Inexact Coordinate Descent Method for Nonsmooth Separable Minimization

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Outline:

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3. Convergence Analysis
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A Problem Overview:

◊ An old problem in optimization

\[ Ax \approx b, \]

where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \) are given.

◊ Linear least square problem

\[
\min_{x} \| Ax - b \|_2^2,
\]

◊ Box-constrained convex QP

\[
\min_{l \leq x \leq u} \| Ax - b \|_2^2,
\]
○ $\ell_1$- regularization

$$\min_{x} \|Ax - b\|_2^2 + c\|x\|_1,$$

where $c > 0$ is a user chosen regularization parameter.

○ Recent interests have focussed on finding solutions that are parsimonious/sparse.

○ Ex: the “Basis Pursuit” model for signal denoising.
\textbf{Structured Nonsmooth Optimization}

- Objective function is \textit{sum of smooth func and nonsmooth separable (convex polyhedral) func.}

$$\min_{x} f(x) + cP(x), \quad P(x) = \sum_{j} P_{j}(x_{j})$$

where $c > 0$, $f$ smooth, $P$ nonsmooth, $(\text{epi}P = \{(x, \zeta) \mid P(x) \leq \zeta\}$ is a polyhedral set).

- Ex1: box-constrained QP

$$f(x) = \|Ax - b\|^2_2,$$

$$P(x) = \begin{cases} 
0 & \text{if } l \leq x \leq u \\
\infty & \text{else.}
\end{cases}$$

- Ex2: $\ell_1$- regularization

$$f(x) = \|Ax - b\|^2_2, \quad P(x) = \|x\|_1$$
Inexact (block) Coordinate Descent Method

○ Decent Direction

○ Choose $J(\neq \emptyset) \subset \{1, \ldots, n\}$,
  
  \[ \text{sym pd } H \in \mathbb{R}^{n\times n} \]

○ Solve:

\[
\min_d \nabla f(x)^T d + \frac{1}{2} d^T H d + cP(x + d)
\]

\[ \text{s.t. } d_j = 0 \ \forall j \notin J, \]

Using the convexity of $P$, it can be seen that

\[
(f + cP)(x + \alpha d) \leq (f + cP)(x) - \alpha \frac{1}{2} d^T H d + o(\alpha),
\]

for $0 < \alpha < 1$, whenever $d \neq 0$.

● if $J = \{1, \ldots, n\}$, $P(x) = \begin{cases} 0 & \text{if } l \leq x \leq u, \\ \infty & \text{else.} \end{cases}$

  then $d$ is a scaled gradient-projection direction for box-constrained minimization;

● if $f$ is quadratic, $H = \nabla^2 f(x)$, then $d$ is a (block) coordinate minimization direction.
○ **Stepsize**

○ Choose a stepsize $\alpha$ so that $x^{\text{new}} = x + \alpha d$ achieves sufficient descent.

Armijo rule:

Choose $\alpha$ to be the largest element of

$$\{\alpha_{\text{init}} \beta^k\}_{k=0,1,...}$$

satisfying

$$(f + cP)(x + \alpha d) \leq (f + cP)(x) - \alpha \sigma d^T H d,$$

where $0 < \beta < 1$, $0 < \sigma < \frac{1}{2}$, and $\alpha_{\text{init}} > 0$.

This rule, like that for SQP, requires only function evaluations.

○ By choosing $\alpha_{\text{init}}$ based on previous stepsizes, the number of evaluations can be kept small.
Choose J

- Gauss-Seidel
  
  $J$ cycles through $\{1\}, \{2\}, \ldots, \{n\}$ or, more generally, $J$ collectively covers $1, 2, \ldots, n$ for every fixed number of consecutive iterations.

- Gauss-Southwell

Owing to the convex separable nature of $P$, “natural” residual:

$$R(x) = (R(x)_j)_{j=1}^n,$$

$$R(x)_j = \arg \min_{d_j} g_j^T d_j + \frac{1}{2} d_j^T H_{jj} d_j + c P_j (x_j + d_j)$$

where $g = (g_j)_{j=1}^n$, $g = \nabla f(x)$.

