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Aims and Objectives

Game theory provides a Mathematical tool for multi-person decision making. This module is an introduction to game theory, studying basic concepts, models and solutions of games and their applications.
Topics:

A game is a model for multi-person decision making. The most important concept is equilibrium which provides solutions to games.

Games of complete information: Static games with finite or infinite strategy spaces, Nash equilibrium of pure and mixed strategy; Dynamic games, backward induction solutions, information sets, subgame-perfect equilibrium, repeated games.

Games of incomplete information: Bayesian equilibrium; First price sealed auction, second price sealed auction, and other auctions; Dynamic Bayesian games; Perfect Bayesian equilibrium; Signaling games.

Cooperative games: Bargaining theory; Cores of n-person cooperative games; The Shapley value and its applications in voting, cost sharing, etc.
Assessment

20% on one mid-term test (1 hour).

80% on the final examination (2.5 hours).

Recommended Texts

[1] Robert Gibbons, *A Primer in Game Theory* (70%)


[3] James W. Friedman, *Game Theory with Applications to Economics* (15%)
1 Static Games of Complete Information

The Prisoners’ Dilemma.

Two suspects are arrested and charged with a crime. The police lack sufficient evidence to convict the suspects, unless at least one confesses. The suspects are held in separate cells and told “If only one of you confesses and testifies against your partner, the person who confesses will go free while the person does not confess will surely be convicted and given a 20-year jail sentence. If both of you confess, you will both be convicted and sent to prison for 5 years. Finally, if neither of you confesses, both of you will be convicted of a minor offence and sentenced to 1 year in jail”. What should the suspects do?
This problem can be represented in a bi-matrix:

<table>
<thead>
<tr>
<th>Prisoner1</th>
<th>Holdout</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Holdout</td>
<td>$-1, -1$</td>
<td>$-20, 0$</td>
</tr>
<tr>
<td>Confess</td>
<td>$0, -20$</td>
<td>$-5, -5$</td>
</tr>
</tbody>
</table>

Prisoner 1 is also called the row player.

Prisoner 2 is also called the column player.

Each entry of the bi-matrix has two numbers: the first number is the payoff of the row player and the second is that of the column player.

Both players have a dominant strategy, Confess, in the sense that no matter what one expects the other to do, confess is the best possible.
Definition 1.1 *The normal-form (also called strategic-form) representation of an n-player game specifies the players’ strategy spaces* $S_1, \ldots, S_n$ *and their payoff functions* $u_1, \ldots, u_n$. *We denote this game by* 

$$G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}.$$ 

*Let* $(s_1, \ldots, s_n)$ *be a combination of strategies, one for each player. Then* $u_i(s_1, \ldots, s_n)$ *is the payoff to player* $i$ *if for each* $j = 1, \ldots, n$, *player* $j$ *chooses strategy* $s_j$. 

The payoff of a player depends not only on his own action but also on the actions of others! This interdependence is the essence of games!

*In The Prisoners’ Dilemma:* 

$G = \{S_1, S_2; u_1, u_2\}$, 

$S_1 = \{ \text{Holdout, Confess } \} = S_2$, 

$u_1(H, C) = -20, \quad u_2(H, C) = 0, \quad \ldots$. 
When there are only two players and each player has a finite number of strategies, then the payoff functions can be represented by a bimatrix, e.g.

\[
\begin{array}{c|cc}
& L & R \\
\hline
T & u_1(T, L), u_2(T, L) & u_1(T, R), u_2(T, R) \\
B & u_1(B, L), u_2(B, L) & u_1(B, R), u_2(B, R) \\
\end{array}
\]

\[
= \begin{array}{c|cc}
& L & R \\
\hline
T & 2, 1 & 0, 2 \\
B & 1, 2 & 3, 0 \\
\end{array}
\]

**Splitting the bill:**

Two friends are going to dinner and plan to split the bill no matter who orders what. There are two meals, a cheap one priced at $10, which gives each of them $12 of pleasure, and an expensive dinner priced at $20, which gives them each $18 of pleasure. What will they order?
Representation of the game:

\[ G = \{ S_1, S_2; u_1, u_2 \}, \]

\[ S_1 = \{ \text{Cheap, Expensive} \} = S_2, \]

The payoff functions \( u_1 \) and \( u_2 \) are given in the bimatrix:

\[
\begin{array}{c|cc}
\text{Peter} & \text{Cheap} & \text{Expensive} \\
\hline
\text{John} & 2,2 & -3,3 \\
\text{Cheap} & 3,-3 & -2,-2 \\
\text{Expensive} & & \\
\end{array}
\]

When John orders cheap and Peter orders expensive, both have to pay $15, John gets $12 of pleasure while Peter gets $18. The net pleasure is \( 12 - 15 = -3 \) for John and \( 18 - 15 = 3 \) for Peter.

Notations:

\[ s = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) = (s_i, s_{-i}) \]
\[ s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \]
\[ S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n \]
Definition 1.2 In a normal-form game \( G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\} \), let \( s'_i, s''_i \in S_i \). Strategy \( s'_i \) is strictly dominated by strategy \( s''_i \) (or strategy \( s''_i \) strictly dominates strategy \( s'_i \)), if for each feasible combination of the other players’ strategies, player \( i \)’s payoff from playing \( s'_i \) is strictly less than the payoff from playing \( s''_i \), i.e.,

\[
u_i(s'_i, s_{-i}) < u_i(s''_i, s_{-i}), \quad \forall s_{-i} \in S_{-i}.
\]

Rational players do not play strictly dominated strategies since they are always not optimal no matter what strategies others would choose.

In the Prisoner’s Dilemma Game, Holdout is a strictly dominated strategy while Confess is a strictly dominant strategy.

In the game of splitting the bill, Expensive is a strictly dominant strategy for both players.
Iterated elimination of strictly dominated strategies:

<table>
<thead>
<tr>
<th></th>
<th>Player2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
</tr>
<tr>
<td>Player1</td>
<td></td>
</tr>
<tr>
<td>Up</td>
<td>1, 0</td>
</tr>
<tr>
<td>Down</td>
<td>0, 3</td>
</tr>
</tbody>
</table>

For player 1, neither Up or Down is strictly dominated.

For player 2, Right is strictly dominated by Middle.

Both players know that Right will not be played by player 2. Thus they can play the game as if it were the following game:
Now, Down is strictly dominated by Up for player 1. Down can be eliminated from player 1’s strategy space, resulting in

\[
\begin{array}{c|cc}
\text{Player2} & \text{Left} & \text{Middle} \\
\hline
\text{Player1} & \text{Up} & 1,0 & 1,2 \\
\text{Down} & 0,3 & 0,1 \\
\end{array}
\]

Left is strictly dominated by Middle for player 2. Eliminate Left from player 2’s strategy space leaving (Up, Middle) as the outcome of the game.
Iterated elimination of strictly deminated strategies may not always work.

In the following game, no strategy is strictly domi-
minated for any player.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0,4</td>
<td>4,0</td>
<td>5,3</td>
</tr>
<tr>
<td>M</td>
<td>4,0</td>
<td>0,4</td>
<td>5,3</td>
</tr>
<tr>
<td>B</td>
<td>3,5</td>
<td>3,5</td>
<td>6,6</td>
</tr>
</tbody>
</table>

Player1

Player2
Definition 1.3 In the $n$-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the best response for player $i$ to a combination of other players’ strategies $s_{-i} \in S_{-i}$, denoted by $R_i(s_{-i})$, is referred to as the set of maximizers of

$$\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Remarks: $R_i(s_{-i}) \subset S_i$ can be an empty set, a singleton, or a finite or infinite set. We call $R_i$ the best-response correspondence for player $i$.

Definition 1.4 In the $n$-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, the strategies $(s_1^*, \ldots, s_n^*)$ are a Nash equilibrium if

$$s_i^* \in R_i(s_{-i}^*) \quad \forall i = 1, \ldots, n,$$

equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \quad \forall i = 1, \ldots, n.$$
Find a Nash equilibrium

Let $G(R_i)$ denote the graph of $R_i$, defined by

$$G(R_i) = \{(s_i, s_{-i}) \mid s_i \in R_i(s_{-i}), s_{-i} \in S_{-i}\}.$$ 

Then

$$(s_1^*, \ldots, s_n^*) \in \cap_{i=1}^n G(R_i)$$

iff

$$s_i^* \in R_i(s_{-i}^*), s_{-i}^* \in S_{-i}, \quad i = 1, \ldots, n,$$

iff $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium.

Two ways for finding Nash equilibria.

(i) For any guess $(s_1^*, s_2^*) \in S_1 \times S_2$, compute $R_1(s_2^*)$ and $R_2(s_1^*)$. Then, $(s_1^*, s_2^*)$ is a Nash equilibrium if

$$s_1^* \in R_1(s_2^*) \text{ and } s_2^* \in R_2(s_1^*).$$

(ii) Compute $R_1(s_2)$ for all $s_2 \in S_2$ and $R_2(s_1)$ for all $s_1 \in S_1$. Then, any $(s_1^*, s_2^*) \in G(R_1) \cap G(R_2)$ is a Nash equilibrium.
Finding Nash equilibria for a bimatrix game:
Underline the payoff to player \( j \)’s best response to each of player \( i \)’s strategies. If both \( u_1(s_1^*, s_2^*) \) and \( u_2(s_1^*, s_2^*) \) are underlined, then \((s_1^*, s_2^*)\) is an intersection of \( G(R_1) \) and \( G(R_2) \), thus is a Nash equilibrium.

Example:

<table>
<thead>
<tr>
<th></th>
<th>( s_1^* )</th>
<th>( s_2^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T</strong></td>
<td>0, 4</td>
<td>4, 0</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>4, 0</td>
<td>0, 4</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>3, 5</td>
<td>3, 5</td>
</tr>
</tbody>
</table>

Player2

The unique Nash equilibrium is \((B, R)\).
The Battle of the Sexes

Mary and Peter are deciding on an evening’s entertainment, attending either the opera or a prize fight. Both of them would rather spend the evening together than apart, but Peter would rather they be together at the prize fight while Mary would rather they be together at the opera.

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Fight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Fight</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Both (Opera, Opera) and (Fight, Fight) are Nash equilibria. Thus a game can have multiple Nash equilibria.
A game with 3 players

<table>
<thead>
<tr>
<th></th>
<th>Player 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A₃</td>
</tr>
<tr>
<td>A₁A₂</td>
<td>−1, 0, 2</td>
</tr>
<tr>
<td>A₁B₂</td>
<td>5, −2, 3</td>
</tr>
<tr>
<td>B₁A₂</td>
<td>0, 2, 1</td>
</tr>
<tr>
<td>B₁B₂</td>
<td>2, −3, −3</td>
</tr>
</tbody>
</table>

This game can also be represented as

<table>
<thead>
<tr>
<th></th>
<th>A₃</th>
<th>B₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A₂</td>
<td>−1, 0, 2</td>
<td>3, 5, 1</td>
</tr>
<tr>
<td>B₂</td>
<td>5, −2, 3</td>
<td>−2, 1, −5</td>
</tr>
<tr>
<td>B₁</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A₂</td>
<td>0, 2, 1</td>
<td>2, 2, 5</td>
</tr>
<tr>
<td>B₂</td>
<td>2, −3, −3</td>
<td>−1, 4, 0</td>
</tr>
</tbody>
</table>

The Nash equilibrium is (B₁, B₂, B₃).
The relation between Nash equilibrium and iterated elimination of strictly dominated strategies:

**Proposition 1.1** If the strategies \((s_1^*, \ldots, s_n^*)\) are a Nash equilibrium in an \(n\)-player normal-form game \(G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}\), then each \(s_i^*\) cannot be eliminated in iterated elimination of strictly dominated strategies.

**Proof.** Suppose \(s_i^*\) is the first of the strategies \((s_1^*, \ldots, s_n^*)\) to be eliminated or being strictly dominated. Then there exists \(s_i''\) not yet eliminated from \(S_i\), strictly dominates \(s_i^*\), i.e.,

\[
u_i(s_1, \ldots, s_{i-1}, s_i^*, s_{i+1}, \ldots, s_n) < u_i(s_1, \ldots, s_{i-1}, s_i'', s_{i+1}, \ldots, s_n)
\]

for all strategies \((s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)\) that have not been eliminated from the other player’s strategy spaces. Since \(s_i^*\) is the first equilibrium
strategy to be eliminated, we have

\[
u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*)
\]
\[
< u_i(s_1^*, \ldots, s_{i-1}^*, s_i^!, s_{i+1}^*, \ldots, s_n^*)
\]
which contradicts to the fact that \(s_i^*\) is a best re-

sponse to \((s_1^*, \ldots, s_{i-1}^*, s_i^+, s_{i+1}^*, \ldots, s_n^*)\).

The proposition tells us that any Nash equilibrium can survive the iterated elimination of strictly domi-
nated strategies (IESDS) hence must be an outcome of IESDS, i.e.,

\[
\{\text{Nash equilibria}\} \subseteq \{\text{Outcomes of IESDS}\}.
\]
Example:

<table>
<thead>
<tr>
<th></th>
<th>Player1</th>
<th>Player2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Up</td>
<td>Left</td>
</tr>
<tr>
<td>1, 0</td>
<td>1, 2</td>
<td>0, 1</td>
</tr>
<tr>
<td>2, 3</td>
<td>0, 1</td>
<td>2, 0</td>
</tr>
</tbody>
</table>

Right is strictly dominated by Middle, thus the game can be reduced to

<table>
<thead>
<tr>
<th></th>
<th>Player1</th>
<th>Player2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Up</td>
<td>Left</td>
</tr>
<tr>
<td>1, 0</td>
<td>1, 2</td>
<td></td>
</tr>
<tr>
<td>2, 3</td>
<td>0, 1</td>
<td></td>
</tr>
</tbody>
</table>

There is no strictly dominated strategy, hence IESDS has 4 outcomes

\[\{(U, L), (U, M), (D, L), (D, M)\}\].

However, there are only 2 Nash equilibria

\[\{(U, M), (D, L)\}\].
Proposition 1.2 In the $n$-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s^*_1, \ldots, s^*_n)$, then these strategies are the unique Nash equilibrium of the game.

Proof. By Proposition 1.1, equilibrium strategies can never be eliminated in IESDS. Since $(s^*_1, \ldots, s^*_n)$ are the only strategies which are not eliminated, $s^*_i$ is thus the only possible equilibrium strategy for player $i$. Hence, we cannot find two different Nash equilibria.

To prove $(s^*_1, \ldots, s^*_n)$ are indeed a Nash equilibrium, we use proof by contradiction. Suppose $s^*_i$ is not a best response for player $i$ to $(s^*_1, \ldots, s^*_{i-1}, s^*_{i+1}, \ldots, s^*_n)$. Let the relevant best response be $b_i \neq s^*_i$, i.e., for any $s_i \in S_i$,
\begin{align*}
u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \\
\leq u_i(s_1^*, \ldots, s_{i-1}^*, b_i, s_{i+1}^*, \ldots, s_n^*) \quad (1.1)
\end{align*}

Because the only strategy in $S_i$ which survives eliminations is $s_i^*$, thus $b_i$ must be strictly dominated by some strategy $t_i$ at some stage of the process of iterated elimination. So we have

\begin{align*}
u_i(s_1, \ldots, s_{i-1}, b_i, s_{i+1} \ldots, s_n) \\
< u_i(s_1, \ldots, s_{i-1}, t_i, s_{i+1} \ldots, s_n)
\end{align*}

for all strategies $(s_1, \ldots, s_{i-1}, s_{i+1} \ldots, s_n)$ that have not been eliminated from the other player’s strategy spaces. Since $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ are not eliminated,

\begin{align*}
u_i(s_1^*, \ldots, s_{i-1}^*, b_i, s_{i+1}^*, \ldots, s_n^*) \\
< u_i(s_1^*, \ldots, s_{i-1}^*, t_i, s_{i+1}^* \ldots, s_n^*)
\end{align*}

which contradicts to (1.1), the best response property of $b_i$. \qed
Splitting a dollar

Players 1 and 2 are bargaining over how to split one dollar. Both players simultaneously name shares they would like to have, $s_1$ and $s_2$, where $0 \leq s_1, s_2 \leq 1$. If $s_1 + s_2 \leq 1$, then the players receive the shares they named; if $s_1 + s_2 > 1$, then both players receive zero. What are the pure-strategy Nash equilibria of this game?

Given any $s_2 \in [0, 1)$, the best response for player 1 is $R_1(s_2) = 1 - s_2$.

To $s_2 = 1$, the player 1’s best response is the set $[0, 1]$ because player 1’s payoff is 0 no matter what she chooses.

The diagram of $R_1$: 
Similarly, we have the best response for player 2:

\[
s_2 = R_2(s_1) = \begin{cases} 
1 - s_1 & \text{if } 0 \leq s_1 < 1 \\
[0, 1] & \text{if } s_1 = 1
\end{cases}
\]

Intersections of \( s_1 = R_1(s_2) \) and \( s_2 = R_2(s_1) \) are

\[\{(s_1, s_2) : s_1 + s_2 = 1, s_1, s_2 \geq 0\}\] and \((1, 1)\).

They are the pure-strategy Nash equilibria.
**Cournot Model of Duopoly**

If a firm produces $q$ units of a product at a cost of $c$ per unit and can sell it at a price of $p$ per unit, then the firm makes a net profit

$$\pi = pq - cq - c_0,$$

where $c_0$ is the fixed cost.

The firm can decide $p$ and $q$ to maximize its net profit.

Because $c_0$ does not affect the decision of $p$ and $q$, we can ignore $c_0$.

Suppose firms 1 and 2 produce the same product.

Let $q_i$ be the quantity of the product produced by firm $i$, $i = 1, 2$.

Let $Q = q_1 + q_2$, the aggregate quantity of the product.
Since firms produce the same product, the firm which sets the higher price has no market. Thus, they sell the product at the market clearing price

\[ P(Q) = \begin{cases} 
a - Q, & \text{if } Q < a \\
0, & \text{if } Q \geq a. 
\end{cases} \]

Let the cost of producing a unit of the product be \( c \). We assume \( c < a \) and is same for both firms.

How much shall each firm produce?

Firm \( i \) (\( i = 1, 2 \)) shall maximize its net profit

\[ P(q_i + q_j) \cdot q_i - c \cdot q_i \]

for \( q_i \in [0, \infty) \) given \( q_j \).

Formulate the problem into a normal-form game:

(1) The players of the game are the two firms.

(2) Each firm’s strategy space is \( S_i = [0, \infty) \), \( i = 1, 2 \). (Any value of \( q_i \) is a strategy.)
(3) The payoff to firm $i$ as a function of the strategies chosen by it and by the other firm, is simply its profit function:

$$\pi_i(q_i, q_j) = P(q_i + q_j) \cdot q_i - c \cdot q_i$$

$$= \begin{cases} 
q_i[a - (q_i + q_j) - c], & \text{if } q_i + q_j < a \\
-cq_i, & \text{if } q_i + q_j \geq a.
\end{cases}$$

The profit is negative if any quantity is greater than $a$, hence we can assume

$$0 \leq q_1, q_2 \leq a.$$ 

For any $q_i > a - q_j$, the profit $\pi_i = -cq_i < 0$, such $q_i$ cannot be optimum. Therefore, we need only consider $q_i \in [0, a - q_j]$.

The quantity pair $(q_1^*, q_2^*)$ is a Nash equilibrium if, for each firm $i$, $q_i^*$ solves

$$\max_{0 \leq q_i \leq a - q_j^*} q_i[a - (q_i + q_j^*) - c].$$

If $a - c - q_j^* < 0$, then the profit is negative for any $q_i > 0$. Thus $q_i = 0$ is the optimum.
If \( a - c - q^*_j \geq 0 \), the local maximum \( \bar{q}_i \) can be determined by the optimal condition (the first derivative equals zero) \( a - q^*_j - c - 2q_i = 0 \), thus

\[
\bar{q}_i = \frac{1}{2}(a - q^*_j - c).
\]

For a fixed \( q_j \), the function \( q_i[a - (q_i + q_j) - c] \) is concave for \( q_i \), because its second derivative is \(-2 < 0\). Therefore, \( \bar{q}_i \) is the global maximum.

Thus, the best response of firm \( i \) to the given quantity \( q_j \) should be

\[
q_i = R_i(q_j) = \begin{cases} 
\frac{1}{2}(a - q_j - c), & \text{if } q_j \leq a - c \\
0, & \text{if } q_j > a - c.
\end{cases}
\]
The Nash equilibrium \((q_1^*, q_2^*)\) are the best responses, hence can be determined by the intersection of the two response curves, i.e.,

\[
\begin{align*}
q_1 &= \frac{1}{2}(a - q_2 - c) \\
q_2 &= \frac{1}{2}(a - q_1 - c).
\end{align*}
\]

Solving the equations, we obtain

\[
q_1^* = \frac{1}{3}(a - c), \quad q_2^* = \frac{1}{3}(a - c).
\]
Bertrand Model of Duopoly

Suppose now the two firms produce different products. In this case, we cannot use the aggregate quantity to determine market prices as in Cournot’s model. Thus, instead of using quantities as variables, here we use prices as variables. Assume that the cost and the reservation price for the two firms are the same and they are $c$ and $a$ respectively. Assume $c < a$.

If firms 1 and 2 choose prices $p_1$ and $p_2$, respectively, the quantity that consumers demand from firm $i$ is

$$q_i(p_i, p_j) = a - p_i + bp_j$$

where $b > 0$ reflects the extent to which firm $i$’s product is a substitute for firm $j$’s product.

How to find the Nash equilibrium?
The strategy space of firm $i$ consists of all possible prices, thus $S_i = [0, \infty), \ i = 1, 2$.

The profit of firm $i$ is

$$\pi_i(p_i, p_j) = q_i(p_i, p_j) \cdot p_i - c \cdot q_i(p_i, p_j)$$

$$= [a - p_i + bp_j](p_i - c).$$

Thus, the price pair $(p_1^*, p_2^*)$ solves

$$\max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*) = \max_{0 \leq p_i < \infty} (a - p_i + bp_j^*)(p_i - c),$$

whose solution is

$$p_i^* = \frac{1}{2}(a + bp_j^* + c).$$

Hence

$$\begin{cases} p_1^* = \frac{1}{2}(a + bp_2^* + c) \\ p_2^* = \frac{1}{2}(a + bp_1^* + c). \end{cases}$$

Solving the equations, we obtain

$$p_1^* = \frac{a + c}{2 - b}, \ p_2^* = \frac{a + c}{2 - b}.$$

The problem makes sense only if $b < 2!$. \hfill \square
Bertrand Model with Homogeneous Products

In the previous example, we analyzed the Bertrand duopoly model with differentiated products. The case of homogeneous products yields a stark conclusion. Suppose that the quantity that consumers demand from firm $i$ is

$$q_i(p_i, p_j) = \begin{cases} 
  a - p_i & \text{if } p_i < p_j \\
  0 & \text{if } p_i > p_j \\
  (a - p_i)/2 & \text{if } p_i = p_j 
\end{cases}$$

i.e., all customers buy the product from the firm who offers a lower price. Suppose also that there are no fixed costs and that marginal costs are constant at $c$, where $c < a$.

Find the Nash equilibrium $(p_1^*, p_2^*)$. 


Given firm $j$’s price $p_j$, firm $i$’s payoff function is

$$
\pi_i(p_i, p_j) = \begin{cases} 
(a - p_i)(p_i - c) & p_i < p_j \\
\frac{1}{2}(a - p_i)(p_i - c) & p_i = p_j \\
0 & p_i > p_j
\end{cases}
$$

Since the payoff will be negative if $p_i < c$ or $> a$, we can assume the strategy space $S_i = [c, a]$.

We find three cases from the observation of the pay-off curves.
Case 1: Given \( p_j > \frac{a+c}{2} \)

The maximum payoff is reached at \( p_i = \frac{a+c}{2} \). Thus, the best response \( R_i(p_j) = \frac{a+c}{2} \).

Case 2: Given \( c < p_j \leq \frac{a+c}{2} \)

In this case, \( \max_{p_i} \pi_i(p_i, p_j) \) has no solution.

**proof:** It is easy to see from the second figure that

\[
\sup_{p_i} \pi_i(p_i, p_j) = (a - p_j)(p_j - c).
\]

However, no \( p_i \in [c, a] \) can make \( \pi_i(p_i, p_j) = (a - p_j)(p_j - c) \). (For \( p_i \in (c, p_j) \), the function \( \pi_i(p_i, p_j) = (a - p_i)(p_i - c) \) is strictly increasing. For \( p_i > p_j \), \( \pi_i(p_i, p_j) = 0 \). For \( p_i = p_j \), \( \pi_i(p_i, p_j) = \frac{1}{2}(a - p_i)(p_i - c) \).) Thus, there is no maximizer.

