Synchrosqueezed Transforms and Their Applications in Seismic Data Analysis

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Problem Statement

Let $f(x) : \mathbb{R}^d \to \mathbb{R}$ be the given data:

$$f(x) = \sum_{k=1}^{K} \chi_{\Omega_k}(x) \alpha_k(x) e^{2\pi i N_k \phi_k(x)},$$

where $\alpha_k(x)$ and $\phi_k(x)$ are smooth functions, $\chi_{\Omega_k}(x)$ is an indicator function.

**Time-frequency geometry**

Estimate instantaneous frequency $N_k \phi'_k(x)$ (or local wave vector $N_k \nabla \phi_k(x)$ for $d > 1$).

**Mode decomposition**

Estimate each wave-like component $\alpha_k(x) e^{2\pi i N_k \phi_k(x)}$. 
Problem Statement

Let \( f(x) : \mathbb{R}^d \rightarrow \mathbb{R} \) be the given data:

\[
f(x) = \sum_{k=1}^{K} \chi_{\Omega_k}(x) \alpha_k(x) e^{2\pi i N_k \phi_k(x)}.
\]

Difficulties

- Unknown number of components \( K \);
- Non-stationary signals;
- Usually supported in unknown domains \( \Omega_k \);
- Multi-dimensional problem;
- Noisy data;
Literature Review

**Linear time-frequency (T-F) transforms**

- Continuous Gabor transform, wavelet transforms, etc.;
- Continuous wavelet transform (CWT) of a signal $f(x)$:

$$W_f(a, b) = \langle f(x), w_{ab}(x) \rangle = \int f(x) a^{-1/2} w\left(\frac{x-b}{a}\right) dx.$$

- Blurry energy spectrum
- Local maximum or path pursuit?

Figure: Left: True instantaneous frequencies. Right: CWT. Courtesy of [Jean et al Rev. of Geophys., 2014].
Time-frequency optimization

- Matching pursuits (Mallat, Zhang, 1993);
- Basis pursuits (Chen, Donoho, Sauders, 1998);
- Expensive computation;

Figure: Left: Matching pursuits. Right: Basis pursuits. Courtesy of [Jean et al Rev. of Geophys., 2014].
Literature Review

Empirical mode decomposition (EMD)

- Applied to various problems, cited in 11190 articles;
- Heuristic method, difficult to understand;
- Sensitive to noise.

*Figure*: Ensemble EMD. Courtesy of [Jean et al Rev. of Geophys., 2014].
Synchrosqueezed wavelet transform (SSWT)

- Proposed by Daubechies et al. (1996).
- Information function:
  \[ \nu_f(a, b) = \partial_b W_f(a, b)/(2\pi i W_f(a, b)) \]
  
  EX: \( f(x) = A \cos(2\pi \omega x) \), \( \nu_f(a, b) = \omega \).

- Synchrosqueezing process:
  \[ T_f(\nu, b) = \int_{\{a: W_f(a,b) \neq 0\}} |W_f(a, b)| a^{-3/2} \delta(\nu_f(a, b) - \nu) \, da. \]

Figure: Left: CWT. Right: SSWT. Courtesy of [Jean et al Rev. of Geophys., 2014].
Literature Review

Real data:

Figure: Top: Volcanic pre-explosion recordings. Bottom: Short-time Fourier transform. Courtesy of [Jean et al Rev. of Geophys., 2014].
Literature Review

Comparison using real data:

Figure: T-F distribution by CWT, SST, complementary ensemble empirical mode decomposition, basis pursuit. Courtesy of [Jean et al Rev. of Geophys., 2014].
Why Synchrosqueezed Transforms?

Features:

1. Better readability;
2. Local and adaptive transforms;
3. Multi-dimensional transforms;
4. Fast algorithms for forward and inverse transforms;
5. Reasonably robust;

Figure: SST of previous examples.
Mathematical analysis of SST

- Let $w_{ab}(x)$ be a T-F atom (e.g., a wavelet) and
  \[ W_f(a, b) = \langle f(x), w_{ab}(x) \rangle. \]

- Spectral information function:
  \[ \nu_f(a, b) = \partial_b W_f(a, b)/(2\pi i W_f(a, b)) \]

- Key analysis:
  \[ \nu_f(a, b) \approx N \nabla \phi(x) \quad \text{when} \quad f(x) = \alpha(x) e^{2\pi i N \phi(x)}? \]

- Difficulty: maintain accuracy if
  \[ f(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N_k \phi_k(x)}. \]

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**Figure**: Red: wavelet tiling. Blue: wave packet tiling. Green: short-time Fourier tiling. Black: Instantaneous frequencies $N_k \nabla \phi_k(x)$. 
Different 1D SSTs

- Continuous wavelet transform (Daubechies, Lu, Wu, ACHA, 2011)
- Short-time Fourier transform (Thakur, Wu, SIMA, 2011)
- Wave packet transform with a scaling parameter \( s \) (Y., ACHA, 2015)

Estimate \( N_k \) from \( f(x) = \sum_{k=1}^{K} e^{2\pi i N_k x} \).

