Study of Demand Function of Multi-market Segments and Multiple Prices (MSMP) in Pricing

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Abstract

A good model of the relationship between the market demand and prices plays a very important role in economic decision-making. Linear demand functions have been widely used in Economics. They work well when there is a single market segment and only a single price is considered (SSSP). However, difficulties have been encountered when we attempted to design a model for multi-market segments and multiple prices (MSMP). Such a model is needed to set different prices for closely-related products. In this paper, we formulate the demand function of MSMP using a Linear Complementarity Problem (LCP). We will incorporate our model of MSMP demand function into pricing models. This leads to a complementarity constrained optimization problem. It will be shown that a complementarity constrained pricing model can be converted to a linearly constrained one in some instances.

Keywords: Demand-Price Function; Linear Complementarity Problem (LCP); Complementarity Constrained Optimization Problem; Pricing models.
1 Introduction

In today’s economy, as competition becomes intense, product proliferation has become increasingly popular. In order to survive in such a competitive market and obtain higher supplier surpluses, many corporations sell a series of closely-related products to target different market segments, ranging from luxury items, necessities to promotional items. Examples of such markets are the airline industry, hotel industry, electronics industry, etc. Many people have studied markets of multiple segments and relevant issues. We do not attempt a complete survey of this field of research, but just to name a few informative references: Daly (2001), Kons (1999), Liu (2003), Perakis and Sood (2004), Stavins (2001) and Wonnacott and Wonnacott (1990).

The main features of such markets can be described as follows.

Assume that a company is producing a certain commodity or providing a certain service for different groups of customers. To satisfy the different needs of customers or to target different market segments, the company can design different variations of the commodity. These products usually share the same resources. A typical example is an airline selling seats in the same class (for e.g., the economy class), but charging different prices for them. Although seats in a class share a common inventory, different restrictions are attached to them (to target different groups of customers), thus they are regarded as different products. An advantage of such a design is that an airline can charge higher prices to customers who are willing to pay higher prices (to have less restrictions) and in the meantime, it can offer lower prices to attract more customers whose travel plans mainly or partially depend on travel costs. Having a large population of customers is crucial to industries like the airlines because their overhead cost is a major part of the total costs.

Although different products are targeted at different groups of customers, these prod-
ucts may be substitutable because for instance, they share the same resources. Suppose that a company designs a product and thereby sets a price for each market segment. Then the demand from a market segment depends negatively on the price set for this segment and tends to zero as the price goes to infinity. It also depends positively on prices set for other market segments. Thus the demand function for a market segment cannot be concave with respect to the price set for the same segment. Since the quantities to be produced are constrained by the demands from the market segments (which are necessary restrictions in many applications such as pricing problems), a difficulty hereby encountered is that these constraints define a nonconvex set on the quantities produced. Now this nonconvexity property can cause serious difficulties in theoretical analysis and numerical computations of such applications.

To make a model involving a demand function simple and tractable, many researchers often consider simple categories of demand functions. These can include a “linear” function that is truncated before it becomes negative. For example, Federgruen and Heching (1999), Feng and Gallego (1995), Feng and Xiao (2000), Gallego and van Ryzin (1997), Maglaras and Meissner (2003), Raman and Chatterjee (1995) and Weatherford (1997) considered a possible linear demand function that is defined only on nonnegative prices that lead to nonnegative demands. All prices which causes the demand function to be negative are deemed infeasible.

But why must all feasible prices be confined to such a set of prices? We have not found explanations for this. Indeed, we have not found in the existing literature, any definition of a MSMP demand function which tells us how to evaluate the demand outside the set of prices that correspond to nonnegative demands.

The first goal of this paper is to provide a definition of the MSMP demand function for all nonnegative prices. If there is only one market segment, it is natural and trivial to consider a reverse demand function that has been used in Economics for centuries. That
is, setting the demand function to be zero for all prices corresponding to a negative value of demand. However, if there are multiple segments, the definition is no longer trivial. Naturally we will think of simply extending the single market segment reverse demand function for this case, i.e., define the MSMP demand function to be zero on any prices that lead to negative demand for some market segments. We will discuss the problems with such an extension in the next section. Our definition solves these problems by using a map defined by a linear complementarity problem (LCP).

