Dynamic Programming for Stochastic Control Problems with Expectation Constraints

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Outline

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Generalized state constraints

1.Soner (1986) :State constraints

$$\sup_{\nu \in \mathcal{U}} E[U(X_T^\nu)] \text{ under the constraint } X_t^\nu \in \mathcal{O}, \forall t \in [0, T].$$

2. Föllmer and Leukert (1999) : Quantitle hedging ; Soner and Touzi (2002) Target problem

$$\sup_{\nu \in \mathcal{U}} E[U(X_T^\nu)] \text{ under the constraint } P_0(X_T^\nu \in G) \geq m.$$

3. Föllmer and Leukert (2000) : Efficient hedging

$$\sup_{\pi \in \mathcal{U}} E[U(X_T^\nu - \xi)] \text{ under the shortfall constraint } E[l((\xi - X_T^\nu)^+)] \leq m.$$

Cont'

4. Grossman and Zhou (1993), Elie and Touzi (2008) : Drawdown constraint. $X_t^\nu \geq \alpha M_t^\nu$.

5. El Karoui, Jeanblanc and Lacoste (2005) : optimal portfolio strategy with a European guarantee

$$\sup_{\pi \in \mathcal{U}} E[U(X_T^\nu)] \text{ under the constraint } X_T^\nu \geq k;$$

6. And with American constraint

$$\sup_{\nu \in \mathcal{U}} E[U(X_T^\nu)] \text{ under the constraint } X_t^\nu \geq k, \ \forall t \in [0, T].$$

Cont'

7. Federico, Gozz and Sekine : Floor constraint

$$\sup_{\nu,c} E\left[\int_0^T e^{-\rho t} U_1(c(t))dt + e^{-\rho T} U_2(X_T^{\nu,c})\right]$$

under the floor constraint

$$X_t^{\nu,c} \geq K_t = k_0 e^{(r-\alpha)t} \text{ for all } t \in [0, T].$$

Motivation

Bouchard and Nutz (2012) :
Weak Dynamic Programming Principle (DPP), Markovian, expectation constraints.

Nutz and van Handel (2013) :
DPP(tower property), non-Markovian, no state constraints.

Bouchard, Elie and Réveillac (2015) :
BSDEs, expectation constraints on the terminal condition ($E[\Psi(Y_T)] \geq m$).

Our goal : DPP, non-Markovian, expectation constraints.

Approach : measurable selection.

Classical Stochastic Control Problem

Fix a probability space $(\Omega, \mathcal{F}, P)$.
For $(t, x) = [0, T] \times \mathbb{R}^d$, given admissible control set $\mathcal{U}(t, x)$.
For $\nu \in \mathcal{U}(t, x)$, given a process $X^{t,x,\nu}$.

$$V(t, x) = \sup_{\nu \in \mathcal{U}(t,x)} E[f(X_T^{t,x,\nu})].$$

DPP

$$V(t, x) = \sup_{\nu \in \mathcal{U}(t,x)} E[V(\tau, X_\tau^{t,x,\nu})].$$

- Weak DPP : study V_* and V^* (semicontinuous) ;
- Non-Markovian : V depends on the path before time t ;
- Constraints : $X^{t,x,\nu}$ must satisfy some conditions.

Stochastic Control with Expectation Constraints

Consider

$$V(t,\omega,m) = \sup_{\nu \in \mathcal{U}(t,\omega,m)} E[f(X^{t,\omega,\nu})],$$

where

$$\mathcal{U}(t,\omega,m) := \{\nu \in \mathcal{U}(t,\omega) : E[g(X^{t,\omega,\nu})] \leq m\}. \tag{1}$$

Our goal is to prove DPP

$$\begin{aligned} V(t,\omega,m) &= \sup_{\nu \in \mathcal{U}(t,\omega,m)} \sup_{M \in \mathcal{M}^+_{t,\omega,m}(\nu)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)] \\ &= \sup_{\nu \in \mathcal{U}(t,\omega,m)} \inf_{M \in \mathcal{M}^+_{t,\omega,m}(\nu)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)]. \end{aligned}$$

Outline

Notations

Fix the dimension $d \in \mathbb{N} \setminus \{0\}$ and a time horizon $T \in (0, \infty)$.

