Similar to 1.7.

Definition 1.9 A subset N containing a point x in X is called a neighbourhood of x if there exists an open set O in X such that $x \in O \subseteq N$.

Definition 1.10 Let A be a subset of X.

(1) $x \in A$ is called an interior point of A if there exists a neighbourhood U of x inside A.

(2) $x \in X$ is called a limit point of A if $(O \setminus \{x\}) \cap A \neq \emptyset$, for any neighbourhood O of x.

(3) $A^\circ = \{x \in A \mid x \text{ is an interior point of } A\}$ is called the interior of A.

(4) $\overline{A} = A \cup \{x \in X \mid x \text{ is a limit point of } A\}$ is called the closure of A.

Proposition 1.11

(1) A° is an open set and $A^\circ \subseteq A$.

(2) $X \setminus \overline{A} = (X \setminus A)^\circ$ and $X \setminus A^\circ = \overline{X \setminus A}$.

(3) $\overline{A}$ is a closed set and $\overline{A} \supseteq A$.

(4) A is open $\iff A = A^\circ$.

(5) A is closed $\iff A = \overline{A}$.

(6) A° is the largest open set contained in A.

(7) $\overline{A}$ is the smallest closed set containing A.

(8) $A^\circ = \bigcup\{O \mid O \text{ is open and } O \subseteq A\}$.

(9) $\overline{A} = \bigcap\{E \mid E \text{ is closed and } E \supseteq A\}$.

(10) $A^{\circ\circ} = A^\circ$ and $\overline{\overline{A}} = \overline{A}$.

(11) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(12) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(13) Let $B \subseteq A \subseteq X$. Then B is closed in A if and only if $B = E \cap A$ for some closed set E in X.

(14) Let $B \subseteq A \subseteq X$ and let $\overline{B}^A$ be the closure of B in A. Then $\overline{B}^A = \overline{B} \cap A$.

(15) Let $B \subseteq A \subseteq X$ and let $B^{\circ A}$ be the interior of B in A. If A is open, then $B^{\circ A} = B^\circ$.

Proof Exercise.

Definition 1.12 Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $f : X \longrightarrow Y$ be a map. Let $x \in X$. f is said to be continuous at x if for any open neighbourhood U of $f(x)$, there exists an open neighbourhood O of x such that $f[O] \subseteq U$.
f is said to be continuous on X if f is continuous at each point of X.

Proposition 1.13 Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $f : X \longrightarrow Y$ be a map. The following statements are equivalent.
(1) f is continuous on X.

(2) For any open set $U \subseteq Y$, $f^{-1}[U]$ is open in X.

(3) For any closed set $E \subseteq Y$, $f^{-1}[E]$ is closed in X.

(4) For any $A \subseteq X$, $f[\overline{A}] \subseteq \overline{f[A]}$.

Proof Exercise.

Proposition 1.14 Let $f : X \longrightarrow Y$ and $f : Y \longrightarrow Z$ be maps. If f and g are continuous, then $f \circ g$ is continuous.

Proof This follows easily by 1.13(2).

Definition 1.15 Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be topological spaces and $f : X \longrightarrow Y$ a map.
(1) f is called an open map if for any open set U in X, $f[U]$ is open in Y.

(2) f is called a closed map if for any closed set E in X, $f[E]$ is closed in Y.

(3) f is called a homeomorphism if f is bijective, continuous and f^{-1} is continuous.

Proposition 1.16 Let $f : X \longrightarrow Y$ be a bijective continuous mapping. The following statements are equivalent.
(1) $f^{-1} : Y \longrightarrow X$ is a homeomorphism.

(2) $f : X \longrightarrow Y$ is an open map.

(3) $f : X \longrightarrow Y$ is a closed map.

Proof Exercise.

Proposition 1.17 (The Combination Principle) Let $A_1, \ldots, A_n$ be a finite collection of closed subsets of X such that $\bigcup_{i=1}^{n} A_i = X$. Then $f : X \to Y$ is continuous if and only if $f \mid A_i$ is continuous for each $i = 1, \ldots, n$.

Proof The 'only if' part is obvious. Suppose $f \mid A_i$ is continuous for each $i = 1, \ldots, n$. Let E be a closed subset of Y. Then $f^{-1}[E] = \bigcup_{i=1}^{n}(f^{-1}[E] \cap A_i) = \bigcup_{i=1}^{n}(f \mid A_i)^{-1}[E]$. Since $f \mid A_i$ is continuous, $(f \mid A_i)^{-1}[E]$ is closed in A_i. As A_i

is closed in X, $(f \mid A_i)^{-1}[E]$ is closed in X. Hence $f^{-1}[E]$ is closed in X. This shows that f is continuous on X.

B. Product and Sum Topologies

Definition 1.18 Let $\{X_\lambda\}_{\lambda\in I}$ be a collection of topological spaces and let $p_\alpha : \prod_{\lambda\in I} X_\lambda \longrightarrow X_\alpha$ be the natural projection.
The product topology on $\prod_{\lambda\in I} X_\lambda$ is defined to be the one generated by the subbasis $\{p_\alpha^{-1}[O_\alpha] \mid O_\alpha \text{ is open in } X_\alpha,\ \alpha \in I\}$.

Note that when $\prod_{\lambda\in I} X_\lambda$ is given the product topology the projection p_α is continuous and open. In the case that the product is finite the collection $\{\prod_{\alpha\in I} O_\alpha \mid O_\alpha \text{ is open in } X_\alpha\}$ is a basis for the product topology on $\prod_{\lambda\in I} X_\lambda$.

Proposition 1.19 Let $f : X \longrightarrow \prod_{\lambda\in I} X_\lambda$ be a map. Then f is continuous if and only if $p_\alpha \circ f$ is continuous for all $\alpha \in I$.

Proof Exercise.

The following results are immediate consequences.

Proposition 1.20 The diagonal map $d : X \longrightarrow X \times X$, defined by $d(x) = (x, x)$, and the twisting map $\tau : X \times Y \longrightarrow Y \times X$, defined by $\tau(x, y) = (y, x)$, are continuous.

Proposition 1.21 Given a family of maps $\{f_\lambda : X_\lambda \to Y_\lambda\}_{\lambda\in I}$, the product map $\prod_{\lambda\in I} f_\lambda : \prod_{\lambda\in I} X_\lambda \longrightarrow \prod_{\lambda\in I} Y_\lambda$, defined by $(\prod_{\lambda\in I} f_\lambda)(x_\lambda) = (f_\lambda(x_\lambda))$, is continuous.

Definition 1.22 Let $\{X_\alpha\}$ be a family of topological spaces. The topological sum $\sqcup X_\alpha$ is the topological space whose underlying set is $\cup X_\alpha$ equipped with the topology $\{O \subseteq \cup X_\alpha \mid O \cap X_\alpha \text{ is open in } X_\alpha \text{for each } \alpha\}$.

