Synchrosqueezed Transforms and Applications

Haizhao Yang
Department of Mathematics, Stanford University

Collaborators: Ingrid Daubechies*, Jianfeng Lu‡ and Lexing Ying†

* Department of Mathematics, Duke University
‡ Department of Mathematics and Chemistry and Physics, Duke University
† Department of Mathematics and ICME, Stanford University

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Medical study (Y., ACHA, 14)

- A superposition of two ECG signals.
  \[ f(t) = \alpha_1(t)s_1(2\pi\phi_1(t)) + \alpha_2(t)s_2(2\pi\phi_2(t)). \]
- Spike wave shape functions \( s_1(t) \) and \( s_2(t) \).

**Figure**: Complicated wave shape functions.

**Figure**: Good decomposition.
A superposition of several wave fields.
- Nonlinear components, bounded supports.

Figure: One target component with structure noise and Gaussian random noise. Courtesy of Fomel and Hu for providing data.
Materials science (Y., Lu and Ying, preprint)

Atomic crystal analysis

- Observation: an assemblage of wave-like components;
- Goal: Crystal segmentation, crystal rotations, crystal defects, crystal deformations.
Art forensics (Y., Lu, Brown, Daubechies, Ying, preprint)

Painting canvas analysis

- Observation: a superposition of wave-like components;
- Goal: count threads and estimate texture deformation.

**Figure**: Top: a X-ray image of canvas. Left: horizontal thread count. Right: horizontal thread angle.
1D mode decomposition

Known: A superposition of wave-like components

\[ f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t)e^{2\pi i N_k \phi_k(t)}. \]

Unknown: Number \( K \), components \( f_k(t) \), smooth instantaneous amplitudes \( \alpha_k(t) \), smooth instantaneous frequencies \( N_k \phi'_k(t) \).

Existing methods:

- Empirical mode decomposition methods (Huang et al. 98, 09);
- Synchrosqueezed wavelet transform (Daubechies et al. 09, 11);
  Synchrosqueezed wave packet transform (Y. 14);
- Data-driven time-frequency analysis (Hou et al. 11, 12, 13);
- Regularized nonstationary autoregression (Fomel 13);
1D wave packets

Given a mother wave packet $w(t)$ and a scaling parameter $s \in (1/2, 1)$, the family of wave packets $\{w_{ab}(t): a \geq 1, b \in \mathbb{R}\}$ is defined as

$$w_{ab}(t) = a^{s/2} w(a^s (t - b)) e^{2\pi i (t-b)a},$$

or equivalently, in the Fourier domain as

$$\hat{w}_{ab}(\xi) = a^{-s/2} e^{-2\pi ib \xi} \hat{w}(a^{-s}(\xi - a)).$$

1D wave packet transform

The 1D wave packet transform of a function $f(t)$ is a function

$$W_f(a, b) = \langle w_{ab}, f \rangle = \int w_{ab}(t) f(t) dt$$

for $a \geq 1, b \in \mathbb{R}$. 
A simple example

A plane wave with an instantaneous frequency $N$:

$$f(t) = e^{2\pi i N t}.$$ 

Its wave packet transform:

$$W_f(a, b) = \int_{\mathbb{R}} e^{2\pi i N t} a^{s/2} w(a^s(t-b)) e^{-2\pi i (t-b)^a} \, dt$$

$$= a^{-s/2} e^{2\pi i N b} \hat{w}(a^{-s}(N - a)).$$

The oscillation of $W_f(a, b)$ in $b$ reveals $N$:

$$\frac{\partial_b W_f(a, b)}{2\pi i W_f(a, b)} = \frac{a^{-s/2} \partial_b e^{2\pi i N b} \hat{w}(a^{-s}(N - a))}{2\pi i a^{-s/2} e^{2\pi i N b} \hat{w}(a^{-s}(N - a))} = N.$$
Definition: Instantaneous frequency estimate

\[ \omega_f(a, b) = \frac{\partial_b W_f(a, b)}{2\pi i W_f(a, b)} \]

for \( W_f(a, b) \neq 0 \).