This means that \( R_i(p_j) = \emptyset \).

Case 3: Given \( p_j = c \)

\( \pi_i(p_i, c) = 0 \) for any \( p_i \in [c, a] \). Thus any \( p_i \) is a maximizer, and \( R_i(c) = [c, a] \).
The best-response correspondences are sketched below:

The only intersection of the two correspondences is \((c, c)\). This shows that if the firms choose prices simultaneously, then the unique Nash equilibrium is that both firms charge the price \(p_i = c\).
The problem of the commons

If citizens respond only to private incentives, public goods will be underprovided and public resources overutilized.

Example:

$n$ farmers in a village, $(n \geq 2)$. For $i = 1, \ldots, n$, $g_i$ is the number of goats owned by farmer $i$.

Let $G = g_1 + \ldots + g_n$.

The cost of buying and caring for a goat is $c$.

The value to a farmer of grazing a goat on the green when a total number of $G$ goats are grazing is $v(G)$ per goat.

$$v(G) = \begin{cases} 0, & \text{if } G \geq G_{max} \\ > 0, & \text{if } G < G_{max}. \end{cases}$$

When $G < G_{max}$, $v'(G) < 0$, $v''(G) < 0$. 
The farmers simultaneously choose how many goats to own. Assume goats are continuously divisible; let $S_i = [0, G_{max})$.

The payoff to farmer $i$ is

$$g_i \cdot v(g_1 + \ldots + g_n) - c \cdot g_i.$$

Thus, if $(g_1^*, \ldots, g_n^*)$ is to be a Nash equilibrium, then, for each $i$, $g_i^*$ must maximize

$$g_i \cdot v(g_i + g_{-i}^*) - c \cdot g_i,$$

where $g_{-i}^* = g_1^* + \ldots + g_{i-1}^* + g_{i+1}^* + \ldots + g_n^*$. 
Differentiate the function above with respect to $g_i$ resulting in

$$v(g_i^* + g_{-i}^*) + g_i^* \cdot v'(g_i^* + g_{-i}^*) - c = 0 \quad (1.2)$$

for $i = 1, 2, \ldots, n$. Solve this system of equations, we obtain Nash equilibrium $(g_1^*, \ldots, g_n^*)$.

For example, take

$$v(x) = a - x^2, \quad a > c.$$  

Then $v'(x) = -2x$. The equations (1.2) are

$$a - (g_1 + \ldots + g_n)^2 - 2g_i(g_1 + \ldots + g_n) - c = 0$$

for $i = 1, \ldots, n$. Obviously $g_1 = \ldots = g_n$. Thus

$$a - n^2g_1^2 - 2ng_1^2 - c = 0.$$  

We obtain Nash equilibrium

$$g_i^* = \sqrt{\frac{a - c}{n^2 + 2n}}, \quad i = 1, \ldots, n.$$
A comparison between the game solution and the social optimum solution

If the whole village decides the total number of goats to graze, then the social optimum $G^{**}$ should solve

$$
\max_{0 \leq G < G_{\text{max}}} \ G v(G) - Gc,
$$

which implies

$$
v(G^{**}) + G^{**} v'(G^{**}) - c = 0. \quad (1.3)
$$

On the other hand, by adding together $n$ equations in (1.2), we obtain

$$
nv(G^*) + G^* v'(G^*) - nc = 0,
$$

which implies

$$
v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0, \quad (1.4)
$$

where $G^* = g_1^* + \ldots + g_n^*$. 

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**Claim:** $G^* > G^{**}$.

Suppose $G^* \leq G^{**}$. Then $v(G^*) \geq v(G^{**})$ since $v' < 0$. Similarly, $0 > v'(G^*) \geq v'(G^{**})$ since $v'' < 0$.

Next $G^*/n < G^* \leq G^{**}$, since $n \geq 2$.

Thus

$$\frac{G^*}{n} v'(G^*) \geq \frac{G^*}{n} v'(G^{**}) > G^{**} v'(G^{**}).$$

This implies

$$v(G^*) + \frac{G^*}{n} v'(G^*) > v(G^{**}) + G^{**} v'(G^{**}),$$

which contradicts to (1.4) and (1.3). Therefore

$$G^* > G^{**}.$$ 

This means that too many goats are grazed in the Nash equilibrium, compared to the social optimum.
For example, take

\[ v(x) = a - x^2, \quad a > c. \]

Then \( v'(x) = -2x \).

\[ a - (G^*)^2 + \frac{G^*}{n}(-2G^*) - c = 0. \]

\[ a - (G^{**})^2 + G^{**}(-2G^{**}) - c = 0. \]

These imply

\[ G^* = \frac{\sqrt{a - c}}{\sqrt{1 + 2/n}} > G^{**} = \frac{\sqrt{a - c}}{\sqrt{3}}. \]

In fact,

\[ \lim_{n \to \infty} \frac{G^*}{G^{**}} = \sqrt{3} \approx 1.73. \]

This means that, when there is a large number of farmers, who choose the number of goats simultaneously by their own, the village grazes 73% more goats than the social optimum.
Final-Offer Arbitration

Suppose the wage dispute of a firm and a union is settled by the final-offer arbitration. First the firm and the union simultaneously make offers, denoted by $w_f$ and $w_u$ respectively ($w_f < w_u$). Second, the arbitrator chooses one of the two offers as the settlement. Assume that the arbitrator has an ideal settlement $x$ and simply chooses as the settlement the offer that is closer the $x$: the arbitrator chooses $w_f$ if $x < (w_f + w_u)/2$, and chooses $w_u$ if $x > (w_f + w_u)/2$. Both players believe that $x$ is randomly distributed according to a cumulative probability distribution denoted by $F(x)$, with associated probability density function denoted by $f(x) (= F'(x))$. Let $x_m$ be the median of $x$, i.e., $F(x_m) = 1/2$.

Question: Why not choose $(w_f + w_u)/2$ as the final settlement?
The objective of the firm is to minimize the expected wage settlement and the objective of the union is to maximize it.

Because

\[
Prob(w_f \text{ is chosen}) = Prob(x < \frac{w_f + w_u}{2}) = F\left(\frac{w_f + w_u}{2}\right),
\]

\[
Prob(w_u \text{ is chosen}) = 1 - F\left(\frac{w_f + w_u}{2}\right),
\]

the expected wage settlement is

\[
\phi(w_f, w_u) = w_f F\left(\frac{w_f + w_u}{2}\right) + w_u \left[1 - F\left(\frac{w_f + w_u}{2}\right)\right].
\]

\((w^*_f, w^*_u)\) are a Nash equilibrium if and only if they solve

\[
\min_{w_f} \phi(w_f, w_u^*) \quad \text{and} \quad \max_{w_u} \phi(w_f^*, w_u),
\]

respectively.
From the optimality conditions $\frac{\partial \phi}{\partial w_f} = 0$ and $\frac{\partial \phi}{\partial w_u} = 0$, it follows that

$$\frac{1}{2}(w_u - w_f) f\left(\frac{w_f + w_u}{2}\right) = F\left(\frac{w_f + w_u}{2}\right)$$

$$\frac{1}{2}(w_u - w_f) f\left(\frac{w_f + w_u}{2}\right) = 1 - F\left(\frac{w_f + w_u}{2}\right)$$

So we have $F\left(\frac{w_f + w_u}{2}\right) = \frac{1}{2}$, which implies

$$\frac{w_f + w_u}{2} = x_m.$$ 

Furthermore we have

$$w_u - w_f = \frac{1}{f(x_m)}.$$ 

Solving the above two equations, we obtain the Nash equilibrium

$$w_f^* = x_m - \frac{1}{2f(x_m)}, \quad w_u^* = x_m + \frac{1}{2f(x_m)}.$$
Matching Pennies

Each player has a penny and must choose whether to display it with heads or tails facing up. If the two pennies match (i.e., both are heads up or both are tails up) then player 2 wins player 1’s penny; if the pennies do not match then 1 wins 2’s penny.

\[
\begin{array}{c|cc}
\text{Player2} & \text{Heads} & \text{Tails} \\
\hline
\text{Player1} & -1, 1 & 1, -1 \\
\text{Heads} & 1, -1 & -1, 1 \\
\text{Tails} & & \\
\end{array}
\]

There is no Nash equilibrium in this game. The reason is that the solution to such a game necessarily involves uncertainty about what the other players will do.

In order to find equilibria to such problems, we introduce the notion of a mixed strategy.
Mixed Strategies

Definition 1.5 In the normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. Then each strategy $s_{ik}$ in $S_i$ is called a pure strategy for player $i$. A mixed strategy for player $i$ is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK})$, where $p_{i1} + \ldots + p_{iK} = 1$ and $p_{ik} \geq 0$.

In Matching Pennies, $S_i = \{Heads, Tails\}$. So Heads and Tails are the pure strategies; a mixed strategy for player $i$ is the probability distribution $(q, 1 - q)$, where $q$ is the probability of playing Heads, $1 - q$ is the probability of playing Tails, and $0 \leq q \leq 1$. The mixed strategy $(1, 0)$ is simply the pure strategy Heads; and the mixed strategy $(0, 1)$ is simply the pure strategy Tails.
Recall that the definition of Nash equilibrium guarantees that each player’s pure strategy is a best response to the other players’ pure strategies. To extend the definition to include mixed strategies, we simply require that each player’s mixed strategy be a best response to the other players’ mixed strategies.

We illustrate the notion of Nash equilibria by considering the case of two players.

Let

\[ S_1 = \{ s_{11}, s_{12}, \ldots, s_{1J} \} \]

and

\[ S_2 = \{ s_{21}, s_{22}, \ldots, s_{2K} \} \]

be the respective sets of pure strategies for players 1 and 2. We use \( s_{1j} \) and \( s_{2k} \) to denote arbitrary pure strategies from \( S_1 \) and \( S_2 \) respectively.

If player 1 believes that player 2 will play the strate-
gies \((s_{21}, s_{22}, \ldots, s_{2K})\) with the probabilities \(p_2 = (p_{21}, p_{22}, \ldots, p_{2K})\), then player 1’s expected payoff from playing the pure strategy \(s_{1j}\) is

\[v_1(s_{1j}, p_2) = \sum_{k=1}^{K} p_{2k}u_1(s_{1j}, s_{2k}),\]

and player 1’s expected payoff from playing the mixed strategy \(p_1 = (p_{11}, p_{12}, \ldots, p_{1J})\) is

\[v_1(p_1, p_2) = \sum_{j=1}^{J} p_{1j}v_1(s_{1j}, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j}p_{2k}u_1(s_{1j}, s_{2k}).\]

If player 2 believes that player 1 will play the strategies \((s_{11}, s_{12}, \ldots, s_{1J})\) with the probabilities \(p_1 = (p_{11}, p_{12}, \ldots, p_{1J})\), then player 2’s expected payoff from playing the pure strategies \((s_{21}, \ldots, s_{2K})\) with the probability \(p_2 = (p_{21}, \ldots, p_{2K})\) is

\[v_2(p_1, p_2) = \sum_{k=1}^{K} p_{2k}v_2(p_1, s_{2k}) = \sum_{k=1}^{K} \sum_{j=1}^{J} p_{2k}p_{1j}u_2(s_{1j}, s_{2k}).\]
Definition 1.6 In the two-player normal-form game $G = \{S_1, S_2; u_1, u_2\}$, the mixed strategies $(p_1^*, p_2^*)$ are a Nash equilibrium if each player’s mixed strategy is a best response to the other player’s mixed strategy:

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*)$$

and

$$v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2)$$

for all probability distributions $p_1$ and $p_2$ on $S_1$ and $S_2$ respectively.
Find mixed-strategy Nash equilibria

We only consider two-player games in which each player has 2 pure strategies, so that we can use the graphical approach.

Let $p_1 = (r, 1 - r)$ be a mixed strategy for player 1 and $p_2 = (q, 1 - q)$ a mixed strategy for player 2. Then

$$v_1(p_1, p_2) = rv_1(s_{11}, p_2) + (1 - r)v_1(s_{12}, p_2).$$

For each given $p_2$, i.e., for each given $q$, we compute the value of $r$, denoted $r^*(q)$, such that $p_1 = (r, 1 - r)$ is a best response for player 1 to $p_2 = (q, 1 - q)$. That is, $r^*(q)$ is the set of solutions to

$$\max_{0 \leq r \leq 1} v_1(p_1, p_2).$$

$r^*(\cdot)$ is called the best-response correspondence. It may contain more than one value.
Observing $v_1(p_1, p_2)$, we obtain
\[
r^*(q) = \begin{cases} 
1, & \text{if } v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \\
0, & \text{if } v_1(s_{11}, p_2) < v_1(s_{12}, p_2) \\
[0, 1], & \text{if } v_1(s_{11}, p_2) = v_1(s_{12}, p_2)
\end{cases}
\]

Similarly, maximizing
\[
v_2(p_1, p_2) = qv_2(p_1, s_{21}) + (1 - q)v_2(p_1, s_{22})
\]
yields
\[
q^*(r) = \begin{cases} 
1, & \text{if } v_2(p_1, s_{21}) > v_2(p_1, s_{22}) \\
0, & \text{if } v_2(p_1, s_{21}) < v_2(p_1, s_{22}) \\
[0, 1], & \text{if } v_2(p_1, s_{21}) = v_2(p_1, s_{22})
\end{cases}
\]

A mixed-strategy Nash equilibrium is an intersection of the two best-response correspondences $r^*(q)$ and $q^*(r)$. 
Find a Nash equilibrium for the Game of Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player1</td>
<td>Heads</td>
<td>−1, 1</td>
</tr>
<tr>
<td></td>
<td>Tails</td>
<td>1, −1</td>
</tr>
</tbody>
</table>

Let $p_1 = (r, 1 - r)$ be a mixed strategy in which player 1 plays Heads with probability $r$. Let $p_2 = (q, 1 - q)$ be a mixed strategy for player 2. Then

\[
v_1(s_{11}, p_2) = q \cdot (-1) + (1 - q) \cdot 1 = 1 - 2q
\]

\[
v_1(s_{12}, p_2) = q \cdot 1 + (1 - q) \cdot (-1) = -1 + 2q.
\]

Player 1 chooses Heads, i.e., $r^*(q) = 1 \iff v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \iff q < 1/2$. Hence

\[
r^*(q) = \begin{cases} 
1, & \text{if } 0 \leq q < 1/2 \\
0, & \text{if } 1/2 < q \leq 1 \\
[0, 1], & \text{if } q = 1/2
\end{cases}
\]
Graph of $r^*(q)$:

To determine $q^*(r)$, we use

\[
\begin{align*}
v_2(p_1, s_{21}) &= r \cdot 1 + (1 - r) \cdot (-1) = 2r - 1 \\
v_2(p_1, s_{22}) &= r \cdot (-1) + (1 - r) \cdot 1 = -2r + 1.
\end{align*}
\]

Player 2 chooses Heads, i.e., $q^*(r) = 1 \iff v_2(p_1, s_{21}) > v_2(p_1, s_{22}) \iff r > 1/2$, hence

\[
q^*(r) = \begin{cases} 
1, & \text{if } 1/2 < r \leq 1 \\
0, & \text{if } 0 \leq r < 1/2 \\
[0, 1], & \text{if } r = 1/2
\end{cases}
\]
We draw the graphs of $r^*(q)$ and $q^*(r)$ together.

The graphs of the best response correspondences $r^*(q)$ and $q^*(r)$ intersect at only one point $q = \frac{1}{2}$ and $r = \frac{1}{2}$. Thus $p_1^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, $p_2^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ are the only Nash equilibrium in mixed strategies for the game. This means that randomly choosing Tails and Heads with equal probability is the best strategy.
If there are more than 2 strategies for a player, we can first eliminate strictly dominated strategies. The strategies played with positive probability in a mixed strategy Nash equilibrium survive the elimination. If there are only 2 strategies left for each player, then we can use the approach discussed before.

Example.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>4, 4</td>
<td>4, 0</td>
<td>5, 1</td>
</tr>
<tr>
<td>Player2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>2, 0</td>
<td>0, 4</td>
<td>5, 3</td>
</tr>
<tr>
<td>B</td>
<td>3, 5</td>
<td>3, 5</td>
<td>6, 6</td>
</tr>
</tbody>
</table>

Eliminate the dominated strategies M and C, resulting in
Let $p_1 = (r, 1 - r)$ be the mixed strategy in which 1 plays T with probability $r$. Let $p_2 = (q, 1 - q)$ be the mixed strategy in which 2 plays L with probability $q$.

The expected payoffs for player 1:

\[
v_1(T, p_2) = 4q + 5(1 - q) = 5 - q
\]

\[
v_1(B, p_2) = 3q + 6(1 - q) = 6 - 3q.
\]

Player 1 chooses T $\iff$ $5 - q > 6 - 3q$ $\iff$ $q > 1/2$. Thus the best response $r^*(q)$ for player 1 is
\[ r^*(q) = \begin{cases} 
0, & \text{if } 0 \leq q < 1/2 \\
1, & \text{if } 1/2 < q \leq 1 \\
[0, 1], & \text{if } q = 1/2 
\end{cases} \]

The expected payoffs for Player 2:

\[ v_2(p_1, L) = 4r + 5(1 - r) = 5 - r \]
\[ v_2(p_1, R) = r + 6(1 - r) = 6 - 5r. \]

Player 2 chooses L \iff 5 - r > 6 - 5r \iff r > 1/4. Thus the best response \( q^*(r) \) for Player 2 is

\[ q^*(r) = \begin{cases} 
0, & \text{if } 0 \leq r < 1/4 \\
1, & \text{if } 1/4 < r \leq 1 \\
[0, 1], & \text{if } r = 1/4 
\end{cases} \]
The best-response correspondences intersect at three points:

\((r = 0, q = 0), (r = 1, q = 1), (r = 1/4, q = 1/2)\).

In the original game, these are two pure-strategy Nash equilibria \((T, L)\) and \((B, R)\), and a mixed-strategy Nash equilibrium \(\{p_1 = (1/4, 0, 3/4), p_2 = (1/2, 0, 1/2)\}\).
In general, we have the following existence theorem.

**Theorem 1.1** *(Nash, 1950)* In the $n$-player normal-form game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, if $n$ is finite and $S_i$ is finite for every $i$, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

The proof of Nash’s Theorem involves a *fixed-point theorem*. It undergoes two steps: (1) showing that a Nash equilibrium is a fixed point of a certain correspondence; (2) using an appropriate fixed-point theorem to show that this correspondence must have a fixed point. The relevant correspondence is the $n$-player best-response correspondence.
Summary

In this chapter we consider games of the following simple form: first the players simultaneously choose actions; then the players receive payoffs that depend on the combination of actions just chosen. Here, we assume that the information is complete, i.e., each player’s payoff function (the function that determines the player’s payoff from the combination of actions chosen by the players) is common knowledge among all the players.

The two basic issues in game theory are: how to describe a game and how to solve the resulting game-theoretic problem. We define the normal-form representation of a game and the notion of a strictly dominated strategy. We show that some games can be solved by applying the iterated elimination of strictly dominated strategies, but also
that in other games this approach produces a very imprecise prediction about the play of the game (sometimes as imprecise as “anything could happen”). We then motivate and define *Nash equilibrium* — a solution concept that produces much tighter predictions in a very broad class of games.

We analyze three applications: Cournot’s model of imperfect competition, Bertrand’s model of imperfect competition, and the problem of the commons. In each application we first translate an informal statement of the problem into a normal-form representation of the game and then solve for the game’s Nash equilibrium. (Each of these applications has a unique Nash equilibrium, but we also discuss examples in which this is not true.)
We define the notion of a *mixed strategy*, which we interpret in terms of one player’s uncertainty about what another player will do. We show how to find the mixed-strategy Nash equilibrium for two-player two-strategy games graphically, where the Nash equilibrium is the intersection of the best response correspondences. We then state and discuss Nash’s (1950) Theorem, which guarantees that a Nash equilibrium (possibly involving mixed strategies) exists in a broad class of games.
2 Dynamic Games of Complete Information

By **complete information**, we mean that the payoff functions are common knowledge. We will consider 2 cases:

(1) *Dynamic games with complete and perfect information*

By **perfect information**, we mean that at each move in the game, the player with the move knows the full history of the play of the game thus far.

(2) *Dynamic games with complete but imperfect information*

At some move the player with the move does not know the history of the game.
2.1 Dynamic Games with Complete and Perfect Information

**Example 1.** Player 1 chooses an action L or R. Player 2 observes player 1’s action and then chooses an action $L'$ or $R'$. Each path (a combination of two actions) in the following tree followed by two payoffs, the first for player 1 and the second for player 2.
This type of games may take the following form:

1. Player 1 chooses an action $a_1$ from the feasible set $A_1$.
2. Player 2 observes $a_1$ and then chooses an action $a_2$ from the feasible set $A_2$.
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.

The key features are:

(i) the moves occur in sequence;
(ii) all previous moves are observed before the next move is chosen, (perfect information);
(iii) the players’ payoffs from each combination of moves are common knowledge, (complete information).
**Backwards Induction**

At the second stage, player 2 observes the action (say $a_1$) chosen by player 1 at the first stage, then chooses an action by solving

$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$

Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This is player 2’s best response to player 1’s action. (E.g., $R_2(L) = R'$, $R_2(R) = L'$.)

Knowing the player 2’s best response, player 1 should solve

$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$

Assume it has a unique solution, $a_1^*$. (E.g., $a_1^* = R$, $R_2(a_1^*) = L'$.)

We call $(a_1^*, R_2(a_1^*))$ the **backwards-induction outcome** of the game.
A comparison If both players choose their actions simultaneously, then the Nash equilibrium \((a_1^{**}, a_2^{**})\) is the intersection of the two best responses, i.e. it solves

\[
\begin{align*}
a_2^{**} &= R_2(a_1^{**}) \\
a_1^{**} &= R_1(a_2^{**})
\end{align*}
\]

In the backwards-induction outcome, we determines \(a_1^*\) by maximizing \(u_1(a_1, R_2(a_1))\) and let

\[
a_2^* = R_2(a_1^*).
\]

Note: \(a_1^*\) may not maximize \(u_1(a_1, a_2^*)\).

Example: If we let both players in Example 1 choose actions simultaneously, then they play the following game:

\[
\begin{array}{c|cc}
 & L' & R' \\
\hline
L & 3,1 & 1,2 \\
R & 2,1 & 0,0
\end{array}
\]
The best actions of this static game are \((a_1^{**}, a_2^{**}) = (L, R')\), whereas the best actions of the dynamic game (as in the original example 1) are \((a_1^*, a_2^*) = (R, L')\).

So the backwards-induction outcome in a dynamic game could be different from the Nash equilibrium of the corresponding game played simultaneously.

**A three-move game:**

1. Player 1 chooses \(L\) or \(R\).
   
   \(L\): ends the game with payoff of 2 to player 1 and 0 to player 2.

2. Player 2 observes \(R\), then chooses \(L'\) or \(R'\).

   \(L'\): ends the game with payoff of 1 to both.
3. Player 1 observes player 2’s choice and his earlier choice in the first stage. If the earlier choices are $R$ and $R'$, player 1 chooses $L''$ or $R''$. Both end the game.

$L''$: gives payoff 3 to player 1 and 0 to player 2;
$R''$: gives payoff 0 to player 1 and 2 to player 2.

Game tree:
We compute the backwards-induction outcome of this game, starting from the last stage.

1. For player 1 at the third stage, $L''$ is optimal in which player 1 gets 3 and player 2 gets 0.

2. For player 2 at the second stage,
   - If he chooses $R'$, then $L''$ will be chosen by player 1 at the third stage, and player 2’s payoff will be 0 at $L''$.
     - If he chooses $L'$, he gets 1.
     \[\Rightarrow\] Therefore, he should choose $L'$.

3. For player 1 at first stage,
   - If he chooses $R$, he would get 1 in the second stage.
     - If he chooses $L$, he gets 2.
     \[\Rightarrow\] Hence, player 1 should choose $L$. 
The backwards-induction outcome:

Player 1 plays $L$ at the first stage, and the game ends.

**Stackelberg Model of Duopoly**

Consider a dominant firm moving first and a follower moving second.

1. Firm 1 chooses a quantity $q_1 \geq 0$.
2. Firm 2 observes $q_1$ and then chooses a quantity $q_2 \geq 0$.
3. The payoff to firm $i$ is the profit

\[
\pi_i(q_1, q_2) = q_i[P(Q) - c],
\]

where $Q = q_1 + q_2$ and

\[
P(Q) = \begin{cases} 
    a - Q, & \text{if } Q < a \\
    0, & \text{if } Q \geq a
\end{cases}
\]
Find the backwards-induction outcome.