Note that when $\sqcup X_\alpha$ is given the sum topology the inclusion $i_\alpha : X_\alpha \longrightarrow \sqcup X_\alpha$ is continuous. In general it may not be possible to fit together the topologies on a family of spaces $\{X_\alpha\}$ to obtain a topology on the union restricting to the original topology on each X_α. For example, if two spaces X_α and X_β overlap they may not induce the same topologies on $X_\alpha \cap Y_\beta$. A family $\{X_\alpha\}$ of spaces is said to be compatible if the two topologies on $X_\alpha \cap X_\beta$ are identical for all pair α, β.

Proposition 1.23 Let $\{X_\alpha\}$ be a compatible family of spaces such that $X_\alpha \cap X_\beta$ is open in X_α and X_β, for each α, β. Then X_α has the subspace topology from $\sqcup X_\alpha$.

Proof Exercise.

For example if $\{X_\alpha\}$ is a family of disjoint topological spaces, then each X_α is a subspace of $\sqcup X_\alpha$.

Proposition 1.24 If $\{f_\alpha : X_\alpha \longrightarrow Y\}$ is a family of maps such that f_α and f_β agree on $X_\alpha \cap X_\beta$ for all α, β, then there is a unique continuous map $\sqcup f_\alpha : \sqcup X_\alpha \longrightarrow Y$ such that $(\sqcup f_\alpha) \mid X_\alpha = f_\alpha$, for all α.

Proof Exercise.

C. Separation Axioms and Compactness

Definition 1.25 Let X be a topological space.

(1) X is T_0 if for each pair of distinct points, at least one has a neighbourhood not containing the other.

(2) X is T_1 or Fréchet if for any distinct $x, y \in X$, there exist open neighbourhoods U of x and V of y such that y is not in U and x is not in V.

(3) X is T_2 or Hausdorff if for any distinct $x, y \in X$, there exist open neighbourhoods U of x and V of y such that $U \cap V = \emptyset$.

(4) X is said to be regular if for any closed set E and any point x not in E, there exist an open neighbourhood U of E and an open neighbourhood V of x such that $U \cap V = \emptyset$.

(5) X is said to be normal if for any two disjoint closed sets E and F, there exist an open neighbourhood U of E and an open neighbourhood V of F such that $U \cap V = \emptyset$.

(6) X is T_3 if X is T_1 and regular.

(7) X is T_4 if X is T_1 and normal.

It is easy to see that X is T_1 if and only if every singleton set in X is closed. Hence we have $T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$.

Examples

1. Let $X = \{0, 1\}$ and $\mathcal{T} = \{\emptyset, X, \{0\}\}$. Then $(X, \mathcal{T})$ is T_0 but not T_1.
2. Every metric space is T_4.
3. Let X be an infinite set with the complement finite topology.
4. Then X is T_1 but not T_2.

5. Let $K = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ and let $\mathcal{B} = \{(a,b) \subseteq R \mid a < b\} \cup \{(a,b) \backslash K \mid a < b\}$. Then $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Clearly $\mathbb{R}$ with this topology is T_2 but it is not regular.

6. $\mathbb{R}_l$ is T_4.

7. $\mathbb{R}_l \times \mathbb{R}_l$ is regular but not normal. In general, a product of normal spaces may not be normal.

8. A subspace of a Hausdorff space is Hausdorff and a subspace of a regular space is regular. However, a subspace of a normal space may not be normal.

9. A product of Hausdorff spaces is Hausdorff and a product of regular spaces is regular.

Urysohn's Lemma If A and B are nonempty disjoint closed subsets in a normal space X, then there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f[A] = 0$ and $f[B] = 1$.

Proof See [2] p.207.

Definition 1.26 A cover of a topological space X is a collection $\mathcal{C} = \{O_\alpha\}_{\alpha \in I}$ of subsets of X such that $\cup_{\alpha \in I} O_\alpha = X$. It is said to be an open cover if each O_α is open in X. If the index set I is finite (or countable), then $\{O_\alpha\}_{\alpha \in I}$ is called a finite (or countable) cover. A subcover of a cover $\mathcal{C}$ of X is a subcollection $\mathcal{S}$ of $\mathcal{C}$ such that $\mathcal{S}$ is a cover of X.

Definition 1.27 A space X is said to be compact if every open cover of X has a finite subcover. $A \subseteq X$ is said to be compact if A with the subspace topology is a compact space.

Examples

1. Any finite space is compact.

2. An indiscrete space is compact.

3. $[a,b]$ is compact.

4. An infinite set provided with the complement finite topology is a compact space.

5. $\mathbb{R}$ is not compact. Neither does $\mathbb{R}_l$

Proposition 1.28 $A \subseteq X$ is compact if and only if every cover of A by open subsets of X has a finite subcover.

Proof Exercise.

Proposition 1.29

(1) A closed subspace of a compact space is compact.

(2) A compact subset of a Hausdorff space is closed.

(3) A continuous image of compact space is compact.

(4) A bijective continuous map from a compact space to a T_2 space is a homeomorphism.

Proof Exercise.

Proposition 1.30 Let $\{X_\lambda\}_{\lambda \in I}$ be a collection of spaces. If an infinite number of X_λ are non-compact, then any compact subset in $\prod_{\lambda \in I} X_\lambda$ has empty interior.

Proof Exercise.

Proposition 1.31 Let X be compact. Then X is T_2 if and only if X is T_4.

Proof Exercise.

Tychronoff's Theorem A product of compact spaces is compact.

Proof See [2] p.229.

Let X be the closed interval $[a, b]$ and for each $\alpha \in I$ let X_α be a copy of X. Then $\prod_{\alpha \in I} X_\alpha$ is compact. In particular $[a, b]^n$ is compact. Moveover we have the following result.

Heine-Borel Theorem Let A be a subset of $\mathbb{R}^n$. Then A is compact if and only if A is closed and bounded.

Extreme Value Theorem Let X be compact and $f : X \to \mathbb{R}$ be a continuous function. Then f attains its maximum and minimum on X.

Lebesgue Covering Lemma Let (X, d) be a compact metric space and $\{O_\alpha\}$ an open cover of X. Then there exists a positive number δ, called a Lebesgue number of the cover, such that each open ball of radius $\leq \delta$ is contained in at least one O_α.

Definition 1.32 A space X is said to be locally compact if every point of X has a compact neighbourhood.

Examples

1. A compact space is locally compact.
2. $\mathbb{R}$ is locally compact. (For any $x \in \mathbb{R}, [x-1, x+1]$ is a compact neighbourhood of x).
3. $\mathbb{R}^n$ is locally compact.