Definition: Synchrosqueezed wave packet transform (SSWPT)

\[ \mathcal{T}_f(\omega, b) = \int_{\mathbb{R}} |W_f(a, b)|^2 \delta(\Re \omega_f(a, b) - \omega) \, da. \]

Comparison of supports

A plane wave \( f(t) = e^{2\pi i N t} \), for a fixed \( b \),

\[ \text{supp} \, W_f(a, b) \approx (N - N^s, N + N^s); \]

\[ \text{supp} \, \mathcal{T}_f(\omega, b) \text{ concentrates at } \omega = N. \]
SS for sharpened representation

Figure: The supports of the 1D wave packet transform and 1D SSWPT of a synthetic benchmark signal.
Theory of 1D SSWPT

Theorem: (Y. 14 ACHA)

If

\[ f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)} \]

and \( f_k(t) \) are well-separated, then

- \( \mathcal{T}_f(a, b) \) has well-separated supports \( Z_k \) concentrating \( (N_k \phi'_k(b), b) \);
- \( f_k(t) \) can be accurately recovered by applying an inverse transform on \( \mathcal{I}_{Z_k}(a, b) \mathcal{T}_f(a, b) \).

where \( \mathcal{I}_{Z_k}(a, b) \) is an indication function.
Robustness properties of 1D SSWPT

- Bounded perturbation;
- Gaussian random noise (colored);
- Possible compactly supported in space.

Theorem: (Y. and Ying, 14, preprint)

- A non-linear wave \( f(x) = \alpha(x)e^{2\pi i N \phi(x)} \), \( \phi(x) = O(1) \);
- A zero mean Gaussian random noise \( e \) with covariance \( \epsilon_1^q \) for some \( q > 0 \);
- A wave packet \( w_{ab}(x) \) compactly supported in the Fourier domain;
- Main results: if \( s(x) = f(x) + e \), then after thresholding, with a probability at least

\[
\left( 1 - e^{-O(N^{2-3s} \epsilon_1^{-q})} \right) \left( 1 - e^{-O(N^{-2-s} \epsilon_1^{-q})} \right),
\]

\[
\omega_s(a, b) = \frac{\partial_b W_s(a, b)}{2\pi i W_s(a, b)} \approx N \phi'(b)
\]
Properties of 1D SSWPT\textsuperscript{12}

- When \( s = 1 \), wave packets become wavelets;
- When \( s = 1/2 \), wave packets become wave atoms;
- Larger \( s \), more accurate to estimate instantaneous frequencies;
- Smaller \( s \), more robust to estimate instantaneous frequencies;
- Smaller \( s \), better resolution to distinguish wave-like components in the high frequency domain.
- Smaller \( s \), better for the general mode decomposition problem:

\[
    f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) s_k(2\pi N_k \phi_k(t)) \\
    = \sum_{k=1}^{K} \alpha_k(t) \sum_{n} \hat{s}_k(n) e^{2\pi i N_k \phi_k(t)}
\]

\textsuperscript{1}Y. ACHA, 14.
\textsuperscript{2}Y. and Ying, arXiv:1410.5939, 14.
Difference of wavelets and wave packets

The size of the essential support of $\hat{w}_{ab}(\xi)$ is $O(a^s)$.

**Figure:** In the frequency domain: $s = 1$, wavelet tiling (blue); Sampling bump functions (black); Fourier transforms of plane waves (red).

**Figure:** $s < 1$, wave packets.
Difference of SS wavelets and SS wave packets

**Figure**: Seismic trace benchmark signal: $s = 0.5$; $s = 0.625$; $s = 1$. Top: whole domain. Bottom: high frequency part.
Robustness properties of 1D SSTs

Smaller scaling parameter $s$ in the SSWPT, better robustness.

Figure: Noisy synthetic benchmark signal. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$. 
Robustness properties of 1D SSTs

Higher redundancy in the time-frequency transform, better robustness.

Figure: 16 times redundancy. From left to right: $s = 0.625$, $s = 0.75$, and $s = 0.875$. 
Volcanic signal tremor

Figure: From left to right: $s = 1$ (SSWT); $s = 0.75$; $s = 0.625$. Top: Normal SST. Bottom: Enhance the energy in the high frequency part.
2D Synchrosqueezed (SS) transforms

2D wave packets +SS = 2D SS wave packet (SSWPT)
2D general curvelets = 2D SS curvelet (SSCT)

2D wave packets
2D wave packets \( \{w_{ab}(x) : a, b \in \mathbb{R}^2, |a| \geq 1\} \) are defined as
\[
w_{ab}(x) = |a|^s w(|a|^s(x - b))e^{2\pi i(x - b) \cdot a},
\]
or equivalently in Fourier domain
\[
\hat{w}_{ab}(\xi) = |a|^{-s} e^{-2\pi i b \cdot \xi} \hat{w}(|a|^{-s}(\xi - a)).
\]
Notations:

1. The scaling matrix
   \[
   A_a = \begin{pmatrix}
   a^t & 0 \\
   0 & a^s
   \end{pmatrix}.
   \]

2. The rotation angle \( \theta \) and rotation matrix
   \[
   R_\theta = \begin{pmatrix}
   \cos \theta & -\sin \theta \\
   \sin \theta & \cos \theta
   \end{pmatrix}.
   \]

