Basis partition of the space of linear programs through a differential equation ∗

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Submitted June 18, 2008, Revised May 22, 2010

Abstract: The space of linear programs (LP) can be partitioned into a finite number of sets, each corresponding to a basis. This partition is thus called the basis partition. The closed-form solution on the space of LP can be determined with the basis partition if we can characterize the basis partition. A differential equation on the Grassmann manifold which represents the space of LP provides a powerful tool for characterizing the basis partition. In paper [6], the author presented some basic concepts and properties of this differential equation. This paper continues the research of [6] and presents three useful properties.

Keywords: Linear programming, Space of linear programs, Basis partition, Grassmannian/Grassmann manifold, Projection matrix, Differential equation.

AMS subject classification: 90C05, 14X15.

*Research is supported in part by Singapore MOE AcRF T1 R-146-000-121-112.
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1 Introduction

We refer to the collection of all linear programming (LP) instances as the space of linear programs, denoted by SLP. For each LP instance we define an optimal basis as the index set of constraints which are active at every optimal solution. Since an LP instance has a strictly complementary solution, the optimal basis can be defined uniquely. For each index set $B$, we denote by $SLP(B)$ the set of LP instances which have the common optimal basis $B$. The family $\{SLP(B) : \text{all index sets } B\}$ is a partition of SLP. We refer to this partition as the basis partition.

The notion of basis comes along with the notion of complementarity. Thus, the basis partition can be defined on any space of optimization problems and complementarity problems which have a finite number of coefficients. In principle, the ideas and results of the basis partition on the space of LP can be extended to the space of linear complementarity problems, the space of quadratic programs, the space of semidefinite programs, and other more general conic programs.

If we know the optimal basis of an LP instance, we can find the solution of the LP instance by solving a system of linear equations, i.e., we can find the solution in terms of coefficients of the LP instance. This means, we can find a closed-form solution on the SLP if the basis partition is known explicitly.

Many problems involve an infinite number of optimization instances, just to name a few, parametric optimization, bilevel optimization, stochastic programming, and constrained dynamical systems, e.g., dynamical systems with state variables and control variables where control variables are determined by an optimization problem whose coefficients depend on state variables. Closed-form solutions on the space of optimization instances are useful, sometimes even necessary, for solving such problems. A way, perhaps the only way, to generate closed-form solutions on the space of optimization instances is to explicitly determine the basis partition of the space.

However, to know the basis partition explicitly is a very difficult issue, far more difficult than to solve an individual LP instance. We have now very powerful tools for solving individual LP instances, such as the simplex method and the interior point method. However, we have known very little about the basis partition, because no effective tools have been developed to find its structures.

In our previous paper [6], a novel tool, a differential equation on the space of projection
matrices (namely, the Grassmann manifold), is presented to characterize the basis partition of the space of LP. We have presented basic concepts and some properties of the basis partition and the differential equation in [6]. This paper continues this research and presents three main results:

(i) We establish a one-to-one correspondence between a path and an equilibrium-eigenvector pair.

(ii) We find an LP representation for a path in terms of the corresponding equilibrium and eigenvector.

(iii) We characterize sources and sinks of attraction regions and their boundaries (viewed as stable/unstable manifolds) through a simple calculation of dimensions of these sources and sinks.

These results provide the foundations for investigation of the basis partition. We wish that eventually we can discover and characterize structures of the basis partition in simple and explicit forms, and can design algorithms, with the discovered structures, to solve problems which need the closed-form solution map on the space of linear programs.

The rest of this paper is organized as follows. Section 2 collates concepts and notions which display the field we are studying. Section 3 presents some properties of the solution of the differential equation. Section 4 introduces some known results about center/stable/unstable manifolds. Sections 5, 6 and 7 present the three main results of this paper.

Throughout this paper we use the following notations. For any vectors $x, s \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, we denote $x \circ s = (x_1s_1, \ldots, x_ns_n)^T$, $x^\alpha = (x_1^\alpha, \ldots, x_n^\alpha)^T$, and $\lfloor x \rfloor = \text{diag}(x)$. We use the symbol $\mathbf{1}$ for the vector of all ones regardless of its dimension. For any map $f : \mathcal{V} \to \mathcal{W}$, we denote by $Df(p) : T_p\mathcal{V} \to T_{f(p)}\mathcal{W}$ the Fréchet derivative of $f$ at $p \in \mathcal{V}$.

2 Preliminaries

This section collates concepts and notions which display the field we are studying.

Consider the linear program:

$$\min \ c^T x$$
\begin{align*}
\text{s.t.} \quad Ax &= b \\
\quad x &\geq 0 \tag{2.1}
\end{align*}

and its dual
\begin{align*}
\max \quad b^T y \\
\text{s.t.} \quad A^T y + s &= c \\
\quad s &\geq 0 \tag{2.2}
\end{align*}

where $A \in \mathbb{R}^{m \times n}$ is of full row rank, and $b, c, x$ and $s$ are vectors of appropriate dimensions. We say that this linear program is of dimension $(n, m)$.

**Definition 2.1** We refer to a set of coefficients $(A, b, c)$ as a **strictly feasible instance** (in short, **instance**) of linear programming if the primal and dual problems, (2.1) and (2.2), have strictly feasible solutions, i.e. feasible solutions with $x > 0$ and $s > 0$. We denote by $SLP(n, m)$ the set of all strictly feasible instances of dimension $(n, m)$. We call $SLP(n, m)$ the space of linear programs.

**Definition 2.2** An index set $B \subset \{1, \ldots, n\}$ is said to be the **optimal basis** of $(A, b, c)$ if for each $i \in B$ the dual constraint $a_i^T y \leq c_i$ is satisfied at equality for every dual optimal solution $y$ and for each $i \notin B$ the primal constraint $x_i \geq 0$ is satisfied at equality for every primal optimal solution $x$.

Note that an instance need not be nondegenerate. A basis can be any subset, even an empty set or a full set.

It is known that for any feasible instance $(A, b, c)$, there exists a **unique** optimal basis $(B, N)$. Furthermore, there exists an optimal solution $(x, y)$ such that
\begin{align*}
Ax &= b, \quad A_B^T y = c_B, \quad A_N^T y < c_N, \quad x_B > 0, \quad x_N = 0.
\end{align*}

Such a solution is called a **strictly complementary optimal solution**.

Since each instance $(A, b, c)$ possesses a unique basis $B$, we can partition $SLP(n, m)$ into $\{SLP(B) : B \subset \{1, \ldots, n\}\}$, where $SLP(B)$ is the set of all $(A, b, c)$ whose basis is $B$. This partition is referred to as the **basis partition** of $SLP(n, m)$.

A novel tool we use to characterize the basis partition is a differential equation which is defined on the **space of projection matrices**
\[
\text{Gr}(m, n) := \{M \in S^m : MM = M, \ \text{rank}(M) = m\}.
\]
where $S^n$ is the set of all symmetric $n \times n$-matrices. The space of projection matrices is also known as the Grassmann manifold. The differential equation we use to characterize the basis partition is, cf. [5] and [6],

$$\dot{M} = h(M),$$

where the derivative is taken with respect to $t \in (-\infty, +\infty)$ and

$$h(M) := M[M1] + [M1]M - 2M[M1]M.$$

We denote by $M(t)$ or $M(t, M_0)$ the solution of $\dot{M} = h(M)$ with $M(0) = M_0$. For clarity of notation, sometimes we use the map $\phi : R \times \text{Gr}(m, n) \rightarrow \text{Gr}(m, n)$ which is defined by $\phi(t, M_0) = M(t, M_0)$. The map $\phi$ is called the flow of $\dot{M} = h(M)$ in the literature of dynamical systems. Furthermore, for any $t \in R$, we define $\phi_t : \text{Gr}(m, n) \rightarrow \text{Gr}(m, n)$ by $\phi_t(M) = \phi(t, M)$. It is known that $\phi_t \in C^\infty$ since $h \in C^\infty$.

A close relationship between $SLP(n, m)$ and $\text{Gr}(m, n)$ and the basis partitions on them was shown in [6]. Here we briefly summarize this relationship.

Throughout the paper, we define the map $\Pi$ from the set of all full rank $m \times n$-matrices to $\text{Gr}(m, n)$ by

$$\Pi(A) = A^T(AA^T)^{-1}A,$$

and the map $\Gamma : SLP(n, m) \rightarrow \text{Gr}(m, n)$ by

$$\Gamma(A, b, c) = \Pi(A[x]),$$

where $(x, s, y)$ is the analytic center of $(A, b, c)$, i.e., the unique solution of the system

$$Ax = b, \quad A^Ty + s = c, \quad x \circ s = 1, \quad x > 0, s > 0.$$

Conversely, for any $M \in \text{Gr}(m, n)$ we can construct an instance $(A, b, c)$ such that $\Gamma(A, b, c) = M$, see Lemma 2.7 in [6]. Thus, the map $\Gamma$ is surjective but not injective.

For any strictly feasible instance $(A, b, c) \in SLP(n, m)$, the map from $(A, b, c)$ to the analytic center $(x, s, y)$ and the map $\Pi$ are both $C^\infty$. Therefore, the map $\Gamma$ is $C^\infty$.

For any strictly feasible instance, the perturbed KKT system:

$$x \circ s = e^{-t}1$$
$$Ax = b$$
$$A^Ty + s = c$$
$$x > 0, \quad s > 0$$

(2.5)
has unique solution for any $t \in R$. We refer to this solution $(x(t), s(t))$, $t \in R$, as the **central path** of the LP instance. The limit of the central path $(\bar{x}, \bar{s}) = \lim_{t \to +\infty} (x(t), s(t))$ is a pair strictly complementary optimal solution of the primal and dual problems.

Any instance $(A, b, c) \in SLP(n, m)$ defines a path $M(t)$ in two ways: (i) $M(t)$ is the solution of $\dot{M} = h(M)$ with $M(0) = \Gamma(A, b, c)$, and (ii) $M(t) = \Pi(A[x(t)])$ where $x(t)$ is the central path of $(A, b, c)$. By Theorem 2.9 in [6], these two ways define the same path. Furthermore, Theorem 3.3 in [6] shows that the limit point $\bar{M} = \lim_{t \to +\infty} M(t)$ is an *equilibrium*, i.e. $h(\bar{M}) = 0$, which exhibits

$$\bar{M}1 = \begin{pmatrix} 1_B \\ 0_N \end{pmatrix}$$

where the index set $B$ is precisely the optimal basis of the instance $(A, b, c)$.

On the space $Gr(m, n)$, we can define a basis partition \{G(B) : B \subset \{1, \ldots, n\}\}, where $G(B)$ is the set of all $M \in Gr(m, n)$ such that the limit point $\bar{M} = \lim_{t \to +\infty} \phi_t(M)$ satisfies $\bar{M}1 = \begin{pmatrix} 1_B \\ 0_N \end{pmatrix}$.

It is remarkable that the basis partition of $Gr(m, n)$ is completely and solely defined by the dynamical system $\dot{M} = h(M)$, while, in the meantime, the partitions of $SLP(n, m)$ and $Gr(m, n)$ are related by $\Gamma(SLP(B)) = G(B)$. By virtue of this relationship, we can study the basis partition of $SLP(n, m)$ via the basis partition of $Gr(m, n)$. This approach will prove to be an essential advance in the study of the basis partition, due to the well-structured space $Gr(m, n)$ and the dynamical system $\dot{M} = h(M)$ on it.

We visualize the basis partition of $Gr(1, 3)$ as follows. Each line passing the origin of $R^3$ is one element in $Gr(1, 3)$. Such a line is represented as a point on the top half of the sphere, as shown in the picture below. On the boundary of the half sphere, i.e. the circle in the picture, two opposite points are on a same line, thus they represent one element. This means, two opposite points on the circle are regarded as one identical point.
where equilibrium points are

\[
\bar{M}_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

\[
\bar{M}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{M}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{M}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
\bar{M}_{12} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{M}_{23} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \bar{M}_{31} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

and all points on the circle in the form of

\[
\bar{M} = uu^T \quad \text{for} \ u \in \mathbb{R}^3 \ \text{with} \ u^T u = 1 \ \text{and} \ u^T 1 = 0.
\]

The basis partition of \( \text{Gr}(1, 3) \) consists of 8 sets \( \{ G(B) : B \subset \{1, 2, 3\} \} \) which are described as follows:

- For the basis \( B = \{i\}, \ i = 1, 2, 3, \ G(\{i\}) \) is an attraction region which is the open set containing the stable equilibrium \( \bar{M}_i \);

- For the basis \( B = \{i, j\}, \ G(\{i, j\}) \) is the line segment containing the unstable equilibrium \( \bar{M}_{ij} \);

- For the basis \( B = \{1, 2, 3\}, \ G(\{1, 2, 3\}) \) is the singleton \( \{ \bar{M}_0 \} \);

- For the basis \( B = \emptyset, \ G(\emptyset) \) is the circle. (Note that for any point \( \bar{M} \) on the circle, \( \bar{M} 1 = 0 \), thus \( B = \emptyset \).)
An annotation is necessary: Two opposite points on the circle are regarded as one identical point. Thus, the bottom boundary of \( G\{1\} \), namely the bottom one third of the circle, is also the boundary on the top of \( G\{2\} \) and \( G\{3\} \), i.e. this boundary is the boundary between \( G\{1\} \) and \( G\{2\} \) and between \( G\{1\} \) and \( G\{3\} \).

### 3 Some properties of paths \( M(t) \)

**Lemma 3.1** Let \( M(t) \) be a path satisfying \( \dot{M} = h(M) \). For any \( t_0, t_1, t_2, t_3 \in \mathbb{R} \), if \( \Gamma(A, b, c) = M(t_0) \), then \( \Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) = M(t_0 - t_1 + t_2 + t_3) \).

**Proof.** Let \( M(t \mid t_0) \) denote the solution of \( \dot{M}(t \mid t_0) = h(M(t \mid t_0)) \) with \( M(0 \mid t_0) = M(t_0) \).

Then it is known that \( M(t \mid t_0) = M(t + t_0) \) for any \( t \) and \( t_0 \).

For any \((A, b, c) \in \Gamma^{-1}(M(0 \mid t_0))\), let \((x(t), s(t), y(t))\) be the central path of \((A, b, c)\), which satisfies

\[
\begin{align*}
Ax(t) &= b \\
A^Ty(t) + s(t) &= c \\
x(t)s(t) &= e^{-t}1.
\end{align*}
\]  

By Theorem 2.9 in [6], we have

\[
M(t \mid t_0) = [x(t)]A^T(A[x(t)]^2A^T)^{-1}A[x(t)].
\]

Let \((\tilde{x}(t), \tilde{s}(t), \tilde{y}(t))\) be the central path of \((e^{t_1}A, e^{t_2}b, e^{t_3}c)\). Then it satisfies

\[
\begin{align*}
e^{t_1}A\tilde{x}(t) &= e^{t_2}b \\
e^{t_1}A^T\tilde{y}(t) + \tilde{s}(t) &= e^{t_3}c \\
\tilde{x}(t)\tilde{s}(t) &= e^{-t}1.
\end{align*}
\]  

By the definition of \( \Gamma \), we have

\[
\begin{align*}
\Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) &= [\tilde{x}(0)](e^{t_1}A)^T((e^{t_1}A)[\tilde{x}(0)]^2(e^{t_1}A)^T)^{-1}(e^{t_1}A)[\tilde{x}(0)] \\
&= [\tilde{x}(0)]A^T(A[\tilde{x}(0)]^2A^T)^{-1}A[\tilde{x}(0)]
\end{align*}
\]  

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Define $\hat{x}(t) = e^{t_1-t_2}\tilde{x}(t)$, $\hat{y}(t) = e^{t_1-t_3}\tilde{y}(t)$ and $\hat{s}(t) = e^{-t_3}\tilde{s}(t)$. It follows from (3.2) that

\[ A\hat{x}(t) = b \]
\[ A^T\hat{y}(t) + \hat{s}(t) = c \]
\[ \hat{x}(t)\hat{s}(t) = e^{-(t-t_1+t_2+t_3)}1. \]

Comparing (3.1) and (3.3), we observe that $\hat{x}(t) = x(t - t_1 + t_2 + t_3)$. Thus,

\[ \Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) = [\hat{x}(0)] A^T(A[\hat{x}(0)]^2 A^T)^{-1}A[\hat{x}(0)] \]
\[ = [\hat{x}(0)] A^T(A[\hat{x}(0)]^2 A^T)^{-1}A[\hat{x}(0)] \]
\[ = [x(-t_1 + t_2 + t_3)] A^T(A[x(-t_1 + t_2 + t_3)]^2 A^T)^{-1}A[x(-t_1 + t_2 + t_3)] \]
\[ = M(-t_1 + t_2 + t_3 | t_0) \]
\[ = M(t_0 - t_1 + t_2 + t_3). \]

This proves the lemma. \hfill \Box

**Lemma 3.2** If $A \in R^{m \times n}$ is of full row rank and $A^T A$ is a projection matrix, then

\[ AA^T = I. \]

**Proof.** Let $M = A^T A$ be a projection matrix. Then we have $MM = M$. This implies $A^T AA^T A = A^T A$. Multiplying $A$ on left and $A^T$ on right, we obtain $(AA^T)^3 = (AA^T)^2$. Since $A$ has full row rank, it follows that $AA^T = I$. \hfill \Box

**Lemma 3.3** Let $\tilde{M}, M_k \in \text{Gr}(m, n)$ with $M_k \rightarrow \tilde{M}$ as $k \rightarrow \infty$.

(i) For any $A \in R^{m \times n}$ with $\tilde{M} = \tilde{A}^T \tilde{A}$, there exist $A_k \in R^{m \times n}$ such that $M_k = A_k^T A_k$ and $A_k \rightarrow \tilde{A}$ as $k \rightarrow \infty$.