4. $\mathbb{R}_l$ is not locally compact. (Exercise. Note that $[a,b)$ is closed in R_l.)
5. A discrete space is locally compact.
6. $\mathbb{Q}$ is not locally compact.

Proposition 1.33 Let X be a locally compact Hausdorff space. Then for any $x \in X$ the collection of all compact neighbourhoods of x is a local basis at x. (That is any neighbourhood of x contains a compact neighbourhood of x.)

Proof Let U be an open neighbourhood of x and E be a compact neighbourhood of x. Then E is regular. Since $U \cap E$ is an open set in E containing x, there exists a set V open in E such that $x \in V \subseteq \overline{V}^E \subseteq U \cap E \subseteq U$. V is open in E implies that $V = E \cap V'$ for some V' open in X. This shows that V is a neighbourhood of x and hence $\overline{V}^E$ is also a neighbourhood of x. Since $\overline{V}^E$ is a closed subset of the compact subset E, $\overline{V}^E$ is compact.

Corollary 1.34 A locally compact Hausdorff space is regular.

Note that a locally compact Hausdorff space may not be normal. (See Royden, *Real Analysis* p.169.)

Proposition 1.35
(1) A closed subspace of a locally compact space is locally compact.

(2) Let $f : X \longrightarrow Y$ be a continuous open surjection. If X is locally compact, then Y is locally compact.

(3) $\prod_{\lambda \in I} X_\lambda$ is locally compact if and only if there exists a finite subset F of I such that X_λ is locally compact $\forall \lambda \in I$ and X_λ is compact $\forall \lambda \in I \setminus F$.

Proof We leave (1) and (2) as exercises. Note that (1) is not true if the subspace is not closed. For example $\mathbb{Q} \subset \mathbb{R}$ is not locally compact. Let's prove (3).
Suppose $\prod_{\lambda \in I} X_\lambda$ is locally compact. Since the projection p_λ is an open map, $X_\lambda = p_\lambda[\prod_{\lambda \in I} X_\lambda]$ is locally compact by (2). If infinitely many X_λ 's are non-compact, then by 1.30 each compact subset of $\prod_{\lambda \in I} X_\lambda$ has empty interior. Hence a point in $\prod_{\lambda \in I} X_\lambda$ cannot have a compact neighbourhood.
Conversely, let F be a finite subset of I such that X_λ is compact $\forall \lambda \in I \setminus F$. Let $(x_\lambda) \in \prod_{\lambda \in I} X_\lambda$. For each $\lambda \in F$, there exists a compact neighbourhood E_λ of x_λ. Then by Tychonoff's theorem, $\prod_{\lambda \in I} O_\lambda$ where $O_\lambda = E_\lambda$ if $\lambda \in F$ and $O_\lambda = X_\lambda$ if $\lambda \in I \setminus F$is a compact neighbourhood of (x_λ).

Definition 1.36
(1) $A \subseteq X$ is said to be dense in X if $\overline{A} = X$.

(2) $A \subseteq X$ is said to be nowhere dense in X if $\overset{o}{\overline{A}} = \emptyset$.

(3) X is said to be of First Category if it is a countable union of nowhere dense subsets of X. It is of Second Category if it is not of First Category.

Theorem 1.37 Let X be a locally compact Hausdorff space. Then the intersection of a countable collection of open dense subsets of X is dense in X.

Baire Category Theorem Let X be a non-empty locally compact Hausdorff space. Then X is of Second Category.

Proof Let $\{E_n \mid n \in \mathbb{Z}^+\}$ be a countable collection of nowhere dense subsets of X. Then each $X \setminus \overline{E_n}$ is open and dense in X. Hence by 1.37, $\bigcap_{n=1}^{\infty}(X \setminus \overline{E_n})$ is dense in X. Therefore

$$X \setminus (\bigcup_{n=1}^{\infty} E_n) \; = \; \bigcap_{n=1}^{\infty}(X \setminus E_n) \; \supseteq \; \bigcap_{n=1}^{\infty}(X \setminus \overline{E_n}) \; \neq \; \emptyset.$$

Definition 1.38 A compactification of a space X is a pair (X^*, i) where X^* is a compact space and $i : X \longrightarrow X^*$ is a homeomorphism of X onto a dense subspace of X^*.

Theorem 1.39 Let $(X, \mathcal{T})$ be a topological space and let ∞ be a point not in X. Then $\mathcal{T}^* = \mathcal{T} \cup \{U \cup \{\infty\} \mid U \subseteq X \text{ and } X - U \text{ is a compact closed subset of } X\}$ is a topology on $X^* = X \cup \{\infty\}$. Furthermore

(1) $(X^*, \mathcal{T}^*)$ is compact.

(2) If X is non-compact, then (X^*, i), where $i : X \rightarrow X^*$ is the inclusion, is a compactification of X.

(3) If X is compact, then ∞ is an isolated point of X^*.

(4) X^* is Hausdorff if and only if X is locally compact and Hausdorff.

Proof Exercise.

Definition 1.40 The space $X^* = X \cup \{\infty\}$ in 1.39 is called the 1-point compactification of X.

Examples

1. The 1-point compactification of the subspace $\mathbb{Z}^+$ of positive integers of $\mathbb{R}$ is homeomorphic to the subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. (Exercise).
2. The n-sphere is the subspace $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ of $\mathbb{R}^{n+1}$. It is a closed and bounded subset of $\mathbb{R}^{n+1}$. Therefore S^n is compact.

The 1-point compactification of $\mathbb{R}^n$ is the n-sphere S^n. A homeomorphism between S^n and $\mathbb{R}^{n^*}$ is provided by the stereographic projection. Let $\eta = (0, \cdots 0, 1) \in \mathbb{R}^{n+1}$ be the north pole of S^n and let $p_n : \mathbb{R}^{n+1} \to \mathbb{R}$ be the projection onto the last factor of $\mathbb{R}^{n+1}$. Identify $\mathbb{R}^n$ as $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$. Then the stereographic projection $s : S^n \setminus \{\eta\} \longrightarrow \mathbb{R}^n$ is given by $s(x) = \eta + \frac{1}{1-p_n(x)}(x - \eta)$. One can easily check that s is a homeomorphism. Now extend s to $\tilde{s}(x) : S^n \longrightarrow$ 1-point compactification $\mathbb{R}^{n^*}$ of $\mathbb{R}^n$ by

$$\tilde{s}(x) = \begin{cases} s(x) & \text{if } x \in S^n \setminus \{\eta\} \\ \infty & \text{if } x = \eta \end{cases} .$$

Then $\tilde{s}$ is continuous at η. Since S^n is compact and $\mathbb{R}^{n^*}$ is Hausdorff, $\tilde{s}$ is a homeomorphism.