Secondly, we will study a pricing model in which the MSMP demand function is used. This pricing model is an LCP constrained optimization problem. We will show that in some situations the optimization problems with LCP constraints can be reduced to the optimization problems with linear constraints. This tremendously simplifies the computations and analysis. As a by-product, this result provides a rigorous justification for the models used in Gallego and Van Ryzin (1997) and Maglaras and Meissner (2003). However, there are also many other situations in which the aforementioned simplification cannot be realized. An example will be given in Section 3. In these situations, the LCP constraints inherited from the MSMP demand function will remain as a core structure in pricing models.

In the remaining sections of our paper, we first formulate our model of the MSMP demand function using LCP in the next section. Then in the third section, we will introduce a general complementarity constrained pricing model and simplify a basic complementarity constrained pricing problem to a linearly constrained one. Some properties of the models will be discussed here. Lastly, in the final section, we propose some potential future research directions.
2 Formulation of MSMP demand function

Suppose that a company produces a commodity and a market survey shows that there are $N$ groups of customers demanding this commodity, i.e. there are $N$ market segments. In order to meet different needs of different groups of customers, the company designs $N$ different products, each targeting one market segment. Let $p_i$ be the price of product $i$, $p_{-i}$ be the price vector of all products except product $i$ and $p = (p_i, p_{-i}) = (p_1, \ldots, p_N)$. Denote the quantity of product $i$ to be produced by $q_i$ and $q = (q_1, \ldots, q_N)$. Let $D_i(p_i, p_{-i})$ denote the function of demand for product $i$ and $D = (D_1, \ldots, D_N)$.

For a demand function to be practically useful, it is required to satisfy some conditions. Here are four basic conditions:

**Condition 1.** $D$ is continuous.

**Condition 2.** $D \geq 0$.

**Condition 3.** For each $i$, $D_i$ is non-increasing in $p_i$ and non-decreasing in $p_j$, $j \neq i$.

**Condition 4.** The total demand cannot increase for any increase in prices, i.e.,

$$\sum_{i=1}^{N} [D_i(p') - D_i(p)] \leq 0, \quad \forall p' \geq p.$$

An immediate consequence of condition 4 is that as prices increase, no $D_i$ can increase if no $D_j$ decreases.

**Remarks on Condition 3:** It is generally observed that the demand for a product is reversely dependent on the product’s price, thus $D_i$ is non-increasing in $p_i$. For any other $j \neq i$, as price $p_j$ rises, the demand for product $j$ may drop. Since product $i$ is substitutable for product $j$, some of the customers who have given up product $j$ may buy product $i$. Therefore, $D_i$ is non-decreasing in $p_j$. 

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Remarks on Condition 4: While Condition 3 considers individual products, Condition 4 considers all products as a whole: the total demand is reversely dependent on the vector of prices, if all other factors are fixed, including for e.g., other sellers’ prices.

In this paper, we consider only a simple demand function which is piecewise linear. Linear models are most frequently used in practice because of simplicity. Furthermore, results for linear models will shed light on nonlinear models. For demand functions, there is another reason for linearity: the demand-price relation is often observed to be convex, but in order to have a convex set defined by the production quantity constraints

$$q_i \leq D_i(p_i, p_{-i}), \quad i = 1, \ldots, N,$$

the demand function must be concave. The linear function is both convex and concave, thus it is the most favorable.