(ii) For any $\tilde{A} \in R^{m \times n}$ with $\tilde{M} = (\tilde{A} A^T)^{-1} \tilde{A}$, there exist $A_k \in R^{m \times n}$ such that $M_k = A_k^T (A_k A_k^T)^{-1} A_k$ and $A_k \rightarrow \tilde{A}$ as $k \rightarrow \infty$.

**Proof.** (i) Because $M_k A^T \rightarrow \tilde{M} A^T$ as $k \rightarrow \infty$ and $\tilde{M} A^T = \tilde{A}^T$ by Lemma 3.2, $M_k A^T$ has full column rank for large $k$ (assumed for all $k$ for simplicity). Define

\[ A_k = (\tilde{A} M_k A^T)^{-1/2} \tilde{A} M_k. \] (3.4)
As \( k \to \infty \), \( M_k \to \bar{M} \), thus
\[
A_k \to (\bar{A} \bar{M} \bar{A}^T)^{-1/2} \bar{A} \bar{M} = \bar{A}.
\]

It remains to show \( A_k^T A_k = M_k \). From the definition (3.4), we have
\[
A_k^T A_k = (\bar{A} M_k \bar{A}^T)^{-1/2} \bar{A} M_k \bar{A}^T (\bar{A} M_k \bar{A}^T)^{-1/2}
= I.
\]

(3.5)

For any \( A_k^T u \in \text{Rang}(A_k^T) \) with \( u \in \mathbb{R}^m \), let \( z = \bar{A}^T (\bar{A} M_k \bar{A}^T)^{-1/2} u \in \mathbb{R}^n \). Then \( A_k^T u = M_k z \in \text{Rang}(M_k) \). Thus, \( \text{Rang}(A_k^T) \subseteq \text{Rang}(M_k) \). Since \( M_k \bar{A}^T \) has full column rank, so does \( A_k^T \). This implies that the subspaces \( \text{Rang}(A_k^T) \) and \( \text{Rang}(M_k) \) have the same dimension \( m \). Therefore,
\[
\text{Rang}(A_k^T) = \text{Rang}(M_k).
\]

(3.6)

Since \( M_k \) is a projection matrix, (3.6) implies
\[
M_k = A_k^T (A_k A_k^T)^{-1} A_k.
\]

Then, by (3.5), we have
\[
M_k = A_k^T A_k.
\]

(ii) Now suppose that \( \bar{M} = \bar{A}^T (\bar{A} \bar{A}^T)^{-1} \bar{A} \). Define \( \hat{A} = (\bar{A} \bar{A}^T)^{-1/2} \bar{A} \). Then \( \hat{A}^T \hat{A} = \bar{M} \). As shown in part (i), there exist \( \hat{A}_k \) such that \( \hat{A}_k^T \hat{A}_k = M_k \) and \( \hat{A}_k \to \hat{A} \) as \( k \to \infty \). Define \( A_k = (\bar{A} \bar{A}^T)^{1/2} \hat{A}_k \). Then we have
\[
A_k \to (\bar{A} \bar{A}^T)^{1/2} \hat{A} = \bar{A},
\]
and
\[
A_k^T (A_k A_k^T)^{-1} A_k = \hat{A}_k^T (\hat{A}_k \hat{A}_k^T)^{-1} \hat{A}_k = \hat{A}_k^T \hat{A}_k = M_k.
\]

\[\Box\]

**Lemma 3.4** Let \( A \in \mathbb{R}^{m \times n} \) and \( W \in \mathbb{R}^{(n-m) \times n} \) be of full row rank and complementary to each other. i.e. \( AW^T = 0 \). Let \( c, d \in \mathbb{R}^n \) be any vectors. The path \( M(t) \) is associated with \((A, Ac, d)\) if and only if the path \( I - M(t) \) is associated with \((W, Wd, c)\). In particular,
\[
\Gamma(W, Wd, c) = I - \Gamma(A, Ac, d).
\]
**Remark:** Rigorously speaking, \((A, Ac, d) \in SLP(n, m)\) and \((W, Wd, c) \in SLP(n, n - m)\). Thus, the two \(\Gamma\) in the above equation are defined on different spaces.

**Proof.** Let \(\tilde{A} \in R^{m \times n}\) and \(\tilde{W} \in R^{(n-m) \times n}\) be of full row rank and satisfy \(\tilde{A}\tilde{W}^T = 0\). We have

\[
I = (\tilde{A}^T \tilde{W}^T) \left( \begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \right) \\
= (\tilde{A}^T \tilde{W}^T) \left( \begin{pmatrix} \tilde{A}^T & 0 \\ 0 & \tilde{W}\tilde{W}^T \end{pmatrix} \right)^{-1} \left( \begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \right) \\
= \tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} + \tilde{W}^T(\tilde{W}\tilde{W}^T)^{-1}\tilde{W}. 
\]

(3.7)

Next, we show that

\[
Ax = Ac \\
A^Ty + s = d \\
x \circ s = e^{-t}1 \\
x, s > 0, 
\]

and

\[
W^Tu + x = c \\
Ws = Wd \\
x \circ s = e^{-t}1 \\
x, s > 0, 
\]

have the same solution \((x, s)\), (with appropriate \(y\) and \(u\)).

Let \((x, s, y)\) be the solution of (3.8). Let

\[
\begin{pmatrix} v \\ u \end{pmatrix} = (A^T, W^T)^{-1}(c - x).
\]

That is,

\[
c - x = A^Tv + W^Tu.
\]

The first equation in (3.8) leads to

\[
0 = A(c - x) = AA^Tv + AW^Tu = AA^Tv,
\]

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which yields \( v = 0 \). It follows that

\[
c - x = W^T u.
\]

Thus, \((x, u)\) satisfies the first equation in (3.9). Multiplying the second equation in (3.8) with \( W \), we obtain the second equation in (3.9). Therefore, \((x, s, u)\) is a solution of (3.9). Analogously, we can show that any solution \((x, s)\) of (3.9) is a solution of (3.8), \((y \text{ and } u \text{ are determined accordingly})\).

Now, the path \( M(t) \) which is associated with \((A, Ac, d)\), namely with the solution of (3.8), is defined by

\[
M(t) = [x]^T A^T (A [x]^2 A^T)^{-1} A [x],
\]

and the path \( \hat{M}(t) \) which is associated with \((W, W d, c)\), namely with the solution of (3.9), is defined by

\[
\hat{M}(t) = [s]^T W^T (W [s]^2 W^T)^{-1} W [s].
\]

Let \( \hat{A} = A [x] \) and \( \hat{W} = W [s] \). Using \([x] [s] = [x \circ s] = e^{-t} I\), we have \( \hat{A} \hat{W}^T = e^{-t} AW^T = 0 \).

Thus, by (3.7), \( \hat{M}(t) = I - M(t) \).

4 Stable, unstable and center manifolds of the dynamic system

Note that the equilibria we are considering are not hyperbolic, thus we cannot apply the stable/unstable manifold theorem to the dynamic system \( \dot{M} = h(M) \). We need the Center Manifold Theorem, as Theorem 4.1 below. We cite it from [2] which is the first publication of the full proof of the Center Manifold Theorem. Also note that we will use center, stable and unstable manifolds locally, i.e. in a neighborhood of an equilibrium \( \bar{M} \).

We first present some properties for general dynamic systems in \( \mathbb{R}^n \). With a linear transformation, we can write a dynamic system in the form

\[
\begin{align*}
\dot{x} &= Ax + a(x, y, z) \\
\dot{y} &= By + b(x, y, z) \\
\dot{z} &= Cz + c(x, y, z),
\end{align*}
\]

(4.1)
where $x \in R^{k_x}$, $y \in R^{k_y}$, $z \in R^{k_z}$, and $A$, $B$, and $C$ are constant matrices such that all the eigenvalues of $A$, $B$ and $C$ have zero, negative and positive real parts, respectively. The function $a$, $b$ and $c$ are $C^l$ ($1 \leq l < \infty$) with $(a,b,c)(0) = 0$ and $(Da,Db,Dc)(0) = 0$. (Here, $Da$ is Jacobian matrix of $a$.) Thus, the origin $(x, y, z) = 0$ is an equilibrium. We will investigate invariant manifolds in a neighborhood of this equilibrium.

A set $S \subset R^{k_x} \times R^{k_y} \times R^{k_z}$ is said to be a local invariant manifold for (4.1) if for $(x_0, y_0, z_0) \in S$, the solution $(x, y, z)(t)$ of (4.1) with $(x, y, z)(0) = (x_0, y_0, z_0)$ is in $S$ for $|t| < T$ where $T > 0$. If we can always choose $T = \infty$, then we say that $S$ is an invariant manifold.

We cite the following theorem from [2] (as a special case where $\theta$ is absent). We do not claim that this theorem presents the state-of-the-art result. But it is enough for our use.

**Theorem 4.1** For system (4.1) with $3 \leq l < \infty$, there exist local invariant manifolds

$$W^s = \{ (x, y, z) \mid x = u^s(y), \| y \| < \delta, z = w^s(y) \},$$

$$W^u = \{ (x, y, z) \mid x = u^u(z), y = v^u(z), \| z \| < \delta \},$$

where $u^s, w^s, u^u, v^u$ are real vector-valued functions defined and $C^{l-2}$ in some neighborhood $N_\delta(0)$ for $\delta > 0$ sufficiently small; $u^s, w^s, u^u, v^u$ and their first-order derivatives vanish at the origin; $W^s, W^u$ are (locally) unique.

For system (4.1) with $2 \leq l < \infty$, there exist local invariant manifolds

$$W^c = \{ (x, y, z) \mid \| x \| < \delta, y = v^c(x), z = w^c(x) \},$$

$$W^{cs} = \{ (x, y, z) \mid \| (x, y) \| < \delta, z = w^{cs}(x, y) \},$$

$$W^{cu} = \{ (x, y, z) \mid \| (x, z) \| < \delta, y = v^{cu}(x, z) \},$$

where $v^c, w^c, w^{cs}, v^{cu}$ are real vector-valued functions defined and $C^{l-1}$ in some neighborhood $N_\delta(0)$ for $\delta > 0$ sufficiently small; $v^c, w^c, w^{cs}, v^{cu}$ and their first-order derivatives vanish at the origin; $(W^c, W^{cs}, W^{cu}$ need not be unique).

The invariant manifolds $W^s, W^u, W^c, W^{cs}, W^{cu}$ are called, respectively, the stable manifold, the unstable manifold, the center manifold, the center-stable manifold, and the center-unstable manifold.

As pointed out in the introduction of [3], any solution of (4.1) which does not start on the center-stable manifold at $t = 0$ (and therefore is never on the center-stable manifold) must
eventually (for \( t > T, T \) sufficiently large) remain outside any sufficiently small neighborhood of the origin. (This can be shown by using the techniques of [2] Section 7.) Thus the question of stability of the origin is really of interest only on the center-stable manifold, and on this manifold the system (4.1) reduces to

\[
\begin{align*}
\dot{x} &= Ax + a(x, y) \\
\dot{y} &= By + b(x, y),
\end{align*}
\]

(4.2)

where \( a \) and \( b \) and their first order derivatives vanish at the origin. With respect to (4.2), the stable manifold and a center manifold is given by

\[
\begin{align*}
\mathcal{W}^s &= \{(x, y) \mid x = u^s(y), \|y\| < \delta\}; \\
\mathcal{W}^c &= \{(x, y) \mid \|x\| < \delta, y = v^c(x)\},
\end{align*}
\]

where \( u^s, v^c \) and their first order derivatives vanish at \((x, y) = 0\). The flow on the center manifold \( \mathcal{W}^c \) is governed by the system

\[
\dot{p} = Ap + a(p, v^c(p)).
\]

(4.3)

The next theorem tells us that (4.3) contains all the necessary information needed to determine the asymptotical behavior of small solutions of (4.2). The theorem is cited from [1] Theorem 1.2.

**Theorem 4.2** (a) Suppose that the zeros solution of (4.3) is stable (asymptotical stable) (unstable). Then the zeros solution of (4.2) is stable (asymptotical stable) (unstable).

(b) Suppose that the zeros solution of (4.3) is stable. Let \((x(t), y(t))\) be a solution of (4.2) with \((x(0), y(0))\) sufficiently small. Then there exists a solution \(p(t)\) of (4.3) such that as \(t \to +\infty\),

\[
\begin{align*}
x(t) &= p(t) + O(e^{-\gamma t}), \\
y(t) &= v^c(p(t)) + O(e^{-\gamma t}),
\end{align*}
\]

(4.4)

where \( \gamma > 0 \) is a constant.

We define the global stable and unstable manifolds at equilibrium \((x, y, z) = 0\) as follows:

\[
\begin{align*}
\mathcal{W}_g^s &= \{(x, y, z) \mid \text{solution of (4.1) starting from } (x, y, z) \text{ reaches } \mathcal{W}^s \text{ for some } t > 0\}, \\
\mathcal{W}_g^u &= \{(x, y, z) \mid \text{solution of (4.1) starting from } (x, y, z) \text{ reaches } \mathcal{W}^u \text{ for some } t < 0\}.
\end{align*}
\]
The global stable and unstable manifolds may not be submanifolds, i.e. may not be continuous with respect to the topological structure of the ambient manifold. In this paper, we use only local topological structures of the stable and unstable manifolds, which have been shown by Theorem 4.1. Thus, in this paper, we do not investigate whether the global stable and unstable manifolds are submanifolds or not.

For the dynamic system $\dot{M} = h(M)$, according to Theorem 4.1, basic properties of the stable/unstable manifolds are as follows: Let $\mathcal{E}^-(\tilde{M})/\mathcal{E}^+(\tilde{M})$ be the generalized eigenspace which is the span of all eigenvectors for negative/positive real-part eigenvalues of $Dh(\tilde{M})$. The stable/unstable manifold $W^s(\tilde{M})/W^u(\tilde{M})$ has the same dimension as $\mathcal{E}^-(\tilde{M})/\mathcal{E}^+(\tilde{M})$ and is tangent at $\tilde{M}$ to $\mathcal{E}^-(\tilde{M})/\mathcal{E}^+(\tilde{M})$, i.e.

$$T_{\tilde{M}}W^s(\tilde{M}) = \mathcal{E}^-(\tilde{M}), \quad T_{\tilde{M}}W^u(\tilde{M}) = \mathcal{E}^+(\tilde{M}).$$

(4.5)

Note that $Dh(\tilde{M})$ has only the eigenvalue 0, one negative eigenvalue $-1$ and one positive eigenvalue 1. Thus, we can simply write

$$\mathcal{E}^0(\tilde{M}) = \{ U \in T_{\tilde{M}}(\text{Gr}(m, n)) \mid Dh(\tilde{M})U = 0 \},$$

(4.6)

$$\mathcal{E}^+(\tilde{M}) = \{ U \in T_{\tilde{M}}(\text{Gr}(m, n)) \mid Dh(\tilde{M})U = U \},$$

(4.7)

$$\mathcal{E}^-(\tilde{M}) = \{ U \in T_{\tilde{M}}(\text{Gr}(m, n)) \mid Dh(\tilde{M})U = -U \}.$$  

(4.8)

For the dynamic system $\dot{M} = h(M)$, we are able to present a center manifold explicitly. This center manifold consists of equilibrium points. First, we present some properties of the set of equilibria.

**Definition 4.3** For any $B \subset \{1, \ldots, n\}$ and $0 \leq m_B \leq m$, we define

$$\mathcal{EC}(B, m_B) = \{ M = \begin{pmatrix} M_B & 0 \\ 0 & M_N \end{pmatrix} \in \text{Gr}(m, n) \mid M_B1_B = 1_B, M_N1_N = 0, \text{rank}(M_B) = m_B \}.$$ 

We refer to $\mathcal{EC}(B, m_B)$ as an equilibrium cluster.

**Theorem 4.4** For the system $\dot{M} = h(M)$, the following statements hold true.

(i) Every equilibrium belongs to an equilibrium cluster.