Choose $j$ to satisfy

$$\|R(x)_j\|_\infty \geq \omega \|R(x)\|_\infty, \ 0 < \omega \leq 1$$
• Ex: \((H = I)\)

\[ P \equiv 0, \; R(x)_j = -g_j. \]

\[ P(x) = \begin{cases} 0 & \text{if } l \leq x \leq u, \\ \infty & \text{else} \end{cases}, \]

\[ R(x)_j = \text{median}\{l_j - x_j, -g_j, u_j - x_j\}. \]

• \( P(x) = \|x\|_1, \)

\[ R(x)_j = -\text{median}\{g_j - c, x_j, g_j + c\}. \]
Convergence Analysis

◊ Global Convergence

◊ Proposition:
Let \( \{x^k\} \) be generated by InexactCD-Gauss-Southwell, \( x^{k+1} = x^k + \alpha^k d^k \).

Assume that \( P \) is lsc, \( \{d^k\} \) is bounded, and \( \alpha^k \) is chosen by the Armijo rule.

Then every cluster point of \( \{x^k\} \) is a stationary point.
\textbf{Convergence Rate}

○ Error Bound
\[ \text{dist}(x, S) \leq \kappa_1 \|R(x)\|_\infty \text{ whenever } \|R(x)\|_\infty \leq \epsilon_1, \]
for some \( \kappa_1 > 0, \epsilon_1 > 0 \), where \( S \) denotes the set of stationary points and \( \text{dist}(x, S) = \min_{s \in S} \|x - s\|_2 \).

Corollary:

For the case of smooth problems with polyhedral constraints (ref \(*\))
\[ f(x) - \nu \leq \kappa_2 \|R(x)\|_\infty^2 \text{ whenever } \|R(x)\|_\infty \leq \epsilon_2, \]
for some \( \kappa_2 > 0, \epsilon_2 > 0 \), where \( \nu = \lim_{k \to \infty} f(x^k) \).

But this key bound used previously for convergence rate analysis fails for the general case.

\* Luo, Z.-Q. and Tseng, P., ”Error bounds and convergence analysis of feasible descent methods: a general approach”
Example:

\[
\min_{x \in \mathbb{R}} x^2 - x + |x|
\]

New key bound:

\[(f + cP)(x) - \nu \leq \kappa_3 \| R(x) \|^2 + h(R(x))\]

whenever \( \| R(x) \|_\infty \leq \epsilon_3 \),

for some \( \kappa_3 > 0, \epsilon_3 > 0 \),

where \( \nu = \lim_{k \to \infty} (f + cP)(x^k) \) and \( h(x) \) is a nonnegative linear function.
Theorem 1:
Assume $f(x) = g(Ex)$ where $E \in \mathbb{R}^{m \times n}$, $g$ is strongly convex on $\mathbb{R}^m$ with
\[ \|\nabla g(x) - \nabla g(y)\| \leq L\|x - y\|. \]
Let $\{x^k\}$ be generated by InexactCD-Gauss-Seidel (Armijo rule) with $\gamma \|z\|^2 \leq z^T H^k z$, $\limsup_{k,j} \alpha^k_j \leq \frac{\gamma}{L}$ and $\{d^k\}$ bdd.