The best response function $R_2(q_1)$ for firm 2 to quantity $q_1$ solves

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2)$$

$$= \max_{q_2 \geq 0} \begin{cases} q_2[a - q_1 - q_2 - c], & \text{if } q_1 + q_2 < a \\ -cq_2, & \text{if } q_1 + q_2 \geq a \end{cases}$$

which yields

$$R_2(q_1) = \begin{cases} \frac{a-q_1-c}{2} & \text{if } q_1 < a - c \\ 0 & \text{if } q_1 \geq a - c. \end{cases}$$

Firm 1 knows $R_2(q_1)$ and thus solves

$$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)),$$

where

$$\pi_1 \ (q_1, R_2(q_1)) = \begin{cases} q_1[a - q_1 - \frac{a-q_1-c}{2} - c] & \text{if } q_1 < a - c \\ q_1[a - q_1 - c] & \text{if } a > q_1 \geq a - c \\ -cq_1 & \text{if } q_1 \geq a. \end{cases}$$
For $q_1 > a - c$, the profit is always negative. Thus we need only to find the maximum for

$$q_1[a - q_1 - \frac{a - q_1 - c}{2} - c] = \frac{1}{2}q_1(a - q_1 - c).$$

The optimality condition

$$a - q_1 - c - q_1 = 0$$

yields

$$q_1^* = \frac{a - c}{2},$$

which maximizes the profit $\pi_1(q_1, R_2(q_1))$.

Firm 2’s quantity is

$$q_2^* = R_2(q_1^*) = \frac{a - q_1^* - c}{2} = \frac{a - c}{4}.$$

The market price is

$$P = a - \frac{3(a - c)}{4} = c + \frac{a - c}{4}.$$ 

The profits for firms 1 and 2 are respectively:

$$\frac{(a - c)^2}{8} \quad \text{and} \quad \frac{(a - c)^2}{16}.$$
The total profit of firms 1 and 2 is

\[3(a - c)^2/16 = 0.1875(a - c)^2.\]

**Note:** In the Cournot game the optimal quantities are

\[q_1^* = q_2^* = \frac{a - c}{3}.\]

The market price is

\[P = a - \frac{2(a - c)}{3} = c + \frac{a - c}{3}.\]

The profits for firms 1 and 2 are:

\[\frac{(a - c)^2}{9} \quad \text{and} \quad \frac{(a - c)^2}{9}.\]

The total profit of firms 1 and 2 is

\[2(a - c)^2/9 = 0.2222(a - c)^2.\]
A comparison between Cournot game and Stackelberg game

The Cournot game can be interpreted as a game with the following timing:

Stage 1. Firm 1 chooses $q_1$;
Stage 2. Firm 2 chooses $q_2$ without observing $q_1$.

(In the Stackelberg game, firm 2 observes $q_1$ before choosing $q_2$.)

Observation: Firm 2 has more information in Stackelberg game than in Cournot game.

Question: In which game does firm 2 make more profit?
Firm 2’s profit

in Stackelberg game \[ \frac{(a - c)^2}{16} \]
in Cournot game \[ \frac{(a - c)^2}{9} \]

• Having more information can make a player worse off. (This can never happen in single-person decision problems, but can happen in multi-person decision problems, i.e., games.)

**Explanation:**

Firm 1 knows that firm 2 first observes \( q_1 \) and then chooses \( q_2 \), and also knows that firm 2 is rational and must choose the best response to \( q_1 \).

The knowledge of this behavior of firm 2 is utilized by firm 1. This makes firm 1 better off and firm 2 worse off.
2.2 Two-Stage Games of Complete But Imperfect Information

I. Players 1 and 2 simultaneously choose actions $a_1$ and $a_2$ from the feasible sets $A_1$ and $A_2$, respectively.

II. Players 3 and 4 observe the outcome of the first stage $(a_1, a_2)$ and then simultaneously choose actions $a_3$ and $a_4$ from the feasible sets $A_3$ and $A_4$ respectively.

III. Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.

Subgame-Perfect Outcome

For each given $(a_1, a_2)$, player 3 and player 4 try to find the Nash equilibrium in stage 2. Assume the second-stage game has a unique Nash equilibrium $(a_3(a_1, a_2), a_4(a_1, a_2))$. 

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Then, player 1 and player 2 play a simultaneous-move game with payoffs

\[ u_i(a_1, a_2, a_3(a_1, a_2), a_4(a_1, a_2)), \quad i = 1, 2. \]

Suppose \((a_1^*, a_2^*)\) is the Nash equilibrium of the simultaneous-move game. Then

\[ (a_1^*, a_2^*, a_3(a_1^*, a_2^*), a_4(a_1^*, a_2^*)) \]

is the subgame-perfect outcome of the 2-stage game.

**Bank Runs**

Two investors have each deposited $5 millions with a bank. The bank has invested these deposits in a long-term project.

If the bank is forced to stop its investment before the project matures, a total of $8 millions can be recovered.
If the bank allows the investment to reach maturity, the project will pay out a total of $16 millions.

There are two dates at which the investors can make withdrawals at the bank:

- Date 1 is before the bank’s investment matures;
- Date 2 is after.

Payoffs:

<table>
<thead>
<tr>
<th></th>
<th>Withdraw</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Date 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Withdraw</td>
<td>4, 4</td>
<td>5, 3</td>
</tr>
<tr>
<td>Don’t</td>
<td>3, 5</td>
<td>Next stage</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Withdraw</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Date 2</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Withdraw</td>
<td>8, 8</td>
<td>10, 6</td>
</tr>
<tr>
<td>Don’t</td>
<td>6, 10</td>
<td>9, 9</td>
</tr>
</tbody>
</table>
We work backwards:

At date 2:

The unique Nash equilibrium is: both withdraw and each obtains $8 millions.

At date 1:

They play the following game:

<table>
<thead>
<tr>
<th></th>
<th>Withdraw</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Withdraw</td>
<td>4, 4</td>
<td>5, 3</td>
</tr>
<tr>
<td>Don’t</td>
<td>3, 5</td>
<td>8, 8</td>
</tr>
</tbody>
</table>

The game has 2 pure-strategy Nash equilibria:

1) Both investors withdraw, leading to a payoff (4, 4);

2) Both don’t, leading to a payoff (8, 8).

Thus, the original 2-stage game has 2 subgame-perfect outcomes:
1) Both withdraw at date 1 to obtain (4, 4).
2) Both don’t withdraw at date 1, but do withdraw at date 2, yielding (8, 8).

**Question:** Can the bank design a payoff such that WITHDRAW at date 2 is the unique outcome?

**Tariff Game**

Two countries, denoted by \( i = 1, 2 \). Each government can choose a tariff rate. Each country has a firm that produces output for both home consumption and export. Consumers can buy from the home market from either the home firm or the foreign firm through import.

- If the total quantity on the market in country \( i \) is \( Q_i \), then the market price is
  \[
  P_i(Q_i) = a - Q_i.
  \]
• The firm in country $i$ (i.e. firm $i$) produces $h_i$ for home consumption and $e_i$ for export.

Thus,

$$Q_i = h_i + e_j.$$ 

• The firm has constant marginal cost $c$, (no fixed cost).

• Firm $i$ pays tariff $t_j e_i$ to the government $j$.

*The timing of the game:*

1. The governments simultaneously choose tariff rates $t_1$ and $t_2$.
2. The firms observe the tariff rates and simultaneously choose $(h_1, e_1)$ and $(h_2, e_2)$. 
3. The payoff to firm $i$ is its profit:

$$
\pi_i (t_i, t_j, h_i, e_i, h_j, e_j) = [a - (h_i + e_j)]h_i + [a - (h_j + e_i)]e_i - c(h_i + e_i) - t_j e_i.
$$

4. The payoff to government $i$ is welfare. It is the sum of the consumers’ surplus enjoyed by the consumers in country $i$ (which is $Q_i^2/2$), the profit earned by firm $i$, and the tariff collected from firm $j$:

$$
w_i (t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j.
$$

Suppose the tariff rates are chosen as $t_1, t_2$.

If $((h_1^*, e_1^*), (h_2^*, e_2^*))$ is a Nash equilibrium in the second-stage subgame, then for each $i$, $(h_i^*, e_i^*)$ must solve

$$
\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*).
$$
Rewrite

\[ \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*) = \left[ a - (h_i + e_j^*) - c \right] h_i \]
\[ + \left[ a - (h_j^* + e_i) - c - t_j \right] e_i. \]

We can simply maximize the first term for \( h_i \) and the second term for \( e_i \) separately.

Assume
\[ e_j^* \leq a - c, \quad \text{(otherwise } h_i = 0); \]
\[ h_j^* \leq a - c - t_j, \quad \text{(otherwise } e_i = 0). \]

The maximizers \( h_i^* \) and \( e_i^* \) must satisfy
\[ h_i^* = \frac{1}{2}(a - e_j^* - c) \]
\[ e_i^* = \frac{1}{2}(a - h_j^* - c - t_j), \]
for \( i = 1, 2 \) (and \( j = 2, 1 \)).

Solving this system of 4 equations, we obtain
\[ h_i^* = \frac{a - c + t_i}{3}; \]
Next, we move to the first stage of the game.

Denote

\[ w_i^*(t_i, t_j) = w_i(t_i, t_j, h_i^*, e_i^*, h_j^*, e_j^*). \]

If \((t_i^*, t_j^*)\) is a Nash equilibrium of the game between the two governments, then \(t_i^*\) must solve

\[
\max_{t_i \geq 0} w_i^*(t_i, t_j^*),
\]

where

\[
w_i^*(t_i, t_j^*) = \frac{1}{2}[h_i^* + e_j^*]^2 + h_i^*[a - c - (h_i^* + e_j^*)] + e_i^*[a - c - (h_j^* + e_i^*) - t_j^*] + t_i e_j^*
\]

\[
= \frac{1}{2} \left[ \frac{2(a - c) - t_i}{3} \right]^2 + \left[ a - c + \frac{t_i}{3} \right]^2 + \left[ a - c - 2t_j^* \right]^2 + t_i \left[ a - c - 2t_i \right].
\]
From
\[
\frac{d}{dt_i} w_i^*(t_i, t_j^*) = \frac{1}{3}(a - c - 3t_i) = 0
\]
we obtain
\[
t_i^* = \frac{a - c}{3}.
\]
Then we substitute \( t_i^* \) into the expressions of \( h_i^* \) and \( e_i^* \).

The subgame-perfect outcome of the tariff game:
\[
t_1^* = t_2^* = \frac{a - c}{3};
\]
\[
h_1^* = h_2^* = \frac{4(a - c)}{9};
\]
\[
e_1^* = e_2^* = \frac{a - c}{9}.
\]
Remark: If the government had chosen tariff rate to be zero without playing the game for themselves, then the total quantity in the market is \( \frac{2(a-c)}{3} \), which is more than \( Q_i^* = h_i^* + e_j^* = \frac{5(a-c)}{9} \). That is, the consumers’ surplus in the case of zero tariffs is higher than that with subgame-perfect outcome.

In fact, zero tariffs are socially optimal in the sense that \( t_1 = t_2 = 0 \) is the solution to:

\[
\max_{t_1, t_2 \geq 0} w_1^*(t_1, t_2) + w_2^*(t_2, t_1).
\]

So, there is an incentive for the governments to sign a treaty to have zero tariffs (i.e., free trade).
2.3 Extensive-form representation of games

Definition 2.1 The extensive-form representation of a game specifies

(1) the players in the game;

(2a) when each player has the move,

(2b) what each player can do at each of his or her opportunities to move,

(2c) what each player knows at each of his or her opportunities to move;

(3) the payoffs received by each player for each combination of moves that could be chosen by the players.
Example 2.1:

1. Player 1 chooses an action $a_1$ from the feasible set $A_1 = \{L, R\}$.

2. Player 2 observes $a_1$ and then chooses an action $a_2$ from $A_2 = \{L', R'\}$.

3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$ as shown in the game tree below.
The game begins with a **decision node** for player 1. After 1’s choice of *L* or *R*, 2’s decision node is reached. A **terminal node** is reached after 2’s move, and payoff is received.

A dynamic game of complete and perfect information is a game in which the players move *in sequence*, all previous moves are observed before the next move is chosen, and payoffs are common knowledge. Such games can be easily represented by a game tree.

When information is not perfect, some previous moves are not observed by the player with the current move. To present this kind of ignorance of previous move and to describe *what each player knows* at each of his/her move, we introduce the notion of a player’s information set.
Definition 2.2 An information set for a player is a collection of decision nodes satisfying:

1. The player needs to move at every node in the information set.

2. When the play of the game reached a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.

- Point 2 implies that the player must have the same set of feasible actions at each decision node in an information set.

- A game is said to have imperfect information if some of its information sets are non-singletons.

- In an extensive-form game, a collection of decision nodes, which constitutes an information set, is connected by a dotted line.
As an example, let’s state a two-person simultaneous-move (static) game as follows:

1. Player 1 chooses $a_1 \in A_1$.
2. Player 2 does not observe player 1’s move but chooses an $a_2 \in A_2$.
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.

Then, we need an information set to describe player 2’s ignorance of player 1’s actions.

**Example 2.2:** Normal-form of the Prisoners’ Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>$L_2$</th>
<th>$R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>1, 1</td>
<td>5, 0</td>
</tr>
<tr>
<td>$R_1$</td>
<td>0, 5</td>
<td>4, 4</td>
</tr>
</tbody>
</table>
Extensive-form of the Prisoners’ Dilemma:

Thus, static game of complete information can be represented by dynamic games of complete but imperfect information.

Example 2.3:

(1) Player 1 chooses an action $a_1$ from $A_1 = \{L, R\}$. 
(2) Player 2 observes $a_1$ and then chooses an action $a_2$ from $A_2 = \{L', R'\}$.

(3) Player 3 observe whether or not $(a_1, a_2) = (R, R')$ and then chooses an action $a_3$ from $A_3 = \{L'', R''\}$.

Player 3 has a non-singleton information set and a singleton.
Now we want to find equilibria of a dynamic game. An equilibrium is a combination of strategies. We will first define strategies for a dynamic game, and then define the normal-form representation of the dynamic game.

**Definition 2.3** A strategy for a player is a complete plan of actions. It specifies a feasible action for the player in every contingency in which the player might be called on to act.

If a player left a strategy with his/her lawyer before the game began, the lawyer could play the game for the player without ever needing further instructions as to how to play.

For the game in Example 2.1, player 1 has 2 strategies: $L$ and $R$;
player 2 has 2 actions: $L'$ and $R'$, but 4 strategies:

$$(L', L'), (L', R'), (R', L'), (R', R'),$$

where $(R', L')$ means:

if player 1 plays $L$ then player 2 plays $R'$, if player 1 plays $R$ then player 2 plays $L'$.

A strategy $(R', L')$ can be regarded as a function $f : A_1 \to A_2$, meaning $f(L) = R'$ and $f(R) = L'$.

The normal-form representation is

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>(L', L')</td>
</tr>
<tr>
<td>$L$</td>
<td>3, 1</td>
</tr>
<tr>
<td>$R$</td>
<td>2, 1</td>
</tr>
</tbody>
</table>

The Nash equilibria are $(L, (R', R'))$ and $(R, (R', L'))$.

The second equilibrium constitutes the backward-induction outcome. But the first one doesn’t.
Differences between an equilibrium and an outcome.

Consider the following 2-stage game of complete and perfect information:

1. Player 1 chooses an action $a_1$ from $A_1$;
2. Player 2 observes $a_1$ and then chooses $a_2$ from $A_2$;
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.

The best response $R_2(a_1)$ solves $\max_{a_2 \in A_2} u_2(a_1, a_2)$. $a_1^*$ solves $\max_{a_1 \in A_1} u_1(a_1, R_2(a_1))$.

The backwards-induction outcome is $(a_1^*, R_2(a_1^*))$.
But the Nash equilibrium is $(a_1^*, R_2(\cdot))$.
Note:

(i) $R_2(a_1^*)$ is an **action** but not a strategy.  
$R_2(\cdot)$ is a **strategy**.

$R_2(\cdot)$ is a function.

$R_2(a_1^*)$ is the value of the function $R_2$ at $a_1^*$.

(ii) $a_1^*$ in a NE $(a_1^*, R_2(\cdot))$ need not maximize $u_1(a_1, R_2(a_1))$.

In Example 2.1,
\( a_1^* = R \) and \( R_2(a_1^*) = L' \) are actions;

\((R, L')\) is the backwards-induction outcome;

\( R_2 = (R', L') \) is a strategy for player 2 (the best response);

\((R, (R', L'))\) is the Nash equilibrium.

\((L, (R', R'))\) is also a NE. But \( L \) does not maximize \( u_1(a_1, R_2(a_1)) \).

Consider the 2-stage game of complete but imperfect information,

1. players 1 and 2 simultaneously choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \) respectively;

2. players 3 and 4 observe \((a_1, a_2)\) and then simultaneously choose \( a_3 \in A_3 \) and \( a_4 \in A_4 \) respectively;

3. payoffs are \( u_i(a_1, a_2, a_3, a_4) \) for \( i = 1, 2, 3, 4 \).
For any \((a_1, a_2)\), players 3 and 4 play the second stage game with Nash equilibrium
\[
(a_3(a_1, a_2), a_4(a_1, a_2)).
\]
Here \(a_3\) and \(a_4\) are understood as mappings
\[
(a_3, a_4) : A_1 \times A_2 \to A_3 \times A_4.
\]
Players 1 and 2 play the first stage game with pay-offs \(u_i(a_1, a_2, a_3(a_1, a_2), a_4(a_1, a_2))\) to obtain the Nash equilibrium \((a_1^*, a_2^*)\).

The subgame-perfect outcome is
\[
(a_1^*, a_2^*, a_3(a_1^*, a_2^*), a_4(a_1^*, a_2^*)).
\]

But the Nash equilibrium is
\[
(a_1^*, a_2^*, a_3(\cdot, \cdot), a_4(\cdot, \cdot)).
\]
In the Tariff Game, governments are players 1 and 2, and firms are players 3 and 4.

Governments’ actions are \( t_i \in [0, \infty) \);

Firms’ actions are \((h_i, e_i) \in [0, \infty)^2\).

The subgame-perfect outcome is

\[
t_1^* = t_2^* = (a - c)/3,
\]

\[
h_1^* = h_2^* = 4(a - c)/9, \quad e_1^* = e_2^* = (a - c)/9.
\]

Governments’ strategies are \( t_i \in [0, \infty) \).

Firms’ strategies are functions of \((t_1, t_2)\), e.g.,

\[
(h_i, e_i) = \left(\frac{a - c + t_i}{3}, \frac{a - c - 2t_j}{3}\right).
\]

The Nash equilibrium is

\[
t_1^* = t_2^* = (a - c)/3,
\]

\[
h_i^* = \frac{a - c + t_i}{3}, \quad e_i^* = \frac{a - c - 2t_j}{3}, \quad i = 1, 2.
\]
2.4 Subgame-perfect Nash equilibrium

Are all Nash equilibria good solutions?

Consider Example 2.1 again.

\[
\begin{array}{c|c|c|c|c}
\text{Player 1} & L' & R' & L' & R' \\
\text{Player 2} & \text{Player 2} & \text{Player 2} & \text{Player 2} & \text{Player 2} \\
\end{array}
\]

From its normal-form representation

<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>L'</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>R'</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|c|c|c|c}
\text{Player 1} & L & L' & R' & R' \\
\text{Player 2} & \text{Player 2} & \text{Player 2} & \text{Player 2} & \text{Player 2} \\
\end{array}
\]

<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>(L', L')</th>
<th>(L', R')</th>
<th>(R', L')</th>
<th>(R', R')</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>3,1</td>
<td>3,1</td>
<td>1,2</td>
<td>1,2</td>
<td></td>
</tr>
<tr>
<td>R</td>
<td>2,1</td>
<td>0,0</td>
<td>2,1</td>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>
we can find 2 Nash equilibria:

\[(R, (R', L')) \text{ and } (L, (R', R')).\]

The NE \((R, (R', L'))\) is a good solution, because \(R'\) and \(L'\) are optimal actions at left and right decision nodes, respectively. Also it constitutes the backward induction outcome \((R, L')\).

The NE \((L, (R', R'))\) is not reasonable. It plays \(R'\) at the right decision node, which is obviously worse than \(L'\).

- What are reasons for \((L, (R', R'))\) to be a NE?

One can think the strategy \((R', R')\) by player 2 as a **threat** to player 1. Player 2 may say:

“If player 1 plays \(R\), player 2 will play \(R'\) to give 0 for both;

If player 1 plays \(L\), player 2 still plays \(R'\) to give 1 to player 1 and 2 to player 2.”
If player 1 believes this threat, he will choose $L$ to get 1.

If player 1 does not allow this threat to work, he chooses $R$, then player 2 will not carry the threat out since $R'$ is not optimal to player 2 in this case. Hence, this threat is not credible.

To rule out such bad equilibria, we will define a stronger solution called “subgame perfect Nash equilibrium”.

**Definition 2.4** A subgame in an extensive-form game

- a) begins at a decision node $n$ that is a singleton information set (but is not the game’s first decision node);
- b) includes all the decision and terminal nodes following node $n$ in the game tree (but no nodes that do not follow $n$);
c) does not cut any information sets (i.e., if a decision node \( n' \) follows \( n \) in the game tree, then all other nodes in the information set containing \( n' \) must also follow \( n \), and so must be included in the subgame).

- The game in Example 2.1 has 2 subgames.
- The prisoners’ Dilemma in Example 2.2 has no subgame, since player 2’s decision nodes are in the same non-singleton information set.
- The game in Example 2.3 has only 1 subgame, beginning at player 3’s decision node following \( R \) and \( R' \). Note: None of subtrees beginning at player 2’s decision nodes forms a subgame, because it violates (c).

**Definition 2.5 (Selten, 1965):** A Nash equilibrium is **subgame-perfect** if the players’ strategies constitute a Nash equilibrium in every subgame.
It can be shown that any finite dynamic game of complete information has a subgame-perfect Nash equilibrium, perhaps in mixed-strategies.

**A Nash equilibrium need not be a subgame-perfect Nash equilibrium**

In Example 2.1, the Nash equilibrium \((R, (R', L'))\) is subgame-perfect, because \(R'\) and \(L'\) are the optimal strategies in the left and right subgames, respectively, where player 2 is the only player.

On the other hand, the Nash equilibrium \((L, (R', R'))\) is **not** subgame-perfect, because when player 1 chooses \(R\), \(R'\) is not optimal to player 2 in the right subgame.

- Nash equilibria that rely on *non-credible* threats or promises can be eliminated by the requirement of subgame perfection.
Examples

Game (i): Extensive-form:

Backwards-induction outcome: 1 plays $A$, then the game ends.

Normal-form: ($na$ stands for ‘no action’)

Nash equilibrium is $(A, (na, C))$. 
It is subgame-perfect because \( C \) is the best choice for player 2 at the subgame:

\[
\begin{array}{c}
2 \\
C \\
1 \\
2 \\
D \\
3 \\
1 \\
\end{array}
\]

Game (ii): Extensive-form:

Player 2 observes whether player 1 plays \( A \) or not. If \( A \) is played, Player 2 will choose \( E \) with payoff

\[
\begin{array}{c}
2 \\
D \\
6 \\
-1 \\
E \\
2 \\
0 \\
L \\
4 \\
4 \\
R \\
-2 \\
5 \\
L \\
R \\
5 \\
-2 \\
R \\
1 \\
1 \\
\end{array}
\]
(2, 0). If $A$ is not played, player 2 considers a game of imperfect information:

\[
\begin{array}{c|cc}
 & L & R \\
\hline
B & 4, 4 & -2, 5 \\
C & 5, -2 & 1, 1 \\
\end{array}
\]

which results in the Nash equilibrium $(C, R)$ with payoff $(1, 1)$. Player 1’s payoffs are 2 by Playing $A$ and 1 by playing $B$ or $C$. Thus player 1 chooses $A$.

The outcome: 1 plays $A$, then 2 plays $E$.

Normal-form:

\[
\begin{array}{c|cccc}
 & (D, L) & (D, R) & (E, L) & (E, R) \\
\hline
A & 6, -1 & 6, -1 & 2, 0 & 2, 0 \\
B & 4, 4 & -2, 5 & 4, 4 & -2, 5 \\
C & 5, -2 & 1, 1 & 5, -2 & 1, 1 \\
\end{array}
\]

The unique Nash equilibrium is $(A, (E, R))$. This equilibrium is subgame-perfect because $E$ is the Nash equilibrium in the subgame.
Game (iii): Extensive-form:

Outcome: 1 plays $B$, then the game ends.