- $\Omega = \{\omega \in C([0, T]; \mathbb{R}^d) : \omega_0 = 0\}$: the canonical space of continuous paths ;
- B : the canonical process $B_t(\omega) = \omega_t$;
- $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$: the filtration generated by B ;
- $\mathcal{F} = \mathcal{F}_T$;
- P_0 : the Wiener measure on $(\Omega, \mathcal{F})$;
- $\mathfrak{P}(\Omega)$: the set of all probability measures on $(\Omega, \mathcal{F})$;
- $\mathcal{T}$: the collection all stopping times with values in $[0, T]$;
- $\mathcal{T}^t$: the collection all stopping times with values in $[t, T]$.

Expectation

$\bar{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$.

Define integrals for all measurable functions $\xi : \Omega \to \bar{\mathbb{R}}$.
If $E^P[\xi^+]$ or $E^P[\xi^-]$ is finite

$$E^P[\xi] := E^P[\xi^+] - E^P[\xi^-];$$

If $E^P[\xi^+] = E^P[\xi^-] = +\infty$,

$$E^P[\xi] := -\infty.$$

$E^P[\cdot]$: taking expectations w.r.t. probability P, and if $P = P_0$, simply write $E[\cdot]$.

Problem Formulation

For each $(t,\omega) \in [0,T] \times \Omega$, given $\mathcal{P}(t,\omega) \subseteq \mathfrak{P}(\Omega)$.
$\mathcal{P}(t,\omega)$: admissible probabilities without constraints.

Assume $\mathcal{P}(t,\omega) = \mathcal{P}(t,\tilde{\omega})$ if $\omega = \tilde{\omega}$ on $[0,t]$.
If $P \in \mathcal{P}(t,\omega)$, $P(\Omega_t^\omega) = 1$, where $\Omega_t^\omega := \{\omega' \in \Omega : \omega' = \omega \text{ on } [0,t]\}$.

Given two functions $\xi, \eta : \Omega \to \bar{\mathbb{R}}$.
ξ is upper semianalytic, η is lower semianalytic.

Define $\mathcal{P}(\eta, m) = \{P \in \mathfrak{P}(\Omega) : E^P[\eta] \leq m\}$
and set $\mathcal{P}(t,\omega,m) = \mathcal{P}(t,\omega) \cap \mathcal{P}(\eta,m)$.
$\mathcal{P}(t,\omega,m)$: admissible probabilities with constraint level m.

$$V(t,\omega,m) = \sup_{P \in \mathcal{P}(t,\omega,m)} E^P[\xi].$$

Auxiliary Martingales (Dynamically Monitor Constraint)

For each $P \in \mathcal{P}(t, \omega, m)$, there exists a process M on $[t, T] \times \Omega$ such that

- $M_t = m$;
- $M_T(\tilde{\omega}) \geq \eta(\tilde{\omega})$ for P-a.e. $\tilde{\omega} \in \Omega$;
- M is a martingale under P.

$\mathcal{M}^+_{t,\omega,m}(P)$: the collection of all such processes.

Fix $\tau \in \mathcal{T}^t, M \in \mathcal{M}^+_{t,\omega,m}(P), \tilde{\omega} \in \Omega$.

Denote $\mathcal{P}(\tau, \tilde{\omega}, M_\tau) = \mathcal{P}(\tau(\tilde{\omega}), \tilde{\omega}, M_\tau(\tilde{\omega}))$, and similarly $V(\tau, \tilde{\omega}, M_\tau) = V(\tau(\tilde{\omega}), \tilde{\omega}, M_\tau(\tilde{\omega}))$.

DPP

$$V(t,\omega,m) = \sup_{P\in\mathcal{P}(t,\omega,m)} \sup_{M\in\mathcal{M}^+_{t,\omega,m}(P)} E^P[V(\tau,\cdot,M_\tau)]$$
$$= \sup_{P\in\mathcal{P}(t,\omega,m)} \inf_{M\in\mathcal{M}^+_{t,\omega,m}(P)} E^P[V(\tau,\cdot,M_\tau)].$$

Conditional Probability Distributions

For any $P \in \mathfrak{P}(\Omega), \tau \in \mathcal{T}$, there is a regular conditional probability distribution (r.c.p.d) $\{P_\tau^\omega\}_{\omega \in \Omega}$ of P given $\mathcal{F}_\tau$.