(ii) If $(B_1, m_{B_1}^1) \neq (B_2, m_{B_2}^2)$, then $\mathcal{EC}(B_1, m_{B_1}^1) \cap \mathcal{EC}(B_2, m_{B_2}^2) = \emptyset$. 

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(iii) \( EC(B, m_B) \) is a connected \( C^\infty \)-submanifold of \( \text{Gr}(m, n) \).

(iv) \( EC(B, m_B) \) is a \( C^\infty \) center manifold, i.e. \( EC(B, m_B) = W^s(M) \) for any \( \bar{M} \in EC(B, m_B) \).

**Proof.** (i) follows immediately from Theorem 3.2 [6].

(ii) For any \( M^i \in EC(B_i, m_{B_i}^i), i = 1, 2 \),

\[
M^1 1 = \begin{pmatrix} 1_{B_1} \\ 0 \end{pmatrix}, \quad M^2 1 = \begin{pmatrix} 1_{B_2} \\ 0 \end{pmatrix}.
\]

If \( B_1 \neq B_2 \), then \( M^1 1 \neq M^2 1 \). Thus, \( M^1 \neq M^2 \). If \( B_1 = B_2 = B \), but \( m^1_B \neq m^2_B \), then \( \text{rank}(M^1_B) \neq \text{rank}(M^2_B) \), which also implies that \( M^1 \neq M^2 \). This shows \( EC(B_1, m^1_B) \cap EC(B_2, m^2_B) = \emptyset \).

(iii) Denote \( n_B = |B|, n_N = n - n_B, m_N = m - m_B \). Assume \( n_B \geq m_B \geq 0 \) and \( n_N \geq m_N \geq 0 \). We can write

\[
EC(B, m_B) := \{ M = \begin{pmatrix} M_B & 0 \\ 0 & M_N \end{pmatrix} \mid M_B \in \text{Gr}(m_B, n_B), M_B 1_B = 1_B, M_N \in \text{Gr}(m_N, n_N), M_N 1_N = 0_N \}.
\]

For the submatrix \( M_B \), we have

\[
\{ M_B \in G(n_B, m_B) \mid M_B 1_B = 1_B \} = \text{Gr}(m_B, n_B) \cap \{ M_B \in S^{n_B} \mid M_B 1_B = 1_B \}.
\]

Both \( \text{Gr}(m_B, n_B) \) and \( \{ M_B \in S^{n_B} \mid M_B 1_B = 1_B \} \) are connected subsets of the linear space \( S^{n_B} \). Thus, \( \{ M_B \in G(n_B, m_B) \mid M_B 1_B = 1_B \} \) is connected. Similarly, \( \{ M_N \in G(n_N, m_N) \mid M_N 1_N = 0_N \} \) is connected. Therefore, \( EC(B, m_B) \) is connected.

\( EC(B, m_B) \) is a subset of the \( C^\infty \)-manifold \( \text{Gr}(m, n) \) determined by equality constraints \( M_B 1_B = 1_B \) and \( M_N 1_N = 0_N \). Thus it is a \( C^\infty \)-submanifold.

(iv) For every \( M^0 \in EC(B, m_B) \), the solution \( M(t, M^0) = M^0 \) for all \( t \). Thus, \( EC(B, m_B) \) is invariant. By Lemma 4.6 in [6], a tangent to \( EC(B, m_B) \) at \( \bar{M} \) is an eigenvector of \( Dh(\bar{M}) \) for the eigenvalue \( \lambda = 0 \). This means that \( T_{\bar{M}} EC(B, m_B) = E^0(\bar{M}) \). Thus \( EC(B, m_B) \) is a center manifold at \( \bar{M} \).

**Remark:** At an equilibrium, there may be many center manifolds and most of them are lower order differentiable. The above theorem shows that \( EC(B, m_B) \) is a center manifold and \( C^\infty \). We believe that \( EC(B, m_B) \) is the unique center manifold. However, we do not use the uniqueness of the center manifold in this paper, thus we do not investigate this issue.
5 Correspondence between equilibrium-eigenvector pairs and paths

The partition of $\text{Gr}(m, n)$ is defined by the dynamic system $\dot{M} = h(M)$ as follows. An attraction region consists of all paths which converge to a stable equilibrium. The boundary of attraction regions consists of paths which converge to unstable equilibria. Typically, at an equilibrium $\bar{M}$, $Dh(\bar{M})$ has positive and negative eigenvalues. One can imagine that a path $M(t)$ will converge to an equilibrium in an eigenvector direction for a negative eigenvalue as $t \to +\infty$ and converge to an equilibrium in an eigenvector direction for a positive eigenvalue as $t \to -\infty$. Since the spectra of $Dh(\bar{M})$ consists of only three eigenvalues: $-1$, $0$ and $1$, we had a bold conjecture that each eigenvector corresponds to a unique path. Note that $\text{Gr}(m, n)$ comprises all paths and the partition is determined by paths. If the above conjecture is true, then the structure of the partition can be determined by the structure of the eigenspaces of $Dh(\bar{M})$ at equilibria.

It is well known that each regular (nonequilibrium) point, as an initial point, determines a unique path. However, an equilibrium point, as an initial point, does not determine a path, because the vector field at an equilibrium is zero. In order to determine a path starting from an equilibrium, we need, in addition, a direction. Now the question is whether there exists a unique path for each equilibrium-direction pair. For most dynamical systems, the answer is negative. Fortunately, for the dynamical system $\dot{M} = h(M)$, due to its simple spectral structure, we will be able to show the existence and uniqueness of a path starting from an equilibrium along an eigenvector direction.

First, we show this property for the simplest case where the Jacobian of a dynamic system has only one nonzero eigenvalue.

**Lemma 5.1** ¹ Let $N_0$ be an open neighborhood of $0$ in $\mathbb{R}^n$. Suppose that $g : N_0 \to \mathbb{R}^n$ is $C^l$, $l \geq 2$, and satisfies $g(0) = 0$ and $Dg(0) = 0$. Let $x(t, x^0)$ be the solution of

$$x' = \lambda x + g(x), \quad x(0, x^0) = x^0,$$

where $\lambda \neq 0$ is a constant. Then the following statements hold.

(i) There exists a $\delta > 0$ such that for any $x^0 \in \mathbb{R}^n$ with $\|x^0\| \leq \delta$, the solution $x(t, x^0)$ of

¹Perhaps this result is known, but I could not find it. Thus, I present a proof in Appendix 1.
(5.1) satisfies
\[ x(t, x^0) = e^{\lambda t} v + O(e^{2\lambda t} \|x^0\|^2), \]  
(5.2) for \( \lambda t \to -\infty \) (i.e., \( t \to -\infty \) if \( \lambda > 0 \) and \( t \to \infty \) if \( \lambda < 0 \)), where
\[ v = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds. \]  
(5.3) Furthermore,
\[ \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds = O(\|x^0\|^2). \]  
(5.4)

(ii) Let \( \Psi \) be a map defined by \( \Psi(x^0) = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds \). There exist open neighborhoods of 0, \( N \) and \( M \) in \( \mathbb{R}^n \), such that the map \( \Psi : N \to M \) is invertible. The map \( \Psi \) and its inverse \( \Psi^{-1} \) are \( C^1 \).

For almost all of equilibria of \( \dot{M} = h(M) \), the Jacobian \( Dh(M) \) has more than one eigenvalue, thus Lemma 5.1 cannot be used directly. Our strategy is to apply Lemma 5.1 to certain stable/unstable submanifolds each of which is associated with only one eigenvalue. This can be achieved because the spectral structure of \( Dh(M) \) is extremely simple. There are only three eigenvalues: 0, 1 and -1.

Now we begin to show that, for the dynamical system \( \dot{M} = h(M) \), any equilibrium-eigenvector pair, i.e. a pair consisting of an equilibrium \( \tilde{M} \) and an eigenvector \( U \) of \( Dh(\tilde{M}) \), determines a unique path.

**Theorem 5.2** For any equilibrium \( \tilde{M} \in \text{Gr}(m, n) \) of \( \dot{M} = h(M) \), the following hold true.

(i) There exists a diffeomorphism \( \Theta_0 : N_M \to N_U \), where \( N_M \) is an open neighborhood of \( \tilde{M} \) in \( W^s(\tilde{M}) \) and \( N_U \) an open neighborhood of the origin in \( E^-(\tilde{M}) \).

(ii) There exists a bijection \( \Theta : \mathcal{W}_g^s(\tilde{M}) \to E^-(\tilde{M}) \), such that \( U = \Theta(M^0) \) if and only if
\[ M(t, M^0) = \tilde{M} + e^{\lambda t} U + O(e^{2\lambda t}), \]  
(5.5) for \( \lambda = -1 \) and \( t \to +\infty \). The map \( \Theta_0 \) is a restriction of \( \Theta \).

The same statements hold true for \( \mathcal{W}_g^u(\tilde{M}), E^+(\tilde{M}) \), \( \lambda = 1 \) and \( t \to -\infty \).
Proof. For ease of reading, we write several lemmas in the proof.

In the proof, we will only consider \( \lambda = -1 \). For \( \lambda = 1 \), the proof is precisely the same, with only the stable manifold changed to the unstable manifold.

Before Lemma 5.5, we will consider only local properties, i.e., all statements are referred to small neighborhoods of some points in respective spaces.

By Theorem 4.1 and as stated in (4.5), there exists a manifold \( \mathcal{W}^s(\bar{M}) \) tangent to \( \mathcal{E}^-(\bar{M}) \) at \( \bar{M} \) and invariant under the dynamic system \( \dot{M} = h(M) \). Because \( h \) is \( C^\infty \), so is \( \mathcal{W}^s(\bar{M}) \). Let \( k \) be the dimension of \( \mathcal{W}^s(\bar{M}) \). Let \( \varphi : \mathcal{N}_M \to \mathcal{N}_x \) be a diffeomorphism with \( \varphi(\bar{M}) = 0 \), where \( \mathcal{N}_M \subset \mathcal{W}^s(\bar{M}) \) is an open neighborhood of \( \bar{M} \) and \( \mathcal{N}_x \subset R^k \) is an open neighborhood of 0. ((\( \mathcal{N}_M, \varphi \)) is a chart of the smooth manifold \( \mathcal{W}^s(\bar{M}) \)).

From \( \dot{M} = h(M) \) we can derive an ODE \( x' = f(x) \) on \( R^k \) in terms of \( x = \varphi(M) \):

\[
x' = D\varphi(M)\dot{M} = D\varphi(M)h(M)
\]

At \( x = 0 \), we have

\[
f(0) = D\varphi(\varphi^{-1}(0))h(\varphi^{-1}(0)) = D\varphi(\bar{M})h(\bar{M}) = 0,
\]

The Jacobian of \( f \) is

\[
Df(x) = D^2\varphi(\varphi^{-1}(x))D\varphi^{-1}(x)h(\varphi^{-1}(x)) + D\varphi(\varphi^{-1}(x))Dh(\varphi^{-1}(x))D\varphi^{-1}(x).
\]

Because \( \bar{M} = \varphi^{-1}(0) \) and \( h(\bar{M}) = 0 \),

\[
Df(0) = D\varphi(\bar{M})Dh(\bar{M})D\varphi^{-1}(0). \tag{5.6}
\]

For any \( v \in R^k \) and small \( t \in R \), \( \varphi^{-1}(tv) \) is a curve on \( \mathcal{W}^s(\bar{M}) \) passing through \( \varphi^{-1}(0) = \bar{M} \). Thus, the vector \( \frac{d}{dt}\varphi^{-1}(tv)|_{t=0} \) is a tangent of \( \mathcal{W}^s(\bar{M}) \) at \( \bar{M} \). Since \( \frac{d}{dt}\varphi^{-1}(tv)|_{t=0} = D\varphi^{-1}(0)v \) and \( D\varphi^{-1}(x) = [D\varphi(\varphi^{-1}(x))]^{-1} = [D\varphi(M)]^{-1} \), we see that \( [D\varphi(\bar{M})]^{-1}v \in T_\bar{M}(\mathcal{W}^s(\bar{M})) \).

Conversely, for any \( U \in T_\bar{M}(\mathcal{W}^s(\bar{M})) \), there exists a curve \( M(t) \) with \( M(0) = \bar{M} \) and \( \dot{M}(0) = U \). Let \( \xi(t) = \varphi(M(t)) \). Then \( \xi(t) \) is a curve in \( \mathcal{V} \) passing through \( x = 0 \). Let \( v = \xi'(0) \). Then \( v \in R^k \) and \( v = D\varphi(M(0))\dot{M}(0) = D\varphi(\bar{M})U \). Thus, \( U = [D\varphi(\bar{M})]^{-1}v \). This shows

\[
T_\bar{M}(\mathcal{W}^s(\bar{M})) = \{[D\varphi(\bar{M})]^{-1}v \mid v \in R^k \} \tag{5.7}
\]
Since $W^s(M)$ is tangent to $E^-(M)$ at $M$, i.e.

$$T_M(W^s(M)) = E^-(M),$$

we have

$$E^-(M) = \{[D\varphi(M)]^{-1}v \mid v \in \mathbb{R}^k\}. \tag{5.8}$$

This shows that $[D\varphi(M)]^{-1}v$ is an eigenvector of $Dh(M)$ for $\lambda = -1$, i.e.,

$$Dh(M)[D\varphi(M)]^{-1}v = \lambda [D\varphi(M)]^{-1}v, \quad \forall v \in \mathbb{R}^k.$$

This yields

$$Dh(M)[D\varphi(M)]^{-1} = \lambda [D\varphi(M)]^{-1}.$$ 

It follows from (5.6) that

$$Df(0) = D\varphi(M) Dh(M)[D\varphi(M)]^{-1} = \lambda D\varphi(M)[D\varphi(M)]^{-1} = \lambda I.$$

Now we denote $g(x) = f(x) - \lambda x$ and write $x' = f(x)$ as

$$x' = \lambda x + g(x).$$

Then $g$ is analytic on the open set $\mathcal{N}_x$ of $\mathbb{R}^k$ and satisfies $g(0) = 0$ and $Dg(0) = Df(0) - \lambda I = 0$.

**Lemma 5.3** There exists an open neighborhood $\mathcal{N}_U$ of the origin in $E^-(M)$, such that for any $U \in \mathcal{N}_U$ there exists a unique $M^0 \in \mathcal{N}_M$ satisfying (5.5).

**Proof.** For any eigenvector $U$ of $Dh(M)$ for $\lambda = -1$, by (5.8) there is a $v \in \mathbb{R}^k$ such that $U = [D\varphi(M)]^{-1}v$. If $\|U\|$ is small, then $|v|$ is small. By Lemma 5.1 (ii), there exists a unique $x^0$ such that the solution $x(t, x^0)$ of $x' = \lambda x + g(x)$ with $x(0, x^0) = x^0$ satisfies

$$x(t, x^0) = e^{\lambda t}v + O(e^{2\lambda t}), \quad \forall \lambda t \leq 0. \tag{5.9}$$

We may assume that $|v|$ is so small that $x(t, x^0) \in \mathcal{N}_x$ for all $t \in [0, +\infty)$. Let $M^0 = \varphi^{-1}(x^0)$ and $M(t, M^0) = \varphi^{-1}(x(t, x^0))$. Then $M(t, M^0) \in \mathcal{N}_M$ for all $t \in [0, +\infty)$. It is easy to verify
that \( M(t, M^0) \) is the solution of \( \dot{M} = h(M) \) with \( M(0, M^0) = M^0 \), and satisfies

\[
M(t, M^0) = \varphi^{-1}(x(t, x^0)) \\
= \varphi^{-1}(0) + D\varphi^{-1}(0)x(t, x^0) + O(\|x(t, x^0)\|^2) \\
= \tilde{M} + [D\varphi(\tilde{M})]^{-1}x(t, x^0) + O(\|x(t, x^0)\|^2) \\
= \tilde{M} + e^M[D\varphi(\tilde{M})]^{-1}v + O(e^{2M}) \\
= \tilde{M} + e^MU + O(e^{2M}).
\]

The uniqueness of the initial point \( M^0 \) which satisfies the above can be shown as follows.

Let \( M^0 \) and \( \tilde{M}^0 \) be initial points which correspond to the same \( \tilde{M} \) and \( U \). By Theorem 4.1, the stable manifold \( W_s(\tilde{M}) \) is unique. We may assume that \( M^0 \) and \( \tilde{M}^0 \) are in \( W_s(\tilde{M}) \), in particular, in the same neighborhood \( N_M \) of \( \tilde{M} \) with the diffeomorphism \( \varphi \). Let \( x^0 = \varphi(M^0) \) and \( \tilde{x}^0 = \varphi(\tilde{M}^0) \). Then, the solutions \( x(t, x^0) \) and \( x(t, \tilde{x}^0) \) converge to the same point \( 0 = \varphi(\tilde{M}) \) in the same direction \( v = [D\varphi^{-1}(0)]^{-1}U \). By Lemma 5.1 (ii), we have \( x^0 = \tilde{x}^0 \). This implies \( M^0 = \varphi^{-1}(x^0) = \varphi^{-1}(\tilde{x}^0) = \tilde{M}^0 \). Therefore, the initial point \( M^0 \) is uniquely determined by \( \tilde{M} \) and \( U \). This proves the lemma 5.3.