3. A unit vector \( e_\theta = (\cos \theta, \sin \theta)^T \) with a rotation angle \( \theta \).

2D general curvelets

2D general curvelets \( \{w_{a \theta b}(x), \, a \in [1, \infty), \, \theta \in [0, 2\pi), \, b \in \mathbb{R}^2\} \) are defined as

\[
w_{a \theta b}(x) = a^{t+s} e^{2\pi i a(x-b) \cdot e_\theta} w(A_a R_\theta^{-1}(x - b)),
\]
or equivalently in Fourier domain

\[
\widehat{w_{a \theta b}}(\xi) = \widehat{w}(A_a^{-1} R_\theta^{-1}(\xi - a \cdot e_\theta)) e^{-2\pi i b \cdot \xi} a^{-\frac{t+s}{2}}.
\]
2D wave packets and 2D general curvelets

Figure: Essential support of the Fourier transform of: continuous wave packets; continuous general curvelets; a discrete general curvelet with parameters \((s, t)\), roughly of size \(a^s \times a^t\).
Theory for 2D SS wave packet transforms

**Theorem 1: (Y. and Ying SIIMS 13)**
A non-linear wave $f(x) = \alpha(x)e^{2\pi i \phi(x)}$, a wave packet $w_{ab}(x)$, define a transform:

$$W_f(a, b) = \langle s(x), w_{ab}(x) \rangle = \int s(x)w_{ab}(x) \, dx.$$  

$$\omega_f(a, b) = \frac{\nabla_b W_f(a, b)}{2\pi i W_f(a, b)} \approx \nabla \phi(b)$$

**Theorem 2: (Y. and Ying preprint 14)**
A zero mean Gaussian random noise $e$ with covariance $\epsilon_1^q$ for some $q > 0$. If $s(x) = f(x) + e$, then after thresholding, with a probability at least

$$\left(1 - e^{-O(N^{2s} \epsilon_1^{-q})}\right) \left(1 - e^{-O(N^{-2s} \epsilon_1^{-q})}\right) \left(1 - e^{-O(N^{-2} \epsilon_1^{-q})}\right),$$

$$\omega_s(a, b) = \frac{\nabla_b W_s(a, b)}{2\pi i W_s(a, b)} \approx N\nabla \phi(b)$$
Theory for 2D SS curvelet transforms

Theorem 1: (Y. and Ying SIMA 14)
A non-linear wave \( f(x) = \alpha(x)e^{2\pi i \phi(x)} \), a general curvelet \( w_{a \theta b}(x) \), define a transform:

\[
W_f(a, \theta, b) = \langle s(x), w_{a \theta b}(x) \rangle = \int s(x)w_{a \theta b}(x)\,dx.
\]

\[
\omega_f(a, \theta, b) = \frac{\nabla_b W_f(a, \theta, b)}{2\pi i W_f(a, \theta, b)} \approx \nabla \phi(b)
\]

Theorem 2: (Y. and Ying preprint 14)
A zero mean Gaussian random noise \( e \) with covariance \( \epsilon_1^q \) for some \( q > 0 \).
If \( s(x) = f(x) + e \), then after thresholding, with a probability at least

\[
\left(1 - e^{-O(N^{2-2t} \epsilon_1^{-q})}\right) \left(1 - e^{-O(N^{-2s} \epsilon_1^{-q})}\right) \left(1 - e^{-O(N^{-2} \epsilon_1^{-q})}\right),
\]

\[
\omega_f(a, \theta, b) = \frac{\nabla_b W_f(a, \theta, b)}{2\pi i W_f(a, \theta, b)} \approx \nabla \phi(b)
\]
Synchrosqueezing for sharpened representation:

\[ T_f(\omega, b) = \int_{\{a: W_f(a, b) \neq 0\}} W_f(a, b) \delta(\omega_f(a, b) - \omega) \, da. \]

Figure: An example of a superposition of two 2D waves using 2D SSWPT.
Figure: $T_f(\omega, b)$ of the same example in the last figure. Left: noiseless. Middle: SNR $= 3$. Right: SNR $= -3$. 
Difference of 2D SSWPT and 2D SSCT

Usually $s = t$ is better than $s < t$, except for the banded wave-like components.

Figure: Left: A superposition of two banded waves; Middle: 2D SSWPT; Right: 2D SSCT.
**SynLab**: a MATLAB toolbox

- 1D SS Wave Packet Transform
- 2D SS Wave Packet/Curvelet Transform

**Applications:**

- Geophysics: seismic wave field separation and ground-roll removal.
- Atomic crystal image analysis.
- Art forensic.

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