Normal-form:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AE$</td>
<td>2, 3</td>
<td>0, 2</td>
</tr>
<tr>
<td>$AF$</td>
<td>4, 1</td>
<td>0, 2</td>
</tr>
<tr>
<td>$BE$</td>
<td>1, 4</td>
<td>1, 4</td>
</tr>
<tr>
<td>$BF$</td>
<td>1, 4</td>
<td>1, 4</td>
</tr>
</tbody>
</table>

Nash equilibria: $(BE, D)$ and $(BF, D)$. 

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$(BE, D)$ is not subgame-perfect, because $E$ is not Nash equilibrium in the subgame:

$$(BE, D)$$

$$(BF, D)$$ is subgame-perfect, because $F$ and $(D, F)$ are Nash equilibria in subgames:
One may argue that if $B$ is played by player 1 then player 1’s actions $E$ and $F$ will not be played. Thus, we can represent the game as

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$AE$</td>
<td>2, 3</td>
<td>0, 2</td>
</tr>
<tr>
<td>$AF$</td>
<td>4, 1</td>
<td>0, 2</td>
</tr>
<tr>
<td>$B$</td>
<td>1, 4</td>
<td>1, 4</td>
</tr>
</tbody>
</table>

But this is not correct, because “$B$” is not a strategy. A strategy should be a complete plan specifying what to play in every subgame. But “$B$” does not specify what player 1 will play in subgames.

Also, we cannot check if $(B, D)$ is subgame-perfect.
2.5 **Sequential Bargaining**

Players 1 and 2 are bargaining over one dollar. They discount payoffs received a period later by a factor $\delta$ with $0 < \delta < 1$.

**The three-period bargaining game:**

(1a) In the first period, player 1 proposes $s_1(1)$ for himself and $s_2(1)$ for player 2.

(1b) Player 2 either accepts the offer to end the game or rejects the offer to continue the game.

(2a) In the second period, player 2 proposes $s_1(2)$ for player 1 and $s_2(2)$ for himself.

(2b) Player 1 either accepts the offer to end the game or rejects the offer to continue the game.

(3) In the third period, players 1 and 2 receive shares $\bar{s}_1$ and $\bar{s}_2$ of the dollar.
Here the settlement \((\bar{s}_1, \bar{s}_2)\), with \(\bar{s}_1 \geq 0, \bar{s}_2 \geq 0\) and \(\bar{s}_1 + \bar{s}_2 \leq 1\), is given exogenously.

The present value of payoff to player \(i\) is \(\delta^{t-1}s_i(t)\) if the bargaining is ended in period \(t\).

**Backwards-induction**

In general, in period \(t\), \(s_1(t)\) and \(s_2(t)\) are offered to players 1 and 2. The offers satisfy

\[
s_1(t) \geq 0, \quad s_2(t) \geq 0, \quad s_1(t) + s_2(t) = 1.
\]

**Claim:** Suppose it is known that players 1 and 2 will receive \((u_1, u_2)\) in period \(t+1\), where \((u_1, u_2) \geq 0\) and \(u_1 + u_2 \leq 1\). Then, in period \(t\), player \(i\)’s best strategy is to offer \(s_j(t) = \delta u_j\) to player \(j\) and player \(j\) will accept the offer.

**Proof.** If player \(i\)’s offer in period \(t\) is rejected, the players receive \((u_1, u_2)\) in period \(t + 1\), which is \(\delta(u_1, u_2)\) in period \(t\). Thus, \(i\)’s offer in period \(t\) will
be accepted by $j$ if and only if $s_j(t) \geq \delta u_j$.

Now, $i$’s choices are:

$(1)s_j(t) = \delta u_j, \quad (2)s_j(t) < \delta u_j.$

If (1), then $j$ will accept the offer and $i$ will receive $1 - \delta u_j$.

If (2), then $j$ will reject the offer and $i$ will receive $\delta u_i$, which is $\leq \delta (1 - u_j) < 1 - \delta u_j$.

Since (1) is better than (2), $i$ should offer $s_j(t) = \delta u_j$ to $j$ and $j$ will accept it.

Now we can find the backward-induction outcome as follows.

In the second period, player 2 is at the move ($i = 2$ and $j = 1$). Because the payoff to player 1 in period 3 is $\bar{s}_1$, by the Claim, player 2 will offer $s_1(2) = \delta \bar{s}_1$ to player 1 and thus $s_2(2) = 1 - \delta \bar{s}_1$ to himself.
Thus, in the first period \((i = 1 \text{ and } j = 2)\), player 1 will offer \(\delta(1 - \delta \bar{s}_1)\) to player 2 and \(1 - \delta(1 - \delta \bar{s}_1)\) to himself, and player 2 will accept the offer. Then, the game ends.

The backwards-induction outcome of the three-period bargaining game:

Player 1 offers the settlement
\[
\begin{cases} 
1 - \delta(1 - \delta \bar{s}_1) & \text{to player 1} \\
\delta(1 - \delta \bar{s}_1) & \text{to player 2}
\end{cases}
\]
Player 2 will accept the offer. Then the game ends.

**Remark:** Note that the backwards-induction outcome is independent of \(\bar{s}_2\).
The infinite-horizon bargaining game:

This game is the same as the three-period bargaining game except that the exogenous settlement in step (3) is replaced by an infinite sequence of steps (3a), (3b), (4a), (4b) and so on. Bargaining continues until one player accepts an offer.

The present value of payoff to player $i$ is $\delta^{t-1}s_i(t)$ if the bargaining is settled in period $t$.

An insight: The game beginning in period 3 is identical to the game beginning in period 1, because it is an infinitely repeated game and both start from the player 1’s offer, and so on.
**Idea:** Denote by $G$ the infinite-horizon bargaining game. Let $(\bar{u}_1, \bar{u}_2)$ be the (optimal) payoffs players can receive in the backwards-induction outcome of $G$. Since $(\bar{u}_1, \bar{u}_2) = \delta^{t-1}(s_1(t), s_2(t))$ for some $t \in \{1, 2, \ldots, \infty\}$, we have $(\bar{u}_1, \bar{u}_2) \geq 0$ and $\bar{u}_1 + \bar{u}_2 \leq 1$.

The game $G$ can be played as follows.

First, player 1 proposes $s(1)$.

Then, player 2 either accepts the offer to end the game or rejects the offer and proposes $s(2)$.

In the third period, player 1 either accepts the offer to end the game or rejects the offer $s(2)$. If the offer is rejected, the player will go over to the the infinite-horizon bargaining game $G$ (beginning in period 3) and receive $(\bar{u}_1, \bar{u}_2)$.

Players can regard these payoffs as a settlement in period 3, i.e. $(\bar{s}_1, \bar{s}_2) = (\bar{u}_1, \bar{u}_2)$, then they play a
three-period game. As shown before, in this three-period game, Player 1 should offer
\[
\begin{cases}
1 - \delta(1 - \delta \bar{u}_1) & \text{to himself} \\
\delta(1 - \delta \bar{u}_1) & \text{to Player 2}
\end{cases}
\]
in period 1. Player 2 accepts the offer and the game ends.

This outcome can be regarded as the outcome of the infinite-horizon game. Thus, the optimal payoff of the infinite-horizon game received by player 1 is \(1 - \delta(1 - \delta \bar{u}_1)\). Note that \(\bar{u}_1\) is defined as the optimal payoff of the infinite-horizon game received by player 1. Thus, we have
\[
\bar{u}_1 = 1 - \delta(1 - \delta \bar{u}_1).
\]
This leads to
\[
\bar{u}_1 = \frac{1}{1 + \delta}.
\]
Because the first-period offer is accepted (and the game ends), the payoff \((\bar{u}_1, \bar{u}_2)\) of the game equals the first-period offer \((s_1(1), s_2(1))\) (no discount).
Since $s_1(1) + s_2(1) = 1$, we have

$$\bar{u}_2 = 1 - \bar{u}_1 = \frac{\delta}{1 + \delta}.$$ 

**Outcome:** In period 1, Player 1 offers $s_1^*(1) = \frac{1}{1+\delta}$ to himself and $s_2^*(1) = \frac{\delta}{1+\delta}$ to Player 2. Player 2 accepts the offer and the game ends.
Rigorous proof (Note: there may be multiple optimal payoffs \((\bar{u}_1, \bar{u}_2)\))

Let \(s_h\) be the highest payoff player 1 can receive in any backwards-induction outcome of the game as a whole.

We can also regard \(s_h\) as the third-period payoff for player 1. Then the result of the three-period bargaining game says that, using \(s_h\) as the exogenous settlement \(s\),

\[
f(s_h) = 1 - \delta + \delta^2 s_h
\]

is an optimal payoff for player 1 in period 1. Hence \(f(s_h) \leq s_h\), since \(s_h\) is also the maximum payoff in period 1.

Because any first-period optimal payoff for player 1 can be represented in the form of \(f(s)\) with some third-period optimal payoff \(s\) (as we see in the three-period bargaining game), there exists an optimal payoff \(s_3\) such that \(s_h = f(s_3)\). Because \(f(s)\) is an
increasing function of $s$ and $s_3 \leq s_h$, 

$$s_h = f(s_3) \leq f(s_h).$$

Therefore, 

$$f(s_h) = s_h,$$

i.e., 

$$1 - \delta + \delta^2 s_h = s_h.$$ 

Thus, 

$$s_h = \frac{1}{1 + \delta}.$$ 

Let $s_l$ be the lowest payoff player 1 can receive in any backwards-induction outcome of the game as a whole. Similarly, 

$$f(s_l) = s_l.$$ 

Therefore 

$$s_h = s_l = \frac{1}{1 + \delta},$$

which implies that $s^* = \frac{1}{1 + \delta}$ is the unique outcome of backwards-induction of the game.
**Backwards-induction outcome:** In the first period, player 1 offers \( s^* = \frac{1}{1 + \delta} \) to himself and \( 1 - s^* = \frac{\delta}{1 + \delta} \) to player 2. Player 2 accepts the offer, and ends the game.

**Remark:**

The game has infinitely many periods, but ends at the first period.

The player with the first move gains a higher payoff.
2.6 Infinitely repeated games

Let $\pi_t$ be the payoff in stage $t$. Given a discount factor $\delta \in (0, 1)$, the **present value** of the sequence of payoffs $\{\pi_1, \pi_2, \ldots\}$ is

$$\pi_1 + \delta \pi_2 + \delta^2 \pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1} \pi_t.$$

**Infinitely repeated game** (two-person):

1. In the first stage, the players play the stage game $G$ and receive payoffs $\pi_{1,1}$ and $\pi_{2,1}$;
2. In the $t$-th stage, the players observe the actions chosen in the preceding $(t - 1)$ stages, and then play $G$ to receive $(\pi_{1,t}, \pi_{2,t})$.
3. The payoffs of the infinitely repeated game is the present value of the sequence of payoffs:

$$\left( \sum_{t=1}^{\infty} \delta^{t-1} \pi_{1,t}, \sum_{t=1}^{\infty} \delta^{t-1} \pi_{2,t} \right).$$
Playing the stage game $G$ does not mean having to play an equilibrium of $G$. It simply means that each player chooses an action in $G$ and receives a payoff.

A strategy played by Player $i$ is in the form:

$$\{a_{i1}, a_{i2}, a_{i3}, \ldots\},$$

where $a_{it}: A_1 \times \ldots \times A_{t-1} \rightarrow A_t$.

The payoff received at stage $t$ is $\pi_{it} = u_i(a_{it}, a_{jt})$, where $u_i(a_{it}, a_{jt})$ is the payoff of the stage game $G$.

**Example:**

Let the stage game $G$ be the game of Prisoners' Dilemma:

<table>
<thead>
<tr>
<th></th>
<th>$L_2$</th>
<th>$R_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>1, 1</td>
<td>5, 0</td>
</tr>
<tr>
<td>$R_1$</td>
<td>0, 5</td>
<td>4, 4</td>
</tr>
</tbody>
</table>
Some strategies of infinitely repeated game

(1) **Non-cooperative strategy** \((NC_i)\):

Play \(L_i\) in every stage.

(2) **Trigger strategy** \((T_i)\):

- Play \(R_i\) in the first stage.
- In the \(t\)-th stage, if the outcome of all \((t - 1)\) preceding stages has been \((R_1, R_2)\), then play \(R_i\); otherwise, play \(L_i\).

Remarks on the trigger strategy:

- Player \(i\) cooperated until someone fails to cooperate, which triggers a switch to noncooperation forever after.
- If both players adopt the trigger strategy, then the outcome of the infinitely repeated game is \((R_1, R_2)\) in every stage.
Claim: $(NC_1, NC_2)$ is a Nash equilibrium.

Proof. Assume player $i$ plays $L_i$ in every stage. Then player $j$ can only be worse off if he plays $R_j$ at any stage (receiving 0 instead of 1). Thus player $j$’s best response is also “to play $L_j$ in every stage”.

\[\square\]

Claim: $(T_1, T_2)$ is a Nash equilibrium if and only if $\delta \geq 1/4$.

Proof. Assume player $i$ has adopted the trigger strategy. We seek to show player $j$’s best response is also to adopt the trigger strategy.

Case 1: The outcome in a previous stage is not $(R_1, R_2)$. In this case, player $i$ plays $L_i$ forever. Player $j$’s best response is thus to play $L_j$ forever.
Case 2: In the first stage or a stage that all the preceding outcomes have been \((R_1, R_2)\).

If player \(j\) does not play the trigger strategy, then he plays \(L_j\) in this stage. Player \(i\) still plays \(R_i\) in this stage but \(L_i\) forever from the next stage. Thus player \(j\) also has to play \(L_j\) from the next stage onwards. This means player \(j\)'s payoff from this stage onwards is

\[
5 + \delta \cdot 1 + \delta^2 \cdot 1 + \ldots = 5 + \frac{\delta}{1 - \delta}.
\]

If player \(j\) plays the trigger strategy, then he should play \(R_j\) in this stage, and the outcome from this stage onwards will be \((R_1, R_2)\) in every stage. Thus player \(j\)'s payoff from this stage onwards amounts to

\[
4 + 4\delta + 4\delta^2 + \ldots = \frac{4}{1 - \delta}.
\]
Therefore, playing the trigger strategy in this case is optimal iff

\[
\frac{4}{1 - \delta} \geq 5 + \frac{\delta}{1 - \delta}
\]

\[
\iff 4 \geq 5 - 5\delta + \delta
\]

\[
\iff \delta \geq \frac{1}{4}.
\]

Summarizing cases 1 and 2, the trigger strategies constitute a Nash equilibrium for the game iff \( \delta \geq \frac{1}{4} \).

In the above game, the lower bound for the discount factor is \( \bar{\delta} = \frac{1}{4} \). We would like to see how \( \bar{\delta} \) depends on the stage game. Let’s consider a general case

\begin{center}
\begin{tabular}{c|cc}
Player1 & \( L_2 \) & \( R_2 \) \\
\hline
\( L_1 \) & \( x, x \) & \( z, 0 \) \\
\( R_1 \) & \( 0, z \) & \( y, y \) \\
\end{tabular}
\end{center}

with \( x < y < z \).
Players receive $x$ if they do not cooperate and $y$ if they cooperate. A player receives $z$ in the period when he breaks the cooperation.

We seek to analyze how $\bar{\delta}$ depends on $x, y, z$.

Note that $x \leftrightarrow 1$, $y \leftrightarrow 4$ and $z \leftrightarrow 5$. We see that playing the trigger strategy is optimal iff

$$\frac{y}{1 - \delta} \geq z + \frac{x\delta}{1 - \delta}$$

$$\iff y \geq z - z\delta + x\delta$$

$$\iff \delta \geq \frac{z - y}{z - x}.$$

The lower bound $\bar{\delta} = \frac{z - y}{z - x}$ satisfies $0 < \bar{\delta} < 1$,

- $\bar{\delta} \to 1$ as $y \to x$. (The gain from cooperation is small, thus the trigger strategy works only for a small range of $\delta \in [\bar{\delta}, 1)$.)

- $\bar{\delta} \to 0$ as $y \to z$. (The gain from deviation is small, thus the trigger strategy works for almost any $\delta \in (0, 1)$.)
**Claim:** The trigger-strategy Nash equilibrium in the infinitely repeated Prisoners’ Dilemma is subgame perfect.

**Proof.** In an infinitely repeated game, a subgame is characterized by its previous history. The subgames can be grouped as follows:

(i) Subgames whose previous histories are always finite sequence of \((R_1, R_2)\).

(ii) Subgames whose previous histories contain other outcomes different from \((R_1, R_2)\).

For a subgame in Case (i), the players’ strategies in such a subgame are again the trigger strategy, which is a Nash equilibrium for the whole game and thus for the subgame as well.

For a subgame in Case (ii), the players’ strategies are simply to repeat \((L_1, L_2)\) all the time in the subgame, which is also a Nash equilibrium. □
Collusion between Cournot Duopolists

— In the Cournot model, both firms produce $q_c = \frac{a-c}{3}$, and their profits are $\pi_c = \frac{(a-c)^2}{9}$.

— If there is only one firm to produce the product for the market (i.e., a monopoly), then it maximizes $q(a - q - c)$, resulting in the monopoly quantity $q_m = \frac{a-c}{2}$ and profit $\pi_m = \frac{(a-c)^2}{4}$.

If there are two firms to collude to produce $\frac{q_m}{2}$ each, then they jointly produce the monopoly quantity $q_m$. Their profits are $\frac{\pi_m}{2} = \frac{(a-c)^2}{8}$.

— If firm $i$ produces $\frac{q_m}{2}$, then the best response for firm $j$ is to produce $q_d = \frac{3(a-c)}{8}$, which maximizes $q_j(a - q_j - \frac{q_m}{2} - c)$.

Firm $j$’s payoff in this case is $\pi_d = \frac{9(a-c)^2}{64}$; Firm $i$’s payoff in this case is $\frac{3(a-c)^2}{32}$.
Consider the infinitely repeated game based on the Cournot stage game when both firms have the discount factor $\delta$.

**Trigger strategy:**

1. Produce half the monopoly quantity, $\frac{q_m}{2}$, in the first period.

2. In the $t$-th period, produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in all the preceding $(t - 1)$ periods; otherwise, produce the Cournot quantity $q_c$.

**Claim:** For the infinitely repeated game with the Cournot stage game, both playing the trigger strategy is a Nash equilibrium if and only if $\delta \geq \frac{9}{17}$.

**Proof.** Suppose firm $i$ has adopted the trigger strategy.

If a quantity other than $\frac{q_m}{2}$ has been chosen by any firm before the current period, then firm $i$ chooses
$q_c$ from this period onwards. The best response for firm $j$ is also to choose $q_c$ from this period onwards, since $(q_c, q_c)$ is the unique Nash equilibrium for the stage game.

Let $t$ be the first period in which firm $j$ deviates from $\frac{q_m}{2}$. Then firm $i$ produces $\frac{q_m}{2}$ in this period but $q_c$ from period $(t+1)$ onwards. As the best response to firm $i$’s quantities, firm $j$ should produce $q_d$ in $t$-th period and $q_c$ from period $(t+1)$ onwards. Thus, firm $j$’s the present value of the payoffs from period $t$ onwards amounts to

$$\pi_d + \frac{\delta}{1 - \delta} \pi_c.$$ 

On the other hand, if firm $j$ never deviates from $\frac{q_m}{2}$, then its present value of payoffs from period $t$ onwards is

$$\frac{1}{1 - \delta} \cdot \frac{\pi_m}{2}.$$
Therefore, trigger strategy is the best response for firm $j$ to firm $i$’s trigger strategy iff

\[
\frac{1}{1 - \delta} \frac{\pi_m}{2} \geq \pi_d + \frac{\delta}{1 - \delta} \pi_c
\]

\[
\iff \frac{1}{1 - \delta} \frac{(a - c)^2}{8} \geq \frac{9(a - c)^2}{64} + \frac{\delta}{1 - \delta} \frac{(a - c)^2}{9}
\]

\[
\iff \frac{1}{8} \geq \frac{9}{64} - \frac{9}{64} \delta + \frac{\delta}{9}
\]

\[
\iff \delta \geq \frac{1}{17}.
\]
3 Static Game of Incomplete Information

Games of *incomplete information* are also called *Bayesian games*.

In a game of incomplete information, at least one player is uncertain about another player’s payoff function.

3.1 Cournot competition under asymmetric information

Consider the Cournot duopoly model, except:

Firm 1’s cost function is $c_1(q_1) = cq_1$.

Firm 2’s cost function is

$$c_2(q_2) = \begin{cases} 
  c_H q_2, & \text{with probability } \theta \\
  c_L q_2, & \text{with probability } 1 - \theta 
\end{cases}$$

where $c_L < c_H$ are low cost and high cost.
The information is asymmetric:

Firm 1’s cost function is known by both; however, firm 2’s cost function is only completely known by itself. Firm 1 knows only the marginal cost of firm 2 to be $c_H$ with probability $\theta$ and $c_L$ with probability $1 - \theta$.

The two firms simultaneously choose

$$(q_1^*, q_2^*(c_H), q_2^*(c_L)).$$

$q_2^*(c_H)$ solves

$$\max_{q_2} [a - q_1^* - q_2 - c_H]q_2$$

and $q_2^*(c_L)$ solves

$$\max_{q_2} [a - q_1^* - q_2 - c_L]q_2.$$

Firm 1 should maximize its expected profit, i.e., $q_1^*$ maximizes

$$\theta[a - q_1 - q_2^*(c_H) - c]q_1 + (1 - \theta)[a - q_1 - q_2^*(c_L) - c]q_1.$$
It is easy to obtain

\[ q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \]

\[ q_2^*(c_L) = \frac{a - q_1^* - c_L}{2} \]

\[ q_1^* = \frac{\theta [a - q_2^*(c_H) - c] + (1 - \theta) [a - q_2^*(c_L) - c]}{2} \]

Substituting \( q_2^*(c_H) \) and \( q_2^*(c_L) \) into the above, we obtain

\[ q_1^* = \frac{1}{4} [a - 2c + q_1^* + \theta c_H + (1 - \theta)c_L] \]

Thus, the equilibrium of the game is

\[ q_1^* = \frac{a - 2c + \theta c_H + (1 - \theta)c_L}{3}, \]

\[ q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1 - \theta}{6} (c_H - c_L), \]

\[ q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta}{6} (c_H - c_L). \]
In the Cournot model with complete information (see question 2 in tutorial 2), if firms 1 and 2’s costs are \((c, c_*)\) (firm 1 knows the cost \(c_*\) of firm 2), then firm 2 should produce \(\frac{a - 2c_* + c}{3}\). Hence, in the case of asymmetric information, firm 2 produces more (less) and thus has higher (lower) profit (because \(\pi^* = (q^*)^2\)) in the game of asymmetric information than in the game of complete information when the cost is high (low).

### 3.2 Static Bayesian games and Bayesian Nash equilibrium

In the above example, firm 2 has two payoff functions:

\[
\pi_2(q_1, q_2; c_L) = [(a - q_1 - q_2) - c_L]q_2,
\]

\[
\pi_2(q_1, q_2; c_H) = [(a - q_1 - q_2) - c_H]q_2.
\]
Firm 1 has only one payoff function

\[ \pi_1(q_1, q_2; c) = E_{q_2}[(a - q_1 - q_2) - c]q_1. \]

We say firm 2’s type space is \( T_2 = \{c_L, c_H\} \) and firm 1’s type space is \( T_1 = \{c\} \).

In general, let player \( i \)'s possible payoff functions be \( u_i(a_1, \ldots, a_n; t_i) \), where \( t_i \in T_i \) is called player \( i \)'s type. Let \( t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \) be the types of other players and \( T_{-i} \) the set of all the \( t_{-i} \). Player \( i \) knows his own type but only the probability distribution \( P_i(t_{-i} \mid t_i) \) on \( T_{-i} \), which is \( i \)'s belief about other players’ types, given \( i \)'s knowledge of his own \( t_i \).
Definition 3.1 The normal-form representation of an $n$-player static Bayesian game specifies the players’ action spaces $A_1, \ldots, A_n$, their type spaces $T_1, \ldots, T_n$, their beliefs $P_1, \ldots, P_n$, and their payoff functions $u_1, \ldots, u_n$. Player $i$’s type, $t_i$, is privately known by player $i$, determines player $i$’s payoff function, $u_i(a_1, \ldots, a_n; t_i)$, and is a member of the set of possible types, $T_i$. Player $i$’s belief $P_i(t_{-i} | t_i)$ describes $i$’s uncertainty about the $n - 1$ other players’ possible types, $t_{-i}$, given $i$’s own type, $t_i$. We denote this game by

$$G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; P_1, \ldots, P_n; u_1, \ldots, u_n\}.$$
The timing of a static Bayesian game:

1. Nature draws a type vector $t = (t_1, \ldots, t_n)$, $t_i \in T_i$.
2. Nature reveals $t_i$ to player $i$, but not to any other players;
3. The players simultaneously choose actions, player $i$ choosing $a_i \in A_i$;
4. Payoffs $u_i(a_1, \ldots, a_n; t_i)$ are received.