**Lemma 5.4** For any \( M^0 \in N_M \), there exists a unique \( U \in N_U \) satisfying (5.5).

**Proof.** For any \( M^0 \in N_M \subset W^s(\tilde{M}) \), let \( x^0 = \varphi(M^0) \). Let the system \( \dot{x} = \lambda x + g(x) \) correspond to \( \dot{M} = h(M) \) restricted to \( W^s(\tilde{M}) \). Define \( v = x^0 + \int_0^\infty e^{-\lambda s}g(x(s, x^0))ds \). By Lemma 5.1, we have

\[
x(t, x^0) = e^Mv + O(e^{2M}).
\]

Let \( U = D\varphi^{-1}(0)v \). Then the solution \( M(t, M^0) \) of \( \dot{M} = h(M) \) with \( M(0, M^0) = M^0 \) satisfies

\[
M(t, M^0) = \tilde{M} + e^MU + O(e^{2M}).
\]

The uniqueness of \( U \in E^-(\tilde{M}) \) satisfying (5.5) is obvious. If

\[
M(t, M^0) = \tilde{M} + e^MU + O(e^{2M}) \text{ and } M(t, M^0) = \tilde{M} + e^\tilde{M}U + O(e^{2M}),
\]

then we have \( U - \tilde{U} = O(e^\lambda) \to 0 \), as \( t \to +\infty \). Thus \( U = \tilde{U} \). This completes the proof of Lemma 5.4.

**Proof of Theorem 5.2 (i).**
Define \( \Theta_0 : N_M \rightarrow N_U \) by \( M^0 \mapsto U \) which satisfy (5.5).

Lemmas 5.3 and 5.4 show that \( \Theta_0 : N_M \rightarrow N_U \) is bijective.

We can write \( \Theta_0 = \Upsilon \circ \Psi \circ \varphi \), where \( \varphi : M^0 \mapsto x^0 \) is defined at the beginning of the proof of Theorem 5.2, \( \Psi : x^0 \mapsto v \) is defined in Lemma 5.1 and \( \Upsilon : v \mapsto U = [D\varphi(M)]^{-1}v \). Since the dynamic system \( \dot{M} = h(M) \) is \( C^\infty \), all the three maps and their inverses are \( C^\infty \). Thus, \( \Theta_0 \) and its inverse are \( C^\infty \). This shows that \( \Theta_0 : N_M \rightarrow N_U \) is a diffeomorphism. This completes the proof of Theorem 5.2 part (i).

Lemma 5.5 For any \( M^0 \in W_s^g(\bar{M}) \), \( e^{-\lambda t} \Theta_0(M(t, M^0)) \) is a constant, namely, independent of \( t \). Here we assume that \( M(t, M^0) \in N_M \) so that \( \Theta_0 \) can be defined.

Proof. For any \( \hat{t}, \tilde{t} \in \mathbb{R} \), denote \( \hat{M}^0 = M(\hat{t}, M^0), \tilde{M}^0 = M(\tilde{t}, M^0) \), \( \hat{U} = \Theta_0(\hat{M}^0) \) and \( \tilde{U} = \Theta_0(\tilde{M}^0) \). By the definition of \( \Theta_0 \), \( (\hat{M}^0, \hat{U}) \) and \( (\tilde{M}^0, \tilde{U}) \) satisfy (5.5), namely,

\[
\begin{align*}
M(t, \hat{M}^0) &= \hat{M} + e^{\lambda t} \hat{U} + O(e^{2\lambda t}) \quad (5.10) \\
M(t, \tilde{M}^0) &= \tilde{M} + e^{\lambda t} \tilde{U} + O(e^{2\lambda t}). \quad (5.11)
\end{align*}
\]

By the semigroup property,

\[
\begin{align*}
M(t, \hat{M}^0) &= M(t, M(\hat{t}, M^0)) \\
&= M(t + \hat{t}, M^0) \\
&= M(t + \hat{t} - \tilde{t}, M(\tilde{t}, M^0)) \\
&= M(t + \hat{t} - \tilde{t}, \tilde{M}^0).
\end{align*}
\]

Thus, it follows from (5.10) and (5.11) that

\[
e^{\lambda t} \hat{U} + O(e^{2\lambda t}) = e^{\lambda(t+\tilde{t}-\hat{t})} \tilde{U} + O(e^{2\lambda t}).
\]

Multiplying \( e^{-\lambda t} \) on both sides, we obtain

\[
\hat{U} = e^{\lambda(t-\tilde{t})} \tilde{U} + O(e^{\lambda t}).
\]

Letting \( t \rightarrow +\infty \) (\( \lambda = -1 \)), we have

\[
e^{-\lambda t} \hat{U} = e^{-\lambda t} \tilde{U}.
\]

This shows that \( (\hat{t}, \hat{U}) \) and \( (\tilde{t}, \tilde{U}) \) define the same image. The proof of Lemma 5.5 is completed. \(\square\)

By Lemma 5.5, the following definition is independent of the choice of \( \hat{t} \). Thus, the map \( \Theta \) is well defined.
Definition 5.6 We define the map $\Theta : \mathcal{W}_g(\bar{M}) \to \mathcal{E}^-(\bar{M})$ as follows. For any $M^0 \in \mathcal{W}_g(\bar{M})$, choose an $\hat{t} \geq 0$ satisfying $\hat{M}^0 := M(\hat{t}, M^0) \in N_M$. Let $\hat{U} = \Theta_0(\hat{M}^0)$ and define

$$\Theta(M^0) = e^{-\lambda \hat{U}} \in \mathcal{E}^-(\bar{M}).$$

(5.12)

Lemma 5.7 $\Theta : \mathcal{W}_g(\bar{M}) \to \mathcal{E}^-(\bar{M})$ is bijective.

Proof. Surjective: For any $U \in \mathcal{E}^-(\bar{M})$, let $\alpha > 0$ be small so that $\hat{U} = \alpha U \in N_U$. Denote $\hat{M}^0 = \Theta_0^{-1}(\hat{U})$ and $M^0 = M(-\lambda^{-1} \ln \alpha, \hat{M}^0)$. Then, $\hat{M}^0 \in N_M$. Since $M(\lambda^{-1} \ln \alpha, M^0) = \hat{M}^0 \in N_M$, thus $M^0 \in \mathcal{W}_g(\bar{M})$. Let $\hat{t} = \lambda^{-1} \ln \alpha$. Then by the definition of $\Theta$,

$$\Theta(M^0) = e^{-\lambda \hat{t} \hat{U}} = e^{-\lambda \hat{t} \hat{U}} \Theta_0(\hat{M}^0) = \alpha^{-1} \Theta_0(\hat{M}^0) = \alpha^{-1} \hat{U} = U.$$

This shows that $\Theta$ is surjective.

Injective: For any $M^0, M^1 \in \mathcal{W}_g(\bar{M})$, we can choose $\hat{t} \geq 0$ such that $\hat{M}^0 = M(\hat{t}, M^0) \in N_M$ and $\hat{M}^1 = M(\hat{t}, M^1) \in N_M$. Then $\Theta(M^0) = e^{-\lambda \hat{t} \hat{U}} \Theta_0(\hat{M}^0)$ and $\Theta(M^1) = e^{-\lambda \hat{t} \hat{U}} \Theta_0(\hat{M}^1)$. If $\Theta(M^0) = \Theta(M^1)$, then $\Theta_0(\hat{M}^0) = \Theta_0(\hat{M}^1)$. Since $\Theta_0$ is bijective, we have $\hat{M}^0 = \hat{M}^1$, which implies

$$M^0 = M(-\hat{t}, \hat{M}^0) = M(-\hat{t}, \hat{M}^1) = M^1.$$

This shows that $\Theta$ is injective.

Lemma 5.8 $U = \Theta(M^0)$ if and only if $(M^0, U)$ satisfy (5.5).

Proof. Suppose $U = \Theta(M^0)$. Choose $\hat{t} \geq 0$ such that $\hat{M}^0 = M(\hat{t}, M^0) \in N_M$. Denote $\hat{U} = \Theta_0(\hat{M}^0)$. By definition of $\Theta$, $\Theta(M^0) = e^{-\lambda \hat{U}}$. Thus, $U = e^{-\lambda \hat{U}}$. By definition of $\Theta_0$, we have

$$M(t, \hat{M}^0) = \hat{M} + e^{\lambda \hat{U}} + O(e^{2\lambda \hat{U}}).$$

Thus,

$$M(t, M^0) = M(t - \hat{t}, \hat{M}^0) = \hat{M} + e^{\lambda(t - \hat{t}) \hat{U}} + O(e^{2\lambda(t - \hat{t})}) = \hat{M} + e^{\lambda \hat{U}} + O(e^{2\lambda \hat{U}}).$$

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Conversely, suppose that $M^0 \in W^s_g(\bar{M})$ and $U \in \mathcal{E}^-(\bar{M})$ satisfy (5.5). Choose $\hat{t} \geq 0$ such that $\hat{M}^0 = M(\hat{t}, M^0) \in N_M$. Let $\hat{U} = \Theta_0(M^0)$. Then by definition of $\Theta$, $\Theta(M^0) = e^{-\lambda \hat{U}}$. By definition of $\Theta_0$, we have

$$M(t, \hat{M}^0) = \tilde{M} + e^\lambda \hat{U} + O(e^{2\lambda t}).$$

It follows that

$$M(t, M^0) = M(t - \hat{t}, \hat{M}^0) = \tilde{M} + e^{\lambda (t-\hat{t})} \hat{U} + O(e^{2\lambda (t-\hat{t})}) = \tilde{M} + e^\lambda e^{-\lambda \hat{t}} \hat{U} + O(e^{2\lambda t}).$$

Comparing with (5.5), we obtain

$$e^\lambda U + O(e^{2\lambda t}) = e^\lambda e^{-\lambda \hat{t}} \hat{U} + O(e^{2\lambda t}).$$

This implies $U = e^{-\lambda \hat{t}} \hat{U}$. That is, $U = \Theta(M^0)$.

Lemmas 5.7 and 5.8 prove Theorem 5.2 part (ii).

The proof of Theorem 5.2 is completed.

Remark: If there is a topology and differential structure on $W^s_g(\bar{M})$ such that, for any $t \in \mathbb{R}$, the map $M(t, \cdot) : W^s_g(\bar{M}) \to W^s_g(\bar{M})$ and its inverse are $C^\infty$, then the map $\Theta : W^s_g(\bar{M}) \to \mathcal{E}^-(\bar{M})$ defined by (5.5) is a diffeomorphism. This can be shown as follows. Theorem 5.2 has shown that $\Theta : W^s_g(\bar{M}) \to \mathcal{E}^-(\bar{M})$ is bijective. So, it suffices to show that $\Theta$ and its inverse are (locally) $C^\infty$ at any point $M^0 \in W^s_g(\bar{M})$. Suppose $\hat{M}^0 = M(\hat{t}, M^0) \in N_M$, i.e. $\hat{M}^0$ is close to $\bar{M}$. We can write $\Theta = \sigma \circ \Theta_0 \circ M(\hat{t}, \cdot)$, where $\sigma$ is defined by $U = \sigma(\hat{U}) = e^{-\lambda \hat{t}} \hat{U}$. Since $\sigma, \Theta_0, M(\hat{t}, \cdot)$ are (local) diffeomorphisms on respective neighborhoods, $\Theta$ and its inverse are locally $C^\infty$.

Theorem 5.2 shows that every point $M^0 \in W^s_g(\bar{M})$ determined a path which converges to $\bar{M}$ in a direction $U$. Next, in Theorem 5.10, we will show that if a point $M^0$ determines a path converging to an equilibrium $\tilde{M}$, then $M^0 \in W^s_g(\bar{M})$.

The following property is used to prove Theorem 5.10.

**Theorem 5.9** Let $(x(t), y(t), z(t))$ be the solution of (4.1) with $(x(0), y(0), z(0)) = (x^0, y^0, z^0)$ and $\gamma > 0$ be a constant.

---

2This result is probably known. But I could not find it in the literature. My colleague Weixiao Shen kindly provided a proof, see Appendix 2
(i) If the curve \((x(t), y(t), z(t))\) intersects every open neighborhood of the origin and \((x(t), y(t)) = O(e^{-\gamma t})\) for \(t \to +\infty\), then \((x^0, y^0, z^0) \in W^s_\gamma(0)\).

(ii) If the curve \((x(t), y(t), z(t))\) intersects every open neighborhood of the origin and \((x(t), z(t)) = O(e^{\gamma t}), t \to -\infty\), then \((x^0, y^0, z^0) \in W^s_\gamma(0)\).

**Theorem 5.10** 3 For any equilibrium \(\bar{M}\) of the dynamic system \(\dot{M} = h(M)\),

\[
W^s_\gamma(\bar{M}) = \{ M^0 \in \text{Gr}(m, n) | \lim_{t \to +\infty} M(t, M^0) = \bar{M} \}
\]

\[
W^u_\gamma(\bar{M}) = \{ M^0 \in \text{Gr}(m, n) | \lim_{t \to -\infty} M(t, M^0) = \bar{M} \}.
\]

**Proof.** We only prove the first equation. The second equation is an analogue.

Since \(Dh(\bar{M})\) restricted to \(E^- (\bar{M})\) has only eigenvalues with negative real parts and \(E^- (\bar{M}) = T_{\bar{M}}(W^s(\bar{M}))\), \(M(t, M^0)\) converges to \(\bar{M}\) for any \(M^0 \in W^s(\bar{M})\). For any \(M^0 \in W^s_\gamma(\bar{M})\), \(M(t, M^0)\) converges to \(W^s(\bar{M})\) and thus also converges to \(\bar{M}\). This proves

\[
W^s_\gamma(\bar{M}) \subseteq \{ M^0 \in \text{Gr}(m, n) | \lim_{t \to +\infty} M(t, M^0) = \bar{M} \}.
\]

Conversely, suppose that \(M(t, M^0) \to \bar{M}\), as \(t \to +\infty\). (The case of \(t \to -\infty\) can be shown similarly with \(W^s\) and \(W^s\) changed to \(W^u\) and \(W^u\).)

First we assume that \(M^0\) is sufficiently close to \(\bar{M}\). As argued in Introduction of [3], \(M^0\) must be in a center-stable manifold \(W^{cs}(\bar{M})\). We can construct a diffeomorphism from \(W^{cs}(\bar{M})\) to a neighborhood of the origin in \(R^k_0 \times R^k_s\), which maps the system \(\dot{M} = h(M)\) to a system like (4.2) with the solution \((x(t), y(t))\) corresponding \(M(t)\). In particular, \(M(t) \to \bar{M}\) implies \((x(t), y(t)) \to (0, 0)\). By Theorem 4.2 (b), the solution \((x(t), y(t))\) satisfies (4.4) with a solution \(p(t)\) on the center manifold. By Theorem 4.4 (iv), \(W^s(\bar{M}) = EC(B, m_B)\). This means that all points in the center manifold \(W^c(\bar{M})\) are equilibria. Thus, any solution on the center manifold is a constant. By (4.4), \(y(t) \to 0\) implies \(p(t) \to 0\). Since \(p(t)\) is a constant, it can only be \(p(t) \equiv 0\). Thus, (4.4) yields \((x(t), y(t)) = O(e^{-\gamma t})\). By Theorem 5.9, \((x(t), y(t))\) is in the stable manifold (for \(t \geq \hat{t}\)). Correspondingly, \(M(t) \in W^s(\bar{M})\). In particular, \(M^0 \in W^s(\bar{M})\).

Now, suppose that \(M^0\) is not close to \(\bar{M}\). Since \(M(t, M^0) \to \bar{M}\), there exists a sufficiently large \(\hat{t}\) such that \(\hat{M}^0 = M(\hat{t}, M^0)\) is sufficiently close to \(\bar{M}\). As above, we can show that \(\hat{M}^0 \in W^s(\bar{M})\), i.e. \(\hat{M}(\hat{t}, M^0) \in W^s(\bar{M})\). This implies that \(M^0 \in W^s_\gamma(\bar{M})\).

\[\Box\]

3The author would like to thank his colleague Weixiao Shen for enlightening him on the proof of this theorem.
Since every point $M \in \text{Gr}(m, n)$ defines a unique path (if $M$ is an equilibrium, then it defines a constant path $M(t) \equiv M$) and every path converges to an equilibrium in a unique direction as $t \to +\infty$ and to another equilibrium in a unique direction as $t \to -\infty$, we have the following corollary.