Our model is an extension of the fundamental model of reverse demand for a single product: $D(p) = b - ap$ if $0 \leq p \leq b/a$ and $D(p) = 0$ if $p > a$. For the MSMP model, the main piece of $D$, where $D$ is not constant, is represented as

$$d(p) \equiv b - Ap,$$

where

$$A = \begin{pmatrix}
\alpha_1 & -\gamma_{2 \rightarrow 1} & \cdots & -\gamma_{N \rightarrow 1} \\
-\gamma_{1 \rightarrow 2} & \alpha_2 & \cdots & -\gamma_{N \rightarrow 2} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{1 \rightarrow N} & -\gamma_{2 \rightarrow N} & \cdots & \alpha_N
\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}.$$

Condition 3 holds iff

$$\alpha_i \geq 0, \quad \gamma_{i \rightarrow i'} \geq 0, \quad \gamma_{i' \rightarrow i} \geq 0.$$

For condition 4 to hold, $A$ must be strictly column dominant. We will see why later.

The following assumption is made throughout this paper.
Assumption 1  The parameters defining $b$ and $A$ satisfy the relationships $b_i \geq 0$, $\alpha_i > 0$, and $\gamma_{i'\rightarrow i} > 0$, $\forall i, i' \in \{1, \ldots, N\}$. In addition, $A$ is strictly column dominant, that is, $\alpha_i > \sum_{i'} \gamma_{i'\rightarrow i} \gamma_{i'\rightarrow i}$ for all $i \in \{1, \ldots, N\}$.

Raz and Porteus (2003) discussed some methods used to estimate the above parameters facing the seller.

The assumption that $A$ is strictly column dominant implies that $A$ is a P-matrix (see Cottle, Pang and Stone (1992)).

In order to define a function $D$ satisfying Condition 2, we denote

$$\Omega = \{ p \in R_N^+ \mid b - Ap \geq 0 \}, \quad \text{where} \quad R_N^+ = \{ p \in R^N \mid p \geq 0 \}.$$ 

By the assumption $b \geq 0$, $\Omega \neq \emptyset$. We can define

$$D(p) = d(p), \quad \forall p \in \Omega.$$ 

For the $N$-dimensional demand function, we cannot simply define $D(p) = 0$ for all $p \in R_N^+ \setminus \Omega$, because $d(p)$ need not be 0 on $\partial \Omega$ (the boundary of $\Omega$), thus $D$ so defined is not continuous.

In order to determine for which $p \in R_N^+ \setminus \Omega$ we should define $D_i(p) = 0$, we define a map $B$: $B(p) = p$ if $p \in \Omega$ and $B(p) \in \partial \Omega$ if $p \in R_N^+ \setminus \Omega$. Then we can define $D_i(p) = 0$ if $d_i(B(p)) = 0$. The function $D$ so defined satisfies condition 2. (Later we will discuss how to define $D_i(p)$ if $d_i(B(p)) > 0$.)

If $d_i(\bar{p}_{i}, \bar{p}_{-i}) = 0$, then $d_i(p_i, \bar{p}_{-i}) < 0$ for all $p_i > \bar{p}_i$. Thus we expect that $D_i(p_i, \bar{p}_{-i}) = 0$ for all $p_i > \bar{p}_i$. That is, $D$ should map $(p_i, \bar{p}_{-i})$ for all $p_i > \bar{p}_i$ to $(\bar{p}_i, \bar{p}_{-i})$. We found that the map defined below can satisfy this requirement.
Definition 2 For any \( p \in R_+^N \), \( B(p) \) is defined as the solution of the LCP\( (p) \): find \( x (= B(p)) \) such that

\[
0 \leq b - Ax \perp p - x \geq 0.
\] (1)

Since \( A \) is a P-matrix under our assumption, the LCP has a unique solution (see Cottle, Pang and Stone (1992) or Facchinei and Pang (2003)). Therefore, the map \( B : R_+^N \rightarrow R^N \) is well defined, i.e., it is a point-to-point map. The following lemma shows that the map \( B \) defined above is our desirable map.

Lemma 3 For any \( p \in R_+^N \), the following holds true.

(i) \( B(p) \geq 0 \).

(ii) \( B(p) = p \) if \( p \in \Omega \) and \( B(p) \in \partial \Omega \) if \( p \in R_+^N \setminus \Omega \).

