Ch 9.2: Autonomous Systems and Stability

In this section we draw together and expand on geometrical ideas introduced in Section 2.5 for certain first order equations and Section 9.1 for second order linear homogeneous systems with constant coefficients. These ideas concern the qualitative study of differential equations and the concept of stability.
Initial Value Problem

We are concerned with systems of two simultaneous differential equations of the form

\[ \begin{aligned}
0 \dot{\lambda} &= (0 \lambda) \dot{\lambda}, & 0 x &= (0 \lambda) x \\
\end{aligned} \]

containing \( \lambda \), satisfying the initial conditions of \( \lambda \) and \( x \), defined in some interval \( I \). If \((0 \lambda, 0 x) \) is a point in \( D \), then by Theorem 7.1.1 there exists a unique solution \( x(\tau) \), \( y(\tau) \), defined in some interval \( I \). We assume that the functions \( F \) and \( G \) are continuous and have continuous partial derivatives in some domain of \( xy \)-plane.

We assume that the functions \( F \) and \( G \) are continuous and have differential equations of the form

\[ \begin{aligned}
(\lambda) x(\tau) & = \frac{dx}{d\tau} \quad \text{and} \quad (\lambda) y(\tau) = \frac{dy}{d\tau}
\end{aligned} \]

Initial Value Problem
We can write the initial value problem in vector form:

\[
\begin{pmatrix}
I \\
0
\end{pmatrix}
\int_0^t \begin{pmatrix}
1 \\
0
\end{pmatrix} \, dt + \begin{pmatrix}
0 \\
1
\end{pmatrix} \phi = \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

where

\[
x = \begin{pmatrix}
x \\
y
\end{pmatrix}, \quad f(x) = \begin{pmatrix}
F(x, y) \\
G(x, y)
\end{pmatrix}, \quad x_0 = \begin{pmatrix}
x_0 \\
y_0
\end{pmatrix}, \quad \text{and}
\]

\[
\phi(t) = \begin{pmatrix}
\phi(t) \\
\psi(t)
\end{pmatrix}
\]

is a curve traced out by a point in the \( xy \)-plane (phase plane).

In vector form, the solution \( x = \phi(t) \) is a curve traced out by a point in the \( xy \)-plane (phase plane).

\[
0 \lambda + 0 x = x, \quad \int (\lambda x) dt + \int (\lambda x) dp = \phi(t) \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad \lambda + x = x
\]

where

\[
x = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad \phi(t) = \begin{pmatrix}
1 \\
0
\end{pmatrix} \lambda + \begin{pmatrix}
0 \\
1
\end{pmatrix} x = x
\]

We can write the initial value problem in vector form:

\[
0 \lambda = \begin{pmatrix}
0 \\
1
\end{pmatrix} \lambda, \quad 0 x = \begin{pmatrix}
0 \\
1
\end{pmatrix} x, \quad (\lambda x) dt = \lambda p, \quad (\lambda x) dp = \lambda p
\]

In vector form:

\[
0 \lambda = \begin{pmatrix}
0 \\
1
\end{pmatrix} \lambda, \quad 0 x = \begin{pmatrix}
0 \\
1
\end{pmatrix} x, \quad (\lambda x) dt = \lambda p, \quad (\lambda x) dp = \lambda p
\]
For our initial value problem, note that the functions $F$ and $G$ depend on $x$ and $y$, but not $t$. Such a system is said to be autonomous.

The system $x' = Ax$, where $A$ is a constant matrix, is an example of an autonomous system. However, if one or more of the elements of the coefficient matrix $A$ is a function of $t$, then the system is nonautonomous.

The geometrical qualitative analysis of Section 9.1 can be extended to two-dimensional autonomous systems in general, but is not as useful for nonautonomous systems.
Phase Portraits for Autonomous Systems

Our autonomous system has a direction field that is independent of time. Hence a single phase portrait displays important qualitative information about all solutions of the system. They pass through the same trajectory, regardless of the time at which they pass through \((x_0, y_0)\) in the phase plane.