**Corollary 5.11** There are one-to-one correspondences between

(i) a point $M \in \text{Gr}(m, n)$,

(ii) a path $M(t)$,

(iii) an equilibrium-eigenvector pair $(\bar{M}, U)$ for $\lambda = -1$, and

(iv) an equilibrium-eigenvector pair $(\bar{M}, U)$ for $\lambda = 1$.

For an equilibrium cluster $\mathcal{E}(B, m_B)$, we define

$$
\mathcal{W}^s(B, m_B) := \{ M \in \text{Gr}(m, n) : \lim_{t \to +\infty} \phi_t(M) \in \mathcal{E}(B, m_B) \},
$$

$$
\mathcal{W}^u(B, m_B) := \{ M \in \text{Gr}(m, n) : \lim_{t \to -\infty} \phi_t(M) \in \mathcal{E}(B, m_B) \}.
$$

We call both of $\mathcal{W}^s(\bar{M})$ and $\mathcal{W}^u(B, m_B)$ the stable manifold. Their distinction is obvious and thus no confusion will be caused. In fact, $\mathcal{W}^s(B, m_B)$ is a center-stable manifold $\mathcal{W}^{cs}(\bar{M})$ at a point $\bar{M} \in \mathcal{E}(B, m_B)$.

Every stable equilibrium $M(B)$ is itself an equilibrium cluster (a singleton). For $\mathcal{E}(B, m_B) = \{ M(B) \}$, all paths in the stable manifold $\mathcal{W}^s(B, m_B)$ converge to the stable equilibrium $M(B)$. Thus, this stable manifold is the attraction region $G(B)$. Stable manifolds associated with unstable equilibrium clusters are boundaries of attraction regions.

A stable manifold $\mathcal{W}^s(B, m_B)$ consists of paths which converge to equilibria in $\mathcal{E}(B, m_B)$ in directions of eigenvectors. Thus, the stable manifold $\mathcal{W}^s(B, m_B)$ is isomorphic to the set of equilibrium-eigenvector pairs

$$
\Sigma^-(B, m_B) := \{(M, U) \mid M \in \mathcal{E}(B, m_B), \ U \in \mathcal{E}^-(M) \}.
$$

We refer to $\Sigma^-(B, m_B)$ as the sink of the stable manifold $\mathcal{W}^s(B, m_B)$. Similarly, an unstable manifold $\mathcal{W}^u(B, m_B)$ is isomorphic to the source

$$
\Sigma^+(B, m_B) := \{(M, U) \mid M \in \mathcal{E}(B, m_B), \ U \in \mathcal{E}^+(M) \}.
$$

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The analysis of the sinks/sources is easier than the analysis of the stable/unstable manifolds, because equilibria and eigenvectors have simple structures. We will investigate $\Sigma^{-}(B, m_B)$ and $\Sigma^{+}(B, m_B)$ in details in Section 6.

### 6 LP representation of equilibrium-eigenvector pairs

We can map an LP instance $(A, b, c)$ to a projection matrix $M = \Gamma(A, b, c)$, then define a path $M(t)$ by $\dot{M} = h(M)$ with $M(0) = \Gamma(A, b, c)$. This path $M(t)$ converges to an equilibrium $\bar{M}$ in a direction $\bar{U}$ in the sense of (5.5). We will say that $(A, b, c)$ is mapped to $(\bar{M}, \bar{U})$ and denote this map by

$$(\bar{M}, \bar{U}) = \Lambda(A, b, c).$$

Conversely, given an equilibrium-eigenvector pair $(\bar{M}, \bar{U})$, in this section we will construct an instance $(A, b, c)$ which is mapped to $(\bar{M}, \bar{U})$. We will call such an instance $(A, b, c)$ an LP representation of $(\bar{M}, \bar{U})$.

The motivations of LP representations are two-fold. First, the characterization of the partition on the space of $(A, b, c)$ is our ultimate goal. Thus, we need an LP representation of $(\bar{M}, \bar{U})$ to carry the partition of the space of $(\bar{M}, \bar{U})$ onto the partition of the space of $(A, b, c)$. Second, the central path $x(t)$ and the optimal solution $x^*$ of an LP instance $(A, b, c)$ are helpful in constructing the corresponding path $M(t)$ and the limiting equilibrium $\bar{M}$. For example, if we want to find the sink of a given source, we can construct an LP representation $(A, b, c)$ of the given source $(M_-, U_-)$ and find the optimal solution $x^*$ of $(A, b, c)$, then construct the sink point $M_+$ by Theorem 3.3 in [6].

One will see many applications of the LP representation to the characterization of attraction regions and their boundaries in our coming papers.

In this section, we will show how to construct an LP instance for a path in terms of the corresponding equilibrium-eigenvector pair. Finding an LP representation of $(\bar{M}, \bar{U})$ turns out to be nontrivial due to the fact that the pair $(\bar{M}, \bar{U})$ is a limiting feature of a path.

For any equilibrium $M$ and vector $d$, we denote

$$h_M(d) := M[d] + [d]M - 2M[d]M.$$
Theorem 6.1 For any $\tilde{M} \in \text{Gr}(m, n)$ with $\tilde{M}1 = 0$ and any eigenvector $\tilde{U} = h_{\tilde{M}}(d)$, with $\tilde{M}d = d$, of $Dh(\tilde{M})$ for $\lambda = 1$, let $\tilde{A} \in R^{m \times n}$ satisfy $\tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} = \tilde{M}$. Then

$$\Lambda(\tilde{A}, \tilde{A}d, 1) = (\tilde{M}, \tilde{U}).$$

Proof. First, we show that the instance $(A, b, c) = (\tilde{A}, \tilde{A}d, 1)$ is strictly feasible. Note that $\tilde{M}1 = 0$ implies $\tilde{A}1 = 0$. Thus, for any $\alpha \in R$, $x = d + \alpha 1$ satisfies $\tilde{A}x = \tilde{A}d$. Choosing $\alpha > 0$ sufficiently large, we can have $x = d + \alpha 1 > 0$. Now, $y = 0$ and $s = 1 > 0$ satisfies $\tilde{A}_T y + s = 1$. Thus, $(\tilde{A}, \tilde{A}d, 1)$ has strictly feasible solutions.

For given $\tilde{M}$ and $\tilde{U}$, by Theorem 5.2 (ii), there exists a unique (initial) point $M^0$ such that the unique solution $M(t)$ of $\dot{M} = h(M)$ with $M(0) = M^0$ satisfies (5.5). In order to show $\Lambda(A, \tilde{A}d, 1) = (\tilde{M}, \tilde{U})$, we need only to show that the path $M(t)$ is defined by $(A, \tilde{A}d, 1)$, namely, $\Gamma(A, \tilde{A}d, 1) = M^0$.

First, we assume that $\tilde{A}_T \tilde{A} = \tilde{M}$.

Because $M(-k) \to \tilde{M}$ as $k \to \infty$, by Lemma 3.3, there exist $A_k \in R^{m \times n}$ satisfying $A_k^T A_k = M(-k)$ and $A_k \to \tilde{A}$ as $k \to \infty$. Then, it follows from (5.5) that

$$A_k^T A_k 1 = M(-k)1 = e^{-k}\tilde{U}1 + O(e^{-2k}) = e^{-k}d + O(e^{-2k}).$$

Since $A_k^T A_k$ is a projection matrix, by Lemma 3.2, we have $A_k A_k^T = I$. Thus,

$$A_k 1 = e^{-k}A_k d + O(e^{-2k}).$$

Let $f_k = e^k A_k 1$. Since $A_k \to \tilde{A}$, we have $f_k \to \tilde{A}d$ as $k \to \infty$.

Because $A_k^T A_k = M(-k)$, by Lemma 2.7 in [6], we have $\Gamma(A_k, A_k 1, 1) = M(-k)$, namely, $\Gamma(A_k, e^{-k}f_k, 1) = M(-k)$. By Lemma 3.1, $\Gamma(A_k, f_k, 1) = M(0) = M^0$.

Since $\Gamma : SLP(n, m) \to \text{Gr}(m, n)$ is continuous, we have

$$M^0 = \lim_{k \to \infty} \Gamma(A_k, f_k, 1) = \Gamma(A, \tilde{A}d, 1).$$

Now, consider any $\tilde{A}$ satisfying $\tilde{A}_T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} = \tilde{M}$. Let $Q = (\tilde{A}\tilde{A}^T)^{-1/2}$ and $\tilde{A} = Q\tilde{A}$. Then $\tilde{A}_T \tilde{A} = \tilde{M}$. As we showed above, $\Lambda(\tilde{A}, \tilde{A}d, 1) = (\tilde{M}, \tilde{U})$. 

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Because for any invertible matrix \( Q \in \mathbb{R}^{m \times m} \), \((QA, Qb, c)\) and \((A, b, c)\) define the same path \( M(t) \), we have
\[
\Lambda(\bar{A}, \bar{A}d, 1) = \Lambda(Q\bar{A}, Q\bar{A}d, 1) = \Lambda(\bar{A}, \bar{A}d, 1) = (\bar{M}, \bar{U}).
\]

\[\square\]

**Theorem 6.2** For any \( \bar{M} \in \text{Gr}(m, n) \) with \( \bar{M}1 = 1 \) and any eigenvector \( \bar{U} = h_{\bar{M}}(d) \), with \( \bar{M}d = 0 \), of \( Dh(\bar{M}) \) for \( \lambda = 1 \), let \( \bar{A} \in \mathbb{R}^{m \times n} \) satisfy \( \bar{A}At^{-1} = \bar{M} \). Then
\[
\Lambda(\bar{A}, \bar{A}1, -d) = (\bar{M}, \bar{U}).
\]

**Proof.** Let \( \bar{M} = I - \bar{M} \). Then \( \bar{M}1 = 0 \) and \( \bar{M}d = d \). Choose a full row rank \( W \in \mathbb{R}^{(n-m) \times n} \) with \( \bar{A}WT = 0 \). Then, we have \( \bar{M} = W^T(WW^T)^{-1}W \). Let \( \bar{U} = h_{\bar{M}}(-d) \). Then, by Theorem 6.1, the path \( \bar{M}(t) \) defined by \((W, -Wd, 1)\) satisfies
\[
\bar{M}(t) = \bar{M} + e^t\bar{U} + O(e^{2t}) = \bar{M} + e^t h_{\bar{M}}(-d) + O(e^{2t}) \quad \text{for} \quad t \to -\infty.
\]

Let \( M(t) = I - \bar{M}(t) \). Since \( \bar{A}WT = 0 \), by Lemma 3.4, the path \( M(t) \) is associated with \((\bar{A}, \bar{A}1, -d)\).

Since \( h_{I-M}(d) = h_M(d) \), we have
\[
M(t) = \bar{M} - e^t h_{\bar{M}}(-d) + O(e^{2t}) = \bar{M} + e^t h_{\bar{M}}(d) + O(e^{2t}) = \bar{M} + e^t \bar{U} + O(e^{2t}).
\]

\[\square\]

In general, we have

**Theorem 6.3** Let \( \bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix} \) with \( \bar{M}_B1_B = 1_B \) and \( \bar{M}_N1_N = 0 \) and \( \bar{U} = h_{\bar{M}}(d) = \begin{pmatrix} h_{\bar{M}_B}(d_B) & 0 \\ 0 & h_{\bar{M}_N}(d_N) \end{pmatrix} \) with \( \bar{M}_Bd_B = 0 \) and \( \bar{M}_Nd_N = d_N \). Here \( \bar{U} \) is an eigenvector of \( Dh(\bar{M}) \) for \( \lambda = 1 \). Let \( A_{JB} \in \mathbb{R}^{m_B \times n_B} \) and \( A_{KN} \in \mathbb{R}^{m_N \times n_N} \) satisfy \( A_{JB}^T(A_{JB}A_{JB}^T)^{-1}A_{JB} = \bar{M}_B \) and \( A_{KN}^T(A_{KN}A_{KN}^T)^{-1}A_{KN} = \bar{M}_N \). Then we can construct an instance \((A, b, c)\),
\[
A = \begin{pmatrix} A_{JB} & 0 \\ 0 & A_{KN} \end{pmatrix}, \quad b = \begin{pmatrix} A_{JB}1_B \\ A_{KN}d_N \end{pmatrix}, \quad c = \begin{pmatrix} -d_B \\ 1_N \end{pmatrix},
\]

such that
\[
\Lambda(A, b, c) = (\bar{M}, \bar{U}).
\]

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As in the proof of Theorem 6.1, we need only to consider

\[
M_B(t) = \bar{M} + e^t h_{\bar{M}}(d_B) + O(e^2t), \quad M_N(t) = \bar{M} + e^t h_{\bar{M}}(d_N) + O(e^2t), \quad t \to -\infty. (6.1)
\]

Let \(x_B(t)\) and \(x_N(t)\) be the central paths of these two instances, i.e. they satisfy the system (2.5). Then it is easy to see that \(x(t) = (x_B(t), x_N(t))\) is the central path of the instance \((A, b, c)\) defined in the theorem. Therefore, the path defined by \((A, b, c)\) can be written as

\[
M(t) = [x(t)]^T A^T (A[x(t)]^2 A^T)^{-1} A[x(t)]
\]

\[
= \begin{pmatrix}
M_B(t) & 0 \\
0 & M_N(t)
\end{pmatrix},
\]

where \(M_B(t) = \Pi(A_{JB}[x_B(t)])\) and \(M_N(t) = \Pi(A_{KN}[x_N(t)])\).

Since \(x_B(t)\) and \(x_N(t)\) are the central paths of \((A_{JB}, A_{JB}1_B, -d_B)\) and \((A_{KN}, A_{KN}d_N, 1_N)\), the paths \(M_B(t)\) and \(M_N(t)\) defined by them satisfy (6.1). Hence,

\[
M(t) = \bar{M} + e^t h_{\bar{M}}(d) + O(e^2t), \quad t \to -\infty.
\]

\[
\square
\]

There is another type of eigenvectors for \(\lambda = 1\) which we consider in the following theorem.

**Theorem 6.4** Let \(\bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix} \) with \(\bar{M}_B \mathbf{1}_B = \mathbf{1}_B\) and \(\bar{M}_N \mathbf{1}_N = 0\) and \(\bar{U} = \begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix} \) with \(\bar{M}_B U_0 = 0\) and \(U_0 \bar{M}_N = U_0\). Here \(\bar{U}\) is an eigenvector of \(Dh(\bar{M})\) for \(\lambda = 1\). Let \(A_{JB} \in R^{m_B \times n_B}\) and \(A_{KN} \in R^{m_N \times n_N}\) satisfy \(A_{JB}^T(A_{JB} A_{TB}^T)^{-1} A_{JB} = \bar{M}_B\) and \(A_{KN}^T(A_{KN} A_{TK}^T)^{-1} A_{KN} = \bar{M}_N\). Then we can construct an instance \((A, b, c)\),

\[
A = \begin{pmatrix}
A_{JB} \\
A_{KN} U_0^T \\
0
\end{pmatrix}, \quad b = \begin{pmatrix}
A_{JB} \mathbf{1}_B \\
0
\end{pmatrix}, \quad c = \begin{pmatrix}
0 \\
\mathbf{1}_N
\end{pmatrix},
\]

such that

\[
\Lambda(A, b, c) = (\bar{M}, \bar{U}).
\]

**Proof.** As in the proof of Theorem 6.1, we need only to consider \(A_{JB}\) and \(A_{KN}\) satisfying \(A_{JB}^T A_{JB} = \bar{M}_B\) and \(A_{KN}^T A_{KN} = \bar{M}_N\).