- $P_\tau^\omega \in \mathfrak{P}(\Omega)$ for each ω;
- $\omega \in \Omega \to P_\tau^\omega(A) \in [0,1]$ is $\mathcal{F}_\tau$-measurable for any $A \in \mathcal{F}$;
- $E^{P_\tau^\omega}[\xi] = E^P[\xi | \mathcal{F}_\tau](\omega)$ for P-a.e. $\omega \in \Omega$;
- $P_\tau^\omega(\Omega_\tau^\omega) = 1$ for any $\omega \in \Omega$

where $\Omega_\tau^\omega := \{\omega' \in \Omega : \omega' = \omega \text{ on } [0, \tau(\omega)]\}$.

Concatenated Probability

Given $P \in \mathfrak{P}(\Omega)$ and a family $(Q^\omega)_{\omega \in \Omega}$ such that

- $\omega \in \Omega \to Q^\omega \in \mathfrak{P}(\Omega)$ is $\mathcal{F}_\tau$-mearurable ;
- $Q^\omega(\Omega_\tau^\omega) = 1$ for all $\omega \in \Omega$.

Define $P \otimes_\tau Q^\cdot$ by

$$P \otimes_\tau Q^\cdot(A) := \int_\Omega Q^\omega(A) P(d\omega), \quad \forall A \in \mathcal{F}.$$

Then

$$E^{P \otimes_\tau Q^\cdot}[\eta] = E^P[E^{Q^\omega}[\eta]].$$

Assumption 1

Let $(t,\bar{\omega}) \in [0,T] \times \Omega$ be arbitrary, $\tau \in \mathcal{T}^t, P \in \mathcal{P}(t,\bar{\omega})$.

1. (1) Measurability : The graph $[[\mathcal{P}]] := \{(t,\omega,P) : (t,\omega) \in [0,T] \times \Omega, P \in \mathcal{P}(t,\omega)\}$ is an analytic subset of $[0,T] \times \Omega \times \mathfrak{P}(\Omega)$.
2. (2) Invariance : There is a family of r.c.p.d (P^ω_τ) of P given $\mathcal{F}_\tau$ such that $P^\omega_\tau \in \mathcal{P}(\tau,\omega)$ for P-a.e. $\omega \in \Omega$.
3. (3) Stability under pasting : Let $(Q^\omega)_{\omega\in\Omega}$ be such that $\omega \to Q^\omega$ is $\mathcal{F}_\tau$-measurable and $Q^\omega \in \mathcal{P}(\tau,\omega)$ for P-a.e. $\omega \in \Omega$, then $P \otimes_\tau Q^\cdot \in \mathcal{P}(t,\bar{\omega})$.

Theorem 1

Under Assumption 1,

$$
\begin{aligned}
V(t,\omega,m) &= \sup_{P\in\mathcal{P}(t,\omega,m)} \sup_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^{P}[V(\tau,\cdot,M_\tau)] \\
&= \sup_{P\in\mathcal{P}(t,\omega,m)} \inf_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^{P}[V(\tau,\cdot,M_\tau)].
\end{aligned}
$$

Some Results

Lemma (1)

If ξ is upper semianalytic, the function $P \in \mathfrak{P}(\Omega) \to E^P[\xi] \in \bar{\mathbb{R}}$ is upper semianalytic.

Lemma (2)

Let $D = \{(t, \omega, m, P) : (t, \omega, m) \in [0, T] \times \Omega \times \mathbb{R}, P \in \mathcal{P}(t, \omega, m)\}$, then D is an analytic subset of $[0, T] \times \Omega \times \mathbb{R} \times \mathfrak{P}(\Omega)$.

Proof The function $L(m, P) := E^P[\eta] - m$ is lower semianalytic by Lemma 1. Observe that

$$D = \{(t, \omega, m, P) : P \in \mathcal{P}(t, \omega)\} \cap \{(t, \omega, m, P) : L(m, P) \leq 0\}.$$

The first part is analytic by Assumption 1(1), and the second part is analytic due to lower semianalytic property of L.