Thus all solutions to an initial value problem of the form

\[
0 \dot{x} = (0 \dot{y}) x, \quad 0 \dot{y} = (0 \dot{x}) y,
\]

which has a direction field that is independent of time,

\[
(\dot{x}, \dot{y}) G = \dot{y} / \dot{x}, \quad (\dot{x}, \dot{y}) H = \dot{x} / \dot{y}
\]

Our autonomous system
Stability and Instability

For the following definitions, we consider the autonomous system
\[ x' = f(x) \]
and denote the magnitude of \( x \) by \( \| x \| \).

The points, if any, where \( f(x) = 0 \) are called critical points. A critical point \( x_0 \) is said to be stable if, for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that every solution \( x = \phi(t) \) satisfying \( \| \phi(0) - x_0 \| < \delta \) exists for all positive \( t \) and satisfies \( \| \phi(t) - x_0 \| < \epsilon \) for all \( t \geq 0 \).

Otherwise, \( x_0 \) is unstable.

A critical point \( x_0 \) is said to be equilibrium if, for all positive \( t \) and satisfies \( \| x - (t)\phi \| \) exists for all positive \( t \) such that every solution \( x \) satisfying \( (t)\phi = x \) for all \( t \geq 0 \) there is a constant, or equilibrium, solutions of the system of equations.

These points \( \phi = 0 \) also, and hence critical points correspond to these points, if any, where \( f(x) = 0 = \nabla f(x) \) and denote the magnitude of \( x \) by \( \| x \| \).

For the following definitions, we consider the autonomous system
\[ x' = f(x) \]
Asymptotic Stability

A critical point $x_0$ is said to be asymptotically stable if it satisfies

$$\lim_{t \to \infty} x = (t)\phi \quad \text{as} \quad x \to x_0$$

Thus trajectories that start “sufficiently close” to $x_0$ not only stay close to $x_0$ but must eventually approach $x_0$ as $t \to \infty$.

Thus asymptotic stability is a stronger property than stability.

However, note that

$$\lim_{t \to \infty} x = (t)\phi \quad \text{does not imply stability}.$$

Thus the one in Figure (a) below but not for the one in Figure (b) below.
The Oscillating Pendulum

The concepts of asymptotic stability, stability, and instability can be easily visualized in terms of an oscillating pendulum. Suppose a mass $m$ is attached to one end of a weightless rigid rod of length $L$, with the other end of the rod supported at the origin $O$. See figure below.

The gravitational force $mg$ acts downward, while the centripetal force $c |\dot{\theta}|$, $c > 0$, always opposes the direction of motion. The position of the pendulum is described by the angle $\theta$ with respect to the horizontal, and $\dot{\theta}$ is the velocity of the mass. The damping force $|\dot{\theta}|$ always opposes the direction of motion.

The Oscillating Pendulum
The principle of angular momentum states that the time rate of change of angular momentum about any point is equal to the moment of the resultant force about that point. The angular momentum about the origin is $mL^2\left(\frac{d\theta}{dt}\right)$, and hence the governing equation is

$$\Theta = \sum (\frac{1}{\theta_0} p_0) \frac{d}{dt} \sum \gamma m L$$

Here, $L$ and $L \sin \theta$ are the moment arms of the resistive and gravitational forces. This equation is valid for all four sign possibilities of $\theta$ and $d\theta/dt$.

The angular momentum about the origin is $mLw$, and the moment of the resultant force about any point is equal to the change of angular momentum about that point.

The principle of angular momentum states that the time rate of change of angular momentum is equal to the moment of the resultant force about any point.
Rewriting our equation in standard form, we obtain

$$L g m l = \omega^2 x$$

To convert this equation into a system of two first order equations, we let

$$\begin{align*}
    x &= \theta \\
y &= \theta' 
\end{align*}$$

Then, we have

$$\begin{align*}
    0 &= \lambda x - x \sin \omega - \omega, \\
    0 &= \lambda 
\end{align*}$$

These points correspond to two physical equilibrium positions, one with the mass directly above point of support ($\theta = 0$), and the other with the mass directly below point of support ($\theta = \pi$). Intuitively, the first is stable and the second is unstable.
Stability of Critical Points: Damped Case

If mass is slightly displaced from lower equilibrium position, it will oscillate with decreasing amplitude, and slowly approach equilibrium position as damping force dissipates initial energy. This type of motion illustrates asymptotic stability.