Note that \(A_{JB} U_0 = 0, U_0 \mathbf{1}_N = U_0 \bar{M}_N \mathbf{1}_N = 0\) and \(U_0^T \mathbf{1}_B = U_0^T \bar{M}_B \mathbf{1}_B = 0\). One can verify that

\[
x(t) = \begin{pmatrix}
\mathbf{1}_B \\
e^{-t} \mathbf{1}_N
\end{pmatrix}, \quad s(t) = \begin{pmatrix}
e^{-t} \mathbf{1}_B \\
\end{pmatrix}, \quad y(t) = \begin{pmatrix}
e^{-t} A_{JB} \mathbf{1}_B \\
0
\end{pmatrix}
\]

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are the central path of \((A,b,c)\) defined by (6.2), i.e. satisfying (2.5). Using this central path, we can calculate the path \(M(t)\) as follows:

\[
A[x(t)] = \begin{pmatrix} A_{JB} & 0 \\ A_{KN}U_0^T & e^{-t}A_{KN} \end{pmatrix},
\]

\[
A[x(t)]^2A^T = \begin{pmatrix} I_J & 0 \\ 0 & e^{-2t}I_K + A_{KN}U_0^T U_0A_{KN}^T \end{pmatrix}
= \begin{pmatrix} I_J & 0 \\ 0 & e^{-t}I_K \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & I_K + e^{2t}A_{KN}U_0^T U_0A_{KN}^T \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & e^{-t}I_K \end{pmatrix},
\]

thus

\[
(A[x(t)]^2A^T)^{-1} = \begin{pmatrix} I_J & 0 \\ 0 & e^tI_K \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & I_K + O(e^{2t}) \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & e^tI_K \end{pmatrix}.
\]

\[
M(t) = [x(t)]A^T \begin{pmatrix} I_J & 0 \\ 0 & e^tI_K \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & I_K + O(e^{2t}) \end{pmatrix} \begin{pmatrix} I_J & 0 \\ 0 & e^tI_K \end{pmatrix} A[x(t)]
= \begin{pmatrix} A^T_{JB} & e^tU_0A_{KN}^T \\ 0 & A_{KN}^T \end{pmatrix} \begin{pmatrix} A_{JB} & 0 \\ e^tA_{KN}U_0^T & A_{KN} \end{pmatrix} + O(e^{2t})
= \begin{pmatrix} \bar{M}_B & e^tU_0 \\ e^tU_0^T & \bar{M}_N \end{pmatrix} + O(e^{2t})
= \bar{M} + e^t\bar{U} + O(e^{2t}).
\]

The above two theorems show the LP representations for two types of eigenvectors. For general eigenvector \(\bar{U} = \begin{pmatrix} h_{\bar{M}_B}(d_B) \\ U_0^T + h_{\bar{M}_N}(d_N) \end{pmatrix}\), based on our numerical experiments, we conjecture that

\[
A = \begin{pmatrix} A_{JB} & 0 \\ A_{KN}U_0^T & A_{KN} \end{pmatrix}, \quad b = \begin{pmatrix} A_{JB}1_B \\ A_{KN}d_N \end{pmatrix}, \quad c = \begin{pmatrix} -d_B \\ 1_N \end{pmatrix},
\]

is an LP representation. However, we have not been able to give a rigorous proof yet.

Now, we turn to eigenvectors with \(\lambda = -1\).

**Theorem 6.5** Let

\[
\bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix} \in \text{Gr}(m,n), \quad \bar{U} = \begin{pmatrix} -h_{\bar{M}_B}(U_01_N) \\ U_0^T \\ -h_{\bar{M}_N}(U_0^T1_B) \end{pmatrix} \in T_{\bar{M}}\text{Gr}(m,n)
\]

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where $\tilde{M}_B \in \text{Gr}(m_B, n_B)$, $\tilde{M}_N \in \text{Gr}(m_N, n_N)$ and $U_0 \in R^{n_B \times n_N}$ satisfy

$$\tilde{M}_B 1_B = 1_B, \quad \tilde{M}_N 1_N = 0, \quad \tilde{M}_B U_0 = U_0, \quad U_0 \tilde{M}_N = 0.$$ 

Here $\bar{U}$ is an eigenvector of $Dh(\tilde{M})$ for $\lambda = -1$. Let $A_{JB} \in R^{m_B \times n_B}$ and $A_{KN} \in R^{m_N \times n_N}$ satisfy $A_{JB}^T (A_{JB} A_{JB}^T)^{-1} A_{JB} = M_B$ and $A_{KN}^T (A_{KN} A_{KN}^T)^{-1} A_{KN} = M_N$. Then we can construct an instance $(\tilde{A}, \tilde{b}, \tilde{c})$,

$$\bar{A} = \begin{pmatrix} A_{JB} & A_{JB}U_0 \\ 0 & A_{KN} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} A_{JB} 1_B \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{c} = \begin{pmatrix} 0 \\ 1_N \end{pmatrix},$$

such that

$$\Lambda(\bar{A}, \bar{b}, \bar{c}) = (\bar{M}, \bar{U}).$$

**Proof.** We need only consider matrix $A_{JB}$ with $A_{JB}^T A_{JB} = \tilde{M}_B$, i.e. $A_{JB} A_{JB}^T = I$, because for any nonsingular $Q$ matrices $A_{JB}$ and $QA_{JB}$ define the same $\tilde{M}_B$. Similarly, we assume $A_{KN}^T A_{KN} = \tilde{M}_N$.

For any small $\epsilon > 0$, define

$$A^\epsilon = \begin{pmatrix} A_{JB} - \epsilon C_B & A_{JB}U_0 \\ 0 & A_{KN} - \epsilon C_N \end{pmatrix}, \quad b^\epsilon = \begin{pmatrix} A_{JB} 1_B + \epsilon A_{JB}U_0 1_N \\ 0 \end{pmatrix}, \quad c^\epsilon = \begin{pmatrix} \epsilon 1_B \\ 1_N \end{pmatrix},$$

where

$$C_B = A_{JB}[U_0 1_N](I - \tilde{M}_B), \quad C_N = A_{KN}[U_0^T 1_B](I - \tilde{M}_N).$$

Denote

$$\bar{x}^\epsilon = \begin{pmatrix} 1_B + \epsilon U_0 1_N \\ 0 \end{pmatrix}, \quad \bar{y}^\epsilon = \begin{pmatrix} \epsilon A_{JB} 1_B \\ 0 \end{pmatrix}, \quad \bar{s}^\epsilon = \begin{pmatrix} 0 \\ 1_N - \epsilon U_0^T 1_B \end{pmatrix}.$$ 

From $(I - \tilde{M}_B) 1_B = 0$ and $(I - \tilde{M}_B) U_0 = 0$ it follows that $C_B 1_B = 0$ and $C_B U_0 = 0$. This leads to

$$A^T \bar{x}^\epsilon = b^\epsilon.$$ 

Using $A_{JB}^T A_{JB} 1_B = \tilde{M}_B 1_B = 1_B$ and $C_B^T A_{JB} 1_B = (I - \tilde{M}_B)[U_0 1_N] 1_B = (I - \tilde{M}_B) U_0 1_N = 0$, we have

$$A^T \bar{y}^\epsilon = \epsilon \left( A_{JB}^T - \epsilon C_B^T \right) A_{JB} 1_B$$

$$= \epsilon \left( 1_B \\ U_0^T 1_B \right).$$

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This shows that
\[ A^T \bar{y} + \bar{s} = c'. \]
For sufficiently small \( \epsilon > 0 \), both \( \bar{x}_B \) and \( \bar{s}_N \) are positive. Thus \((\bar{x}, \bar{s}, \bar{y}')\) is a strictly complementary optimal solution to \((A', b', c')\) and has the optimal basis \((B, N)\).

For any \( \epsilon > 0 \), denote by \((x'(\cdot), s'(\cdot))\) the central path of \((A', b', c')\). Define \( t_\epsilon = -\ln \epsilon \), i.e. \( \epsilon = e^{-t_\epsilon} \). Using \( C_B 1_B = 0, A_{KN} 1_N = 0, A_{KN} U_0^T = 0 \) and \( C_N 1_N = A_{KN} [U_0^T 1_B] 1_N = A_{KN} U_0^T 1_B = 0 \), one can verify that
\[
A' \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix} \right) = b', \quad A'^T 0 + \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix} \right) = c', \quad \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix} \right) \circ \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix} \right) = e^{-t_\epsilon} 1.
\]
This shows that \((x'(t_\epsilon), s'(t_\epsilon))\) is a point on the central path of \((A', b', c')\) at \( t = t_\epsilon \).

This means
\[
x'(t_\epsilon) = \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix}, \quad s'(t_\epsilon) = \left( \begin{pmatrix} 1_B \\ e^{-t_\epsilon} 1_N \end{pmatrix} \right) \right).
\]
(Note that \((x'(t), s'(t)) = \left( \begin{pmatrix} 1_B \\ e^{-t} 1_N \end{pmatrix}, \left( \begin{pmatrix} 1_B \\ e^{-t} 1_N \end{pmatrix} \right) \right)\) need not be true for \( t \neq t_\epsilon \).)

For each \( \epsilon > 0 \), let the path \( M'(t) \) be defined by the instance \((A', b', c')\). By Theorem 5.2, there exists an equilibrium \( M' = \lim_{t \to +\infty} M'(t) \in Gr(m, n) \) and an eigenvector \( U' \in T_M Gr(m, n) \) such that
\[
M'(t) = M' + e^{-t} U' + O(e^{-2t}), \quad \text{as } t \to +\infty. \tag{6.3}
\]

Since the path \(M'(t)\) is defined by \((A', b', c')\), we have
\[
M'(t_\epsilon) = [x'(t_\epsilon)] A^T [A' [x'(t_\epsilon)]^2 A'^T]^{-1} A' [x'(t_\epsilon)]
\]
\[
A' [x'(t_\epsilon)]^2 A'^T = \begin{pmatrix} I_J + O(\epsilon^2) & O(\epsilon^3) \\ O(\epsilon^3) & \epsilon^2 (I_K + O(\epsilon^2)) \end{pmatrix}
\]
\[
= \begin{pmatrix} I_J & 0 \\ 0 & \epsilon I_K \end{pmatrix} (I + O(\epsilon^2)) \begin{pmatrix} I_J & 0 \\ 0 & \epsilon I_K \end{pmatrix}
\]
Thus,
\[
(A'[x'(t_\epsilon)]^2 A'^T)^{-1} = \begin{pmatrix} I_J & 0 \\ 0 & \epsilon^{-1} I_K \end{pmatrix} (I - O(\epsilon^2)) \begin{pmatrix} I_J & 0 \\ 0 & \epsilon^{-1} I_K \end{pmatrix}
\]
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Now, we have

\[
M'(t_e) = \left( \begin{array}{cc}
A_{JB}^T - \epsilon C_B^T & 0 \\
\epsilon U_0 A_{JB}^T & A_{KN} - \epsilon C_N^T
\end{array} \right) \left( \begin{array}{cc}
A_{JB} - \epsilon C_B & \epsilon A_{JB} U_0 \\
0 & A_{KN} - \epsilon C_N
\end{array} \right) + O(\epsilon^2)
\]

\[
= \left( \begin{array}{cc}
M_B - \epsilon h_{MB}(U_0 1_N) & \\
\epsilon U_0^T & \tilde{M}_N - \epsilon h_{MN}(U_0^T 1_B)
\end{array} \right) + O(\epsilon^2)
\]

\[
= \tilde{M} + \epsilon \tilde{U} + O(\epsilon^2).
\]

(Note that (6.4) is only shown to be true at \( t = t_e \), but not for all \( t \).)

By Theorem 3.3, (3.9) and (3.10) in [6] (note that \( A^e \) satisfies the assumption \( A_{KB}^e = 0 \)), \( \tilde{M}^e \), the limit of \( M'(t) \), is determined by \( A^e \) and \( (\bar{x}^e, \bar{s}^e) \). More precisely, we have

\[
\tilde{M}^e = \left( \begin{array}{cc}
\tilde{M}_B^e & 0 \\
0 & \tilde{M}_N^e
\end{array} \right),
\]

where

\[
\tilde{M}_B^e = [\bar{x}_B^e] A_{JB}^e (A_{JB}^e [\bar{x}_B^e]^2 A_{JB}^e)^{-1} A_{JB}^e [\bar{x}_B^e]
\]

\[
\tilde{M}_N^e = [\bar{s}_N^e]^{-1} A_{KN}^e (A_{KN}^e [\bar{s}_N^e]^{-2} A_{KN}^e)^{-1} A_{KN}^e [\bar{s}_N^e]^{-1}.
\]

For \( \tilde{M}_B^e \),

\[
A_{JB}^e [\bar{x}_B^e] = (A_{JB} - \epsilon A_{JB}[U_0 1_N](I - \tilde{M}_B))(I + \epsilon [U_0 1_N])
\]

\[
= A_{JB} + \epsilon A_{JB}[U_0 1_N] M_B + O(\epsilon^2),
\]

\[
(A_{JB}^e [\bar{x}_B^e]^2 A_{JB}^e)^{-1} = (I + 2 \epsilon A_{JB}[U_0 1_N] A_{JB}^T + O(\epsilon^2))^{-1}
\]

\[
= I - 2 \epsilon A_{JB}[U_0 1_N] A_{JB}^T + O(\epsilon^2),
\]

\[
\tilde{M}_B^e = (A_{JB} + \epsilon A_{JB}[U_0 1_N] \tilde{M}_B)^T (I - 2 \epsilon A_{JB}[U_0 1_N] A_{JB}^T + \epsilon A_{JB}[U_0 1_N] \tilde{M}_B) + O(\epsilon^2)
\]

\[
= \tilde{M}_B + O(\epsilon^2).
\]

For \( \tilde{M}_N^e \),

\[
A_{KN}^e [\bar{s}_N^e]^{-1} = (A_{KN} - \epsilon C_N)[I_N - \epsilon (U_0^T 1_B)]^{-1}
\]

\[
= (A_{KN} - \epsilon C_N)[I_N + \epsilon (U_0^T 1_B)] + O(\epsilon^2)
\]

\[
= A_{KN} + \epsilon A_{KN}[U_0^T 1_B] \tilde{M}_N + O(\epsilon^2).
\]
Now, similar to the above, we have
\[ \tilde{M}_N = \tilde{M}_N + O(\epsilon^2). \]

Hence,
\[ \tilde{M}^\epsilon = \tilde{M} + O(\epsilon^2). \] (6.5)

By (6.3),
\[ U^\epsilon = \frac{M^\epsilon(t) - \tilde{M}^\epsilon}{e^{-t}} + O(e^{-t}). \]

Using (6.4) and (6.5) for \( \epsilon = e^{-t} \), we have
\[ U^\epsilon = \tilde{U} + O(\epsilon). \] (6.6)

Because the mapping from \((A, b, c)\) to \(M^0 = \Gamma(A, b, c)\) and the mapping from \(M^0\) to \((\tilde{M}, \tilde{U})\) are continuous, the mapping \(\Lambda\) from \((A, b, c)\) to \(\Lambda(A, b, c) = (\tilde{M}, \tilde{U})\) is continuous. Using (6.5) and (6.6), we have
\[ \Lambda(\tilde{A}, \tilde{b}, \tilde{c}) = \lim_{\epsilon \to 0} \Lambda(A^\epsilon, b^\epsilon, c^\epsilon) = \lim_{\epsilon \to 0} (\tilde{M}^\epsilon, U^\epsilon) = (\tilde{M}, \tilde{U}). \]

That is, the instance \((\tilde{A}, \tilde{b}, \tilde{c})\) defines the path \(M(t)\) converging to \(\tilde{M}\) in direction \(\tilde{U}\).

\[ \square \]

7 Sources, sinks and their dimensions

An effective method for characterizing sinks and sources of stable/unstable manifolds is to compute their dimensions.

We denote \(\tilde{n} = m(n - m)\) which is the dimension of \(\text{Gr}(m, n)\). Note that the dimension of an attraction region is the same as the dimension of \(\text{Gr}(m, n)\).

We are particularly interested in stable and unstable manifolds of dimension \(\tilde{n}\) and of dimension \(\tilde{n} - 1\), because these manifolds comprise the attraction regions and the major boundaries of the attraction regions.

**Theorem 7.1**
\[
\begin{align*}
\dim(\Sigma^+(B, m_B)) &= \tilde{n} - (n_N - m_N)m_B & (7.1) \\
\dim(\Sigma^-(B, m_B)) &= \tilde{n} - m_N(n_B - m_B) - (n_B - m_B) - m_N. & (7.2)
\end{align*}
\]
Proof. For simplicity, we omit \((B, m_B)\), writing \(\Sigma^+\), etc, instead of \(\Sigma^+(B, m_B)\), etc.

Since \(\Sigma^+ = \bigcup_{M \in \mathcal{E}C} \mathcal{E}^+(M)\) is the bundle of subspaces \(\mathcal{E}^+(M)\) with the base \(\mathcal{E}C\) and \(\mathcal{E}^+(M)\) have the same dimension, written as \(\dim(\mathcal{E}^+e)\), for all \(M \in \mathcal{E}C\), we have

\[
\dim(\Sigma^+) = \dim(\mathcal{E}C) + \dim(\mathcal{E}^+).
\]

Similarly,

\[
\dim(\Sigma^-) = \dim(\mathcal{E}C) + \dim(\mathcal{E}^-).
\]

By Lemmas 4.3 and 4.5 in [6],

\[
\begin{align*}
\dim(\mathcal{E}^+) &= n_B - m_B + m_N + (n_B - m_B)m_N \\
\dim(\mathcal{E}^-) &= m_B(n_N - m_N)
\end{align*}
\]

By Lemma 4.6 in [6],

\[
\dim(\mathcal{E}C) = (m_B - 1)(n_B - m_B) + (n_N - m_N - 1)m_N.
\]

Elemental calculations amount to (7.1) and (7.2).

Now we can use formulas (7.1) and (7.2) to fully describe the \(\bar{n}\)- and \((\bar{n} - 1)\)-dimensional sources and sinks. We can also describe other dimensional sources and sinks in the similar way, but the description will be more involved.

- \(\bar{n}\)-dimensional sources.