D. Countability, Separability and paracompactness

Definition 1.41 A space X is said to be separable if X contains a countable dense subset.

Examples

1. $\mathbb{R}^n$ is separable. ($\mathbb{Q}^n$ is a countable dense subset.)
2. If $(X, \mathcal{T})$ is separable and $\mathcal{T}'$ is a topology on X such that $\mathcal{T} \subseteq \mathcal{T}'$, then $(X, \mathcal{T}')$ is separable.
3. $\mathbb{R}_l$ is separable. ($\mathbb{Q}$ is a countable dense subset.)

Proposition 1.42

(1) A continuous image of separable space is separable.

(2) An open subspace of a separable space is separable.

(3) A product of countably many separable spaces is separable.

Proof Exercise.

Definition 1.43 A space X is said to be $2nd$ countable if it has a countable basis.

Examples

1. $\mathbb{R}$ is $2nd$ countable. ($\{(a, b) \mid a < b, a, b \in \mathbb{Q}\}$ is a countable basis.)
2. Similarly $\mathbb{R}^n$ is $2nd$ countable.
3. $\mathbb{R}_l$ is not $2nd$ countable. (Let $\mathcal{B}$ be a basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}_l$, pick $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x + 1)$. Note that $x = \inf B_x$. Then the function $f : \mathbb{R} \longrightarrow \mathcal{B}$ given by $f(x) = B_x$ is injective. This shows that $\mathcal{B}$ is not countable.)

Definition 1.44 A space X is said to have a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of open neighbourhoods of x such that any open neighbourhood of x contains a member of $\mathcal{B}$. A space X is said to be $1st$ countable if it has a countable basis at each of its point.

Examples

1. If X is $2nd$ countable, then X is $1st$ countable. Hence $\mathbb{R}^n$ is $1st$ countable.
2. $\mathbb{R}_l$ is $1st$ countable. ($\{[x, x+\frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ is a countable basis at x.)
3. The discrete or indiscrete topology on X is $1st$ countable.
4. Every metric space is $1st$ countable. ($\{B(x, \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ is a countable basis at x.)
5. Let X be an uncountable set and $\mathcal{T}$ be the complement finite topology on X. Then $(X, \mathcal{T})$ is not $1st$ countable. (Exercise.)
6. Let I be an uncountable set. For each $\lambda \in I$, let X_λ be a copy of $\{0,1\}$ with the discrete topology. Note that X_λ is $1st$ countable and is even $2nd$ countable. But $\prod_{\lambda\in I} X_\lambda$ is not $1st$ countable. (Exercise.)

Proposition 1.45

(1) A subspace of a $1st$ ($2nd$) countable space is $1st$ ($2nd$) countable.

(2) A countable product of $1st$ ($2nd$) countable spaces is $1st$ ($2nd$) countable.

(3) Let $f : X \longrightarrow Y$ be a surjective open continuous map. If X is $1st$ ($2nd$) countable, then Y is $1st$ ($2nd$) countable.

Proof (1) and (3) are very easy. Let's prove (2) for $1st$ countability. Let $\{X_i \mid i \in I\}$ be a countable family of $1st$ countable spaces. Let $(x_i) \in \prod_{i\in I} X_i$. For each $i \in I$, let $\mathcal{B}_i$ be a countable basis at x_i. Then $\mathcal{B} = \{\prod_{i\in I} O_i \mid O_i \in \mathcal{B}_i$ and $O_i = X_i\ \forall i \notin J$ where J is a finite subset of $I\}$ is a countable basis at (x_i).

Remarks

1. An arbitrary product of $1st$ ($2nd$) countable spaces may not be $1st$ ($2nd$) countable. (See example 6 above.)
2. Consider $id : \mathbb{R} \longrightarrow \mathbb{R}_l$. It is continuous. Note that $\mathbb{R}$ is $2nd$ countable but $\mathbb{R}_l$ is not.

Definition 1.46 A space X is called Lindelöf if every open cover of X has a countable subcover.

Theorem 1.47 A $2nd$ countable space X is Lindelöf.

Proof Let $\mathcal{B} = \{U_i \mid i \in I\}$ be a countable basis of X. Let $\{O_\alpha\}$ be an open cover of X. Each O_α is a union of members of $\mathcal{B}$. Therefore there exists a countable open cover $\{U_{\alpha i} \mid i \in I\}$ of X by basic open sets in $\mathcal{B}$ such that each U_{α_i} lies in some O_α. Now for each $i \in I$, choose $O_{\alpha_i} \supseteq U_{\alpha_i}$. Then $\{U_{\alpha_i} \mid i \in I\}$ is a countable subcover of $\{O_\alpha\}$.

Examples

1. $\mathbb{R}^n$ is Lindelöf.
2. Compact spaces are Lindelöf.
3. $\mathbb{R}_l$ is Lindelöf. (See Munkres p.192.) Note that $\mathbb{R}_l$ is not $2nd$ countable. Therefore the converse of 1.47 is not true in general.
4. A discrete space on an uncountable set is not Lindelöf.

Theorem 1.48 If X is $2nd$ countable, then X is separable.

Proof Let $\mathcal{B} = \{O_i \mid i \in I\}$ be a countable basis of X. For each $i \in I$, let $x_i \in O_i$. Then $A = \{x_i \mid i \in I\}$ is a countable dense subset of X.

Note that the converse of this theorem is not true. For example take $X = \mathbb{R}_l$.

Theorem 1.49 If X is a separable metric space, then X is $2nd$ countable.

Proof Let A be a countable dense subset of X. We shall prove that $\mathcal{B} = \{B(a, r) \mid r \in \mathbb{Q}^+, a \in A\}$ is a countable basis of X. It suffices to show that any open ball is a union of members of $\mathcal{B}$. Let $x \in B(p, r)$. Since A is dense, there exists $a \in A$ such that $d(a, x) < \frac{1}{2}(r - d(x, p))$. Pick a positive rational number q such that $d(a, x) < q < \frac{1}{2}(r - d(x, p))$. Then $x \in B(a, q) \subseteq B(p, r)$. This proves the assertion.

Theorem 1.50 If X is a Lindelöf metric space, then X is $2nd$ countable.