Suppose the nature draws $t = (t_1, \ldots, t_n)$ according to the prior probability distribution $P(t)$. Then the belief $P_i(t_{-i}|t_i)$ can be computed by using Bayes’ rule

$$P(t_{-i}|t_i) = \frac{P(t_{-i}, t_i)}{P(t_i)} = \frac{P(t_{-i}, t_i)}{\sum_{t'_{-i} \in T_{-i}} P(t'_{-i}, t_i)}.$$

(Usually, we assume $P(t_{-i} | t_i)$ are known, so we need not use $P(t)$ to compute $P(t_{-i} | t_i)$.)
**Definition 3.2** In the static Bayesian game \( G = \{A_1, \ldots, A_n; T_1, \ldots T_n; P_1, \ldots, P_n; u_1, \ldots, u_n\} \), a **strategy** for player \( i \) is a function \( s_i(t_i) \), i.e., \( s_i : T_i \to A_i \). For given type \( t_i \), \( s_i(t_i) \) gives the action. Player \( i \)’s strategy space \( S_i \) is the set of all functions from \( T_i \) into \( A_i \).

In the previous example, \((q_2^*(c_H), q_2^*(c_L))\) is a strategy for firm 2, while \( q_1^* \) is a strategy for firm 1.

**Definition 3.3** In the static Bayesian game \( G = \{A_1, \ldots, A_n; T_1, \ldots T_n; P_1, \ldots, P_n; u_1, \ldots, u_n\} \), the strategies \( s^* = (s_1^*, \ldots, s_n^*) \) are a (pure-strategy) **Bayesian Nash equilibrium** if for each player \( i \) and for each of \( i \)’s type \( t_i \) in \( T_i \), \( s^*_i(t_i) \) solves

\[
\max_{a_i \in A_i} E_{t_{-i}} u_i(s^*_{-i}(t_{-i}), a_i; t_i),
\]

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where

\[
E_{t-i} u_i(s_{-i}^*(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} P_i(t_{-i} | t_i) u_i(s_{-i}^*(t_{-i}), a_i; t_i).
\]

In the previous example, \( q_1^* \) solves

\[
\max_{q_1} \pi_1(q_1, q_2^*(c_H); c) \theta + \pi_1(q_1, q_2^*(c_L); c)(1 - \theta).
\]

\( q_2^*(c_H) \) solves \( \max_{q_2} \pi_2(q_1^*, q_2; c_H) \).

\( q_2^*(c_L) \) solves \( \max_{q_2} \pi_2(q_1^*, q_2; c_L) \).

Hence, they constitute a Bayesian Nash equilibrium for the game.

• In a general finite (action spaces) static Bayesian game, a Bayesian Nash equilibrium exists, perhaps in mixed strategies. (Proof is omitted.)
3.3 Providing a Public Good under Incomplete Information

There are two players, $i = 1, 2$. Players decide simultaneously whether to contribute to the public good, and contributing is a $0 - 1$ decision. Each player derives a benefit of $1$ if at least one of them provides the public good and $0$ if none does; player $i$’s cost of contributing is $c_i$. The payoffs are depicted below.

<table>
<thead>
<tr>
<th></th>
<th>Contribute</th>
<th>Don’t</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contribute</td>
<td>$1 - c_1, 1 - c_2$</td>
<td>$1 - c_1, 1$</td>
</tr>
<tr>
<td>Don’t</td>
<td>$1, 1 - c_2$</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

The benefits of the public good ($1$ each) are common knowledge. However, players only know that the costs (types) have the following distributions:
Player 1 has two types: $c_{1l} = 0.5$ and $c_{1h} = 1.2$. Player 2 has only one type $c_2 = 0.8$.

We can present this game as follows:

• Nature determines whether payoffs are as $G_1$ or $G_2$ below, each game being equally likely.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.5, 0.2</td>
<td>0.5, 1</td>
</tr>
<tr>
<td>D</td>
<td>1, 0.2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

$G_1$, $c_1 = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$-0.2$, 0.2</td>
<td>$-0.2$, 1</td>
</tr>
<tr>
<td>D</td>
<td>1, 0.2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

$G_2$, $c_1 = 1.2$

• Player 1 learns whether Nature has drawn $G_1$ or $G_2$, but player 2 does not.
Player 1 has 4 strategies: CC, CD, DC and DD. Here, CD means:

Contribute if $c_1 = 0.5$,

Don’t contribute if $c_1 = 1.2$.

Player 2 has 2 strategies: C and D.

It is convenient to represent their payoff functions in matrices.

Player 1’s payoffs:

Type1 : $c_1 = 0.5$

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Type2 : $c_1 = 1.2$

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$-0.2$</td>
<td>$-0.2$</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Player 1’s best responses:

$$R_1(C) = DD, \quad R_1(D) = CD.$$
Player 2’s expected payoffs:

\[
\begin{array}{c|cc}
   & C & D \\
\hline
CC & 0.2 & 1 \\
CD & 0.2 & 0.5 \\
DC & 0.2 & 0.5 \\
DD & 0.2 & 0 \\
\end{array}
\]

As an example, we compute player 2’s expected payoff for the combination (CD, C):

\[
u_2(CD, C) = P(c_1 = 0.5)u_2(C, C; c_1 = 0.5) + P(c_1 = 1.2)u_2(D, C; c_1 = 1.2) = 0.5(0.2) + 0.5(0.2) = 0.2.
\]

Player 2’s best responses:

\[
R_2(CC) = D, \quad R_2(CD) = D,
\]
\[
R_2(DC) = D, \quad R_2(DD) = C.
\]

\[
R_1(C) = DD \& R_2(DD) = C \Rightarrow (DD, C) \text{ is a NE};
\]
\[
R_1(D) = CD \& R_2(CD) = D \Rightarrow (CD, D) \text{ is a NE}.
\]
Now consider the game where both players’ type sets are non-singleton. It is common knowledge that the $c_1$ and $c_2$ have the following distributions:

<table>
<thead>
<tr>
<th></th>
<th>$c_1$</th>
<th>0.5</th>
<th>1.2</th>
<th>$c_2$</th>
<th>0.5</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob</td>
<td>1/2</td>
<td>1/2</td>
<td>Prob</td>
<td>1/4</td>
<td>3/4</td>
<td></td>
</tr>
</tbody>
</table>

Each player has two payoff functions:

$$u_i(a_i, a_j(c_j); c_{il})$$

$$= p_{jl}u_i(a_i, a_j(c_{jl}); c_{il}) + p_{jh}u_i(a_i, a_j(c_{jh}); c_{il}),$$

and

$$u_i(a_i, a_j(c_j); c_{ih})$$

$$= p_{jl}u_i(a_i, a_j(c_{jl}); c_{ih}) + p_{jh}u_i(a_i, a_j(c_{jh}); c_{ih}),$$
Compute $u_1(a_1, a_2(c_2); c_1 = 0.5)$:

From

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$p = 1/4$

we compute the expected payoff

<table>
<thead>
<tr>
<th>Player 1</th>
<th>C</th>
<th>D</th>
<th>CC</th>
<th>CD</th>
<th>DC</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>0.25</td>
<td>0.75</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here, e.g.,

\[
u_1(D, CD; c_1l) = p_{2l}u_1(D, C; c_1l) + p_{2h}u_1(D, D; c_1l) = (1/4) \times 1 + (3/4) \times 0 = 1/4.\]
Compute \( u_1(a_1, a_2(c_2); c_1 = 1.2) \):

From

\[
\begin{array}{c|cc}
 & C & D \\
\hline
C & -0.2 & -0.2 \\
D & 1 & 0 \\
\end{array}
\]

we compute the expected payoff

\[
\begin{array}{c|cccc}
 & CC & CD & DC & DD \\
\hline
\text{Player 1} & -0.2 & -0.2 & -0.2 & -0.2 \\
\text{C} & 1 & 0.25 & 0.75 & 0 \\
\text{D} & & & & \\
\end{array}
\]

Best responses:

\[
R_1(CC) = DD, \quad R_1(CD) = CD, \\
R_1(DC) = DD, \quad R_1(DD) = CD.
\]
Compute player 2’s payoffs $u_2(a_1(c_1), a_2; c_2)$:

For $c_2 = 0.5$,

$$\begin{array}{c|cc}
C & D \\
\hline
C & 0.5 & 1 \\
D & 0.5 & 0 \\
\end{array} \quad \Rightarrow \quad \begin{array}{c|cc}
C & D \\
\hline
CC & 0.5 & 1 \\
CD & 0.5 & 0.5 \\
DC & 0.5 & 0.5 \\
DD & 0.5 & 0 \\
\end{array}$$

$p = 1/2$

For $c_2 = 1.2$,

$$\begin{array}{c|cc}
C & D \\
\hline
C & -0.2 & 1 \\
D & -0.2 & 0 \\
\end{array} \quad \Rightarrow \quad \begin{array}{c|cc}
C & D \\
\hline
CC & -0.2 & 1 \\
CD & -0.2 & 0.5 \\
DC & -0.2 & 0.5 \\
DD & -0.2 & 0 \\
\end{array}$$

$p = 1/2$
Best responses:
\[
R_2(\text{CC}) = DD, \quad R_2(\text{CD}) = \{CD, DD\}, \\
R_2(\text{DC}) = \{CD, DD\}, \quad R_2(\text{DD}) = CD.
\]

Find the equilibria:
\[
DD = R_1(\text{CC}) \quad \& \quad CC \notin R_2(DD)
\]
thus “player 1 playing CC” is not in a BNE;
\[
DD = R_1(\text{DC}) \quad \& \quad DC \notin R_2(DD)
\]
thus “player 1 playing DC” is not in a BNE;
\[
CD \in R_1(\text{CD}) \quad \& \quad CD \in R_2(\text{CD})
\]
imply that \((\text{CD}, \text{CD})\) is a BNE;
\[
CD \in R_1(\text{DD}) \quad \& \quad DD \in R_2(\text{CD})
\]
imply that \((\text{CD}, \text{DD})\) is a BNE.
3.4 A First-Price Sealed-Bid Auction

Suppose there are two bidders, \( i = 1, 2 \).

The bidders’ valuations \( v_1 \) and \( v_2 \) for a good are independently and uniformly distributed on \([0, 1]\). Bidders submit their bids \( b_1 \) and \( b_2 \) simultaneously. The higher bidder wins the good and pays her bidding price; the other bidder gets and pays nothing. In the case that \( b_1 = b_2 \), the winner is determined by a flip of a coin.

Formulation of a static Bayesian game

\[ G = \{ A_1, A_2; T_1, T_2; P_1, P_2; u_1, u_2 \} : \]

\[ A_1 = A_2 = [0, \infty), \text{(bids } b_i \in A_i); \]

\[ T_1 = T_2 = [0, 1], \text{(valuations } v_i \in T_i); \]

\[ P_i(v_j) \text{ is the uniform distribution on } [0, 1]; \]
For any \( v_i \in T_i \), player \( i \)'s payoff is
\[
 u_i(b_1, b_2; v_i) = \begin{cases} 
 v_i - b_i & \text{if } b_i > b_j \\
 \frac{v_i - b_i}{2} & \text{if } b_i = b_j \\
 0 & \text{if } b_i < b_j
\end{cases}
\]

Player \( i \)'s strategy is a function \( b_i(v_i) \) from \([0, 1]\) into \([0, \infty)\).

\((b_1(v_1), b_2(v_2))\) is a Bayesian Nash equilibrium if for each \( v_i \in [0, 1] \), \( b_i = b_i(v_i) \) maximizes
\[
 E_{v_j} u_i(b_i, b_j(v_j); v_i) 
 = (v_i - b_i) \Pr \{ b_i > b_j(v_j) \} 
 + \frac{1}{2} (v_i - b_i) \Pr \{ b_i = b_j(v_j) \}.
\]

There may be many Bayesian Nash equilibria of the game. For simplicity we only look for equilibria in the form of \textbf{linear} functions:
\[
b_1(v_1) = a_1 + c_1 v_1, \quad b_2(v_2) = a_2 + c_2 v_2, \]
where \( a_i \geq 0 \), \( c_i > 0 \) and \( a_i < 1 \), \( i = 1, 2 \).
— The determination of Bayesian Nash equilibrium is reduced to the determination of \( a_i, c_i \) \((i = 1, 2)\).

— We confine the search for equilibria to linear functions. However, this is not a restriction on the strategy space, which is still the set of all functions \( b_i : [0, 1] \rightarrow [0, \infty) \). This means that an equilibrium must be better than all functions \( b_i : [0, 1] \rightarrow [0, \infty) \).

— Rationale of assumptions:

\begin{itemize}
  \item \( a_i \geq 0 \) reflects the fact that bids cannot be negative;
  \item \( c_i > 0 \) implies high bids for high valuation.
  \item If \( a_i \geq 1 \), then, together with \( c_i > 0 \), it follows that \( b_i(v_i) > v_i, \forall v_i \in [0, 1] \),
  \begin{align*}
    (b_i(v_i) \geq 1 > v_i, \forall v_i \in [0, 1), \text{ and } b_i(v_i) \geq 1 + c_i > v_i \text{ if } v_i = 1). \end{align*}
\end{itemize}
With such a bid, player $i$ would always end up with negative payoffs. This bid function is certainly non-optimal. Thus we assume $a_i < 1$.

Suppose player $j$ adopts a linear strategy (with $c_j > 0$). Then,

$$Pr(b_i = a_j + c_j v_j) = Pr(v_j = \frac{b_i - a_j}{c_j})$$

$$= \int_{\alpha}^{\alpha} v_j dv_j \quad (\alpha = \frac{b_i - a_j}{c_j})$$

$$= 0.$$

Thus, for any given $v_i \in [0, 1]$, player $i$’s best response $b_i(v_i)$ maximizes

$$\pi_i(b_i) := (v_i - b_i) \text{Prob}\{b_i > a_j + c_j v_j\}.$$ 

For $v_j \in [0, 1]$, the lower and upper bounds of $b_j(v_j)$ are $a_j$ and $a_j + c_j$ respectively. Since it is pointless for player $i$ to bid below player $j$’s minimum bid and foolish for $i$ to bid above $j$’s maximum
bid, we can restrict \( b_i \in [a_j, a_j + c_j] \). Under this restriction,

\[
0 \leq \frac{b_i - a_j}{c_j} \leq 1.
\]

Thus,

\[
Prob\{b_i > a_j + c_j v_j\} = Prob\{v_j < \frac{b_i - a_j}{c_j}\} = \frac{b_i - a_j}{c_j}.
\]

Therefore, player \( i \)'s best response solves

\[
\max_{a_j \leq b_i \leq a_j + c_j} (v_i - b_i) \frac{b_i - a_j}{c_j}.
\]

The unconstrained maximum is

\[
\bar{b}_i = \frac{v_i + a_j}{2}.
\]

If \( \bar{b}_i \in [a_j, a_j + c_j] \), then the best response \( b_i(v_i) = \bar{b}_i \). If \( \bar{b}_i < a_j \), then \( b_i(v_i) = a_j \). If \( \bar{b}_i > a_j + c_j \), then \( b_i(v_i) = a_j + c_j \).
Thus, player $i$’s best response is

$$b_i(v_i) = \begin{cases} 
    a_j & \text{if } v_i \leq a_j \\
    (v_i + a_j)/2 & \text{if } a_j < v_i \leq a_j + 2c_j \\
    a_j + c_j & \text{if } v_i > a_j + 2c_j
\end{cases}$$

Since we want the equilibrium strategy $b_i$ to be a \textit{linear} function on $[0, 1]$, there are only 3 cases:

$$[0, 1] \subseteq \begin{cases} 
    (-\infty, a_j] \\
    [a_j, a_j + 2c_j] \\
    [a_j + 2c_j, \infty)
\end{cases}$$

Case 1 violates the assumption $a_j < 1$. 
Case 3 violates the assumptions $a_j \geq 0$ and $c_j > 0$ which imply $a_j + 2c_j > 0$.

Therefore, $[0, 1] \subseteq [a_j, a_j + 2c_j]$, i.e., $b_i(v_i) = (v_i + a_j)/2$, for $v_i \in [0, 1]$. Hence for $i = 1, 2$, $(j = 2, 1)$,

$$a_i = a_j/2 \quad \text{and} \quad c_i = 1/2.$$ 

This yields

$$a_1 = a_2 = 0, \quad c_1 = c_2 = 1/2.$$ 

Therefore, the unique linear Bayesian Nash equilibrium is

$$b_1(v_1) = v_1/2, \quad b_2(v_2) = v_2/2.$$ 

**General result:**

If the valuations $v_1, v_2$ are uniformly distributed on $[0, 1]$, then among all strictly increasing and differentiable strategies $(b_1, b_2)$, the unique symmetric Bayesian Nash equilibrium is the linear equilibrium $b_1(v_1) = v_1/2$ and $b_2(v_2) = v_2/2$. 
Appendix

If we have somehow guessed that \((b_1(v_1), b_2(v_2)) = (v_1/2, v_2/2)\) should be a Bayesian Nash equilibrium and we wish only to prove it, then it can be done easily.

Suppose that player \(j\) has adopted \(b_j(v_j) = v_j/2\). Player \(i\)’s best response \(b_i(v_i)\) should solve

\[
\max_{b_i \in [0,1/2]} (v_i - b_i) \text{Prob}\{b_i > v_j/2\} = \max_{b_i \in [0,1/2]} 2(v_i - b_i)b_i.
\]

For any \(v_i \in [0,1]\), the unique maximizer is \(b_i(v_i) = v_i/2\). Thus \((b_1(v_1), b_2(v_2)) = (v_1/2, v_2/2)\) is indeed a Bayesian Nash equilibrium.
3.5 A Second-Price Sealed-Bid Auction

There are $n$ potential buyers, or bidders, with valuations $v_1, \ldots, v_n$ for an object. Bidders know their own valuation but do not know the other bidders’ valuations. The bidders simultaneously submit bids $s_i \in [0, \infty)$. The highest bidder wins the object and pays the second highest bid, and the other bidders pay nothing. If more than one bidders bid the highest price, the object is allocated randomly among them. Let $r_i = \max_{j \neq i} s_j$. The bidder $i$’s payoff function is $v_i - r_i$ if $i$ is the winner (i.e., $s_i > r_i$) and 0 if otherwise. (In the case there are several highest bidders, $r_i = s_i$, thus the winner’s payoff is $(v_i - r_i)/k$, where $k$ is the number of bids that equal $s_i$).
For each player $i$, the strategy of bidding his valuation ($s_i = v_i$) weakly dominates all other strategies as shown below.

We compare bidder $i$’s payoffs for different $s_i$:

$$s_i^* = v_i \quad s_i > v_i$$

$$\begin{align*}
0 & \quad \text{if } r_i > s_i \\
v_i - r_i & \quad \text{if } r_i \leq v_i \\
0 & \quad \text{if } v_i < r_i < s_i \\
0 & \quad \text{if } r_i = s_i
\end{align*}$$

$s_i^* = v_i$ dominates $s_i > v_i$.

$$s_i^* = v_i \quad s_i < v_i$$

$$\begin{align*}
v_i - r_i & \quad \text{if } r_i \leq s_i \\
v_i - r_i & \quad \text{if } r_i = s_i \\
0 & \quad \text{if } r_i > v_i \\
v_i - r_i & \quad \text{if } v_i > r_i > s_i
\end{align*}$$

$s_i^* = v_i$ dominates $s_i < v_i$.

The above shows that the strategy $s_i^* = v_i$ weakly dominates any other strategy $s_i \geq 0$. 
By the following Proposition, \((s_1^* = v_1, \ldots, s_n^* = v_n)\) is a Bayesian Nash equilibrium.

**Proposition 3.1** Let \((s_1^*, \ldots, s_n^*)\) be strategies in a static Bayesian game. If for any \(t_i \in T_i, a_i \in A_i\) and \(a_{-i} \in A_{-i},\)

\[
u_i(s_i^*(t_i), a_{-i}; t_i) \geq u_i(a_i, a_{-i}; t_i),
\]

(i.e., \(s_i^*(t_i)\) weakly dominates every \(a_i \in A_i\)), then \((s_1^*, \ldots, s_n^*)\) is a Bayesian Nash equilibrium.

**Proof.** Because \(s_{-i}^*(t_{-i}) \in A_{-i}\), the weak dominance implies

\[
\sum_{t_{-i} \in T_{-i}} \nu_i(s_i^*(t_i), s_{-i}^*(t_{-i}); t_i) Pr(t_{-i}|t_i) \\
\geq \sum_{t_{-i} \in T_{-i}} u_i(a_i, s_{-i}^*(t_{-i}); t_i) Pr(t_{-i}|t_i),
\]

i.e., \(s_i^*(t_i)\) solves

\[
\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(a_i, s_{-i}^*(t_{-i}); t_i) Pr(t_{-i}|t_i).
\]

Thus, \((s_1^*, \ldots, s_n^*)\) is a Bayesian Nash equilibrium.

\(\square\)
3.6 A Double Auction

There are two players: a buyer and a seller.

The buyer’s valuation for the seller’s good is $v_b$, the seller’s is $v_s$. The valuations are private information and are drawn from certain independent distributions on $[0, 1]$.

The seller names an asking price, $p_s$, and the buyer simultaneously names an offer price, $p_b$. If $p_b \geq p_s$, then trade occurs at price $p = (p_b + p_s)/2$; if $p_b < p_s$, then no trade occurs.

Buyer’s payoff is

$$\pi_b(p_b) = \begin{cases} v_b - (p_b + p_s)/2 & \text{if } p_b \geq p_s \\ 0 & \text{if } p_b < p_s \end{cases}$$

Seller’s payoff is

$$\pi_s(p_s) = \begin{cases} (p_b + p_s)/2 - v_s & \text{if } p_b \geq p_s \\ 0 & \text{if } p_b < p_s \end{cases}$$
There are many, many Bayesian Nash equilibria of this game. We will consider two types.

**One-price equilibria**

For any value $x$ in $[0, 1]$, which is given exogenously and is known by both players, the **one-price strategies** are as follows:

- The buyer offers $x$ if $v_b \geq x$ and zero otherwise;
- The seller demands $x$ if $v_s \leq x$ and one otherwise.

Suppose the buyer has adopted the one-price strategy at $x$. The seller’s choices amount to trading at $x$ or not trading. That is, the seller should demand $x$ if $v_s \leq x$, and one (not trading) otherwise. (Why?) Thus, the seller’s best response is the one-price strategy. Similarly, if the seller has adopted the one-price strategy, then the buyer’s best response is also the one-price strategy. Therefore, both playing one-price strategies is a Bayesian Nash equilibrium.
In this equilibrium, trade occurs for the \((v_s, v_b)\) pairs with \(v_s \leq x \leq v_b\).

Trade would be efficient (i.e., both players can benefit from a trade at certain prices) for all \((v_s, v_b)\) pairs such that \(v_s \leq v_b\), but does not occur in the two shaded regions of the figure.
Linear equilibria

Now we look for linear equilibrium strategies

\[ p_i(v_i) = a_i + c_i v_i, \quad i = s, b, \]

with \( a_i \geq 0 \) and \( c_i > 0 \).

The equilibrium is determined by maximizing respective expected payoffs. We assume that \( v_s \) and \( v_b \) are uniformly distributed on \([0, 1]\) and independent.