(iii) Let \( \bar{p} \in \partial \Omega \) with \( d_i(\bar{p}) = (b - A\bar{p})_i = 0 \). Then for any \( p = (p_i, \bar{p}_{-i}) \) with \( p_i > \bar{p}_i \), \( B(p) = \bar{p} \).

(iv) For any \( \bar{p} \in \partial \Omega \), define the set

\[
C(\bar{p}) = \{ p \in R_+^N \setminus \text{int}(\Omega) \mid B(p) = \bar{p} \},
\]

where \( \text{int}(\Omega) \) denotes the interior of the set \( \Omega \). Then \( C(\bar{p}) \) is a convex cone.

Proof. (i) and (ii): If \( p \in \Omega \), then it is easy to see that \( x = p \) solves LCP\( (p) \) and thus \( B(p) = p \). So we have \( B(p) \geq 0 \) and \( B(p) \in \Omega \).

Now suppose \( p \in R_+^N \setminus \Omega \). By the LCP constraints (1), we have \( p \geq B(p) \). Denote \( I = \{ i \mid B_i(p) < p_i \} \) and \( J = \{ 1, \ldots, N \} \setminus I \). For any \( j \in J \), \( B_j(p) = p_j \geq 0 \). By (1), we must have \( b - AB(p)_I = 0 \). Then \( B_I(p) = (A_{II})^{-1}(b_I - A_{IJ}p_J) \). Since \( A_{II} \) does not contain any diagonal entry of \( A \), all entries of \( A_{IJ} \) are negative. Thus, \( b_I - A_{IJ}p_J \geq 0 \). By our
assumptions on \( A \), \( A \) is an M-matrix, thus the inverse of \( A \) and all its principal submatrices are nonnegative (see Berman and Plemmons (1994)). This implies that \( B_I(p) \geq 0 \). We have proven that \( B(p) \geq 0 \).

Since the LCP implies \( b - AB(p) \geq 0 \), together with \( B(p) \geq 0 \), we have \( B(p) \in \Omega \). Now \( p \in R^N_+ \setminus \Omega \) implies \( p \neq B(p) \). Hence there exists a \( j \) such that \( p_j > B_j(p) \). This, by (1), implies \( (b - AB(p))_j = 0 \). Thus, \( B(p) \in \partial \Omega \).

(iii) The nonnegativity constraints in \( \text{LCP}(p) \) are obviously satisfied by \( x = \bar{p} \). Now there is only one component \( i \) for which \( p_i - \bar{p}_i > 0 \), and for this component \( (b - A\bar{p})_i = 0 \). Thus \( b - A\bar{p} \perp p - \bar{p} \). This shows that \( \bar{p} \) solves \( \text{LCP}(p) \). That is, \( B(p) = \bar{p} \).

(iv) Let \( p, p' \in C(\bar{p}) \). Then \( \bar{p} = B(p) = B(p') \). For any \( \lambda \in (0, 1) \), it is easy to check that \( \bar{p} \) is the solution of \( \text{LCP}(\lambda p + (1 - \lambda)p') \). That is, \( \lambda p + (1 - \lambda)p' \in C(\bar{p}) \). Hence, \( C(\bar{p}) \) is convex.

Let \( p \in C(\bar{p}) \). For any \( \lambda \geq 0 \), denote \( p_\lambda = \bar{p} + \lambda(p - \bar{p}) \). Then \( p_\lambda - \bar{p} = \lambda(p - \bar{p}) \geq 0 \) and \( b - A\bar{p} \perp p - \bar{p} \) implies \( b - A\bar{p} \perp p_\lambda - \bar{p} \), thus \( p_\lambda \in C(\bar{p}) \). This shows that \( C(\bar{p}) \) is a cone.

Now what should \( D_i(p) \) be if \( d_i(B(p)) > 0 \)?

Although letting \( D_i(p) = d_i(p) \) if \( d_i(B(p)) > 0 \) will satisfy \( D(p) \geq 0 \), the \( D \) so defined cannot satisfy Condition 4.

Thus we have the following definition.