Proof We shall prove that X is separable. For each $j \in \mathbb{Z}^+$, $\{B(x, \frac{1}{j}) \mid x \in X\}$ is an open cover of X. Since X is Lindelöf, it has a countable subcover $\mathcal{B}_j = \{B(x_{i,j}, \frac{1}{j}) \mid i \in \mathbb{Z}^+\}$. Let $A = \{x_{i,j} \mid i, j \in \mathbb{Z}^+\}$ be the collection of all the centers. Then A is a countable subset of X. Let $B(p, r)$ be an open ball. Pick an $j_o \in \mathbb{Z}^+$ such that $\frac{1}{j_o} < r$. Then $\mathcal{B}_{j_o} = \{B(x_{i,j_o}, \frac{1}{j_o}) \mid i \in \mathbb{Z}^+\}$ covers X. Therefore $p \in B(x_{i_o,j_o}, \frac{1}{j_o})$ for some i_o. Then $d(p, x_{i_o,j_o}) < \frac{1}{j_o} < r$ implies that $x_{i_o,j_o} \in B(p, r)$. Hence $A \cap B(p, r) \neq \emptyset$. That is A is dense in X.

Definition 1.51

(1) A locally finite family of subsets of a topological space X is a family such that each point of X has a neighbourhood meeting only finitely many members of the family.

(2) A refinement $\mathcal{F}$ of a cover $\mathcal{C}$ of X is a cover of X such that each member of $\mathcal{F}$ is contained in some member of $\mathcal{C}$.

(3) A space X is said to be paracompact if every open cover of X has a locally finite open refinement.

Examples

1. A compact space is paracompact.
2. $\mathbb{R}$ is paracompact. (Let $\mathcal{U}$ be an open cover of $\mathbb{R}$. For each closed interval $[N, N+1], N \in \mathbb{Z}$, we have, by compactness, a finite cover $\{U_{N_1}, \ldots, U_{N_n}\}$ by members of $\mathcal{U}$. Take as a refinement $\{U_{N_i} \cap (N, N+1)\}$. This does not cover points in $\mathbb{Z}$. To remedy this, we add in a small open neighbourhood (of diameter less than 1 and small enough to be contained in some member of $\mathcal{U}$) of each N in $\mathbb{Z}$. We then have the required locally finite open refinement.)
3. $\mathbb{R}^n$ is paracompact. (Similar proof as above.)
4. $\mathbb{R}_l$ is paracompact. (Exercise)

Proposition 1.52 A paracompact Hausdorff space is regular.

Proof Let U be an open neighbourhood of a point p. We shall construct a neighbourhood V of p such that $\overline{V} \subseteq U$. For each $q \notin U$, choose open disjoint neighbourhoods V_q, U_q of p, q respectively. Then $\{U_q \mid q \in X \setminus U\} \cup \{U\}$ is an open cover of X and thus has a locally finite open refinement. Let M be a neighbourhood of p meeting only finitely many members of the refinement and let $W_1, \ldots, W_n$ be those not contained in U. Then there exist $q_1, \ldots, q_n \in X \setminus U$ such that $W_i \subseteq U_{q_i}$. Let $V = M \cap \bigcap_{i=1}^n V_{q_i}$. It remains to show that $\overline{V} \subseteq U$. Let W be the union of all members of the above refinement not contained in U. Then $V \cap W \subseteq M \cap (\bigcap_{i=1}^n V_{q_i}) \cap (\bigcup_{i=1}^n U_{q_i}) = \emptyset$. Thus $V \subseteq X \setminus W$. Since $W \supseteq X \setminus U$, we have $\overline{V} \subseteq X \setminus W \subseteq U$.

Proposition 1.53 A paracompact regular space is normal.

Proof Exercise.

Proposition 1.54 A Lindelöf regular space is normal.

Proof Exercise.

Theorem 1.55 If X is paracompact and separable, then X is Lindelöf.

Proof Let $\{U_\alpha\}$ be an open cover of X. Let $\{V_\beta\}$ be a locally finite open refinement of $\{U_\alpha\}$. We may assume each $V_\beta \neq \emptyset$. Because X has a countable dense subset $\{x_i \mid i \in \mathbb{Z}^+\}, \{V_\beta\}$ is at most a countable family. (Each V_β contains at least one x_i. If $\{V_\beta\}$ is an uncountable family, then there is some x_i contained in uncountably many V_β. This contradicts the local finiteness of $\{V_\beta\}$.) For each V_β, choose $U_{\alpha(\beta)} \supseteq V_\beta$. Then $\{U_{\alpha(\beta)}\}$ is a countable subcover of $\{U_\alpha\}$.

Note that a product of two paracompact (Lindelöf) spaces may not be paracompact (Lindelöf). For example, take $\mathbb{R}_l \times \mathbb{R}_l$.

Urysohn's Metrization Theorem A 2nd countable T_3 space is metrizable.

Proof See [2] p.349.

E. Connectedness

Definition 1.56 A space X is said to be connected if it is not the union of two non-empty disjoint open subsets of X. $A \subseteq X$ is connected if it is connected as a subspace of X.

Examples

1. Any indiscrete space is connected.
2. $\mathbb{R}$ is connected. (Exercise)
3. $\mathbb{R}_l$ is not connected.
4. The subspace $\mathbb{Q}$ of all rational numbers of $\mathbb{R}$ is not connected.
5. $\mathbb{R}^n$ is connected.
6. $A \subseteq \mathbb{R}$ is connected if and only if A is an interval.
7. A compact connected T_2 space cannot be a union of countably many but more than one disjoint closed subsets.

Proposition 1.57 The following statements are equivalent:

(1) X is connected.

(2) There is no proper non-empty subset of X which is both open and closed in X.

(3) Every continuous map from X to $\{0,1\}$ is constant, where $\{0,1\}$ is the two point space with the discrete topology.

Proof Exercise.

Proposition 1.58 Let $f : X \longrightarrow Y$ be continuous and A be a connected subset of X. Then $f[A]$ is connected.

Proof This follows directly from 1.57(3).

Proposition 1.59 $\mathbb{R}$ is connected.

Proof A non-empty proper open subset O of $\mathbb{R}$ is a disjoint union of open intervals. Then one of these open intervals has an endpoint a not in O. Hence O is not closed.

Proposition 1.60

(1) If A is a connected subset of X, then $\overline{A}$ is also a connected subset of X.

(2) If A is a connected subset of X and B is subset of X such that $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof Exercise.

Let A be the set $\{(x, \sin(\frac{1}{x})) \mid x > 0\}$. Then $\overline{A}$ and $A \cup \{(0,0)\}$ are connected subsets of $\mathbb{R}^2$.

Corollary 1.61 $A \subseteq \mathbb{R}$ is connected if and only if A is an interval or a singleton set.

Proof An interval I of $\mathbb{R}$ is a subset of $\mathbb{R}$ having at least two points and satisfying the condition that for any $a, b \in I$, the line segment joining a and b is also contained in I. Then I is an interval if and only if I is of the form $(a,b), (a,b], [a,b)$ or $[a,b]$ where $-\infty \leq a < b \leq \infty$. Now the forward implication is clear. Conversely if A is an open interval, then it is homeomorphic to $\mathbb{R}$ which is connected by 1.59. The rest of different types of intervals are also connected by 1.60(2).