Given seller’s linear strategy \( p_s(v_s) \), the buyer’s expected payoff is

\[
E_{v_s} \pi_b(p_b, p_s(v_s) \mid v_b) \\
= \int_{a_s \leq p_s(v_s) \leq p_b} \left( v_b - \frac{p_b + p_s(v_s)}{2} \right) dv_s \\
+ \int_{p_b < p_s(v_s) \leq a_s + c_s} 0 dv_s \\
= \int_{a_s \leq u \leq p_b} \left( v_b - \frac{p_b + u}{2} \right) \frac{du}{c_s} \\
= \frac{1}{c_s} \left( v_b u - \frac{p_b u + u^2}{2} \right) |_{u = a_s}^{u = p_b}
\]
\[
= \frac{1}{c_s} \left( v_b p_b - \frac{p_b^2 + p_s^2}{2} \right) - \frac{1}{c_s} \left( v_b a_s - \frac{p_b a_s + a_s^2}{2} \right)
= \frac{p_b - a_s}{c_s} \left( v_b - \frac{3}{4} p_b - \frac{1}{4} a_s \right).
\]

Maximizing \( E_{v_b} \pi_b(p_b, p_s(v_s) \mid v_b) \) yields the buyer’s best response

\[ p_b = \frac{2}{3} v_b + \frac{1}{3} a_s. \tag{3.1} \]

Analogously, given buyer’s linear strategy \( p_b(v_b) \), the seller’s expected payoff is

\[
E_{v_b} \pi_s(p_s, p_b(v_b) \mid v_s) = \frac{a_b + c_b - p_s}{c_b} \left( \frac{3}{4} p_s + \frac{1}{4} (a_b + c_b) - v_s \right).
\]

Maximizing \( E_{v_b} \pi_s(p_s, p_b(v_b) \mid v_s) \) yields the seller’s best response

\[ p_s = \frac{2}{3} v_s + \frac{1}{3} (a_b + c_b). \tag{3.2} \]

The equation (3.1) implies that

\[ c_b = 2/3, \quad a_b = a_s/3. \]

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and (3.2) implies that
\[ c_s = 2/3, \quad a_s = (a_b + c_b)/3. \]
Therefore, the linear equilibrium strategies are
\[ p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12}, \quad p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}. \]
The trade occurs iff \( p_b \geq p_s \), i.e., iff
\[ v_b \geq v_s + \frac{1}{4}. \]
Comparing the two types of equilibria.

In both cases, the most valuable possible trade (namely, $v_s = 0$ and $v_b = 1$) does occur.

But the one-price equilibrium misses some valuable trades (such as $v_s = 0$ and $v_b = x - \epsilon$, where $\epsilon > 0$ is small) and achieves some trades that are worth next to nothing (such as $v_s = x - \epsilon$ and $v_b = x + \epsilon$).

The linear equilibrium, in contrast, misses all trades worth next to nothing but achieves all trades worth at least $1/4$.

This suggests that the linear equilibrium may dominates the one-price equilibria, in terms of the expected gains that players receive.
A general result:

Myerson and Satterthwaite (1983) show that, for the uniform valuation distributions, the linear equilibrium yields higher expected gains for the players than any other Bayesian Nash equilibria of the double auction (including but far from limited to the one-price equilibria).

This also implies that there is no Bayesian Nash equilibrium of the double auction in which trade occurs if and only if it is efficient (i.e., iff $v_b \geq v_s$).
3.7 Mixed Strategies Revisited

A mixed-strategy Nash equilibrium in a game of complete information can almost always be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information.

Example: The Battle of the sexes game

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Peter</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

has two pure-strategy Nash equilibria (Opera, Opera), (Football, Football), and a mixed-strategy Nash equilibrium in which Mary plays Opera with probability 2/3 and Peter plays Football with probability 2/3.
Suppose Mary and Peter are not completely sure each other’s payoff: if both attend Opera, Mary’s payoff is $2 + t_m$; if both attend Football, Peter’s payoff is $2 + t_p$, where $t_m$ is privately known by Mary and $t_p$ is privately known by Peter, and $t_m$ and $t_p$ are independently drawn from a uniform distribution on $[0, x]$.

This can be expressed as a static Bayesian game $G = \{A_m, A_p; T_m, T_p; P_m, P_p; u_m, u_p\}$, where

$A_m = A_p = \{\text{Opera, Football}\}$;

$T_m = T_p = [0, x]$;

$P_m(t_p) = P_p(t_m) = \frac{1}{x}$ are density functions;

The payoffs are

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mary</td>
<td>$2 + t_m, 1$</td>
<td>$0, 0$</td>
</tr>
<tr>
<td>Football</td>
<td>$0, 0$</td>
<td>$1, 2 + t_p$</td>
</tr>
</tbody>
</table>
In general, Mary’s and Peter’s strategies are 
\( s_m : [0, x] \rightarrow \{O, F\} \) and \( s_p : [0, x] \rightarrow \{O, F\} \).
The strategy \( s_m \) means that some types of Mary play Opera and the other types play Football.

For strategies \( s_m \) and \( s_p \), we denote 
\[
\delta_m = \text{Prob}(\text{Mary plays Opera}) = \text{Prob}(t_m \in [0, x] : s_m(t_m) = O)
\]
\[
\delta_p = \text{Prob}(\text{Peter plays Football}) = \text{Prob}(t_p \in [0, x] : s_p(t_p) = F).
\]

Then 
\[
\text{Prob}(\text{Mary plays Football}) = 1 - \delta_m \\
\text{Prob}(\text{Peter plays Opera}) = 1 - \delta_p.
\]

Given Peter’s strategy \( s_p \), Mary’s expected payoff, if she plays Opera, is 
\[
(2 + t_m) \cdot Pr(\text{Peter chooses } O) + 0 \cdot Pr(F) \\
= (2 + t_m)(1 - \delta_p) + 0\delta_p,
\]
and, if she plays Football, is

\[ 0(1 - \delta_p) + 1\delta_p. \]

Thus Mary playing Opera is optimal iff

\[ (2 + t_m)(1 - \delta_p) \geq \delta_p \]

\[ \iff t_m \geq \frac{\delta_p}{1 - \delta_p} - 2. \]

Let \( m = \frac{\delta_p}{1 - \delta_p} - 2. \) Then Mary’s best response to \( s_p \) is

\[ s_m(t_m) = \begin{cases} 
\text{Opera,} & \text{if } t_m \geq m \\
\text{Football,} & \text{if } t_m < m
\end{cases} \quad (3.3) \]

Similarly, given Mary’s strategy \( s_m \), Peter’s expected payoff:

for Football: \( 0\delta_m + (2 + t_p)(1 - \delta_m); \)

for Opera: \( 1\delta_m + 0(1 - \delta_m). \)

Thus playing Football is optimal iff

\[ (2 + t_p)(1 - \delta_m) \geq \delta_m \]
Let $p = \frac{\delta_m}{1 - \delta_m} - 2$. Then, Peter’s best response to 
$s_m$ is

$$s_p(t_p) = \begin{cases} 
\text{Football,} & \text{if } t_p \geq p \\
\text{Opera,} & \text{if } t_p < p 
\end{cases} \quad (3.4)$$

At the best responses, all $t_m \geq m$ play Opera, thus

$$\delta_m = \frac{x - m}{x}. \quad \text{It follows}$$

$$p = \frac{\delta_m}{1 - \delta_m} - 2 = \frac{x}{m} - 3. \quad (3.5)$$

Similarly, $\delta_p = \frac{x - p}{x}$. Thus

$$m = \frac{\delta_p}{1 - \delta_p} - 2 = \frac{x}{p} - 3. \quad (3.6)$$

(3.3) and (3.4) show that a best response strategy is uniquely defined by a number, e.g. $s_p$ is defined by $p$. Thus we can simply refer to a value of $p$ as a strategy $s_p$.

Given Mary’s strategy $m$, (3.5) shows that Peter’s best response strategy is $p = \frac{x}{m} - 3$. Similarly, (3.6)
shows Mary’s best response \( m \) to Peter’s strategy \( p \).

Hence, \((s^*_m, s^*_p)\) or \((m^*, p^*)\) is a Bayesian Nash equilibrium iff \((m^*, p^*)\) is a solution of \((3.5)\) and \((3.6)\):

\[
p^* = m^* = \frac{\sqrt{9 + 4x} - 3}{2}.
\]

As \(x \to 0\), the Baysian game converges to the original game of complete information. Hence, we ask

**Question:** Will the probabilities \((\delta^*_m, \delta^*_p)\) at the BNE converge to the mixed-strategy equilibrium \((\frac{2}{3}, \frac{2}{3})\) as \(x \to 0\)?

The probability for Mary to play Opera and for Peter to play Football is, as \(x \to 0\),
\[ \delta_m^* = \delta_p^* = \frac{x - p^*}{x} = 1 - \frac{\sqrt{9 + 4x} - 3}{2x} \rightarrow \frac{2}{3}. \]

\[ \left( \frac{\sqrt{9+4x}-3}{2x} = \frac{(9+4x)-9}{2x(\sqrt{9+4x}+3)} = \frac{2}{\sqrt{9+4x}+3} \rightarrow \frac{1}{3}. \right) \]

This shows that mixed-strategies of a game of complete information can be approximated by pure-strategies of a properly designed game of incomplete information.
4 Dynamic Games of Incomplete Information

4.1 Introduction to Perfect Bayesian Equilibrium

Consider the following game

Player 1’s strategies: \( L, M, R \);
Player 2’s strategies: \( L', R' \).
The normal-form representation

<table>
<thead>
<tr>
<th></th>
<th>$L'$</th>
<th>$R'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>$M$</td>
<td>0,2</td>
<td>0,1</td>
</tr>
<tr>
<td>$R$</td>
<td>1,3</td>
<td>1,3</td>
</tr>
</tbody>
</table>

There are two Nash equilibria:

$$(L, L'), \quad (R, R').$$

Note that the above game has no subgames. Thus both $(L, L')$ and $(R, R')$ are subgame-perfect Nash equilibria.

However, $(R, R')$ is based on a noncredible threat: If player 1 believes player 2’s threat of playing $R'$, then player 1 indeed should choose $R$ to end the game with payoff 1 for himself and 3 for player 2, since choosing $L$ or $M$ will give him 0.

On the other hand, if player 1 doesn’t believe the threat by playing $L$ or $M$, then player 2 gets the
move and chooses $L'$. Since $L'$ strictly dominates $R'$ for player 2. The threat of playing $R'$ from player 2 is indeed noncredible.

We need strengthen the equilibrium concept to rule out some subgame-perfect Nash equilibria like $(R, R')$.

**Requirement 1:** At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game. A belief is a probability distribution over the nodes in the information set.

**Requirement 2:** Given their beliefs, the players’ strategies must be sequentially rational. That is, at each information set, the action taken by the player with the move (and the player’s subsequent strategy) must be optimal, given the player’s belief at that information set and the other players’ subsequent strategies, (where a “subsequent strategy”
is a complete plan of action covering every contingency that might arise after the given information set has been reached).

By imposing Requirement 1 on the previous game,

we assume player 2 to believe that $L$ has been played by player 1 with probability $p$.

Given this belief, we can compute player 2’s expected payoff:

$$ \begin{cases} 
  p \cdot 1 + (1 - p) \cdot 2 = 2 - p, & \text{if playing } L', \\
  p \cdot 0 + (1 - p) \cdot 1 = 1 - p, & \text{if playing } R'. 
\end{cases} $$
$R'$ is not optimal at the information set with any belief. Thus, $(R, R')$ cannot satisfy Requirement 2. This shows that Requirements 1 and 2 can already be used to eliminate the equilibrium $(R, R')$ based on noncredible threat.

Allowing arbitrary beliefs will admit undesirable equilibria. Thus, further requirements on players’ beliefs need be introduced.

**Definition 4.1** For a given equilibrium in a given extensive-form game, an information set is **on the equilibrium path** if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is **off the equilibrium path** if it is definitely not to be reached if the game is played according to the equilibrium strategies.
**Requirement 3:** At information sets on the equilibrium path, beliefs are determined by Bayes’ rule and the players’ equilibrium strategies.

In the previous example, suppose the equilibrium \((L, L')\) is played, then player 2’s belief must be \(p = 1\): given player 1’s equilibrium strategy \(L\), player 2 knows which node in the information set has been reached.

Consider a hypothetical situation for the previous game: the game has a mixed-strategy equilibrium in which player 1 plays \(L\) with probability \(q_1\), \(M\) with probability \(q_2\), and \(R\) with probability \(1 - q_1 - q_2\). If 1 played \(L\) or \(M\), and 2 observed it, then Requirement 3 would force player 2’s belief to be
\[ p = \text{Prob}(L \text{ is played} \mid L \text{ or } M \text{ is played}) \]
\[ = \frac{\text{Prob}(\text{‘L is played’} \text{ and } \text{‘L or M is played’})}{\text{Prob}(L \text{ or } M \text{ is played})} \]
\[ = \frac{\text{Prob}(L \text{ is played})}{\text{Prob}(L \text{ or } M \text{ is played})} \]
\[ = \frac{q_1}{q_1 + q_2}. \]

**Requirement 4:** At information sets off the equilibrium path, beliefs are determined by Bayes’ rule and the players’ equilibrium strategies where possible.

**Definition 4.2.** A perfect Bayesian equilibrium consists of strategies and beliefs satisfying Requirements 1 through 4.
Consider the following game:

The game has a unique subgame: it begins at player 2’s only decision node.

We can represent this subgame in the normal form:

$$
\begin{array}{c|cc}
\text{Player 3} & L' & R' \\
\hline
\text{Player 2} & 2,1 & 3,3 \\
& 1,2 & 1,1 \\
\end{array}
$$
The subgame has a unique Nash equilibrium: \((L, R')\).
Since the strategy A of player 1 is worse than her strategy D, given the strategies \((L, R')\) of players 2 and 3, \((D, L, R')\) is thus the unique subgame-perfect Nash equilibrium of the entire game.

It is easy to see that \((D, L, R')\) together with the belief \(p = 1\) satisfy Requirements 1–3. Since there is no information set off this equilibrium path, Requirement 4 is trivially satisfied. Thus, \((D, L, R')\) and \(p = 1\) constitute a perfect Bayesian equilibrium for the game.

Next, consider the strategies \((A, L, L')\). Since \((D, L, L')\) gives payoff 1 to player 1 which is less than the payoff for player 1 when \((A, L, L')\) is played, and since players 2 and 3 get the same payoff whatever they play when player 1 plays A, \((A, L, L')\) is a Nash equilibrium.
When $p = 0$, $L'$ is indeed better than $R'$ for player 3. Player 2 acts optimally to player 3’s subsequent strategy $L'$ by playing $L$. Player 1 also acts optimally to players 2 and 3’s subsequent strategies $(L, L')$ by playing $A$. Thus $(A, L, L')$ and $p = 0$ satisfy Requirements 1–3. Note that the information set of player 3 is not on the equilibrium path (player 1 chooses $A$ to end the game).

But $(A, L, L')$ are not a subgame-perfect Nash equilibrium (since $(L, L')$ are not a Nash equilibrium of the subgame).

This means that Requirements 1–3 are not enough to rule out non-subgame perfect Nash equilibriua!

However, Requirement 4 can rule out $(A, L, L')$ and $p = 0$ as a perfect Bayesian Nash equilibrium: Bayes’ rule results in $p = 1$ when player 2 plays strategy $L$. 
On the other hand, \((A, L, L')\) and \(p = 1\) do not satisfy Requirement 2, since for \(p = 1\), player 3’s optimal strategy is \(R'\) (not \(L'\)).

Another example illustrating Requirement 4.

If player 1’s equilibrium strategy is \(A\), then player 3’s information set is off the equilibrium path. If player 2’s strategy is \(A'\), then Requirement 4 cannot determine player 3’s belief; but if 2’s strategy is
to play $L$ with probability $q_1$, $R$ with $q_2$, $A'$ with $1 - q_1 - q_2$, $q_1 + q_2 > 0$, then Requirement 4 implies $p = \frac{q_1}{q_1 + q_2}$.

We summarize the procedure to determine whether a given equilibrium is a perfect Bayesian equilibrium.

(i) Determine a belief for each information set by Bayes’ rule.

(ii) Check whether the equilibrium is optimal given each belief determined in (i) and the subsequent strategies.
4.2 Signaling Games

Suppose a signaler can be either starving or just hungry, and she can signal that fact to another individual which has food. Suppose that she would like more food regardless of her state, but that the individual with food only wants to give her the food if she is starving. While both players have identical interests when the signaler is starving, they have opposing interests when she is only hungry. When the signaler is hungry, she has an incentive to lie about her need in order to obtain the food. If the signaler regularly lies, then the receiver should ignore the signal and do whatever he thinks best.
A signaling game is a dynamic game of incomplete information involving two players: a Sender (S) and a Receiver (R). The timing of the game is as follows.

1. Nature draws a type $t_i$ for the Sender from a set of feasible types $T = \{t_1, \ldots, t_I\}$ according to a probability distribution $P(t_i)$, where $P(t_i) > 0$ for every $i$ and $P(t_1) + \ldots + P(t_I) = 1$.

2. The Sender observes $t_i$ and then chooses a message $m_j$ from a set of feasible messages $M = \{m_1, \ldots, m_J\}$.

3. The Receiver observes $m_j$ (but not $t_i$) and then chooses an action $a_k$ from a set of feasible actions $A = \{a_1, \ldots, a_K\}$.

4. Payoffs are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$. 
Consider a simple signaling game: \( T = \{t_1, t_2\} \), \( A = \{a_1, a_2\} \), \( P(t_1) = p \), \( M = \{m_1, m_2\} \).

The Sender has 4 strategies:

\[(m_1, m_1), (m_1, m_2), (m_2, m_1), (m_2, m_2),\]

where \((m_2, m_1)\) means that the Sender plays \(m_2\) if Nature draws \(t_1\) and plays \(m_1\) if Nature draws \(t_2\).

The Receiver has 4 strategies:

\[(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2),\]

where \((a_1, a_2)\) means that: play \(a_1\) if the Sender
chooses $m_1$ and play $a_2$ if the Sender chooses $m_2$.

We call the Sender’s strategies $(m_1, m_1), (m_2, m_2)$ pooling because each type sends the same message, and $(m_1, m_2), (m_2, m_1)$ separating because each type sends a different message.

Next, we translate the requirements for a perfect Bayesian equilibrium to the case of signaling games.

**Signaling Requirement 1:** After observing any message $m_j$ from $M$, the Receiver must have a belief about which types could have sent $m_j$. Denote this belief by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j) \geq 0$ for each $t_i \in T$, and

$$\sum_{t_i \in T} \mu(t_i|m_j) = 1.$$  

**Signaling Requirement 2R:** For each $m_j \in M$, the Receiver’s action $a^*(m_j)$ must maximize the Receiver’s expected utility, given the belief $\mu(t_i|m_j)$ about which types could have sent $m_j$.  

That is, $a^*(m_j)$ solves

$$
\max_{a_k \in A} \sum_{t_i \in T} \mu(t_i|m_j)U_R(t_i, m_j, a_k).
$$

**Signaling Requirement 2S:** For each $t_i \in T$, the Sender’s message $m^*(t_i)$ must maximize the Sender’s utility, given the Receiver’s strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves

$$
\max_{m_j \in M} U_S(t_i, m_j, a^*(m_j)).
$$

(These two requirements require both the Receiver and Sender act in an optimal way.)

Now, given the Sender’s optimal strategy $m^*(t_i)$, i.e., $m^*$ is a function from $T$ into $M$, let $T_j = \{t_i \in T : m^*(t_i) = m_j\}$. $T_j$ is the set of all types sending the message $m_j$.

The information set corresponding to $m_j$ is on the equilibrium path iff $T_j \neq \emptyset$. 
Signaling Requirement 3: For each $m_j \in M$, if there exists $t_i \in T$ such that $m^*(t_i) = m_j$, i.e., $T_j \neq \emptyset$, then the Receiver’s belief at the information set corresponding to $m_j$ must follow from Bayes’ rule and the Sender’s strategy:

$$\mu(t_i | m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}, \quad \forall t_i \in T_j. \quad (*)$$

Here $t_i$ stands for “type $t_i$ is chosen by the Nature”, and $m_j$ stands for “message $m_j$ is sent (by strategy $m$)”.

Explanation of $(*)$:

Type $\tilde{t}$ is a random variable. $\tilde{t}$ has a $prior$ probability distribution $P(t) := P(\tilde{t} = t)$ for $t \in T$, according to which the Nature draws type $t \in T$.

Give a strategy $m : T \rightarrow M$, the belief, that $t_i$ is chosen by the Nature when $m_j$ is received, is a conditional probability, or $posterior$ probability:

$$\mu(t_i \mid m_j) = P(\tilde{t} = t_i \mid m(\tilde{t}) = m_j).$$
By Bayes’ rule, we have

\[
\mu(t_i|m_j) = \frac{P(t_i \& m_j)}{P(m_j)} = \frac{P(m_j|t_i)P(t_i)}{\sum_{t \in T_j} P(m_j|t)P(t) + \sum_{t' \notin T_j} P(m_j|t')P(t')}.
\]

\[
P(m_j|t_i) = P(m(\tilde{t}) = m_j|\tilde{t} = t_i)
= P(m(t_i) = m_j)
= \begin{cases} 1 & \text{if } m(t_i) = m_j \\ 0 & \text{if } m(t_i) \neq m_j \end{cases}.
\]

Therefore, if \( t_i \in T_j \), i.e. \( m(t_i) = m_j \), then

\[
\mu(t_i|m_j) = \frac{P(t_i)}{\sum_{t \in T_j} P(t)}.
\]

If \( m(t_i) \neq m_j \), then \( \mu(t_i|m_j) = 0 \). \qed
Remark: Requirement 4 is not applicable to signaling games.

Definition 4.3. A pure-strategy perfect Bayesian equilibrium in a signaling game is a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S), and (3).

Note that a strategy for the Sender is a function from the type space $T$ into the message space $M$; a strategy for the Receiver is a function from the message space $M$ into the action space $A$.

For a perfect Bayesian equilibrium of a signaling game, if the Sender’s strategy is pooling or separating, then we call the equilibrium pooling or separating, respectively.
**Example 1:** Find all the pure-strategy perfect Bayesian equilibria in the following signaling game.

where numbers, $x, y$, are payoffs to the Sender and the Receiver respectively.

In this game,

$$T = \{t_1, t_2\}, \quad M = \{L, R\}, \quad A = \{u, d\}.$$
The Sender’s strategies are: (L,L), (L,R), (R,L) and (R,R), where \((m', m'')\) means that type \(t_1\) chooses \(m'\) and type \(t_2\) chooses \(m''\).

The Receiver’s strategies are: (u,u), (u,d), (d,u), and (d,d), where \((a', a'')\) means that the Receiver plays \(a'\) following L and \(a''\) following R.

We analyze the possibility of the four Sender’s strategies to be perfect Bayesian equilibria.

**Case 1. Pooling on L:**

Suppose the Sender adopts the strategy (L,L).

By Requirement 3, \(p = 1 - p = 0.5\). Given this belief of the Receiver, the Receiver’s expected payoff is \(\frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 3\) for u and \(\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 1\) for d.

Thus, the Receiver’s best response for message L is \(a^*(L) = u\).

For the message R, the Receiver’s belief \(q\) cannot
be determined by Sender’s strategy, thus we can choose any belief \( q \). Furthermore, both \( a^*(R) = u \) and \( a^*(R) = d \) are possible for some \( q \) respectively. Hence we need only to see if sending L is better than sending R for both types \( t_1 \) and \( t_2 \) and for one of \( u \) and \( d \).

If \( a^*(R) = u \), i.e., \((u, u)\) is Receiver’s strategy, then for type \( t_2 \), Sender’s payoff is 3 if L is sent and 4 if R is sent. Hence sending L is not optimal.

If \( a^*(R) = d \), i.e., \((u, d)\) is Receiver’s strategy, then for type \( t_1 \), Sender’s payoff is 2 if L and 3 if R, hence L is not optimal.

Therefore, there is no equilibrium in which Sender plays \((L, L)\).
Case 2. Pooling on R:

Suppose the Sender adopts the strategy (R,R). Then, 
$q = 1 - q = 0.5$. Given this belief, the Receiver’s best response is $a^*(R) = d$ since $1.5 < 2$.

For both $a^*(L) = u$ and $= d$, sending R by type $t_2$ is not optimal ($3 > 2$ and $4 > 2$).

Therefore, there is no equilibrium in which Sender plays (R,R).

Case 3. Separation with $t_1$ playing L

Suppose the Sender adopts the separating strategy (L,R). Then, Signaling Requirement 3 implies $p = 1, q = 0$. For these beliefs, $a^*(L) = d, a^*(R) = d$.

For the type $t_2$, sending L will give the Sender pay-off 4, sending R will give the Sender payoff 2, hence R is not optimal in this case. Therefore, there is no equilibrium in which the Sender plays (L,R).
Case 4. Separation with $t_1$ playing R

Suppose the Sender adopts the separating strategy (R,L). Then, Signaling Requirement 3 implies $p = 0, q = 1$. For these beliefs, $a^*(L) = u, a^*(R) = \{u, d\}$.

If $a^*(R) = u$, then for type $t_1$ sending R is not optimal ($2 > 0$).

Now, let $a^*(R) = d$. For the Sender type $t_1$, sending L will give the Sender payoff 2, sending R will give the Sender payoff 3, and thus R is indeed optimal in this case.

Next, for the Sender type $t_2$, sending L will give the Sender payoff 3, sending R will give the Sender payoff 2, and hence L is the optimal message.