**Figure**: Necessary spacing between different \( N_k \). Top: Wavelet. Middle: Wave packet. Bottom: Short-time Fourier
Different multi-dimensional SSTs

- Synchrosqueezed wave packet transform (SSWPT) (Y., Ying, SIIMS, 2013)
- Synchrosqueezed curvelet transform (SSCT) (Y., Ying, SIMA, 2014)
- Synchrosqueezed monogenic wavelet transform (Clausel, Oberlin, Perrier, ACHA, 2015)

**Figure**: Comparison of localized supports of continuous wavelets (left), wave packets (middle) and curvelets (right) in the Fourier domain. Two dots in each plot show the support of the Fourier transforms of the superposition of two plane waves $e^{2\pi ip \cdot x}$ and $e^{2\pi iq \cdot x}$ with the same wave-number ($|p| = |q|$) but different wave-vectors ($p \neq q$).
SSWPT vs SSCT

Wave field separation problem in geophysics:

Figure: Left: A superposition of two banded waves; Middle: Extracted component by 2D SSWPT; Right: Extracted component by 2D SSCT.
How accurate is SST?

Example: synchrosqueezed transform based on generalized curvelet transform (SSCT)

**Curvelets** $w_{a\theta b}(x)$

Scaling matrix

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}.$$  

**Generalized curvelets** $w_{a\theta b}(x)$

Scaling matrix

$$A_a = \begin{pmatrix} a^t & 0 \\ 0 & a^s \end{pmatrix}$$

with $1/2 \leq s \leq t \leq 1$. 

Mathematical analysis of the SSCT

Theorem [Y., Ying, SIMA 2014]
Suppose $\frac{1}{2} < s \leq \eta < t$ are fixed, $\sigma \geq N^{-\eta}$, and

$$f(x) = e^{-(\phi(x)-c)^2/\sigma^2} \alpha(x)e^{2\pi i N \phi(x)}.$$ 

For any accuracy $\epsilon > 0$, there exists $N_0(\epsilon) = O\left(\max\{\epsilon^{1-t}, \epsilon^{t-\eta}, \epsilon^{2s-1}\}\right)$ such that for any $N > N_0(\epsilon)$ we have

$$\frac{|\nu_f(a, \theta, b) - N\nabla \phi(b)|}{|N\nabla \phi(b)|} \lesssim \sqrt{\epsilon}$$

if $|W_f(a, \theta, b)| \geq a^{-\frac{s+t}{2}} \sqrt{\epsilon}$. 
Mathematical analysis of the SSCT

Lemma 1
If \( a = \Theta(N) \) and \( |\theta - \theta\nabla \phi(b)| \leq \arcsin \left( \frac{1}{\Theta(N)} \right)^{t-s} \), then

\[
W_f(a, \theta, b) \approx a^{-s+t} f(b) \hat{w} \left( A_a^{-1} R_\theta^{-1} (a \cdot u_\theta - N \nabla \phi(b)) \right).
\]

Otherwise, \( W_f(a, \theta, b) \approx \epsilon \).

Figure: Banded wave-like component \( f(x) \). Ellipses represent essential supports of general curvelts: only the green one match the oscillation of \( f(x) \).
Mathematical analysis of the SSCT

**Lemma 2**

If \( a = \Theta(N) \) and \(|\theta - \theta \nabla \phi(b)| \leq \arcsin \left( \frac{1}{\Theta(N)} \right)^{t-s} \), then

\[
\nabla_b W_f(a, \theta, b) = a^{- \frac{s+t}{2}} (2\pi i N \nabla \phi(b) f(b) \hat{w} (A^{-1}_a R^{-1}_\theta (a \cdot u_\theta - N \nabla \phi(b)))) + aO(\epsilon).
\]

**Figure**: Banded wave-like component \( f(x) \). Ellipses represent essential supports of general curvelts: only the green one match the oscillation of \( f(x) \).
Mathematical analysis of the SSCT