Then $\{(f+cP)(x^k)\}$ converges at least Q-linearly and $\{x^k\}$ converges at least R-linearly.
Idea for Proof:

For sufficiently large $k$,

$$(f + cP)(x^{k+1}) - v \leq \kappa \|x^{k+1} - x^k\|^2 + \sum_{j=1}^n \frac{1 - \alpha_j^k}{\alpha_j^k} (\alpha_j^k < A_j^T \mu_j^k, d_j^k) + c(P_j(x_j^k) - P_j(x_j^{k+1}))$$

$$(f + cP)(x^{k+1}) - (f + cP)(x^k) \leq -\frac{L}{2} \|x^{k+1} - x^k\|^2 - \sum_{j=1}^n (\alpha_j^k < A_j^T \mu_j^k, d_j^k) + c(P_j(x_j^k) - P_j(x_j^{k+1}))$$

where $(x_j, \xi_j) \in \text{epi} P_j \iff A_j x_j + a_j \xi_j \leq b_j$, $\kappa > 0$, and $\mu^k$ is some multiplier vector.
- **Theorem 2:**
  Theorem 1 still holds if, in addition, $f$ is separable and *Gauss-Seidel* is replaced by *Gauss-Southwell*.

- **Conjecture:** Theorem 2 still holds without the separability of $f$
Numerical Experience

○ Coded in Matlab (running Matlab 6.5)

○ Time is on a Windows Laptop

○ $H = \text{diag}(\max(\nabla^2 f(x)_{jj}, 1))$

○ Stop when $\|R(x)_j\|_\infty < 10^{-4}$

○ Test functions (except 5) from the More-Garbow-Hillstrom collection
1. Brown almost-linear func(nonconvex)
\[ f(x) = \sum_{i=1}^{n} (x_i + \sum_{j=1}^{n} x_j - (n+1))^2 + ((\prod_{j=1}^{n} x_j) - 1)^2 \]
with \( n = 100 \) and \( x^0 = (1, \ldots, 1) \).

2. Extended Rosenbrock func(nonconvex)
\[ f(x) = \sum_{i=1}^{n/2} (100 * (x_{2i} - x_{2i-1}^2) + (1 - x_{2i-1})^2) \]
with \( n = 100 \) and \( x^0 = (\zeta_j) \) where \( \zeta_{2j-1} = -1.2, \zeta_{2j} = 1 \).

3. Extended Powell singular func(convex)
\[ f(x) = \sum_{i=1}^{n/4} ((x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i} - 1)^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4) \]
with \( n = 1000 \) and \( x^0 = (\zeta_j) \) where \( \zeta_{4j-3} = 3, \zeta_{4j-2} = -1, \zeta_{4j-1} = 0, \zeta_{4j} = 1 \).
4. Variably dimensioned func(convex)
\[ f(x) = \sum_{i=1}^{n} (x_i - 1)^2 + \left( \sum_{i=1}^{n} i(x_i - 1) \right)^2 + \left( \sum_{i=1}^{n} i(x_i - 1) \right)^4 \]
with \( n = 100 \) and \( x^0 = (1 - (j/n)) \).

5. Quadratic func(satisfy assumption)
\[ f(x) = \left( \sum_{i=1}^{n} x_i - n \right)^2 \]
with \( n = 1000 \) and \( x^0 = (1, \ldots, 1) \).

6. Linear func-full rank(satisfy assumption)
\[ f(x) = \left( \sum_{i=1}^{n} \left( x_i - \frac{2}{m} \left( \sum_{j=1}^{n} x_j \right) - 1 \right) \right)^2 + \left( \frac{2}{m} \left( \sum_{j=1}^{n} x_j \right) + 1 \right)^2 \]
with \( n = 1000, m = 1001 \) and \( x^0 = (1, \ldots, 1) \).
Conclusion

1. Faster convergence if the smooth func $f$ is (partially) separable.

2. InexactCD-$Gauss$-$Southwell$ is faster than InexactCD-$Gauss$-$Seidel$, especially, if $f$ is nonseparable.

Future work

1. Prove the conjecture (Linear rate convergence for InexactCD-$Gauss$-$Southwell$ still holds without the separability of $f$).

2. In our test, $n(J) = 1$. Can it be more efficient if we use block coordinate due to the separability structure of $f$?

3. More test on other functions and applications (e.g., regularized nonlinear least square)

4. Convergence acceleration for nonseparable function $f$?
Reference

References


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Table 1: test result