Measurable Selection

Let X and Y be Borel spaces, D an analytic subset of $X \times Y$, and let $f : D \to \bar{\mathbb{R}}$ be upper semianalytic. Define the function $f^* : \text{proj}_X(D) \to \bar{\mathbb{R}}$ by

$$f^*(x) = \sup_{y \in D_x} f(x, y). \tag{2}$$

- **Projection Theorem** :

$$f^* \quad \text{is upper semianalytic.} \tag{3}$$

- **Jankov-von Neumann Theorem** : For every $\epsilon > 0$, there exists an analytically measurable function $\varphi_\epsilon : \text{proj}_X(D) \to Y$ such that $Gr(\varphi_\epsilon) \subseteq D$ and for all $x \in \text{proj}_X(D)$,

$$f[x, \varphi_\epsilon(x)] \geq \begin{cases} f^*(x) - \epsilon & \text{if } \; f^*(x) < +\infty; \\ 1/\epsilon & \text{if } \; f^*(x) = +\infty. \end{cases} \tag{4}$$

Lemma 3

Let $X = [0, T] \times \Omega \times \mathbb{R}$ and $Y = \mathfrak{P}(\Omega)$.

Then $\text{proj}_X(D) = \{(t, \omega, m) : \mathcal{P}(t, \omega, m) \neq \emptyset\}$.

$$V(t, \omega, m) = \sup_{P \in D_{t,\omega,m}} E^P[\xi].$$

Lemma (3)

The function $V(t, \omega, m) : \text{proj}_X(D) \to \bar{\mathbb{R}}$ is upper semianalytic. Moreover, for every $\epsilon > 0$, there exists an analytically measurable function $\varphi_\epsilon : \text{proj}_X(D) \to \mathfrak{P}(\Omega)$ such that for every $(t, \omega, m) \in \text{proj}_X(D)$, $(t, \omega, m, \varphi_\epsilon(t, \omega, m)) \in D$ and

$$E^{\varphi_\epsilon(t,\omega,m)}[\xi] \geq \begin{cases} V(t, \omega, m) - \epsilon & V(t, \omega, m) < \infty; \\ \epsilon^{-1} & V(t, \omega, m) = \infty. \end{cases}$$

Proof of Theorem 1

Step 1 :

$$V(t,\omega,m) \geq \sup_{P\in\mathcal{P}(t,\omega,m)} \sup_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^{P}[V(\tau,\cdot,M_\tau)]. \tag{5}$$

Fix $\epsilon > 0$, $P \in \mathcal{P}(t,\omega,m)$, and take an arbitrary $M \in \mathcal{M}^{+}_{t,\omega,m}(P)$.

$$\bar{\omega} \to \varphi_\epsilon(\tau(\bar{\omega}),\bar{\omega},M_\tau(\bar{\omega}))$$

is $\mathcal{F}^{*}_{\tau}$-measurable. There exists an $\mathcal{F}_\tau$-measurable $Q^{\cdot}_{\epsilon} : \Omega \to \mathfrak{P}(\Omega)$ s.t. $Q^{\bar{\omega}}_{\epsilon} = \varphi_\epsilon(\tau(\bar{\omega}),\bar{\omega},M_\tau(\bar{\omega}))$ for P-a.e. $\bar{\omega} \in \Omega$.

By Assumption 1(2) and equation (9), we have $\mathcal{P}(\tau,\bar{\omega},M_\tau) \neq \emptyset$. Thus for P-a.e $\bar{\omega} \in \Omega$, $Q^{\bar{\omega}}_{\epsilon} \in \mathcal{P}(\tau,\bar{\omega},M_\tau)$ and

$$E^{Q^{\bar{\omega}}_{\epsilon}}[\xi] \geq \begin{cases} V(\tau,\bar{\omega},M_\tau) - \epsilon & V(\tau,\bar{\omega},M_\tau) < \infty \\ \epsilon^{-1} & V(\tau,\bar{\omega},M_\tau) = \infty. \end{cases}$$

Cont'

Then $P \otimes_\tau Q_\epsilon^\cdot \in \mathcal{P}(t,\omega)$, we assert further that $P \otimes_\tau Q_\epsilon^\cdot \in \mathcal{P}(t,\omega,m)$.

$$\begin{aligned} E^{P\otimes_\tau Q_\epsilon^\cdot}[\eta] &= E^P[E^{Q_\epsilon^{\bar{\omega}}}[\eta]] \\ &\le E^P[M_\tau(\bar{\omega})] \\ &= E^P[M_t] \\ &= m. \end{aligned} \tag{6}$$

Cont'