Intermediate Value Theorem

(1) If $f : X \longrightarrow \mathbb{R}$ is continuous and X is connected, then $f[X]$ is an interval.

(2) Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then f assumes all the values between $f(a)$ and $f(b)$.

Proposition 1.62 $[a,b)$ is not homeomorphic to (a,b).

Proposition 1.63 Let $\{A_\alpha\}_{\alpha \in I}$ be a family of connected subsets of X such that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in I} A_\alpha$ is connected.

Proof Use 1.57(3).

Definition 1.64 Two points a, b in a space X are said to be connected, written $a \sim b$, if there is a connected subspace of X containing a and b.

Proposition 1.65 If every pair of points in X are connected, then X is connected.

Proof Let U be both open and closed in X. Suppose $U \neq \emptyset, X$. Then there exist $a \in U, b \in X \setminus U$. By assumption, there exists a connected subset C containing a and b. But then $U \cap C$ is a proper non-empty both open and closed subset of C. This contradicts the fact that C is connected.

Proposition 1.66 $\sim$ is an equivalence relation.

Proof Exercise.

Definition 1.67 The connected component of a point $p \in X$, denoted by $C(p)$, is the equivalence class of $\sim$ containing p.

Proposition 1.68

(1) $C(p)$ is the largest connected set containing p.

(2) $C(p)$ is closed.

Proof

(1) $C(p) = \bigcup\{C : C \text{ is connected and } p \in C\}$. By 1.63 $C(p)$ is connected.

(2) $\overline{C(p)}$ is connected and contains p. Hence by (1), $\overline{C(p)} \subseteq C(p)$.

Proposition 1.69 If X and Y are connected, then $X \times Y$ is also connected.

Proof Let $(a, b), (x, y) \in X \times Y$. Since $X \times \{b\}$ and $\{x\} \times Y$ are connected, we have $(a, b) \sim (x, b) \sim (x, y)$. By 1.65 $X \times Y$ is connected.

In fact one can prove that arbitrary product of connected spaces is connected. This is left as an exercise.

Corollary 1.70 $\mathbb{R}^n, [a, b]^n$ and S^n are connected.

Proof Let $p \in S^n$. As $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n, S^n \setminus \{p\}$ is connected. Hence $S^n = \overline{S^n \setminus \{p\}}$ is connected.

Definition 1.71 X is said to be locally connected at $p \in X$ if for any neighbourhood U of p, there exists a connected neighbourhood V of p contained in U. X is locally connected if it is locally connected at each of its points.

Note that X is locally connected if and only if the collection of all open connected subsets of X is a basis for the topology on X.

Examples

1. Let A be the set $\{(x, \sin(\frac{1}{x})) \mid x > 0\}$. Then $A \cup \{(0, 0)\}$ is not locally connected but connected.
2. Let $A = (0, 1) \cup (2, 3)$. Then A is locally connected but not connected.
3. $\mathbb{R}^n$ and S^n are locally connected.
4. $\mathbb{R}_l$ is not locally connected. (Let U be any neighbourhood of x. Hence $U \supseteq [x, x+2\delta)$ for some $\delta > 0$. Then $[U \cap (\infty, x+\delta)] \bigcup [U \cap [x+\delta, \infty)] = U$ is a disjoint union of two non-empty open subsets of U. Therefore any neighbourhood of x is not connected.)
5. Discrete spaces are locally connected.
6. $\mathbb{Q}$ is not locally connected.

Proposition 1.72 If a space is locally connected at $p \in X$, then p is an interior point of $C(p)$.

Proof By assumption p has a connected neighbourhood U. Therefore $p \in U \subseteq C(p)$.

Corollary 1.73 If X is locally connected, then $C(p)$ is an open subset of X.

Proposition 1.74 Every open subspace of a locally connected space is locally connected.

Proposition 1.75 X is locally connected if and only if the connected components of every open subspace of X are open in X.

Proof Suppose X is locally connected. Let O be an open subspace of X. By 1.73 and 1.74 O is locally connected and each connected component of O is open in O. Since O is open in X, each connected component of O is open in X. Conversely let U be an open neighbourhood of $p \in X$. By assumption the connected component $C_U(p)$ in the subspace U is open in X. Hence $C_U(p)$ is open in X. Therefore $C_U(p)$ is a connected open neighbourhood of p contained in U. This shows that X is locally connected.

Corollary 1.76 Let N be a neighbourhood of a point x in a locally connected space X. Then there exists a connected open neighourhood C of x lying inside N.

Proposition 1.77 Let $\{X_\alpha\}_{\alpha \in I}$ be a family of locally connected spaces such that all but at most finitely many are connected. Then $\prod_{\alpha \in I} X_\alpha$ is locally connected.

Proof Let F be a finite subset of I such that X_α is not connected for all $\alpha \in F$. Let $x \in \prod_{\alpha \in I} X_\alpha$ and O a basic open neighbourhood of x. Note that $O = \prod_{\alpha \in I} O_\alpha$ where O_α is open in X_α and $O_\alpha = X_\alpha$ for all $\alpha \in I \setminus J$ with J some finite subset of I. For each $\alpha \in F \cup J$, there exists an open connected neighbourhood V_α of X_α such that $V_\alpha \subseteq O_\alpha$. Let $U = \prod_{\alpha \in I} U_\alpha$ where $U_\alpha = V_\alpha$ if $\alpha \in F \cup J$ and $V_\alpha = X_\alpha$ if $\alpha \in I \setminus (F \cup J)$. Then U is a product of connected sets and hence it is connected. Also U is a basic open neighbourhood of x lying in O. Note that the converse of this result is also true since local connectedness is invariant under continuous open surjection.

Definition 1.78 A path joining a pair of points a and b of a space X is a continuous map $\alpha : I \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$, where I denotes the closed unit interval $[0, 1]$.

It can be verified easily that the relation of being joined by a path is an equivalence relation. The equivalence classes of this relation are called the path components of X. The path component of a point x in X is denoted by $P(x)$. $P(x)$ is the largest path connected set containing x. However unlike $C(x)$, $P(x)$ may not be a closed set.

Definition 1.79 A space X is said to be path connected if it has exactly one path component. (That is any pair of points in X can be joined by a path.)

Proposition 1.80 A path connected space is connected.

Proposition 1.81 Let $f : X \longrightarrow Y$ be a continuous surjection. If X is path connected, then Y is path connected.

Proof Exercise.

Examples

1. $\mathbb{R}^n$ and S^n are path connected. (It is easy to check that $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ are path connected. Let $\sigma : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n$ be the map given by $\sigma(x) = \frac{x}{\|x\|}$. Then by 1.81 S^n is path connected.)