By (7.1), \(\Sigma^+(B, m_B)\) is an \(\bar{n}\)-dimensional source iff

\[
(n_N - m_N)m_B = 0
\]

iff

\[
n_N - m_N = 0 \quad \text{or} \quad m_B = 0.
\]

If \(n_B > 0\), then \(M_B 1_B = 1_B\) implies that \(M_B\) has at least one nonzero eigenvalue, and thus \(m_B = \text{rank}(M_B) > 0\). Therefore, \(m_B = 0\) implies \(n_B = 0\) (i.e. \(B = \emptyset\)) and \((n_N, m_N) = (n, m)\).

If \(n_N > 0\), then \(M_N 1_N = 0\) implies that \(M_N\) has at least one eigenvalue of zero, and thus \(M_N\) is not of full rank, i.e. \(n_N > m_N\). Therefore, \(n_N - m_N = 0\) implies \(n_N = 0\) and \((n_B, m_B) = (n, m)\) (i.e. \(B = \{1, \ldots, n\}\)). So we have two types of \(\bar{n}\)-dimensional source:
Source I: \(n_B = m_B = 0\) and \((n_N, m_N) = (n, m)\). This implies \(B = \emptyset\), i.e. \(M = M_N\). Thus,

\[
\mathcal{E}C(B, m_B) = \{ M \in \text{Gr}(m, n) \mid M1 = 0 \}
\]

\[
\mathcal{E}^+(M) = \{ h_M(d) \mid d \in \mathbb{R}^n, Md = d \}.
\]

Source II: \(n_N = m_N = 0\) and \((n_B, m_B) = (n, m)\). This implies \(B = \{1, \ldots, n\}\), i.e. \(M = M_B\). Thus,

\[
\mathcal{E}C(B, m_B) = \{ M \in \text{Gr}(m, n) \mid M1 = 1 \},
\]

\[
\mathcal{E}^+(M) = \{ h_M(d) \mid d \in \mathbb{R}^n, Md = 0 \}.
\]

**Remark:** Since these are the only \(\bar{n}\) sources, the two unstable manifolds from these two sources comprise almost the entire \(\text{Gr}(m, n)\), except a \((\bar{n} - 1)\)-dimensional set.

Partitions of source I and source II can substantially characterize the partition of \(\text{Gr}(m, n)\). Moreover, the two sources are related through the exchange of \(M\) and \(I - M\), thus, by virtue of Lemma 3.4, the analysis of one source will sufice.

- \(\bar{n}\)-dimensional sinks.

By (7.2), \(\Sigma^-(B, m_B)\) is an \(\bar{n}\)-dimensional sink iff

\[
m_N(n_B - m_B) + (n_B - m_B) + m_N = 0
\]

iff

\[
n_B = m_B \quad \text{and} \quad m_N = 0.
\]

This implies that \(n_B = m_B = m\) and \(n_N = n - m\). Thus, every \(\bar{n}\)-dimensional sink can contain only a single point which is the stable point \(M\) associated with a basis \(B\), i.e.

\[
M = \text{diag}(M_{ii}), \ M_{ii} = 1 \forall i \in B, \ M_{ii} = 0 \forall i \notin B.
\]

We denote by \(M(B) = \begin{pmatrix} I_B & 0 \\ 0 & 0 \end{pmatrix}\) this stable point. Thus

\[
\mathcal{E}C(B, m_B) = \{ M(B) \}
\]

As shown by Theorem 5.3 in [6], \(M(B)\) is asymptotically stable. By Theorem 5.10, the stable manifold \(W^s(M(B))\) is the attraction region of \(M(B)\). An attraction region has the full dimension as \(\text{Gr}(m, n)\), thus \(T_{M(B)}W^s(M(B)) = T_{M(B)}\text{Gr}(m, n)\). By (4.5), we have

\[
\mathcal{E}^-(M(B)) = T_{M(B)}\text{Gr}(m, n).
\]
Furthermore, by Lemma 4.1 (ii) [6] and the simple structure of \( M(B) = \begin{pmatrix} I_B & 0 \\ 0 & 0 \end{pmatrix} \), it is an easy manipulation to find that
\[
T_{M(B)}(\text{Gr}(m, n)) = \left\{ U = \begin{pmatrix} 0 & U_{BN} \\ U_{BN}^T & 0 \end{pmatrix} : \forall U_{BN} \in \mathbb{R}^{m \times (n-m)} \right\}.
\]

- \((\bar{n} - 1)\)-dimensional sinks. By condition (7.2), a sink is of dimension \( \bar{n} - 1 \) iff
\[
m_N(n_B - m_B) + m_N + (n_B - m_B) = 1.
\]

Thus, there are two types of \((\bar{n} - 1)\)-dimensional sink:

Sink I: \( n_B = m_B \) and \( m_N = 1 \). These imply that \( n_B = m_B = m - 1 \) and \( n_N = n - m + 1 \). Thus the associated equilibrium cluster is
\[
\mathcal{E}(B, m_B) = \left\{ M = \begin{pmatrix} I_{m-1} & 0 \\ 0 & uu^T \end{pmatrix} : u \in \mathbb{R}^{n-m+1}, u^T u = 1, u^T 1 = 0 \right\}
\]
and the eigenspace is
\[
\mathcal{E}^{-}(M) = \left\{ \begin{pmatrix} U_0 \\ U_0^T \end{pmatrix} - h_{uu^T}(U_0^T 1_B) : U_0 u = 0 \right\}.
\]

Sink II: \( n_B = m_B + 1 \) and \( m_N = 0 \). These imply that \( n_B = m + 1, m_B = m \) and \( n_N = n - m - 1 \). Thus the associated equilibrium cluster is
\[
\mathcal{E}(B, m_B) = \left\{ M = \begin{pmatrix} I_{m+1} - vv^T & 0 \\ 0 & 0 \end{pmatrix} : v \in \mathbb{R}^{m+1}, v^T v = 1, v^T 1 = 0 \right\}
\]
and the eigenspace is
\[
\mathcal{E}^{-}(M) = \left\{ \begin{pmatrix} -h_{vv^T}(U_0 1_N) \\ U_0^T \end{pmatrix} : v^T U_0 = 0 \right\}.
\]

These sinks are the sinks of \((\bar{n} - 1)\)-dimensional boundaries, thus are particularly important.

- \((\bar{n} - 1)\)-dimensional sources.

By condition (7.1), a source is of dimension \( \bar{n} - 1 \) iff
\[
(n_N - m_N)m_B = 1.
\]
This implies \( m_B = 1 \) and \( n_N - m_N = 1 \). Thus, we have \( n_B = n - m, m_B = 1, n_N = m \) and \( m_N = m - 1 \). There is only one equilibrium which satisfies these conditions, that is, \( \mathcal{E}(B, m_B) = \{M\} \) where

\[
M = \begin{pmatrix}
\frac{1}{n-m}1_B T_B & 0 \\
0 & I_N - \frac{1}{m}1_N T_N
\end{pmatrix}, \quad 1_B \in \mathbb{R}^{n-m}, 1_N \in \mathbb{R}^m.
\]

The eigenspace is

\[
\mathcal{E}^+(M) = \{h_M(d) : 1_B T_B d_B = 0, 1_N T_N d_N = 0\} + \left\{ \begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix} : 1_B T_B U_0 = 0, U_0 1_N = 0 \right\}
\]

\[
= \left\{ \begin{pmatrix} 1_B d_B^T + d_B 1_B^T \\ 0 \\
0 & 1_N d_N^T + d_N 1_N^T
\end{pmatrix} : 1_B T_B d_B = 0, 1_N T_N d_N = 0 \right\}
\]

\[
+ \left\{ \begin{pmatrix} 0 \\ U_0^T \\ 0 \\
U_0 & 0 \\
0 & 0
\end{pmatrix} : 1_B T_B U_0 = 0, U_0 1_N = 0 \right\}
\]

\[
= \left\{ \begin{pmatrix} 1_B d_B^T + d_B 1_B^T \\ U_0^T \\
0 & 1_N d_N^T + d_N 1_N^T
\end{pmatrix} : 1_B T_B d_B = 0, 1_N T_N d_N = 0, 1_B T_B U_0 = 0, U_0 1_N = 0 \right\}.
\]

**Remarks:** We have completely identified the two sources of all attraction regions. One direction of study is to investigate the induced partition on these sources. We have also completely characterized all sinks of \((\bar{n} - 1)\)-dimensional boundaries. Since boundaries are the primary structure of the partition, this characterization lays a foundation for the constructive investigation of the partition.

**8 Conclusions**

In this paper we present three main results.

Section 5 shows that an equilibrium-eigenvector pair uniquely determines a path which converges to the equilibrium in the direction of the eigenvector. This defines an isomorphism between a stable/unstable manifold \( W^s(B, m_B) / W^u(B, m_B) \) and a sink/source \( \Sigma^- (B, m_B) / \Sigma^+ (B, m_B) \). We are interested in these manifolds because attraction regions and their boundaries are stable manifolds. The structure of sinks/sources is much simpler than that of stable/unstable manifolds. Thus, characterization of attraction regions and their boundaries can be significantly simplified through sinks/sources.

Section 6 further shows that, for each equilibrium-eigenvector pair, we can construct an LP instance which defines the path corresponding to the given equilibrium and eigenvector. LP
representations are important because they bridge $\text{Gr}(m, n)$ and $SLP(n, m)$. This LP representation is particularly useful because it is constructed in terms of equilibria and eigenvectors. Many applications of this LP representation will be seen in our coming papers.

Section 7 shows a method for finding complete descriptions of sources and sinks. We completely present $\bar{n}$- and $(\bar{n} - 1)$-dimensional sources and sinks, because they represent the attraction regions and the major boundaries in the basis partition.

Attraction regions and their boundaries are isomorphic to sinks (as shown in Section 5) which are fully described in Section 7 and represented in Section 6 as sets of LP instances in $SLP(n, m)$. This approach will yield a simple characterization of the basis partition of $SLP(n, m)$. Details of this characterization will appear in our future papers.

Location relations between attraction regions, between an attraction region and its boundaries, and between boundaries, are another important aspect for characterizing the basis partition. The results in this paper will play a fundamental role in discovering these relations.

9 Appendix 1

We will use Grönwall inequality to prove Lemma 5.1. Grönwall inequality has many variants. The following is one of them. These variants have different forms, but they can be proved in similar ways. For completeness, we give the proof below.

Lemma 9.1 (Grönwall inequality). Let $\varphi(t) \geq 0$ be continuous for $t \geq 0$ and $\int_0^\infty \varphi(t)dt < \infty$. Suppose that, for $t \geq 0$, $u(t) \geq 0$ is continuous and satisfies the inequality

$$u(t) \leq K + \int_0^t \varphi(s)u^{1+\alpha}(s)ds$$

for some constants $K > 0$ and $\alpha \geq 0$.

(i) If $\alpha > 0$ and $\alpha K^\alpha \int_0^\infty \varphi(s)ds < 1$, then

$$u(t) \leq \frac{K}{[1 - \alpha K^\alpha \int_0^t \varphi(s)ds]^{1/\alpha}}, \quad \forall t \in [0, \infty).$$

(ii) If $\alpha = 0$, then

$$u(t) \leq K \exp(\int_0^t \varphi(s)ds), \quad \forall t \in [0, \infty).$$
Proof. Let
\[ w(t) = K + \int_0^t \varphi(s)u^{1+\alpha}(s)ds. \]
Then \( u(t) \leq w(t) \) and
\[ w'(t) = \varphi(t)u^{1+\alpha}(t) \leq \varphi(t)w^{1+\alpha}(t). \]
That is,
\[
\frac{w'(t)}{w^{1+\alpha}(t)} \leq \varphi(t). 
\] (9.1)
If \( \alpha > 0 \), integrating on both sides of (9.1) yields
\[
-\frac{1}{\alpha}(w^{-\alpha}(t) - K^{-\alpha}) \leq \int_0^t \varphi(s)ds.
\]
It follows
\[
w(t) \leq \frac{K}{[1 - \alpha K^{\alpha} \int_0^t \varphi(s)ds]^{1/\alpha}}.
\]
on \( t \geq 0 \). This shows part (i) of the lemma because \( u(t) \leq w(t) \).

If \( \alpha = 0 \), integrating on both sides of (9.1) yields
\[
\ln w(t) - \ln K \leq \int_0^t \varphi(s)ds.
\]
It follows that
\[ w(t) \leq K \exp(\int_0^t \varphi(s)ds). \]
This shows part (ii) of the lemma. \( \Box \)

Now we show Lemma 5.1, which is Lemma 9.2 below.

Lemma 9.2 Let \( N_0 \) be an open neighborhood of 0 in \( \mathbb{R}^n \). Suppose that \( g : N_0 \rightarrow \mathbb{R}^n \) is \( C^l \), \( l \geq 2 \), and satisfies \( g(0) = 0 \) and \( Dg(0) = 0 \). Let \( x(t, x^0) \) be the solution of
\[
x' = \lambda x + g(x), \quad x(0, x^0) = x^0, \tag{9.2}
\]
where \( \lambda \neq 0 \) is a constant. Then the following statements hold.

(i) There exists a \( \delta > 0 \) such that for any \( x^0 \in \mathbb{R}^n \) with \( \|x^0\| \leq \delta \), the solution \( x(t, x^0) \) of (9.2) satisfies
\[
x(t, x^0) = e^{\lambda t}v + O(e^{2\lambda t}\|x^0\|^2), \tag{9.3}
\]
for $\lambda t \to -\infty$ (i.e., $t \to -\infty$ if $\lambda > 0$ and $t \to \infty$ if $\lambda < 0$), where
\[
v = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds.
\] (9.4)

Furthermore,
\[
\int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds = O(\|x^0\|^2).
\] (9.5)

(ii) Let $\Psi$ be a map defined by $\Psi(x^0) = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds$. There exist open neighborhoods of $0$, $N$ and $M$ in $\mathbb{R}^n$, such that the map $\Psi : N \to M$ is invertible. The map $\Psi$ and its inverse $\Psi^{-1}$ are $C^l$.

**Proof.** We first prove for the case $\lambda < 0$.

Proof of (i). $x(t, x^0)$ is a solution of (9.2) if and only if it satisfies
\[
x(t, x^0) = e^{\lambda t} x^0 + \int_0^t e^{\lambda(t-s)} g(x(s, x^0)) ds.
\] (9.6)

By the assumption of $g$, there exist positive numbers $\sigma$ and $\delta_0$ such that
\[
\|g(x)\| \leq \sigma \|x\|^2, \quad \forall \|x\| \leq \delta_0.
\]

Because the Jacobian of the right hand side of (9.2) at $x = 0$ is $\lambda I$, $x = 0$ is a stable point of (9.2). Thus, there is $\delta_1 > 0$ such that for any $\|x^0\| \leq \delta_1$ the solution $x(t, x^0)$ satisfies
\[
\|x(t, x^0)\| \leq \delta_1 \quad \forall t \geq 0.
\]

Thus, for $\|x^0\| \leq \min\{\delta_0, \delta_1\}$, we have
\[
\frac{\|x(t, x^0)\|}{e^{\lambda t}} \leq \|x^0\| + \int_0^t e^{-\lambda s} \|g(x(s, x^0))\| ds
\]
\[
\leq \|x^0\| + \int_0^t e^{-\lambda s} \sigma \|x(s, x^0)\|^2 ds
\]
\[
= \|x^0\| + \int_0^t e^{\lambda s} \sigma \left(\frac{\|x(s, x^0)\|}{e^{\lambda s}}\right)^2 ds
\]

By Grönwall inequality ($\alpha = 1$),
\[
\frac{\|x(t, x^0)\|}{e^{\lambda t}} \leq \frac{\|x^0\|}{1 - \|x^0\| \sigma \int_0^t e^{\lambda s} ds}
\]
\[
\leq \frac{\|x^0\|}{1 - \|x^0\| \sigma \lambda^{-1}}
\]
Let \( \delta > 0 \) satisfy \( \delta \sigma |\lambda|^{-1} < 1 \), and denote \( \beta = (1 - \delta \sigma |\lambda|^{-1})^{-1} \). Then for any \( \|x^0\| \leq \delta \), we have

\[
\|x(t, x^0)\|_{e^{\lambda t}} \leq \beta \|x^0\|, \quad \forall \ t \geq 0.
\] (9.7)

This leads to

\[
\int_0^\infty e^{-\lambda s} \|g(x(s, x^0))\| ds \leq \int_0^\infty e^{\lambda s} \sigma \left( \frac{\|x(s, x^0)\|}{e^{\lambda s}} \right)^2 ds \leq \int_0^\infty e^{\lambda s} \sigma^2 \|x^0\|^2 ds = e^{\lambda s} \sigma^2 |\lambda|^{-1} \|x^0\|^2, \quad \forall \ t \geq 0.
\] (9.8)

Now, it follows from (9.6) that

\[
x(t, x^0) = e^{\lambda t} v - e^{\lambda t} \int_0^t e^{-\lambda s} g(x(s, x^0)) ds = e^{\lambda t} v + O(e^{2\lambda t} \|x^0\|^2).
\]

This shows (9.3).