**Definition 4** The MSMP demand function \( D \) is defined by

\[
D(p) = d(B(p)), \quad \forall p \in R^N_+,
\]

where the map \( B \) is as stated in Definition 2.

To show this \( D \) satisfies Condition 4, we first need the following lemma.
Lemma 5 Under Assumption 1, \( p' \geq p \geq 0 \) implies \( B(p') \geq B(p) \).

Proof. Denote \( x = B(p) \) and \( x' = B(p') \). Let \( I = \{ i \mid x_i > x'_i \} \). Then

\[
p'_i \geq p_i \geq x_i > x'_i, \quad \forall i \in I
\]

implies

\[
d_i(x') = (b - Ax')_i = 0, \quad \forall i \in I.
\]

Let \( J = \{1, 2, \ldots, N\} \setminus I \). We can write

\[
\sum_{i \in I} [A(x - x')]_i = \sum_{i \in I} [A_{II}(x_I - x'_I) + A_{IJ}(x_J - x'_J)]_i.
\]

Since \( A_{II} \) is strictly column dominant and \( x_I - x'_I > 0 \), we have

\[
\sum_{i \in I} [A_{II}(x_I - x'_I)]_i = \left( \sum_{i \in I} A_{II} \right) (x_I - x'_I) > 0.
\]

Because \( A_{IJ} < 0 \) and \( x_J - x'_J \leq 0 \), \( A_{IJ}(x_J - x'_J) \geq 0 \). Thus,

\[
\sum_{i \in I} [A(x - x')]_i > 0.
\]

This implies

\[
\sum_{i \in I} [d_i(x) - d_i(x')] < 0.
\]

Thus, there exists a \( k \in I \) such that \( d_k(x) < d_k(x') = 0 \). This contradicts the fact that \( x \) is a solution of LCP\((p)\). \( \square \)

We will prove that all the conditions can be satisfied by our definition of \( D \).

Theorem 6 Under Assumption 1, the demand function \( D : R^+_N \rightarrow R^N \) defined by

\[
D(p) = b - AB(p), \quad \forall p \in R^+_N,
\]

satisfies all conditions 1 – 4.
**Proof.** Since $A$ is a P-matrix, the solution $B(p)$ of LCP($p$) is a continuous function of $p$, cf. Theorem 7.2.1 in Cottle, Pang and Stone (1992). Together with the continuity of $d$, we see that $D(p) = d(B(p))$ is continuous. This shows Condition 1 is satisfied.

Condition 2 is satisfied due to Lemma 3 (i).

The fact that Condition 3 is satisfied follows directly from the definition of the map $B$.

Condition 4 is satisfied if
\[
\sum_{i=1}^{N} [d_i(p') - d_i(p)] \leq 0, \quad \forall p' \geq p
\] holds true. This can be shown as follows. For any $p' \geq p \geq 0$, by Lemma 5, $B(p') \succeq B(p)$. Thus by (2),
\[
\sum_{i=1}^{N} [d_i(B(p')) - d_i(B(p))] \leq 0.
\]
Therefore, (2) implies
\[
\sum_{i=1}^{N} [D_i(p') - D_i(p)] \leq 0, \quad \forall p' \geq p.
\]

Now we go on to show (2) is true. Let $e \in \mathbb{R}^N$ be the vector of all ones. Then we can write
\[
\sum_{i=1}^{N} [d_i(p') - d_i(p)] = e^T A(p - p').
\] Since $A$ is strictly column dominant, $e^T A > 0$. Thus $e^T A(p - p') \leq 0$, which implies (2).

We have proven Condition 4 is satisfied by $D$. \hfill $\square$

To conclude this section, we would like to shed a light on why we define the demand function $D$ via the map $B$, which is obtained from solving an LCP. Is it more natural to define the demand function via the orthogonal projection on $\Omega$, since $\Omega$ is a convex set? We shall explain that a demand function defined by the orthogonal projection does not satisfy all the basic conditions 1–4.