By further derivation, we will have

$$\begin{aligned} E^P[V(\tau,\bar{\omega},M_\tau)\wedge\epsilon^{-1}] &\leq E^P[E^{Q_\epsilon^{\bar{\omega}}}[\xi]]+\epsilon \\ &= E^{P\otimes_\tau Q_\epsilon^{\cdot}}[\xi]+\epsilon \\ &\leq \sup_{P'\in\mathcal{P}(t,\omega,m)} E^{P'}[\xi]+\epsilon \\ &= V(t,\omega,m)+\epsilon. \end{aligned} \tag{7}$$

Let $\epsilon\to 0$,

$$E^P[V(\tau,\bar{\omega},M_\tau)]\leq V(t,\omega,m).$$

Since $M\in\mathcal{M}^+_{t,\omega,m}(P)$ is arbitrary,

$$\sup_{M\in\mathcal{M}^+_{t,\omega,m}(P)} E^P[V(\tau,\cdot,M_\tau)]\leq V(t,\omega,m).$$

$P\in\mathcal{P}(t,\omega,m)$ is arbitrary.

Step2

$$V(t,\omega,m) \leq \sup_{P\in\mathcal{P}(t,\omega,m)} \inf_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^P[V(\tau,\cdot,M_\tau)]. \quad (8)$$

Fix $P \in \mathcal{P}(t,\omega,m)$ and $M \in \mathcal{M}^{+}_{t,\omega,m}(P)$. By Assumption 1(2), there is a family of r.c.p.d $(P^{\bar{\omega}}_{\tau})$ of P given $\mathcal{F}_\tau$ such that $P^{\bar{\omega}}_{\tau} \in \mathcal{P}(\tau,\bar{\omega})$ for P-a.e. $\bar{\omega} \in \Omega$.

Prove $P^{\bar{\omega}}_{\tau} \in \mathcal{P}(\tau,\bar{\omega},M_\tau)$ for P-a.e. $\bar{\omega} \in \Omega$.

$$\begin{aligned} E^{P^{\bar{\omega}}_{\tau}}[\eta] &= E^P[\eta|\mathcal{F}_\tau](\bar{\omega}) \\ &\leq E^P[M_T|\mathcal{F}_\tau](\bar{\omega}) \\ &= M_\tau(\bar{\omega}). \quad (9) \end{aligned}$$

Cont'

Then for P-a.e. $\bar{\omega} \in \Omega$, we have

$$\begin{aligned} E^P[\xi|\mathcal{F}_\tau](\bar{\omega}) &= E^{P_\tau^{\bar{\omega}}}[\xi] \\ &\leq V(\tau, \bar{\omega}, M_\tau). \end{aligned} \tag{10}$$

Taking $P(d\bar{\omega})$-expectations on both sides,

$$E^P[\xi] \leq E^P[V(\tau, \cdot, M_\tau)].$$

Since $M \in \mathcal{M}^+_{t,\omega,m}(P)$ is arbitrary,

$$E^P[\xi] \leq \inf_{M \in \mathcal{M}^+_{t,\omega,m}(P)} E^P[V(\tau, \cdot, M_\tau)].$$

Taking the supremum over $\mathcal{P}(t, \omega, m)$.

Outline

Problem Settings

$\Omega' = \{\omega' \in C([0,T];\mathbb{R}^n) : \omega'_0 = 0\}, \qquad (\Omega', \mathcal{F}', P'_0).$

For each $(t,\omega) \in [0,T] \times \Omega$, given $\mathcal{U}(t,\omega)$
$\mathcal{U}(t,\omega)$: admissible controls without constraints

Choose a control $\nu \in \mathcal{U}(t,\omega)$, given $X^{t,\omega,\nu} : [0,T] \times \Omega' \to \mathbb{R}^d$, the controlled process, satisfying $X^{t,\omega,\nu}_s(\tilde{\omega}) = \omega_s$ for all $s \in [0,t]$, $\tilde{\omega} \in \Omega'$ and $\nu \in \mathcal{U}(t,\omega)$.