2. Let A be the set $\{(x, \sin(\frac{1}{x})) \mid x > 0\} \bigcup \{(0,0)\}$. Then A is not path connected. Note that $A \setminus \{(0,0)\}$ is a path component which is not closed in A.

Proposition 1.82 Let $\{X_\alpha\}_{\alpha \in I}$ be a family of spaces. Then $\prod_{\alpha \in I} X_\alpha$ is path connected if and only if each X_α is path connected.

Proof Exercise.

Definition 1.83 A space is said to be locally path connected at $x \in X$ if each neighbourhood of x contains a path connected neighbourhood. X is said to be locally path connected if it is locally path connected at each of its points.

Note that X is locally path connected if and only if the collection of all path connected open sets of X is a basis for the topology on X.

Examples

1. $\mathbb{R}^n$ is locally path connected since every open ball in $\mathbb{R}^n$ is path connected. Similarly S^n is locally path connected.

2. $(0,1) \cup (2,3)$ is locally path connected but not path connected.

3. For each positive integer n, let L_n be the set of all points on the line segment joining the points $(0,1)$ and $(\frac{1}{n}, 0)$ in $\mathbb{R}^2$ and let $L_\infty = \{(0,y) \mid 0 \le y \le 1\}$. Then the subspace $A = L_\infty \cup \bigcup_{n=1}^{\infty} L_n$ in $\mathbb{R}^2$ is not locally path connected but path connected.

4. If X is locally path connected, then X is locally connected.

5. If L_∞ of A in 3. is replaced by $L_\infty^* = \{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$, then A is locally connected at $(0,1)$ but not locally path connected at $(0,1)$.

Proposition 1.84 Every open subspace of a locally path connected space is locally path connected.

Proposition 1.85 X is locally path connected if and only if the path components of every open subspace of X are open.

Proposition 1.86 $\prod_{\alpha \in I} X_\alpha$ is locally path connected if and only if all X_α are locally path connected and all but at most finitely many are also path connected.

Proof Exercise.

Proposition 1.87 If X is locally path connected and connected, then X is path connected.

Proof By 1.85 a path component of X is open and hence is also closed. Since X is connected, there is exactly one path component.

F. Identifications and Adjunction Spaces

Definition 1.88 A continuous surjective map $p : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is called an identification or a quotient map if $p^{-1}[O]$ is open in X if and only if O is open in Y.

Proposition 1.89 Let $(X, \mathcal{T})$ be a topological space and let Y be a set. Let $p : X \longrightarrow Y$ be a surjective map. Then $\mathcal{T}_p = \{U \subseteq Y \mid p^{-1}[U] \text{ is open in } X\}$ is a topology on Y. Furthermore $p : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}_p)$ is an identification.

Proof Direct verification.

The topology $\mathcal{T}_p$ on Y defined in 1.89 is called the identification or quotient topology induced by p. In fact it is the largest topology on Y that makes p continuous.

Definition 1.90 Let $f : X \longrightarrow Y$ be a map. A subset O in X is said to be saturated with respect to f if $f^{-1}[f[O]] = O$.

If $p : X \longrightarrow Y$ is a map, then any set of the form $p^{-1}[U]$ is saturated with respect to p. Hence a continuous surjective map $p : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is an identification if and only for any open set $O \subseteq X$ saturated with respect to p, $p[O]$ is open in Y.

Proposition 1.91 Let $p : X \longrightarrow Y$ be continuous.

(1) If p is an open surjection (or a closed surjection), then p is an identification.

(2) If there exists a continuous $s : Y \longrightarrow X$ such that $p \circ s = id\,|_Y$, then p is an identification.

Proof Exercise. Note that the converses of (1) and (2) are not true. Also the restriction of an identification may not be an identification.

Proposition 1.92 Let $p : X \longrightarrow Y$ be an identification and $g : Y \longrightarrow Z$ a surjective map. Then $g \circ p$ is an identification if and and only if g is an identification.

Proof Suppose that $g \circ p$ is an identification. One can easily see that the continuity of $g \circ p$ implies the continuity of g. Let $O \subseteq Z$ be such that $g^{-1}[O]$ is open in Y. Because p and g are continuous, $(g \circ p)^{-1}[O] = p^{-1}[g^{-1}[O]]$ is open in X. Since $g \circ p$ is an identification, O is open in Z. Hence g is an identification. Conversely, suppose that g is an identification. Let $O \subseteq Z$ be such that $(g \circ p)^{-1}[O]$ is open in X. Since p and g are identifications, O is open in in Z. Therefore $g \circ p$ is an identification.

Let $(X, \mathcal{T})$ be a space and R be an equivalence relation defined on X. For each $x \in X$, let $[x]$ be the equivalence class containing x. Denote the quotient set by X/R. Let $p : X \longrightarrow X/R$ given by $p(x) = [x]$ be the natural projection. The set X/R with the identification topology $\mathcal{T}_p$ induced by p is called the quotient space of X by R.

Let A be a subset of X. Define an equivalence relation R_A on X by xR_Ay if and only if $x, y \in A$. Then the quotient space X/R_A is the space X with A identified to a point $[A]$ and is usually denoted by X/A. Note that $E \subseteq X$ is saturated (with respect to the projection p) if and only if $E \supseteq A$ or E is disjoint from A.

Corollary 1.93 Let X, Y be spaces with equivalence relations R and S respectively, and let $f : X \longrightarrow Y$ be a relation preserving, (i.e. $x_1Rx_2 \implies f(x_1)Sf(x_2)$) continuous map. Then the induced map $f_* : X/R \longrightarrow Y/S$ given by $f_*([x]) = [f(x)]$ is continuous. Furthermore, f_* is an identification if f is an identification.

Proof Exercise.

Proposition 1.94 Let Y and A be closed subspaces of X, then $Y/(Y \bigcap A)$ is a subspace of X/A. (In fact $Y/(Y \bigcap A)$ is homeomorphic to a subspace of X/A.)

Proof Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & X \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle p} \\ Y/Y \bigcap A & \xrightarrow{\ \overline{i}\ } & X/A \end{array}$$

Clearly $\overline{i}$ is a continuous injection. Thus it suffices to show that $\overline{i} : Y/Y \bigcap A \longrightarrow pi[Y]$ is an open map. Then $Y/Y \cap A$ is homeomorphic to the subspace $pi[Y] = \overline{i}[Y/Y \bigcap A]$ of X/A. Let $U \subseteq Y/Y \bigcap A$ be open. Then $\overline{p}^{-1}[U]$ is open in Y. Therefore there exists a W open in X such that $Y \bigcap W = \overline{p}^{-1}[U]$. Since $\overline{p}^{-1}[U]$ is saturated with respect to $\overline{p}$, either $\overline{p}^{-1}[U] \bigcap A = Y \bigcap A$ or $\overline{p}^{-1}[U] \bigcap A = \emptyset$. In the first case, $\bar{i}[U] = p[Y] \bigcap p[[X \setminus Y] \bigcup W]$. Thus, since $(X \setminus Y) \bigcup W$ is open and saturated, $\bar{i}[U]$ is open in $p[Y]$. In the second case, $\bar{i}[U] = p[Y] \bigcap p[(X \setminus A) \bigcap W]$ which is also open in $p[Y]$ since $X \setminus A$ is saturated and open in X.