Therefore, $[(R,L), (u,d), p=0, q=1]$ is a separating perfect Bayesian equilibrium.
Summary:

Starting from a proposed strategy $s$ for the Sender, we use Bayes’ rule to determine the Receiver’s beliefs. Given the Receiver’s beliefs, determine his best response to the Sender’s strategy $s$. For each strategy $r$ in the receiver’s best response to $s$, check if $s$ is a best response to $r$. If yes, then $(s, r)$, together with the belief, is a perfect Bayesian equilibrium. If $s$ is not a best response to any $r$, then $s$ cannot be a strategy in a PBE.

Repeat the above until all strategies for the Sender are checked.
An alternative way to find perfect Bayesian equilibria.

We first find Nash equilibria from the normal-form representation, then check which equilibria are perfect Bayesian equilibria.

Normal-form representation:

\[
\begin{array}{c|cccc}
  & uu & ud & du & dd \\
  \hline
  LL & 2.5, 3.5 & 2.5, 3.5 & 2.5, 3 & 2.5, 3 \\
  LR & 3, 2.5 & 2, 3 & 2.5, 3 & 1.5, 3.5 \\
  RL & 1.5, 2.5 & 3, 2.5 & 2, 1.5 & 3.5, 1.5 \\
  RR & 2, 1.5 & 2.5, 2 & 2, 1.5 & 2.5, 2 \\
\end{array}
\]

where, e.g.,

\[
U(RL, ud) = Pr(t_1)U(R, d; t_1) + Pr(t_2)U(L, u; t_2) \\
= 0.5(3, 2) + 0.5(3, 3) \\
= (3, 2.5)
\]

Nash equilibrium: (RL, ud).

To check whether they are perfect Bayesian equi-
libria, we need only to find beliefs, satisfying Requirement 3, such that equilibrium strategies are optimal at every information set.

For RL, Bayes’ rule requires $p = 0$ and $q = 1$.

For this belief, $a^*(L) = u$ and $a^*(R) = \{u, d\}$. So, $ud$ is an optimal strategy.

Therefore, $[(RL, ud), p = 0, q = 1]$ is a perfect Bayesian equilibrium.
The following example shows that pure-strategy perfect Bayesian equilibria need not exist.

**Example 2:**

Suppose \((L,L)\) is adopted.

If \(a^*(L) = d, \) \(L\) is not optimal for \(t_2.\)

Then \(p = 1/2.\) It follows \(a^*(L) = u.\)

If \(a^*(R) = d, \) \(L\) is not optimal for \(t_2.\)

Therefore, no equilibrium in this case.

The following example shows that pure-strategy perfect Bayesian equilibria need not exist.
Suppose (R,R) is adopted.

Then \( q = 1/2 \). It follows \( a^*(R) = u \).

For both \( a^*(L) = u \) and \( = d \), R is not optimal for \( t_1 \).

Therefore, (R,R) cannot be optimal and thus no equilibrium in this case.

Suppose (L,R) is adopted.

Then \( p = 1 \) and \( q = 0 \). It follows \( a^*(L) = d \) and \( a^*(R) = d \). But then L is not optimal for \( t_1 \). Therefore, no equilibrium in this case.

Suppose (R,L) is adopted.

Then \( p = 0 \) and \( q = 1 \). It follows \( a^*(L) = u \) and \( a^*(R) = u \). But then R is not optimal for \( t_1 \). Therefore, no equilibrium in this case.

The following game has a “bad” equilibrium which
can be ruled out by perfect Bayesian requirements.

**Example 3:**

The normal-form representation:

<table>
<thead>
<tr>
<th></th>
<th>uu</th>
<th>ud</th>
<th>du</th>
<th>dd</th>
</tr>
</thead>
<tbody>
<tr>
<td>LL</td>
<td>2.5,3.5</td>
<td>2.5,3.5</td>
<td>2.5,3</td>
<td>2.5,3</td>
</tr>
<tr>
<td>LR</td>
<td>3,2.5</td>
<td>2,2</td>
<td>2.5,3</td>
<td>1.5,2.5</td>
</tr>
<tr>
<td>RL</td>
<td>1.5,4</td>
<td>0,2.5</td>
<td>2,3</td>
<td>0.5,1.5</td>
</tr>
<tr>
<td>RR</td>
<td>2,3</td>
<td>-0.5,1</td>
<td>2,3</td>
<td>-0.5,1</td>
</tr>
</tbody>
</table>

Nash equilibria: (LL, ud), (LR, du).

In the equilibrium (LL, ud), Player 2 plays d on the
right hand side (i.e., when she observes R), where however d is strictly dominated by u. Thus, this is a “bad” equilibrium. Now if the Receiver has a belief \((q, 1 - q)\) on the right information set as Requirement 1 and compute the expected payoff for playing \(u\), \(5q + (1 - q)\), and for playing \(d\), \(2q\), then she will find that playing \(d\) when she receives R is not optimal for any \(q\). Therefore there is no perfect Bayesian equilibrium playing (LL, ud).

(LR, du, \(p = 1, q = 0\)) is a perfect Bayesian equilibrium, because when \(p = 1\) and \(q = 0\), playing d and u on the left and right information sets, respectively, are indeed optimal.
4.3 Job-Market Signaling

In the model of job-market signaling, the sender is a worker, the receiver is the market of prospective employers, the type is the worker’s productive ability, the message is the worker’s education choice, and the action is the wage paid by the market. The timing is as follows:

1. Nature determines a worker’s productive ability, $\eta$, which can be either high ($\eta_H$) or low ($\eta_L$), equally likely.

2. The worker learns his or her ability and then chooses a level of education, $e \in \{e_c, e_s\}$. $e_c \sim \text{college education}$; $e_s \sim \text{school education}$.

3. A firm observes the worker’s education (but not the worker’s ability) and then makes a wage offer $w \in \{w_H, w_L\}$, which is accepted by the worker.
The payoffs are: \( w - c(\eta, e) \) to the worker, where \( c(\eta, e) \) is the cost to a worker with ability \( \eta \) of obtaining education \( e \); \(- (y(\eta, e) - w)^2\) to the firm that employs the worker, where \( y(\eta, e) \) is the output of a worker with ability \( \eta \) who has obtained education \( e \).

(Rationale of the firm’s payoff: Competition among firms will drive expected profits to zero. This implies that, in the market, wage should equal production output. A firm will not be able to hire a worker if it offers a too low wage, and cannot earn profit if it pays too high. Thus, the firm’s objective is to set its wage as close to a worker’s production output as possible.)

Assume that \( \eta_H = 10 \) and \( \eta_L = 5 \) with probability 0.5 and 0.5; \( e_c = 10 \) and \( e_s = 5 \); and \( w_H = 34 \) and \( w_L = 24 \). Assume \( c(\eta, e) = 10 - \eta + e \) and \( y(\eta, e) = 2\eta + 2e \).
To find perfect Bayesian equilibria, we consider 4 cases:

$(e_c, e_c)$ is adopted:

Requirement 3 dictates $p = 1/2$. Thus, $a_r(e_c) = w_H$.

For both types $\eta_H$ and $\eta_L$, playing $e_c$ gains higher payoff than playing $e_s$ iff Receiver’s best response is $a_r(e_s) = w_L$. 

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$w_L$ is the best response on RHS
iff $-16q - 196(1 - q) < -36q - 16(1 - q)$
iff $q < 9/10$.

Thus, $(e_c e_c, w_H w_L, p = 1/2, q < 9/10)$ is a perfect-Bayesian equilibrium.

$(e_s, e_s)$ is adopted:

Requirement 3 dictates $q = 1/2$. Thus, $a_r(e_s) = w_L$.

On the LHS, $w_H$ strictly dominates $w_L$, thus $a_r(e_c) = w_H$.

For type $\eta_H$, given $(w_H, w_L)$, the worker’s payoff from playing $e_c$ is higher than playing $e_s$.

Thus, no perfect Bayesian equilibrium in this case.

$(e_c, e_s)$ is adopted:

Requirement 3 dictates $p = 1$ and $q = 0$. Thus $a_r(e_c) = w_H$ and $a_r(e_s) = w_L$. 
For type $\eta_L$, the worker’s payoff from playing $e_c$ is higher than playing $e_s$.

Thus, no perfect-Bayesian equilibrium in this case.

$(e_s, e_c)$ is adopted:

Requirement 3 dictates $p = 0$ and $q = 1$. Thus $a_r(e_c) = w_H$ and $a_r(e_s) = w_H$.

For type $\eta_L$, the worker’s payoff from playing $e_s$ is higher than playing $e_c$.

Thus, no perfect-Bayesian equilibrium in this case.

**Summary:** There is only one perfect-Bayesian equilibrium

$$(e_c e_c, w_H w_L, p = 1/2, q < 9/10).$$
Interpretation of the perfect-Bayesian equilibrium \((e_c, w_H, w_L, p = 1/2, q < 9/10)\): The firm offers wage according to the education level. This is the incentive for workers to pursue high education. Workers, regardless high or low ability, pursue college education.

The firm’s decision is almost insensitive to a worker’s productive ability. The equilibria do not change if the output is changed to \(y(\eta, e) = 3\eta + e\) which is more dependent on the productive ability. (Tutorial question)
Now we assume the wage can be any nonnegative amount. Let the cost $c$ and output $y$ be general functions.

**Proposition 4.1** Let $c(\eta, e) = c_1(\eta) + c_2(e)$. Assume that the worker’s payoffs from playing $e_c$ and $e_s$, respectively, are different. Then there does not exist separating perfect Bayesian equilibrium.
**Proof.** To compare the worker’s payoffs, consider the difference of the worker’s payoffs from playing $e_c$ and $e_s$ (The firm does not know which type sends $e_i$, thus $w^*$ is independent of types.):

$$\Delta(\eta) = \left[w^*(e_c) - c(\eta, e_c)\right] - \left[w^*(e_s) - c(\eta, e_s)\right]$$

for $\eta \in \{\eta_H, \eta_L\}$. $\Delta(\eta) > 0$ (or $< 0$) means that sending $e_c$ (or $e_s$) is optimal for type $\eta$.

Taking the special form of $c$,

$$\Delta(\eta) = \left[w^*(e_c) - c_2(e_c)\right] - \left[w^*(e_s) - c_2(e_s)\right].$$

Since $w^*$ is independent of types, $\Delta$ is independent of $\eta$. Therefore, $\Delta(\eta_H) = \Delta(\eta_L) \neq 0$. This means, either both types sending $e_c$ is better (if $\Delta(\eta_H) = \Delta(\eta_L) > 0$) or both types sending $e_s$ is better (if $\Delta(\eta_H) = \Delta(\eta_L) < 0$). Hence $(e_c, e_s)$ cannot be played in a perfect Bayesian equilibrium.

Similarly, $(e_s, e_c)$ cannot be played in a perfect Bayesian equilibrium.  

\[\square\]
Now let’s consider pooling strategies.

We still assume $c(\eta, e) = c_1(\eta) + c_2(e)$.

If the firm received the signal $e_i$ ($e_i = e_c$ or $e_s$) and believes that this signal is sent by type $\eta_H$ with probability $r$ and by $\eta_L$ with probability $(1 - r)$, then the firm will choose wage $w^*(e_i, r)$ by maximizing his expected payoff

$$-r[y(\eta_H, e_i) - w]^2 - (1 - r)[y(\eta_L, e_i) - w]^2$$

which results in

$$w^*(e_i, r) = ry(\eta_H, e_i) + (1 - r)y(\eta_L, e_i). \ (4.1)$$

- The wage $w^*$ equals the expected worker’s output according to the firm’s belief $(r, 1 - r)$.
**Case 1.** \((e_c, e_c)\) is adopted. Then \(p = 1/2\). Let \((q, 1 - q)\) be the belief on the right information set. Then by (4.1), for each \(\eta \in \{\eta_H, \eta_L\}\), the worker’s payoffs for playing \(e_c\) and \(e_s\) are respectively

\[
\frac{1}{2} y(\eta_H, e_c) + \frac{1}{2} y(\eta_L, e_c) - c_1(\eta) - c_2(e_c)
\]

and

\[
q y(\eta_H, e_s) + (1 - q) y(\eta_L, e_s) - c_1(\eta) - c_2(e_s).
\]

Thus, \((e_c, e_c)\) is the best response iff the former is higher than the latter for some \(q \in [0, 1]\) and for both \(\eta = \eta_H\) and \(\eta_L\).

Refer to \(B(\eta, e) = y(\eta, e) - c_2(e)\) as net benefit of education. It is reasonable to assume \(y\) is increasing function of \(\eta\), then so is \(B\). We further assume \(B(\eta, e) \geq 0\) for any \(\eta\) and \(e\). Then

\[
q \ y(\eta_H, e_s) + (1 - q) y(\eta_L, e_s) - c_2(e_s)
\]

\[
= qB(\eta_H, e_s) + (1 - q)B(\eta_L, e_s).
\]
Hence, $e_c$ is better than $e_s$ iff
\[
\frac{1}{2} B(\eta_H, e_c) + \frac{1}{2} B(\eta_L, e_c) \\
\geq q B(\eta_H, e_s) + (1 - q) B(\eta_L, e_s)
\] (4.2)

for some $q \in [0, 1]$. If
\[
\frac{1}{2} B(\eta_H, e_c) + \frac{1}{2} B(\eta_L, e_c) \geq B(\eta_L, e_s)
\] (4.3)

let $\bar{q}$ be the largest $q \in [0, 1]$ such that (4.2) holds. Then,
\[
\{(e_c, e_c), (w^*(e_c, p), w^*(e_s, q)), p = 1/2, q \in [0, \bar{q}]\}
\]
are perfect Bayesian equilibria.

**Case 2.** $(e_s, e_s)$ is adopted. In this case $q = 1/2$ and $(p, 1 - p)$ can be any probability distribution. By the same reasoning, if
\[
B(\eta_L, e_c) \leq \frac{1}{2}[B(\eta_H, e_s) + B(\eta_L, e_s)],
\] (4.4)

let $\bar{p}$ be the largest $p \in [0, \bar{p}]$ such that
\[
p B(\eta_H, e_c) + (1 - p) B(\eta_L, e_c) \\
\leq \frac{1}{2}[B(\eta_H, e_s) + B(\eta_L, e_s)].
\] (4.5)
Then,
\[ \{(e_s, e_s), (w^*(e_c, p), w^*(e_s, q)), p \in [0, \bar{p}], q = 1/2\} \]
are perfect Bayesian equilibria.

**Interpretation:** In the perfect Bayesian equilibrium in case 1, if (4.3) holds, the worker plays \((e_c, e_c)\), the firm offers \(\frac{1}{2}[y(\eta_H, e_c) + y(\eta_L, e_c)]\) to the worker with college education and \(y(\eta_L, e_s)\) (with \(q = 0\)) to the worker with school education. This means, people, regardless their ability is high or low, will try to pursue college education. The firm will pay the average wage to college graduates and a wage counting for low ability to school graduates. In case 2, if (4.4) holds, all people prefer the school education. The firm will pay the average wage to school graduates and a wage counting for low ability to college graduates.
4.4 Corporate Investment and Capital Structure

An entrepreneur has started a company but needs outside financing to undertake an attractive new project. The entrepreneur has private information about the profitability of the existing company, but the payoff of the new project cannot be disentangled from the payoff of the existing company — all that can be observed is the aggregate profit of the firm. Suppose the entrepreneur offers a potential investor an equity stake in the firm in exchange for the necessary financing.

To translate this problem into a signaling game, suppose that the profit of the existing company can be either high \((\pi = H = 20)\) or low \((\pi = L = 10)\), which are equally likely. Suppose that the project requires investment of \(I = 5\) and will return \(R = 8\).
The potential investor’s alternative rate of return is $r = 40\%$, namely, a return of $I(1 + r) = 7$. The timing and payoffs of the game are then as follows:

1. Nature determines the profit $\pi$ of the existing company.
2. The entrepreneur learns $\pi$ and then offers the potential investor an equity stake $s \in \{s_H, s_L\}$.
3. The investor observes $s$ (but not $\pi$) and then decides either to accept or to reject the offer.
4. If the investor rejects the offer then the investor’s payoff is $I(1 + r)$ and the entrepreneur’s payoff is $\pi$. If the investor accepts $s$ then the investor’s payoff is $s(\pi + R)$ and the entrepreneur’s is $(1 - s)(\pi + R)$.

It is reasonable for an investor to expect the payoff $s(\pi + R)$ to be $R$. Thus, the entrepreneur should
choose \( s \) between

\[
\frac{2}{7} = \frac{8}{20 + 8} \leq s \leq \frac{8}{10 + 8} = \frac{4}{9}.
\]

First, let \( s_H = \frac{4}{9} \) and \( s_L = \frac{2}{7} \).

The game is depicted as follows:
The normal-form representation:

<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>AR</th>
<th>RA</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>12.8, 10.2</td>
<td>12.8, 10.2</td>
<td>15, 7</td>
<td>15, 7</td>
</tr>
<tr>
<td>HL</td>
<td>14.2, 8.8</td>
<td>12.8, 9.7</td>
<td>16.4, 6.1</td>
<td>15, 7</td>
</tr>
<tr>
<td>LH</td>
<td>15, 8</td>
<td>15, 7.5</td>
<td>15, 7.5</td>
<td>15, 7</td>
</tr>
<tr>
<td>LL</td>
<td>16.4, 6.6</td>
<td>15, 7</td>
<td>16.4, 6.6</td>
<td>15, 7</td>
</tr>
</tbody>
</table>

The Nash equilibria:

\[(LL, AR) \quad (LL, RR).\]

The equilibrium (LL,RR) is not perfect Bayesian because on the LHS Accept is better than Reject for any belief.

The unique perfect Bayesian equilibrium is (LL, AR, \( p \in [0, 1] \), \( q = 1/2 \)).

This game ends up with no deal, because the equity stake offered by the entrepreneur is too low.
Now, let $s_H = \frac{4}{9}$ and $s_L = \frac{1}{3}$. The game becomes

The normal-form representation:

<table>
<thead>
<tr>
<th></th>
<th>AA</th>
<th>AR</th>
<th>RA</th>
<th>RR</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>12.8, 10.2</td>
<td>12.8, 10.2</td>
<td>15, 7</td>
<td>15, 7</td>
</tr>
<tr>
<td>HL</td>
<td>13.8, 9.2</td>
<td>12.8, 9.7</td>
<td>16, 6.5</td>
<td>15, 7</td>
</tr>
<tr>
<td>LH</td>
<td>14.3, 8.7</td>
<td>15, 7.5</td>
<td>14.3, 8.2</td>
<td>15, 7</td>
</tr>
<tr>
<td>LL</td>
<td>15.3, 7.7</td>
<td>15, 7</td>
<td>15.3, 7.7</td>
<td>15, 7</td>
</tr>
</tbody>
</table>

The unique perfect Bayesian equilibrium is (LL, AA, $p \in [0, 1]$, $q = 1/2$).
The expected payoffs for the entrepreneur and the investor are 15.333 and 7.667 which are higher than 15 and 7. Both of them make profits.

However, if the existing company’s profit is $\pi = 20$, then the entrepreneur indeed loses money by playing the game which makes a payoff of $56/3$. Although he knows whether $\pi$ is high or low, he cannot quit from the game to avoid losing money, because he has only two choices $\{s_H, s_L\}$. 
4.5 Sequential bargaining under asymmetric information

Consider a firm and a union bargaining over wages. The union’s reservation wage (i.e., the amount that union members earn if not employed by the firm) is \( w_r \). The firm’s profit, \( \pi \), is uniformly distributed on \([\pi_L, \pi_H]\), but the true value of \( \pi \) is privately known by the firm. We simplify the analysis by assuming that \( w_r = \pi_L = 0 \).

The bargaining game lasts at most two periods. In the first period, the union makes a wage offer, \( w_1 \). If the firm accepts this offer then the game ends: the union’s payoff is \( w_1 \) and the firm’s is \( \pi - w_1 \). If the firm rejects this offer then the game proceeds to the second period. The union makes a second wage offer, \( w_2 \). If the firm accepts this offer then the present values of the players’ payoffs (as mea-
sured in the first period) are $\delta w_2$ for the union and 
$\delta (\pi - w_2)$ for the firm, where $\delta$ reflects both dis-
counting and the reduced life of the contract re-
maining after the first period. If the firm rejects 
the union’s second offer then the game ends and 
payoffs are zero for both players.

The unique perfect Bayesian equilibrium of this game:

- The union’s first-period wage offer is

$$w_1^* = \frac{(2 - \delta)^2}{2(4 - 3\delta)} \pi H.$$ 

- If the firm’s profit, $\pi$, exceeds

$$\pi_1^* = \frac{2w_1}{2 - \delta} = \frac{2 - \delta}{4 - 3\delta} \pi H$$

then the firm accepts $w_1^*$; otherwise, the firm 
rejects $w_1^*$.

- If its first-period offer is rejected, the union up-
dates its belief about the firm’s profit: the union 
believes that $\pi$ is uniformly distributed on $[0, \pi_1^*]$. 
• The union’s second-period wage offer (conditional on $w_1^*$ being rejected) is
  \[ w_2^* = \frac{\pi_1^*}{2} = \frac{2 - \delta}{2(4 - 3\delta)}\pi H < w_1^*. \]

• If the firm’s profit, $\pi$, exceeds $w_2^*$ then the firm accepts the offer; otherwise, it rejects it.

Thus, in each period, high-profit firms accept the union’s offer while low-profit firms reject it, and the union’s second-period belief reflects the fact that high-profit firms accepted the first-period offer.

In equilibrium, low-profit firms tolerate a one-period strike in order to convince the union that they are low-profit and so induce the union to offer a lower second-period wage. Firms with very low profits, however, find even the lower second-period offer intolerably high and so reject it, too.
The game tree:

Figure 1:
Analysis:

If the union offers $w_1$ in the first period and the firm expects the union to offer $w_2$ in the second period, then firm’s possible payoffs are $\pi - w_1$ from accepting $w_1$, $\delta(\pi - w_2)$ from rejecting $w_1$ and accepting $w_2$, and zero from rejecting both offers. The firm therefore prefers accepting $w_1$ to accepting $w_2$ if $\pi - w_1 > \delta(\pi - w_2)$, or

$$\pi > \frac{w_1 - \delta w_2}{1 - \delta} \equiv \pi^*(w_1, w_2),$$

and $\pi - w_1 > 0$.

Denote

$$\pi_1 = \max\{\pi^*(w_1, w_2), w_1\}. \quad (4.6)$$

Then, we can derive the firm’s first-period optimal action: accept $w_1$ if $\pi > \pi_1$ and reject $w_1$ if $\pi < \pi_1$.

Now, if $w_1$ is rejected, then the union will believe that $\pi < \pi_1$ which induces the belief (Requirement 3 or 4) that $\pi$ is uniformly distributed on $[0, \pi_1]$. 
If the union offers $w_2$ then the firm’s second-period best response is: accept $w_2$ if and only if $\pi \geq w_2$. Thus, the union should choose $w_2$ (according to Requirement 2) to maximize

$$w_2 \cdot \text{Prob}(\text{firm accepts } w_2 \mid \text{firm rejected } w_1) + 0 \cdot \text{Prob}(\text{firm rejects } w_2 \mid \ldots)$$

$$= w_2 \cdot \text{Prob}(\pi \geq w_2 \mid \pi < \pi_1)$$

$$= w_2 \cdot \text{Prob}(w_2 \leq \pi < \pi_1) / \text{Prob}(\pi < \pi_1)$$

$$= w_2(\pi_1 - w_2) / \pi_1.$$

The optimal wage offer is therefore

$$w_2 = \pi_1 / 2. \quad (4.7)$$

Substituting it in (4.6) results in

$$\pi_1 = \max\{\frac{w_1 - \delta \pi_1 / 2}{1 - \delta}, w_1\}. \quad (4.8)$$

Solving $\pi_1 = \frac{w_1 - \delta \pi_1 / 2}{1 - \delta}$, we obtain

$$\pi_1 = \frac{2w_1}{2 - \delta} \quad (4.9)$$
which is indeed $> w_1$ and thus is the solution of (4.8).

