Synchrosqueezed Wave Packet Transforms and Diffeomorphism Based Spectral Analysis for 1D General Mode Decompositions

Haizhao Yang

Department of Mathematics, Stanford University

July 2014
1 Introduction
   - Problem Statement
   - Existing Methods
   - Summary of Proposed Method

2 Synchrosqueezed Transforms
   - Synchrosqueezed Wavelet Transform
   - Synchrosqueezed Wave Packet Transform

3 Diffeomorphism Based Spectral Analysis

4 Numerical Examples
Data Analysis:

- Superposition of several patterns
- Differ in instantaneous frequency
- Nonlinear and non-stationary with sudden changes

Figure: Decomposition using wavelets
Mode decomposition

Known: A superposition of wave-like components

\[ f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x) e^{2\pi i N_k \phi_k(x)}. \]

Unknown:

- Number \( K \)
- Components \( f_k(x) \)
- Smooth instantaneous amplitudes \( \alpha_k(x) \)
- Smooth instantaneous frequencies \( N_k \phi'_k(x) \)
General mode decomposition problem

Known: A superposition of general wave-like components

\[ f(x) = \sum_{k=1}^{K} f_k(x) = \sum_{k=1}^{K} \alpha_k(x)s_k(2\pi N_k \phi_k(x)), \]

Unknown:
- Basic unknown information
- General wave shape functions \( s_k(x) \),
  i.e. \( \{(n, \hat{s}_k(n) : \hat{s}_k(n) \neq 0\} \)
Empirical mode decomposition (Huang et al. 98,99,09)

- More physical meaning
- Better than Fourier and wavelet analysis
- Need more mathematical understanding

Useful decomposition

Misleading decomposition
Synchrosqueezed wavelet transform (Daubechies et al. 95 11)

- Band-limited wave shape function $s_k$ (Wu 2012)
- No explicit statement about the reconstruction of $s_k$.

Figure: Left: Real Echocardiography (ECG) wave shape and its band-limited approximation. Right: Fourier power spectrum.
Difficulties in general mode decomposition

- Need accurate estimates of high frequency information
- Crossover frequencies, poor scale separation

\[
\begin{align*}
  s_1(2\pi N_1 \phi_1(t)) + s_2(2\pi N_2 \phi_2(t)) &= \sum_n \hat{s}_1(n)e^{2\pi inN_1 \phi_1(t)} \\
  &+ \sum_n \hat{s}_2(n)e^{2\pi inN_2 \phi_2(t)}
\end{align*}
\]

Figure: Instantaneous frequencies: \(s_1\) in blue and \(s_2\) in red.
Innovation 1
Synchrosqueezed wave packet transforms with a scaling parameter $s \in (1/2, 1)$
- Better resolution to estimate instantaneous information in high frequency domain
- General wave shapes $s_k$, NOT band-limited
- Fast algorithm $O(N \log N)$

Innovation 2
Diffeomorphism based spectral analysis (DSA)
- First method to reconstruct $s_k$
- Efficient implementation $O(KMN)$, $M$ is the number of principal Fourier modes of all $s_k$ and $K$ is the number of $s_k$. 
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Synchrosqueezed wavelet transform (SSWT)

Continuous wavelet transform of a signal $s(t)$:

$$W_s(a, b) = \langle s(t), \phi_{ab}(t) \rangle = \int s(t) a^{-1/2} \phi\left(\frac{t - b}{a}\right) dt.$$  

EX: $s(t) = A \cos(\omega t)$.

$$W_s(a, b) = \frac{1}{2\pi} \int \hat{s}(\xi) a^{1/2} \hat{\phi}(a\xi) e^{ib\xi} d\xi = \frac{A}{4\pi} a^{1/2} \hat{\phi}(a\omega) e^{ib\omega}.$$  

Synchrosqueezing for better readability

Figure: Numerical examples by Daubechies et al, signal $f(t) = \sin(8t)$.  

Haizhao Yang
Definitions of SSWT

EX: \( s(t) = A \cos(\omega t) \).

\[
W_s(a, b) = \frac{A}{4\pi} a^{1/2} \hat{\phi}(a\omega) e^{ib\omega} \Rightarrow \frac{\partial_b W_s(a,b)}{iW_s(a,b)} = \omega
\]

**Definition: Instantaneous frequency estimate**

\[
\omega_s(a, b) = \frac{\partial_b W_s(a, b)}{iW_s(a, b)}.
\]

**Definition: Synchrosqueezed wavelet transform**

\[
\mathcal{T}_s(\omega, b) = \int_{\{a: W_s(a,b) \neq 0\}} W_s(a, b) a^{-3/2} \delta(\omega_s(a, b) - \omega) \, da.
\]
Theory of SSWT

Theorem: (Daubechies, Lu, Wu 11 ACHA)

If
\[ f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) e^{2\pi i N_k \phi_k(t)} \]

and \( f_k(t) \) are well-separated, then

- \( \mathcal{T}_f(a, b) \) has well-separated supports \( Z_k \) concentrating \( (N_k \phi'_k(b), b) \);
- \( f_k(t) \) can be accurately recovered by applying an inverse transform on \( \mathcal{I}_{Z_k}(a, b) \mathcal{T}_f(a, b) \).