Examples

1. Let X be a space with a preferred base point $*$, then the reduced suspension SX is the quotient space $(X \times I)/(X \times \{0,1\} \cup \{*\} \times I)$ where $I = [0,1]$ is the unit interval.
2. The reduced cone on X, CX is the quotient space $(X \times I)/(X \times \{0\} \cup \{*\} \times I)$. If $\{*\}$ is a closed subspace, (e.g. X is T_1) $X \times \{0\} \cup \{*\} \times I$ is closed in $X \times I$. Hence by 1.94 $X \times \{1\} \cong X$ is a subspace of CX.

Let $p : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$ be the natural projection which is an identification when $\mathbb{Q}/\mathbb{Z}$ is provided with the quotient topology. The map $p \times id : \mathbb{Q} \times \mathbb{Q} \longrightarrow (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$ is not an identification. In general a product of two identifications may not be an identification. (c.f. A product of two closed maps may not be closed.) However we have the following results.

Propostion 1.95 Let $f : X \longrightarrow Y$ and $g : T \longrightarrow Z$ be identifications. Suppose T is compact and Z is regular. Then $f \times g : X \times T \longrightarrow Y \times Z$ is an identification.

Proposition 1.96 If $f : X \longrightarrow Y$ is an identification and Z is locally compact and regular, then $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is an identification. (This result can also be deduced from the exponential law in function space topology.)

Let X and Y be topological spaces and A be a subset of X. Let $f : A \longrightarrow Y$ be a continuous map. Define an equivalence relation $\sim$ on $Y \sqcup X$ by :

$$a \sim b \quad \text{if and only if} \quad \begin{cases} a = b & \text{for } a, b \in X, \text{or } a, b \in Y \\ f(a) = f(b) & \text{for } a, b \in A \\ a = f(b) & \text{for } a \in Y,\ b \in A \\ f(a) = b & \text{for } a \in A,\ b \in Y\ . \end{cases}$$

Definition 1.97 The quotient space $(Y \sqcup X)/\sim$ is called an adjunction space and is denoted by $Y \bigcup_f X$. The construction means that X is attached to Y by means of $f : A \longrightarrow Y$.

Suppose $\theta : X \longrightarrow Z$ and $\phi : Y \longrightarrow Z$ satisfy $\theta \mid_A = \phi \circ f$, then a map $\theta \cup_f \phi : Y \bigcup_f X \longrightarrow Z$ is defined by

$$(\theta \cup_f \phi)(x) \;=\; \begin{cases} \theta(x) & x \in X \\ \phi(x) & x \in Y\ . \end{cases}$$

Note that this is well-defined since if $x \sim y$ then $\theta(x) = \phi(y)$. Let $p : Y \sqcup X \longrightarrow Y \bigcup_f X$ be the quotient map. Clearly, $(\theta \cup_f \phi) \circ p$ is continuous and hence by 1.92 $\theta \cup_f \phi$ is continuous.

Proposition 1.98 Let A be closed in X. Then the map $i : Y \longrightarrow Y \cup_f X$, defined by $i(y) = p(y)$, is an injective homeomorphism and $i[Y]$ is closed in $Y \cup_f Y$.

Proof Let $B \subseteq Y$ be closed. Then $p^{-1}[i[B]] = B \sqcup f^{-1}[B]$ is closed in $Y \sqcup X$. Hence $i[B]$ is closed in $Y \cup_f X$. Therefore i is a closed injective map. In particular, $i[Y]$ is a closed subspace of $Y \cup_f X$.

Proposition 1.99 Let A be closed in X. Then $p|_{X \setminus A}$ is an injective homeomorphism onto an open subspace of $Y \cup_f X$.

Proof $p|_{X \setminus A}$ is obviously continuous and bijective. Let O be open in $X \setminus A$. Then O is open and saturated in X. Since p is an identification, $p[O]$ is open in $Y \cup_f X$.

Proposition 1.100 If X and Y are normal and $A \subseteq X$ is closed, then $Y \cup_f X$ is normal.

Proof Let P and Q be disjoint closed sets in $Y \cup_f X$. Then $P_1 = P \cap i[Y]$ and $Q_1 = Q \cap i[Y]$ are closed in $i[Y]$. Since Y is normal, $i[Y]$ is normal by 1.83. Therefore there exist neighbourhoods R of P_1 and S of Q_1 such that R, S are open in $i[Y]$ and they have disjoint closures. Note that since $i[Y]$ is closed in $Y \cup_f X$, R and S have the same closures in $i[Y]$ and in $Y \cup_f X$. Let $P_2 = p^{-1}[P \cup \overline{R}] \cap X$ and $Q_2 = p^{-1}[Q \cup \overline{S}] \cap X$. Then P_2 and Q_2 are disjoint and closed in X. Since X is normal, there exist disjoint open neighbourhoods M and N of P_2 and Q_2 respectively. Then $D = p[M \setminus A] \cup R$ and $E = p[N \setminus A] \cup S$ are disjoint open neigbourhoods of P and Q respectively. (They are open since, for example $p^{-1}[D] = R \sqcup (M \setminus A) \cup f^{-1}[R] = M \setminus (A \setminus f^{-1}[R])$ which is open in $Y \sqcup X$.)

Proposition 1.101 The adjunction space construction preserves connectedness, path-connectedness and compactness. When $A \subseteq X$ is closed, the T_1 property is preserved.

Proof Since X and Y are connected (path-connected), their continuous images are also connected (path-connected). As they have non-empty intersection and their union is $Y \cup_f X$, $Y \cup_f X$ is connected (path-connected). Since $Y \cup_f X$ is the continuous image of the compact space $Y \sqcup X$, it is compact. Finally, let $y \in Y$. Then since Y is closed in $Y \cup_f X$, so is $\{y\}$. Similarly if $x \in X \setminus A$, then $p^{-1}[\{x\}]$ is closed in X. Hence $\{x\}$ is closed in $Y \cup_f X$ since p is an identification.

Corollary 1.102 If X and Y are normal Hausdorff and $A \subseteq X$ is closed, then $Y \cup_f X$ is also normal Hausdorff.

References

[**1**] J. Dugundji, *Topology*, Prentice-Hall.
[**2**] J. R. Munkres, *Topology*, Prentice-Hall.