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Denote by $N(p)$, the orthogonal projection of $p$ on $\Omega$. Let us consider $R^2$. Suppose $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \partial \Omega$ with $d_1(\bar{p}) = 0$ and $d_2(\bar{p}) > 0$. Consider $p' = (p_1, \bar{p}_2)$ with $p_1 > \bar{p}_1$. The orthogonal projection, $N(p')$, of $p'$ on $d_1(p) = 0$ satisfies $N(p') > \bar{p}$, and hence $d_2(N(p')) < d_2(\bar{p})$. See Figure 1. If we define $D(p) = d(N(p))$, then $D_2(p_1, \bar{p}_2) < D_2(\bar{p}_1, \bar{p}_2)$ for $p_1 > \bar{p}_1$.

This does not satisfy condition 2: $D_i$ is non-decreasing in $p_j$ for $j \neq i$.

![Figure 1. Illustration of orthogonal projection $N$ and mapping $B$.](image)

## 3 Pricing models in MSMP markets

Demand functions are basic ingredients in pricing models. A general pricing model can be represented as follows, in which (1) the objective of the seller is to maximize his profits; (2) for each product, the amount produced (nonnegative) is constrained by the corresponding market demand; (3) the total amount of products produced is restricted by the fixed resource level; (4) the prices are bounded from below and (5) there may be additional constraints on the prices and amount of products produced.
\[
\max_{(p,q)} f(p,q) \\
\text{s.t. } 0 \leq q \leq D(p) \\
e^T q \leq C \\
p \geq 0 \\
(p, q) \in G
\]  

Remark: Here all products share the same resource, where examples in practice can be inventory level (size of an airplane belonging to an airline), warehouse size in logistics, manufacturing budget in manufacturing business, advertisement budget in sales sector, etc. For simplicity, in this paper we assume that different products consume the same amount of a single resource.

Based on our formulated MSMP demand function \( D(p) = d(B(p)) \), we can write (3) as the following complementarity constrained pricing model:

\[
\max_{(p,x,q)} q^T p - c(q) \\
\text{s.t. } 0 \leq q \leq b - Ax \\
e^T q \leq C \\
0 \leq b - Ax \perp p - x \geq 0 \\
p \geq 0 \\
(p, q) \in G
\]  

where the cost function \( c : \mathbb{R}^N_+ \to \mathbb{R} \) is assumed to be continuous throughout this section. This is an LCP constrained optimization problem. Readers can refer to Luo, Pang and Ralph (1996) for general properties of such problems and methods to solve them.

Lemma 7 For any \( p \in \mathbb{R}^N_+ \), and \( 0 \leq q \leq b - AB(p) \), we have \( q^T B(p) = q^T p \). Here \( B(p) \) is the solution of \( \text{LCP}(p) \).
Proof. For each \( i \), if \( p_i > B(p)_i \), then by LCP\((p)\), \((b - AB(p))_i = 0\). Hence \( q_i = 0 \) and \( B(p)_i q_i = p_i q_i = 0 \). Therefore, \( q^T B(p) = q^T p \) holds true.

\[ \square \]

Lemma 8 Under Assumption 1, \( p \leq A^{-1}b \) for all \( p \in \Omega \). Thus, \( \Omega \) is compact.

Proof. Under Assumption 1, \( A \) is an M-matrix, cf. Berman and Plemmons (1994). Thus \( A^{-1} \) is nonnegative and \( b - Ap \geq 0 \) implies \( p \leq A^{-1}b \).

Proposition 9 Under Assumption 1 and \( C \geq 0 \), if there exists \((p, q) \in G\) such that \( 0 \leq q \leq b - AB(p) \) and \( e^T q \leq C \), then the problem (4) is feasible and bounded, i.e., it has feasible solutions and its objective value is bounded.

Proof. Obviously, for \((p, q) \in G\) satisfying the condition in the proposition, \((p, B(p), q)\) is a feasible solution.