Taking \( t = 0 \) in (9.8), we have \( \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds \leq \sigma \beta^2 |\lambda|^{-1} \|x^0\|^2 \). This shows (9.5).

Proof of (ii). For any fixed \( t \), regard \( x(t, \cdot) : x^0 \to x(t, x^0) \) as a map, where \( x(t, x^0) \) is determined by (9.6). We first estimate the Jacobian of the map \( x(t, \cdot) \).

\[
Dx(t, x^0) = e^{\lambda t} I + \int_0^t e^{\lambda(t-s)} Dg(x(s, x^0)) Dx(s, x^0) ds.
\]

By conditions on \( g \), there exist \( \delta_0 > 0 \) and \( \sigma_1 > 0 \) such that \( \|Dg(x)\| \leq \sigma_1 \|x\| \) for all \( x \in \mathbb{R}^n \) with \( \|x\| \leq \delta_0 \). Then, together with (9.7), we have

\[
\frac{\|Dx(t, x^0)\|}{e^{\lambda t}} \leq 1 + \int_0^t e^{-\lambda s} \|Dg(x(s, x^0))\| \cdot \|Dx(s, x^0)\| ds \leq 1 + \int_0^t \sigma_1 e^{-\lambda s} \|x(s, x^0)\| \cdot \|Dx(s, x^0)\| ds \leq 1 + \int_0^t \sigma_1 \beta e^{\lambda s} \|x^0\| \cdot \frac{\|Dx(s, x^0)\|}{e^{\lambda s}} ds.
\]

By Grönwall inequality (\( \alpha = 0 \)),

\[
\frac{\|Dx(t, x^0)\|}{e^{\lambda t}} \leq \exp(\int_0^t \sigma_1 \beta e^{\lambda s} \|x^0\| ds) \leq \exp(\sigma_1 \beta \delta_0 / |\lambda|), \quad \forall \ t \geq 0, \|x^0\| \leq \delta_0.
\] (9.9)
The Jacobian of $\Psi$ is
\[
D\Psi(x^0) = I + \int_0^{+\infty} e^{-\lambda s} Dg(x(s, x^0))Dx(s, x^0)ds.
\]
Using $\|Dg(x)\| \leq \sigma_1\|x\|$, (9.7) and (9.9), we have
\[
\left\| \int_0^{+\infty} e^{-\lambda s} Dg(x(s, x^0))Dx(s, x^0)ds \right\| \leq \int_0^{+\infty} \sigma_1\beta \exp(\sigma_1\beta\delta_0/|\lambda|)\|x^0\|e^{\lambda s}ds = \sigma_1\beta/|\lambda| \exp(\sigma_1\beta\delta_0/|\lambda|)\|x^0\|.
\]
Therefore, for $\|x^0\| < |\lambda|(\sigma_1\beta)^{-1}\exp(-\sigma_1\beta\delta_0/|\lambda|)$, the Jacobian $D\Psi(x^0)$ is bounded and non-singular.

By the inverse function theorem, there exist an open neighborhood $N = \{x^0 \in R^n \mid \|x^0\| < \delta\}$ for some $\delta > 0$, such that $\Psi$ is an one-to-one map from $N$ to $M = \Psi(N)$, and $M$ is an open neighborhood of $\Psi(0) = 0$ in $R^n$. Furthermore, the map $\Psi$ and its inverse $\Psi^{-1}$ are $C^l$ in $N$ and $M$, respectively. (ii) is proved.

For $\dot{\lambda} > 0$ and $t \to -\infty$, we need only a transformation $t = -\tilde{t}$. Denote $x(t) = \tilde{x}(\tilde{t})$ and $\lambda = -\tilde{\lambda}$. Then $x'(t) = -\tilde{x}'(\tilde{t})$. Thus
\[
\tilde{x}'(\tilde{t}) = -\tilde{\lambda}\tilde{x}(\tilde{t}) + g(\tilde{x}(\tilde{t}))
\]
is equivalent to
\[
x'(t) = -\tilde{\lambda}\tilde{x}(\tilde{t}) - g(\tilde{x}(\tilde{t})) = \lambda x(t) - g(x(t)).
\]
Thus, the above proof applies. $\square$

10 Appendix 2

The following result is probably known. But I could not find it in the literature. My colleague Weixiao Shen$^4$ kindly provided the proof.

**Theorem 10.1** Let $(x(t), y(t), z(t))$ be the solution of (4.1) with $(x(0), y(0), z(0)) = (x^0, y^0, z^0)$ and $\gamma > 0$ a constant.

(i) If the curve $(x(t), y(t), z(t))$ intersects every open neighborhood of the origin and $(x(t), y(t)) = O(e^{-\gamma t})$ for $t \to +\infty$, then $(x^0, y^0, z^0) \in W^s_g(0)$.

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(ii) If the curve \((x(t), y(t), z(t))\) intersects every open neighborhood of the origin and \((x(t), z(t)) = O(e^{\gamma t}), t \to -\infty;\) then \((x^0, y^0, z^0) \in W^s_y(0)\).

**Proof.** We will prove (i). The proof of (ii) is an analogue.

By Theorem 4.1, there exists a \(C^1\) center-stable manifold \(W^{cs} (\subset N_\delta)\). Since the curve \((x(t), y(t), z(t))\) intersects \(N_\delta\), we can choose \(t_0\) with \((x(t_0), y(t_0), z(t_0)) \in N_\delta\). As mentioned in [3], if \((x(t_0), y(t_0), z(t_0)) \not\in W^{cs}\), then there exists a small open neighborhood of the origin which does not contain any point of the curve \((x(t), y(t), z(t))\). Thus, by assumptions of the theorem, \((x(t_0), y(t_0), z(t_0)) \in W^{cs}\). Then, by the invariance of \(W^{cs}\), \((x(t), y(t), z(t)) \in W^{cs}\) for all \(t \geq t_0\). Therefore, \(z(t) = w^{cs}(x(t), y(t))\) and \((x(t), y(t))\) is determined by the system (4.2). Now, we need only to consider the system (4.2).

We can construct a one-to-one \(C^1\) map \(\phi: N_\delta \to N_\delta\) such that

\[
\phi(0,0) = (0,0), \quad \phi(x,0) = (x, v^c(x)), \quad \phi(0,y) = (u^s(y), y),
\]

i.e. \(\phi\) maps \(x\)-axis to \(W^c\) and \(y\)-axis to \(W^s\).

We write \(\phi = \begin{pmatrix} \phi_x \\ \phi_y \end{pmatrix}\) and the Jacobian of \(\phi\) as

\[
D\phi(x, y) = \begin{pmatrix}
\frac{\partial \phi_x(x,y)}{\partial x} & \frac{\partial \phi_x(x,y)}{\partial y} \\
\frac{\partial \phi_y(x,y)}{\partial x} & \frac{\partial \phi_y(x,y)}{\partial y}
\end{pmatrix}.
\]

By (10.1), we have

\[
\frac{\partial \phi_x(x,y)}{\partial x}\bigg|_{(x,y)=(0,0)} = \frac{\partial \phi_x(0,0)}{\partial x} = \frac{\partial x}{\partial x}\bigg|_{x=0} = I,
\]

\[
\frac{\partial \phi_x(x,y)}{\partial y}\bigg|_{(x,y)=(0,0)} = \frac{\partial \phi_x(0,y)}{\partial y} = \frac{\partial u^c(y)}{\partial y}\bigg|_{y=0} = 0,
\]

\[
\frac{\partial \phi_y(x,y)}{\partial x}\bigg|_{(x,y)=(0,0)} = \frac{\partial \phi_y(0,y)}{\partial x} = \frac{\partial y}{\partial x}\bigg|_{x=0} = I,
\]

\[
\frac{\partial \phi_y(x,y)}{\partial y}\bigg|_{(x,y)=(0,0)} = \frac{\partial \phi_y(0,y)}{\partial y} = \frac{\partial v^c(x)}{\partial y}\bigg|_{y=0} = 0.
\]

Therefore,

\[
D\phi(0,0) = I.
\]
Introduce new variables \((\tilde{x}, \tilde{y})\) by \((x, y) = \phi(\tilde{x}, \tilde{y})\).

For small \((x, y)\),
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \phi^{-1}(x, y)
\]
\[
= \phi^{-1}(x, y) - \phi^{-1}(0, 0)
\]
\[
= D\phi^{-1}(0, 0) \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}(\| (x, y) \|^2)
\]
\[
= [D\phi(0, 0)]^{-1} \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}(\| (x, y) \|^2)
\]
\[
= \begin{pmatrix} x \\ y \end{pmatrix} + \mathcal{O}(\| (x, y) \|^2)
\]

Thus, \((x(t), y(t)) = \mathcal{O}(e^{-\gamma t})\) implies \((\tilde{x}(t), \tilde{y}(t)) = \mathcal{O}(e^{-\gamma t})\).

We can write the system (4.2) in terms of \((\tilde{x}, \tilde{y})\) as follows.
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = [D\phi(\tilde{x}, \tilde{y})]^{-1} \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\]
\[
= [D\phi(\tilde{x}, \tilde{y})]^{-1} \left( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a(x, y) \\ b(x, y) \end{pmatrix} \right)
\]
\[
= [D\phi(\tilde{x}, \tilde{y})]^{-1} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \phi(\tilde{x}, \tilde{y}) + [D\phi(0, 0)]^{-1} \begin{pmatrix} a(\phi(\tilde{x}, \tilde{y})) \\ b(\phi(\tilde{x}, \tilde{y})) \end{pmatrix}
\]
\[
= [D\phi(0, 0)]^{-1} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} D\phi(0, 0) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} \tilde{a}(\tilde{x}, \tilde{y}) \\ \tilde{b}(\tilde{x}, \tilde{y}) \end{pmatrix}
\]
\[
= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{a}(\tilde{x}, \tilde{y}) \\ \tilde{b}(\tilde{x}, \tilde{y}) \end{pmatrix}
\]
(10.2)

where
\[
\begin{pmatrix} \tilde{a}(\tilde{x}, \tilde{y}) \\ \tilde{b}(\tilde{x}, \tilde{y}) \end{pmatrix} = \left( [D\phi(\tilde{x}, \tilde{y})]^{-1} - [D\phi(0, 0)]^{-1} \right) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \phi(\tilde{x}, \tilde{y})
\]
\[
+ [D\phi(\tilde{x}, \tilde{y})]^{-1} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \left( \phi(\tilde{x}, \tilde{y}) - D\phi(0, 0) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \right)
\]
\[
+ [D\phi(\tilde{x}, \tilde{y})]^{-1} \begin{pmatrix} a(\phi(\tilde{x}, \tilde{y})) \\ b(\phi(\tilde{x}, \tilde{y})) \end{pmatrix}.
\]

Since \(\phi\) is \(C^l\) with \(l \geq 2\), one can verify that \((\tilde{a}, \tilde{b})\) and its first derivative vanish at \((\tilde{x}, \tilde{y}) = (0, 0)\).
For any point \((0, \tilde{y}) \in \mathcal{N}_\delta\), let \((\hat{x}(t), \hat{y}(t))\) be the solution of (10.2) with \((\hat{x}(t_0), \hat{y}(t_0)) = (0, \tilde{y})\). Then, \((x(t), y(t)) = \phi(\hat{x}(t), \hat{y}(t))\) is a solution of (4.2). Since

\[(x(t_0), y(t_0)) = \phi(0, \tilde{y}) = (u^*(\tilde{y}), \tilde{y}),\]

we have \((x(t_0), y(t_0)) \in \mathcal{W}^s\). Since \(\mathcal{W}^s\) is invariant under the system (4.2), the entire solution \((x(t), y(t)) \in \mathcal{W}^s\) for all \(t \geq t_0\). Thus,

\[(x(t), y(t)) = (u^*(y(t)), y(t)), \quad \forall \ t \geq t_0.\]

This yields

\[(\hat{x}(t), \hat{y}(t)) = \phi^{-1}(x(t), y(t)) = \phi^{-1}(u^*(y(t)), y(t)) = (0, y(t)). \quad \forall \ t \geq t_0.\]

This shows that \(\dot{x}(t) = 0\) for all \(t \geq t_0\). In turn, this implies \(\dot{x}(t) = 0\) for all \(t \geq t_0\). Substituting \((\hat{x}(t_0), \hat{y}(t_0)) = (0, \tilde{y})\) and \(\dot{x}(t_0) = 0\) into the first equation in (10.2), we obtain

\[\hat{a}(0, \tilde{y}) = 0.\]

Now, we write \(\tilde{x}(t) = e^{tA}q(t)\) as solutions of (10.2). We will estimate \(q(t)\) as follows. From the first equation in (10.2), we obtain

\[\dot{q} = e^{-tA}\hat{a}(\tilde{x}, \tilde{y}).\]

This leads to

\[q(t) - q(t_0) = \int_{t_0}^{t} e^{-sA}\hat{a}(\tilde{x}(s), \tilde{y}(s))ds. \quad (10.3)\]

Using \(\hat{a}(0, \tilde{y}) = 0\), \(\frac{\partial\hat{a}}{\partial x}(0, 0) = 0\), the continuity of \(\frac{\partial\hat{a}}{\partial x}\) and \((\tilde{x}(s), \tilde{y}(s)) = O(e^{-\gamma s})\), we have

\[
\|\hat{a}(\tilde{x}(s), \tilde{y}(s))\| = \|\hat{a}(\tilde{x}(s), \tilde{y}(s)) - \hat{a}(0, \tilde{y}(s))\| \\
= \|\tilde{x}(s)\frac{\partial\hat{a}}{\partial x}(\xi, \tilde{y}(s))\| \\
\leq C_1 e^{-\gamma s}\|\tilde{x}(s)\|,
\]

where \(C_1\) is a constant.

It follows from (10.3) that

\[
\|q(t_0)\| \leq \|q(t)\| + \int_{t_0}^{t} \|\tilde{x}(s)\|C_1 e^{-\gamma s}\|\tilde{y}(s)\|ds. \quad (10.4)
\]
It is well known, see e.g. Corollary in [4] Section 1.8, that \( \| e^{tA} \| \leq C_2 |t|^L \) for some constant \( C_2 \) and \( 0 \leq L \leq k_c - 1 \), if all the eigenvalues of \( A \) have zero real parts. (This can be estimated with the Jordan form.) Since

\[
\| q(t) \| \leq \| e^{-tA} \| \| \tilde{x}(t) \| \leq C_2 |t|^L \| \tilde{x}(t) \|,
\]

\( \tilde{x}(t) = O(e^{-\gamma t}) \) implies \( \| q(t) \| = O(e^{-\gamma t}) \). Let \( t \to +\infty \), we obtain from (10.4) that

\[
\| q(t_0) \| \leq \int_{t_0}^{\infty} C_1 e^{-\gamma s} C_2^2 s^{2L} \| q(s) \| ds \leq \left( C_1 C_2^2 \int_{t_0}^{\infty} e^{-\gamma s} s^{2L} ds \right) \sup_{s \geq t_0} \| q(s) \|.
\]

Choose \( \bar{t} > 0 \) so large that \( C_1 C_2^2 \int_{t_0}^{\infty} e^{-\gamma s} s^{2L} ds = 1/2 \). Then for any \( t_0 \geq \bar{t} \), we have

\[
\| q(t_0) \| \leq \frac{1}{2} \sup_{s \geq t_0} \| q(s) \|.
\]  

If \( \| q(t_0) \| \neq 0 \) for some \( t_0 \geq \bar{t} \), (10.5) implies that there exists a sequence \( t_0 < t_1 < t_2 < \ldots \) with

\[
\| q(t_k) \| \geq 2 \| q(t_{k-1}) \| \geq \ldots \geq 2^k \| q(t_0) \|.
\]

This contradicts the convergence of \( q(t) \to 0 \) as \( t \to \infty \). Therefore, \( q(t) \equiv 0 \) must hold true for all \( t \geq \bar{t} \). This in turn implies \( \tilde{x}(t) \equiv 0 \) for all \( t \geq \bar{t} \). Thus,

\[
(x(t), y(t)) = \phi(0, \tilde{y}(t)) = (u^*(\tilde{y}(t)), \tilde{y}(t)),
\]

which means that

\[
x(t) = u^*(\tilde{y}(t)) = u^*(y(t)),
\]

\[
y(t) = \tilde{y}(t).
\]

These, together with

\[
z(t) = w^c(x(t), y(t)) = w^c(u^*(y(t)), y(t)) =: w^*(y(t)),
\]

imply that \( (x(t), y(t), z(t)) \in \mathcal{W}^s \) for all \( t \geq \bar{t} \). Therefore, \( (x^0, y^0, z^0) \in \mathcal{W}^s \).

\[
\]

References