If mass is slightly displaced from upper equilibrium position, it will rapidly fall, and then approach lower equilibrium position. This type of motion illustrates instability.

See figures (a) and (b) below.
Now consider the ideal situation in which the damping coefficient \( \gamma \) (or \( c \)) is zero.

In this case, if the mass is displaced slightly from the lower equilibrium position, it will oscillate indefinitely with constant amplitude about the equilibrium position. Since there is no dissipation in the system, the mass will remain near the equilibrium position but will not approach it asymptotically. This motion is stable but not asymptotically stable. See figure (c) below.

**Stability of Critical Point: Undamped Case**
Consider the autonomous system
\[
\dot{x} = G(x, y), \quad \dot{y} = F(x, y)
\]

This follows that
\[
\frac{\dot{x}}{\dot{y}} = \frac{G}{F}
\]

Thus the trajectories lie on the level curves of \( H(x, y) \).

An equation for the trajectories of
\( H(x, y) = c \), gives

If we can solve this equation using methods from Chapter 2,

which is a first order equation in the variables \( x \) and \( y \),

\[
\frac{\dot{x}}{\dot{y}} = \frac{G}{F}
\]

It follows that
\[
\frac{\dot{x}}{\dot{y}} = \frac{1}{yp} \quad \text{and} \quad \frac{\dot{y}}{\dot{x}} = \frac{1}{xp}
\]

Consider the autonomous system

\textbf{Determination of Trajectories}
Example 1

Consider the system

It follows that

The direction of motion can be inferred from the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in the four quadrants. Thus the trajectories are hyperbolas, as shown below.

The solution of this separable equation is

$$\varphi = C \left( x - \lambda \right) = (\lambda', x) H$$

Thus the trajectories are hyperbolas, as shown below.

The solution of this separable equation is

$$xp x = \lambda p \lambda \quad \iff \quad \lambda / x = xp / \lambda p$$

It follows that

$$x = ip / \lambda p, \lambda = ip / xp$$

Consider the system
Example 2: Separable Equation (1 of 2)

Consider the system

\[
-\frac{dx}{dt} - 2x + y = 0, \quad \frac{dy}{dt} + 2y = 0
\]

It follows that the equilibrium points are found by solving

\[
\lambda = \pm \sqrt{4 + 4} \implies (\lambda', x) H
\]

The solution of this separable equation is

\[
x p(\epsilon x - 2) = \lambda p (\lambda \epsilon - 4) \iff \frac{\lambda \epsilon - 4}{\epsilon x - 2} = \frac{x p}{\lambda p}
\]

and hence (-2, 2) and (2, 2) are the equilibrium points.

Note that the equilibrium points are found by solving

\[
\epsilon = x + x - 2 = 0, \quad 2 - \lambda = \lambda + 4 = (\lambda', x) H
\]

The solution of this separable equation is

\[
x p(\epsilon x - 2) = \lambda p (\lambda \epsilon - 4)
\]

Consider the system

Exemple 2: Separable Equation (1 of 2)
Example 2: Phase Portrait (2 of 2)

We have

\[ c x x y y y x H = \pm \sqrt{c x x y y y x H} \]

A graph of some level curves of \( H \) are given below. Note that \((-2, 2)\) is a center and \((2, 2)\) is a saddle point. Also, one trajectory leaves the saddle point (at \( t = -\infty \), loops around the center, and returns to the saddle point (at \( t = \infty \)).

We have

\[ M \Delta x + a y H(x, y) = 4y - \epsilon x^2 - 12x \]