where \( \mathcal{I}_{Z_k}(a, b) \) is an indication function.
Mode decomposition example by SSWT

Figure: A superposition of two modes with *crossover* frequencies. Courtesy of [Daubechies, Lu, Wu 11].
Synchrosqueezed wave packet transform (SSWPT)

Wave packets \( \{w_{ab}(t) : |a| \geq 1, b \in \mathbb{R} \} \) with \( s \in (1/2, 1) \):

\[
w_{ab}(t) = |a|^{s/2} w(|a|^s (t - b)) e^{2\pi i (t-b)a},
\]

or equivalently, in the Fourier domain as

\[
\hat{w}_{ab}(\xi) = |a|^{-s/2} e^{-2\pi ib\xi} \hat{w}(|a|^{-s}(\xi - a)).
\]

Figure: Comparison of tilings. Top: wavelets. Bottom: wave packets.
Definition: Instantaneous frequency estimate

\[ \omega_s(a, b) = \frac{\partial_b W_s(a, b)}{2\pi i W_s(a, b)}. \]

Definition: Synchrosqueezed wave packet transform

\[ T_s(\omega, b) = \int_{\{a : W_s(a, b) \neq 0\}} |W_s(a, b)|^2 \delta(\Re\omega_s(a, b) - \omega) \, da. \]
Theorem of SSWPT (Y. 13)

If

\[ f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} \alpha_k(t) s_k(2\pi N_k \phi_k(t)) \]

and \( f_k(t) \) has a well-separated Fourier series mode

\[ \hat{s}_k(n) \alpha_k(t) e^{2\pi i n N_k \phi_k(t)} \]

then

- \( T_f(a, b) \) has a well-separated support \( Z_{kn} \) concentrating
  \( (nN_k \phi'_k(b), b) \);
- \( \hat{s}_k(n) \alpha_k(t) e^{2\pi i n N_k \phi_k(t)} \) can be accurately recovered by applying an inverse transform on \( \mathcal{I}_{Z_{kn}}(a, b) T_f(a, b) \), where \( \mathcal{I}_{Z_k}(a, b) \) is an indicator function.
**Figure**: The SSWT of the same ECG signal. Left: low frequency part. Right: high frequency part.
Figure: The SSWPT of an ECG signal.
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General mode decomposition:

\[ f(t) = s_1(2\pi N_1 \phi_1(t)) + s_2(2\pi N_2 \phi_2(t)) = \sum_n \hat{s}_1(n)e^{2\pi inN_1 \phi_1(t)} + \sum_n \hat{s}_2(n)e^{2\pi inN_2 \phi_2(t)} \]
Diffeomorphism based spectral analysis (DSA)

Suppose

\[ f(t) = s_1(2\pi N_1 \phi_1(t)) + s_2(2\pi N_2 \phi_2(t)), \]

we know \( p_k(t) = N_k \phi_k(t), \ k = 1, 2. \)

**Goal**: find \( n \) and \( \hat{s}_k(n) \) such that \( \hat{s}_k(n) \neq 0. \)

Let

\[ h_k(t) = f \circ p_k^{-1}(t) = \sum_{n=\infty}^{\infty} \hat{s}_k(n)e^{2\pi int} + \sum_{j \neq k} \sum_{n=\infty}^{\infty} \hat{s}_j(n)e^{2\pi inN_j \bar{\phi}_j(t)}. \]

**Idea:**

1. The location of the peak of \( |\hat{h}_k(\xi)| \) tell us one candidate \( n \). Hence, solve

   \[ (\tau, j) = \arg \max_{(\xi, k)} |\hat{h}_k(\xi)|. \]

2. Solve

   \[ \hat{s}_j(\tau) = \arg \min_{\beta \in C} \| f(t) - \beta e^{2\pi i\tau N_j \bar{\phi}_j(t)} \|_{L^2}. \]

3. Update \( f(t) = f(t) - \hat{s}_j(\tau)e^{2\pi i\tau N_j \phi_j(t)} \) and repeat this process.
Theory of DSA

Suppose \( f(t) = \sum_{k=1}^{K} \alpha_k(t)s_k(2\pi N_k \phi_k(t)) \) and phase functions \( \{\phi_k(t)\}_{1 \leq k \leq K} \) are well-different. Define

\[
h_k(t) = \frac{f \circ \phi_k^{-1}(t)}{\alpha_k \circ \phi_k^{-1}(t)}.
\]

For a Gabor transform \( \mathcal{F}_T \) with a sufficiently large window, let

\[
(a_0, k_0) = \arg \max_{(a,k)} |\mathcal{F}_T(h_k)(a, b)|,
\]

then \( a_0 \approx nN_{k_0} \) for some \( n \) such that \( \hat{s}_{k_0}(n) \neq 0 \).
General mode decomposition

**Figure**: Blue: Real signals. Red: Reconstructed results. Top: Two recovered modes using a few well separated components. Bottom: Two recovered general modes provided by the DSA method.
Example 1

Figure : Toy Example. Left: SSWPT and DSA. Right: EEMD.
Example 2

Figure: ECG spike shape function. Left: SSWPT and DSA. Right: EEMD.
Example 3

Figure: Real CO\textsubscript{2} concentration data. Left: SSWPT and DSA. Right: EEMD.
Thank you!