Now, the union’s first-period wage offer should be chosen to solve

$$
\max_{w_1} \quad w_1 \cdot \text{Prob} \left\{ \text{firm accepts } w_1 \right\} \\
+ \delta w_2 \text{Prob}\{\text{firm rejects } w_1 \text{ but accepts } w_2\} \\
+ \delta \cdot 0 \cdot \text{Prob} \left\{ \text{firm rejects both } w_1 \text{ and } w_2 \right\}
$$

$$
= \max_{w_1} \quad w_1 \cdot \text{Prob} \left\{ \pi \geq \pi_1 \right\} \\
+ \delta \cdot w_2 \cdot \text{Prob} \left\{ w_2 < \pi < \pi_1 \right\}
$$

$$
= \max_{w_1} \quad w_1 \frac{\pi H - \pi_1}{\pi H} + \delta w_2 \frac{\pi_1 - w_2}{\pi H}
$$

$$
= \max_{w_1} \quad w_1 \frac{\pi H - \pi_1}{\pi H} + \frac{\delta \pi_1^2}{4\pi H}
$$

$$
= \max_{w_1} \quad \frac{1}{\pi H} \left( \pi H w_1 - \frac{2w_1^2}{2 - \delta} + \frac{\delta w_1^2}{(2 - \delta)^2} \right)
$$

**Optimality condition:**

$$
\pi H - \frac{4w_1}{2 - \delta} + \frac{2\delta w_1}{(2 - \delta)^2} = 0.
$$
Optimal first-period wage offer

\[ w_1^* = \frac{(2 - \delta)^2}{2(4 - 3\delta)} \pi H. \]

Substituting it into (4.9) and (4.7), we obtain the other quantities in the equilibrium.
5 Cooperative Games

In cooperative games binding agreements may be made among the players. In addition, we assume that all payoffs are measured in the same units and that there is a transferrable utility which allows side payments to be made among the players. Side payments may be used as inducements for some players to use certain mutually beneficial strategies. Thus, there will be a tendency for players, whose objectives in the game are close, to form alliances or coalitions. Players in a coalition will collectively generate a payoff to the coalition. Thus the central issue is to find a fair distribution of payoffs among players.
Battle of Sexes

<table>
<thead>
<tr>
<th>Mary</th>
<th>Opera</th>
<th>Fight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Fight</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Payoff pairs like (1, 2) and (2, 1) are not fair.

Side payment can transfer utility, thus any payoff pair in $H$ can be achieved by some actions.
Given a set of attainable payoffs, how to determine a fair distribution of payoffs, like \((1.5, 1.5)\) in the above game?

### 5.1 Two-Person Cooperative Games

Imagine two persons isolated on a desert island. Suppose that each has a fixed supply of various commodities and each has a utility function assigning a personal value to any conceivable commodity bundle. The two players’ utility functions need not be the same, nor need their initial holdings be the same. To think of this situation as a game, a player’s payoff is defined as her utility. Suppose they may engage in trade with one another, exchanging anything from their respective holdings that both would agree on. The possibilities for them are depicted below:
\( H \subset R^2 \) denotes the set of attainable payoff pairs and the point \( d \in H \) denotes the threat point, i.e. the outcome that is obtained in the absence of trade, (it is sometimes called the status quo or the no trade point). If a payoff point \( u = (u_1, u_2) \notin H \), then it is impossible for the players to achieve it, but if \( u \in H \), then there are (joint) actions open to them that will result in \( u \) being the payoff. The players will recieve the payoffs \( d = (d_1, d_2) \) if they fail to achieve an agreement (trade).

**Definition 5.1** The pair \( \Gamma = (H, d) \) is a two-person bargaining game if \( H \subset R^2 \) is compact and convex, \( d \in H \), and \( H \) contains at least one element, \( u \), such that \( u \gg d \).

The set of two-person bargaining games is denoted \( W \).
Definition 5.2 Let \((u, v)\) and \((u', v')\) be two payoff pairs. We say \((u, v)\) dominates \((u', v')\) if
\[
u \geq u' \quad v \geq v'.
\]
Payoff pairs which are not dominated by any other pair are said to be Pareto optimal.

E.g., payoff pairs on the curve between points \(A\) and \(B\) in the above figure are Pareto optimal.

Definition 5.3 The Nash bargaining solution is a mapping \(f : W \rightarrow R^2\) that associates a unique element \(f(H, d) = (f_1(H, d), f_2(H, d))\) with the game \((H, d) \in W\), satisfying the following axioms:

1. [Feasibility] \(f(H, d) \in H\).
2. [Individual rationality] \(f(H, d) \geq d\) for all \((H, d) \in W\).
3. [Pareto optimality] \(f(H, d)\) is Pareto optimal.
4. [Invariance under linear transformations] Let $a_1, a_2 > 0, b_1, b_2 \in \mathbb{R},$ and $(H, d), (H', d') \in W$ where $d'_i = a_i d_i + b_i, i = 1, 2,$ and $H' = \{x \in \mathbb{R}^2 \mid x_i = a_i y_i + b_i, i = 1, 2, y \in H\}.$ Then $f_i(H'_i, d'_i) = a_i f_i(H, d) + b_i, i = 1, 2.$

5. [Symmetry] If $(H, d) \in W$ satisfies $d_1 = d_2$ and $(x_1, x_2) \in H$ implies $(x_2, x_1) \in H,$ then $f_1(H, d) = f_2(H, d).$

6. [Independence of irrelevant alternatives] If $(H, d), (H', d') \in W, d = d', H \subset H',$ and $f(H', d') \in H,$ then $f(H, d) = f(H', d').$
The following theorem provides a way to compute the Nash bargaining solution. The proof of the theorem is involved, and thus is omitted.

**Theorem 5.1** A game \((H, d) \in W\) has a unique Nash solution \(u^* = f(H, d)\) satisfying Conditions 1 to 6. The solution \(u^*\) satisfies Conditions 1 to 6 if and only if

\[
(u_1^* - d_1)(u_2^* - d_2) > (u_1 - d_1)(u_2 - d_2)
\]

for all \(u \in H, u \geq d,\) and \(u \neq u^*.\)
Labor-management

Imagine a firm that is a monopolist in the market for its output and in the labor market (hiring) as well. At the same time, a labor union is a monopolist in the labor market (supplying). Letting $L$ denote the level of employment and $w$, the wage rate, suppose the union has a utility function $u(L, w) = \sqrt{L}w$. The firm’s utility is its profit $\pi = L(100 - L) - wL$.

In this situation, the payoff set

$H = \{(u, \pi) \mid u = \sqrt{L}w, \pi = L(100 - L) - wL; \quad L \geq 0, w \geq 0\}.$

The most natural threat on the part of the firm is to cease production, which means the union members will all be without jobs. Similarly, the union can refuse to supply any labor to the firm, and, again, there will be neither production nor jobs. That is, $(0, 0)$ is the threat point.
The Nash bargaining solution maximizes the function

$$(u - d_1)(\pi - d_2) = \sqrt{Lw}[L(100 - L) - wL]$$

for $L \geq 0$ and $w \geq 0$. Optimality conditions are both partial derivatives with respect to $L$ and $w$ being 0:

$$\begin{cases} 
300 - 5L - 3w = 0 \\
100 - L - 3w = 0
\end{cases}$$

which results in

$$L = 50, \quad w = 50/3.$$ 

Thus, the Nash bargaining solution is

$$u^* = 50/\sqrt{3}, \quad \pi^* = 5000/3.$$
5.2 \( n \)-person Cooperative Games

**Definition 5.4** For an \( n \)-person game with the set of players \( N = \{1, 2, \ldots, n\} \), any nonempty subset of \( N \) (including \( N \) itself) is called a coalition. For each coalition \( S \), the characteristic function \( v \) of the game gives the amount \( v(S) \) that the coalition \( S \) can be sure of receiving. The game is denoted by \( \Gamma = (N, v) \).

**Assumption 5.1:** The characteristic function \( v \) satisfies

(i) \( v(\emptyset) = 0 \);

(ii) for any disjoint coalitions, \( K \) and \( L \) contained in \( N \),

\[
v(K \cup L) \geq v(K) + v(L).
\]  
(super-additivity)
Example 5.1: (The Drug Game)

Joe Willie has invented a new drug. Joe cannot manufacture the drug himself. He can sell the drug formula to company 2 or company 3, but cannot sell it to both companies. Company 2 can make a profit of $2 millions if it buys the formula. Company 3 can make a profit of $3 millions if it buys the formula.

Let Joe, companies 2 and 3 be players 1, 2 and 3.

Characteristic function $v$ is defined as

\[
    v(\emptyset) = 0, \quad v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \\
    v(\{1, 2\}) = 2, \quad v(\{1, 3\}) = 3, \quad v(\{2, 3\}) = 0, \\
    v(\{1, 2, 3\}) = 3.
\]

This function satisfies Assumption 5.1.
Example 5.2: (The Garbage Game)

Each of four property owners has one bag of garbage and must dump his or her bag of garbage on somebody’s property. Owners of a coalition will dump their garbage in properties not belonging to the coalition. If \( x \) bags of garbage are dumped on a coalition’s properties, the coalition receives a reward of \(-x\).

Characteristic function:

\[
v(S) = \begin{cases} 
-(4 - |S|) & \text{if } |S| < 4 \\
-4 & \text{if } |S| = 4
\end{cases}
\]

For any disjoint coalitions \( S \) and \( T \),

\[
v(S \cup T) = -4, \text{ if } |S \cup T| = 4;
\]

\[
v(S \cup T) = -(4 - |S \cup T|) = -4 + |S| + |T|, \text{ if } |S \cup T| < 4;
\]

\[
v(S) + v(T) = -4 + |S| - 4 + |T| = -8 + |S| + |T|.
\]

\[
v(S \cup T) \geq v(S) + v(T).
\]

Thus, super-additivity holds.
**Question:** What are fair payoffs players can receive?

**Definition 5.5** An imputation in the game \((N, v)\) is a payoff vector \(x = (x_1, \ldots, x_n)\) satisfying

(i) \(\sum_{i=1}^{n} x_i = v(N)\), (group rational); (ii) \(x_i \geq v(\{i\})\), for all \(i \in N\), (individually rational).

Let \(I(N, v)\) denote the set of all imputations of the game \((N, v)\).

**Definition 5.6** Let \(x, y \in I(N, v)\), and let \(S\) be a coalition. We say \(x\) dominates \(y\) via \(S\) (notation \(x \succ_S y\)) if

(i) \(x_i > y_i\) for all \(i \in S\);

(ii) \(\sum_{i \in S} x_i \leq v(S)\).

We say \(x\) dominates \(y\) (notation \(x \succ y\)) if there is any coalition \(S\) such that \(x \succ_S y\).
Example of Drug Game:

The following are some imputations and their relations:

\[ (1, 0, 2) \prec \{1,2\} \ (1.5, 0.5, 1) \prec \{1,3\} \ (1.7, 0, 1.3) \]

Note that \((2, 0, 1) \prec \{2,3\} \ (0, 1, 2)\) is wrong, because \(1 + 2 = 3 \not\leq v(\{2, 3\}) = 0\).

A dominated imputation \(x\) is unstable. Because if \(x \succ_S y\) then members in \(S\) can gain \(y_i\) instead of \(x_i\) if they form coalition \(S\).

A undominated imputation is a solution.

**Definition 5.7** The set of all undominated imputations for a game \((N, v)\) is called the **core**, denoted by \(C(N, v)\).
Theorem 5.2 The core of a game $(N, v)$ is the set of all $n$-vectors, $x$, satisfying

(a) $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$;

(b) $\sum_{i \in N} x_i = v(N)$.

Proof. Let $S = \{i\}$. Condition (a) reduces to $x_i \geq v(\{i\})$. This, together with condition (b), implies that $x$ is an imputation.

Suppose $x$ satisfies (a) and (b). By contradiction, assume $x$ is dominated. Then there exist an imputation $y$ and a coalition $S$ such that $y \succ_S x$. This implies $y_i > x_i$ for all $i \in S$. Then by (a), $\sum_{i \in S} y_i > v(S)$, which violates condition (ii) in Definition 5.6. Therefore, $x$ is undominated. Thus, $x$ is in the core.

Conversely, suppose that $x$ does not satisfy (a) and (b). If $x$ fails to satisfy (b), it is not even an imputation, hence not in $C(N, v)$. Suppose, then, that
there is some $\emptyset \neq S \subset N$ such that
\[ \sum_{i \in S} x_i = v(S) - \epsilon, \]
where $\epsilon > 0$. Define
\[ z_i = \begin{cases} x_i + \frac{\epsilon}{|S|} & \text{if } i \in S \\ v(\{i\}) + \frac{\alpha}{|N| - |S|} & \text{if } i \notin S \end{cases} \]
where
\[ \alpha = v(N) - v(S) - \sum_{i \in N \setminus S} v(\{i\}). \]
By supperadditivity, $\alpha \geq 0$. It is easily seen that $z$ is an imputation and $z \succ_S x$. Hence $x \not\in C(N, v)$.

\[ \square \]

The above theorem provides a way to check if the core of a game is nonempty: The core of the game $(N, v)$ is nonempty iff
\[ v(N) \geq \min_{x \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^{n} x_i \]
\[ \text{s.t. } \sum_{i \in S} x_i \geq v(S), \quad \forall S \subset N, S \neq N. \]
Example: The core for the Drug Game.

By Theorem 5.2, \(x = (x_1, x_2, x_3) \in C(N, v)\) iff \(x\) satisfies

\[
\begin{align*}
(1) & \quad x_1 \geq 0, \\
(2) & \quad x_2 \geq 0, \\
(3) & \quad x_3 \geq 0, \\
(4) & \quad x_1 + x_2 \geq 2, \\
(5) & \quad x_1 + x_3 \geq 3, \\
(6) & \quad x_2 + x_3 \geq 0, \\
(7) & \quad x_1 + x_2 + x_3 = 3.
\end{align*}
\]

(2), (5) and (7) imply

\[
(8) x_2 = 0, \quad (9) x_1 + x_3 = 3.
\]

(4), (8) and (9) imply

\[
2 \leq x_1 \leq 3, \quad x_3 = 3 - x_1.
\]

Conversely, any imputation

\[
x \in X := \{(x_1, x_2, x_3) \mid x_2 = 0, x_3 = 3 - x_1, 2 \leq x_1 \leq 3\}.
\]

satisfies (1)-(7). Hence, \(C(N, v) = X\).

Remark:

(i) The core \(C(N, v)\) contains infinitely many imputations.
(ii) The core emphasizes that importance of player 1, because he can choose partners, while players 2 and 3 cannot.

(iii) If \( v(\{1, 2\}) = 3 \), then the core is a singleton and the only imputation is \((3, 0, 0)\). This seems overemphasizing the power of choice.

Some Special Games

**Definition 5.8** A game \((N, v)\) is said to be constant-sum if for all \( S \subset N \)

\[
v(S) + v(N - S) = v(N).
\]

E.g. The Garbage Game is constant-sum. (check it)

**Definition 5.9** A game \((N, v)\) is essential if

\[
v(N) > \sum_{i \in N} v(\{i\}).
\]

It is inessential otherwise.
E.g. The Garbage Game is essential:

\[ v(N) = -4, \quad \sum_{i=1}^{4} v\{i\} = -12. \]

**Theorem 5.3** If \((N, v)\) is an inessential game, then for any coalition \(S\)

\[ v(S) = \sum_{i \in S} v\{i\}. \]

**Proof.** Suppose it is not true. Then there exists an \(S\) such that \(v(S) \neq \sum_{i \in S} v\{i\}\). By super-additivity, \(v(S) \geq \sum_{i \in S} v\{i\}\). Thus,

\[ v(S) > \sum_{i \in S} v\{i\}, \]

and

\[ v(N) \geq v(S) + v(N - S) > \sum_{i \in N} v\{i\} \]

which contradicts the definition of an inessential game. \(\Box\)
Theorem 5.4 If a game \((N, v)\) is constant-sum, then its core is either empty or a singleton
\[
\{(v(\{1\}), \ldots, v(\{n\}))\}.
\]

Proof. Assume that \(C(N, v) \neq \emptyset\) and \(x \in C(N, v)\). Then by Theorem 5.2, for any \(i \in N\)
\[
x_i \geq v(\{i\}), \quad \sum_{j \neq i} x_j \geq v(N - \{i\})
\]
and by the constant-sum property
\[
v(N) = \sum_{j \in N} x_j \geq v(\{i\}) + v(N - \{i\}) = v(N)
\]
This implies that \(x_i = v(\{i\}) \forall i \in N\).

Corollary 5.1 If \((N, v)\) is inessential, then it is constant-sum.

Proof. For any coalition \(S\),
\[
v(S) + v(N \setminus S) = \sum_{i \in S} v(\{i\}) + \sum_{i \in N \setminus S} v(\{i\})
= \sum_{i \in N} v(\{i\}) = v(N).
\]
**Corollary 5.2** If \((N, v)\) is an inessential game, then

\[ C(N, v) = \{(v(\{1\}), \ldots, v(\{n\}))\} \]

is a singleton.

**Proof.** \((x_1, \ldots, x_n) = (v(\{1\}), \ldots, v(\{n\}))\) satisfies the conditions in Theorem 5.2. Thus

\[ (v(\{1\}), \ldots, v(\{n\})) \in C(N, v). \]

By Theorem 5.4, there is no other element in the core. Thus, the core is the singleton. \(\square\)

**Corollary 5.3** If a game \((N, v)\) is both essential and constant-sum, then its core is empty.

**Proof.** Because \(\sum_{i \in N} v(\{i\}) < v(N)\), thus

\[(v(\{1\}), \ldots, v(\{n\}))\]

is not an imputation and then is not in the core. Then, by Theorem 5.4, \(C(N, v) = \emptyset\). \(\square\)
Garbage Game has empty core, because it is essential and constant-sum.

**Strategical Equivalence**

**Definition 5.10** Two games $v$ and $u$ are strategically equivalent if there exist constants $a > 0$ and $c_1, \ldots, c_n$, such that for every coalition $S$

$$u(S) = av(S) + \sum_{i \in S} c_i. \quad (5.1)$$

**Lemma 5.1** Suppose that $u$ and $v$ are strategically equivalent satisfying (5.1). Then

(i) $u$ is (in)essential iff $v$ is (in)essential.

(ii) $x$ is an imputation for $v$ iff $ax + c$ is an imputation for $u$, where $c = (c_1, \ldots, c_n)^T$.

(iii) $x \succeq_S y$ (w.r.t. $v$) iff $ax + c \succeq_S ay + c$ (w.r.t. $u$).

(iv) $x \in C(N, v)$ iff $ax + c \in C(N, u)$.  

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Definition 5.11 A characteristic function \( v \) is in \((0, 1)\)-reduced form if

- \( v(\{i\}) = 0 \), for all \( i \in N \).
- \( v(N) = 1 \).

Theorem 5.5 Any essential game \((N, v)\) is strategically equivalent to a game \((N, u)\) in \((0, 1)\)-reduced form.

Proof. Define

\[
a = \frac{1}{v(N) - \sum_{i \in N} v(\{i\})} > 0, \quad c_i = -av(\{i\}).
\]

Let \( u \) be defined by (5.1). Then for any \( i \in N \)

\[
u(\{i\}) = av(\{i\}) + c_i = 0
\]

\[
u(N) = av(N) + \sum_{i \in N} c_i = a[v(N) - \sum_{i \in N} v(\{i\})] = 1.
\]

Thus, \( u \) is in \((0, 1)\)-reduced form. \( \square \)
Theorem 5.6 Classification of small essential games in (0, 1)-reduced form:

(i) Two-person game:

\[ v(\emptyset) = v(\{1\}) = v(\{2\}) = 0, \quad v(N) = 1. \]

(ii) Three-person constant-sum game:

\[ v(\emptyset) = v(\{i\}) = 0, \quad i = 1, 2, 3. \]
\[ v(N) = v(\{i, j\}) = 1, \quad i \neq j \in N. \]

(iii) Three-person game: There exist \( a, b, c \in [0, 1] \) such that

\[ v(\emptyset) = v(\{i\}) = 0, \quad i = 1, 2, 3. \]
\[ v(\{1, 2\}) = a, \quad v(\{1, 3\}) = b, \quad v(\{2, 3\}) = c, \]
\[ v(N) = 1. \]

(Tutorial)
The Shapley Value

Given any $n$-person game $(N, v)$, the Shapley value is an $n$-vector, denoted by $\phi(v)$, satisfying a set of axioms.

The $i$-th component of $\phi(v)$ can be uniquely determined by

$$\phi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n - s - 1)!}{n! \cdot s!(n - s - 1)!} [v(S \cup \{i\}) - v(S)]$$

$$= \frac{1}{n} \sum_{s=0}^{n-1} \frac{1}{n - 1} \sum_{S \subset N \setminus \{i\}, |S|=s} [v(S \cup \{i\}) - v(S)]$$

where $s = |S|$.

Explanation:

Players come randomly. If $i$-th player arrives and finds the members of the coalition $S$ already there, he joins the coalition and receives the amount $v(S \cup \{i\}) - v(S)$, i.e., the marginal amount which he contributes to the coalition, as payoff. The $\phi_i(v)$
is the expected payoff to player $i$. The probability of finding coalition $S$ already there when player $i$ arrives is

$$\gamma(s) = \frac{s!(n - s - 1)!}{n!}.$$ 

So we have the formula.

The probability can be computed as follows. The denominator is the total number of permutations of the $n$ players. The numerator is number of these permutations in which the $|S|$ members of $S$ come first ($|S|$! ways), then player $i$, and then the remaining $n - |S| - 1$ players ($\binom{n - |S| - 1}{n - |S| - 1}$ ways).

**Proposition 5.1** The Shapley value has the following desirable properties:

1. **Individual rationality.** $\phi_i(v) \geq v(\{i\})$ for every $i \in N$.

2. **Efficiency.** The total gain is distributed:

$$\sum_{i \in N} \phi_i(v) = v(N).$$
3. **Symmetry.** If players $i$ and $j$ are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition $S$ not containing $i$ and $j$, then $\phi_i(v) = \phi_j(v)$.

4. **Additivity.** If $v$ and $w$ are characteristic functions, then $\phi(v + w) = \phi(v) + \phi(w)$.

5. **Null player.** If a player $i$ is such that $v(S \cup \{i\}) = v(S)$ for every $S$ not containing $i$, then $\phi_i(v) = 0$.

In fact, given a player set $N$, the Shapley value is the only function, defined on the class of all characteristic functions, that satisfies properties 2, 3, 4 and 5.

**Example 5.3:** Find the Shapley value for Drug Game.

The characteristic function:

$$
v(\emptyset) = v(\{i\}) = 0, \quad i = 1, 2, 3,$$

$$
v(\{1,2\}) = 2, \quad v(\{1,3\}) = 3, \quad v(\{2,3\}) = 0,
$$
\[ v(\{1, 2, 3\}) = 3. \]

**Probability:**

\[ \gamma(0) = \frac{0!(3 - 0 - 1)!}{3!} = \frac{1}{3}, \]
\[ \gamma(1) = \frac{1!(3 - 1 - 1)!}{3!} = \frac{1}{6}, \]
\[ \gamma(2) = \frac{2!(3 - 2 - 1)!}{3!} = \frac{1}{3}. \]

For player 1,

\[
\begin{array}{c|cccc}
S & \emptyset & \{2\} & \{3\} & \{2, 3\} \\
v(S \cup \{1\}) - v(S) & 0 & 2 & 3 & 3 \\
\end{array}
\]

Hence

\[ \phi_1(v) = \frac{2}{6} + \frac{3}{6} + \frac{3}{3} = \frac{5}{6}. \]

For player 2,

\[
\begin{array}{c|cccc}
S & \emptyset & \{1\} & \{3\} & \{1, 3\} \\
v(S \cup \{2\}) - v(S) & 0 & 2 & 0 & 0 \\
\end{array}
\]

Hence

\[ \phi_2(v) = \frac{2}{6} = \frac{1}{3}. \]
Because \( v(N) = \sum_{i \in N} \phi_i(v) \),
\[ \phi_3(v) = v(N) - \phi_1(v) - \phi_2(v) = 3 - \frac{5}{6} - \frac{1}{3} = \frac{5}{6}. \]

Question: What is \( \phi(v) \) if \( v(\{1, 2\}) = 3 \)?

**Example 5.4:** Consider a corporation with four stock-holders having respectively 10, 20, 30 and 40 shares of stock. It is assumed that any decision can be settled by approval of stock-holders holding a simple majority of the shares.

Let the characteristic function has value 1 if a coalition holds \( > 50 \) shares, and 0 if otherwise.

The coalitions with characteristic function value = 1 are:
\[
\{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \text{ and } \{1, 2, 3, 4\}.
\]
\[ S \text{ with } v(S \cup \{i\}) - v(S) = 1 \]

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Probability:

\[ \gamma(1) = \frac{1!(4 - 1 - 1)!}{4!} = \frac{1}{12}, \quad \gamma(2) = \frac{1}{12}. \]

Shapley value:

\[ \phi_1(v) = 1/12 \]
\[ \phi_2(v) = 1/12 + 2/12 = 1/4 \]
\[ \phi_3(v) = 1/12 + 2/12 = 1/4 \]
\[ \phi_4(v) = 2/12 + 3/12 = 5/12. \]

\( \phi_i(v) \) can be interpreted as the voting power of stock-holder \( i \). The voting powers need not be linearly proportional to the numbers of shares helded.