Main asymptotic estimation

If $|W_f(a, \theta, b)| \geq a^{-\frac{s+t}{2}} \sqrt{\epsilon}$

$$v_f(a, \theta, b) = \frac{\nabla_b W_f(a, \theta, b)}{2\pi i W_f(a, \theta, b)} = \frac{N\nabla \phi(b)(g + O(\epsilon))}{g + O(\epsilon)},$$  \hspace{1cm} (1)

with

$$|g| \gtrsim \sqrt{\epsilon}.$$

Hence,

$$\frac{|v_f(a, \theta, b) - N\nabla \phi(b)|}{|N\nabla \phi(b)|} \lesssim \frac{|O(\epsilon)|}{|g + O(\epsilon)|} \lesssim \sqrt{\epsilon}.$$

Note: $N \geq N_0(\epsilon) = O\left(\max\{\epsilon^{\frac{-1}{1-t}}, \epsilon^{\frac{-2}{t-\eta}}, \epsilon^{\frac{-2}{2s-1}}\}\right)$ and $s \leq \eta < t.$
Why robustness? How robust is SST?

Example in canvas painting:
Windowed Fourier transform vs synchrosqueezed transform

Figure: Left: A sample swatch of an X-ray image and its noisy version; Right: Comparison of the spectrum of the windowed Fourier transform and the synchrosqueezed transform.
Robustness Analysis of Synchrosqueezed Transforms

Literature

  ▶ Gaussian white noise;
  ▶ Generalized stationary Gaussian noise;
▶ Y., Preprint, 2014
  ▶ Multi-dimensional
  ▶ Compactly supported in time/space domain
  ▶ Generalized stationary Gaussian noise;
Robustness Analysis of Synchrosqueezed Transforms

Key analysis

- Suppose \( g(x) = f(x) + e(x) = \sum_{k=1}^{K} \alpha_k(x)e^{2\pi i N_k \phi_k(x)} + e(x) \)

- Estimate the probability density function of

\[
\nu_e(a, b) = \frac{\nabla_b W_e(a, b)}{2\pi i W_e(a, b)}
\]

- Quantify the influence of \( \nu_e(a, b) \) on \( \nu_f(a, b) \)

Observations for better statistical stability

- Smaller scaling parameters, better stability
- Larger \( W_f(a, b) \), better stability
- \( \nu_f(a, b) \) is independent of the window function and the time-frequency tiling
Ideas to improve statistical stability

2D robustness test:

Figure: Left: A 2D noiseless component with a constant amplitude 1. Right: A noisy component generated with Gaussian white noise $5N(0, 1)$. 
Ideas to improve statistical stability

Smaller scaling parameters

Figure: Fixing a point $x_2$ and stacking $k_2$. Left: $T_f(k, x)$ with large $t$ and $s$. Right: $T_f(k, x)$ with small $t$ and $s$. 
Ideas to improve statistical stability

Selective coefficients

Figure: Fixing a point $x_2$ and stacking $k_2$. Left: $T_f(k, x)$ generated by reassigning all coefficients. Right: $T_f(k, x)$ generated by reassigning coefficients with the largest magnitude at a space location $x$. 
Ideas to improve statistical stability

Multi-frames by time-frequency dilation, shifting and rotation

Figure: Fixing a point $x_2$ and stacking $k_2$. Left: $T_f(k, x)$ with large $t$ and $s$. Right: $T_f(k, x)$ with small $t$ and $s$. 
Comparison of various SSTs

Figure: Noisy example with $\text{SNR} \approx -13$. Top-left: The real instantaneous frequency. Top-right: The standard SS wavelet transform. Bottom-left: The standard SS short-time Fourier transform. Bottom-right: The SS wave packet transform.
Comparison with multitaper time-frequency reassignment

Figure: Left and middle: The multitaper time-frequency reassignment using an arithmetic mean (MRSA) and a geometric mean (MRSG) with 10 tapers. Right: The SSWPT with $s = 0.75$ and 10 frames.
Quantitative comparison

Earth mover’s distance (EMD) to measure time-frequency localization.

Figure: The EMD as functions in the variable of numbers of frames for different SNRs.
Quantitative comparison

Earth mover’s distance (EMD) to measure time-frequency localization.

Figure: Comparison of the standard SS wavelet (SSWT) SS short-time Fourier (SSSTFT) and the SSWPT with $s = 0.75$ and 10 frames.
Quantitative comparison

Earth mover’s distance (EMD) to measure time-frequency localization.

Figure: Comparison of the multitaper time-frequency reassignment (the numbers of tapers are 5 and 10) and the SSWPT with $s = 0.75$ and 10 frames.
SynLab: a MATLAB toolbox

- Available at http://services.math.duke.edu/~haizhao/Codes.html.
- 1D to 3D synchrosqueezed transforms.
- Comparison with other transforms.

Applications:

- Seismic data analysis.
- Atomic crystal image analysis.
- Art forensic.
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