Given $f, g : \Omega \to \bar{\mathbb{R}}$. f is upper semianalytic, g is lower semianalytic.
Define admissible controls with constraint level m as

$$\mathcal{U}(t,\omega,m) := \{\nu \in \mathcal{U}(t,\omega) : E[g(X^{t,\omega,\nu})] \leq m\}, \tag{11}$$

and the value function

$$V(t,\omega,m) = \sup_{\nu \in \mathcal{U}(t,\omega,m)} E[f(X^{t,\omega,\nu})].$$

Transformation

$X^{t,\omega,\nu} : (\Omega', \mathcal{F}') \to (\Omega, \mathcal{F})$ is measurable.

Define a probability measure $P_{t,\omega,\nu}$ on $(\Omega, \mathcal{F})$ by

$$P_{t,\omega,\nu}(A) = P_0'((X^{t,\omega,\nu})^{-1}(A)), \quad A \in \mathcal{F}.$$

Define $\mathcal{P}(t,\omega) = \{P_{t,\omega,\nu} \in \mathfrak{P}(\Omega) : \nu \in \mathcal{U}(t,\omega)\}$,

and $\mathcal{P}(t,\omega,m) = \{P \in \mathcal{P}(t,\omega) : E^P[g] \leq m\}$.

Transformation

Then by the definition of $P_{t,\omega,\nu}$, we have

$$E^{P_{t,\omega,\nu}}[f] = E[f(X^{t,\omega,\nu})] \text{ and } E^{P_{t,\omega,\nu}}[g] = E[g(X^{t,\omega,\nu})].$$

$$\begin{aligned} V(t,\omega,m) &= \sup_{\nu\in\mathcal{U}(t,\omega,m)} E[f(X^{t,\omega,\nu})] \\ &= \sup_{P\in\mathcal{P}(t,\omega,m)} E^{P}[f]. \end{aligned} \tag{12}$$

Auxiliary Martingales (Dynamically Monitor Constraint)

For each $\nu \in \mathcal{U}(t, \omega, m)$, there exists a process M on $[t, T] \times \Omega$ such that

- $M_t = m$;
- $M_T(X^{t,\omega,\nu}) \geq g(X^{t,\omega,\nu})$ for P_0'-a.s.;
- $M(X^{t,\omega,\nu})$ is a martingale under P_0'.

$\mathcal{M}^{+}_{t,\omega,m}(\nu)$: the collection of all such processes.

Proof of DPP for Control Problem

Check :

$$\mathcal{P}(t,\omega,m) = \{P_{t,\omega,\nu} \in \mathfrak{P}(\Omega) : \nu \in \mathcal{U}(t,\omega,m)\},$$

$$\mathcal{M}^+_{t,\omega,m}(\nu) = \mathcal{M}^+_{t,\omega,m}(P_{t,\omega,\nu}),$$

$$E[V(\tau, X^{t,\omega,\nu}, M_\tau)] = E^{P_{t,\omega,\nu}}[V(\tau,\cdot,M_\tau)].$$

DPP follows directly from Theorem 1.

Theorem 2 and Proof

$$
\begin{aligned}
V(t,\omega,m) &= \sup_{P\in\mathcal{P}(t,\omega,m)} \sup_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^{P}[V(\tau,\cdot,M_\tau)] \\
&= \sup_{P\in\mathcal{P}(t,\omega,m)} \inf_{M\in\mathcal{M}^{+}_{t,\omega,m}(P)} E^{P}[V(\tau,\cdot,M_\tau)].
\end{aligned}
$$

$$
\begin{aligned}
V(t,\omega,m) &= \sup_{\nu\in\mathcal{U}(t,\omega,m)} \sup_{M\in\mathcal{M}^{+}_{t,\omega,m}(\nu)} E[V(\tau,X^{t,\omega,\nu},M_\tau)] \\
&= \sup_{\nu\in\mathcal{U}(t,\omega,m)} \inf_{M\in\mathcal{M}^{+}_{t,\omega,m}(\nu)} E[V(\tau,X^{t,\omega,\nu},M_\tau)].
\end{aligned}
$$

A Special Case

$\mathcal{M}^+_{t,\omega,m} := \cap_{\nu \in \mathcal{U}(t,\omega,m)} \mathcal{M}^+_{t,\omega,m}(\nu)$.

If $\mathcal{M}^+_{t,\omega,m} \neq \emptyset$, then for any $M \in \mathcal{M}^+_{t,\omega,m}$,

$$V(t,\omega,m) = \sup_{\nu \in \mathcal{U}(t,\omega,m)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)]. \tag{13}$$

Thank you for your attention !