By Lemma 7, we can see that for any \((p, x, q)\) feasible to (4), we will always have \( p^T q = x^T q \). Since \( x \in \Omega \) and \( \Omega \) is bounded by Lemma 8, thus \( x \) and \( q \) are both bounded \((0 \leq q \leq b - Ax)\) and in turn the objective value is bounded.

The feasibility condition for (4) is not difficult to satisfy. For instance, if \( G \) contains a point \((p, q = 0)\) for any \( p \in R^N_+ \), then \((p, B(p), 0)\) will be feasible to (4).

However, the set of optimal solutions of (4) can either be empty or have multiple solutions, because the feasible region need not be closed and convex. This will be clearer if we denote

\[ B(G) = \{(x, q) \mid x = B(p), (p, q) \in G, p \geq 0\} \]

and rewrite the problem (4), by virtue of Lemma 7, as follows:

\[
\max_{(x,q)} \quad q^T x - c(q) \\
\text{s.t.} \quad 0 \leq q \leq b - Ax
\]

(5)
\[ e^T q \leq C \]
\[(x, q) \in B(G) \]

The set \( B(G) \) is the image of the set \( G \) under the mapping \( B \) (here \( q \) is left unchanged). Usually, such a set is nonconvex even if \( G \) is convex. Thus the above program is in general nonconvex. Moreover, \( B(G) \) need not be closed even if \( G \) is closed, thus although the above problem is bounded, it may not achieve an optimal solution.

In Figure 2 (where only prices are plotted), the mapped prices in \( B(G) \) consist of two line segments \( U_1U_2 \) and \( U_2U_3 \), excluding the point \( U_3 \). Thus, \( B(G) \) is neither convex nor closed.

![Diagram](image)

Figure 2. \( B(G) \) is neither convex nor closed.

It is not easy to solve problem (4) because it involves LCP constraints. In what follows, we will discuss some cases in which the LCP constraints can be eliminated. More precisely, we wish to reduce the problem (4) to the following problem:
\[
\begin{align*}
\max_{(p,q)} \quad & q^T p - c(q) \\
\text{s.t.} \quad & 0 \leq q \leq b - Ap \\
& e^T q \leq C \\
& p \geq 0. \\
& (p, q) \in G
\end{align*}
\] (6)

The model (6) can be seen as the model (4), with an added restriction that \( p \) must be in \( \Omega \). All \( p \) outside \( \Omega \) are deemed to be infeasible.

Such a simplification is not always possible. Here is a simple example: Suppose \( d_1(p) = 20 - 3p_1 + 2p_2, d_2(p) = 100 + p_1 - 4p_2, c(q) = 80, C = 30 \) and \( G = \{(p, q) | p_1 \geq p_2 \} \).

For problem (6), the constraints \( p_1 \geq p_2 \) and \( d_1(p) \geq 0 \) imply that

\[ p_1 \leq 3p_1 - 2p_2 \leq 20. \]

Thus, the objective value of (6) is bounded from above by

\[ q_1 p_1 + q_2 p_2 \leq q_1 p_1 + q_2 p_1 \leq C p_1 \leq 30 \times 20 = 600. \]

As for problem (4), we can easily verify that \( (p_1, p_2, x_1, x_2, q_1, q_2) = (k, 23, 22, 23, 0, 30) \) for any \( k \geq 23 \) is a feasible solution. The objective value of (4) at this point is

\[ q_1 p_1 + q_2 p_2 = 0 + (30 \times 23) = 690. \]

This shows that the optimal profit obtained using the pricing model (4) is higher than that obtained using the simplified model (6). Therefore, it is wrong to simplify (4) to (6) in this situation.

In situations like the above example, the LCP constraints inherited from the MSMP demand function are unavoidable.

Now we give a sufficient condition under which (4) can be reduced to (6).
**Theorem 10** Under Assumption 1, if for any $p \geq 0$ and $q \geq 0$, $(p, q) \in G$ implies $(B(p), q) \in G$, then the problems (4) and (6) are equivalent in the sense that:

(i) if $(p^*, x^*, q^*)$ is optimal to (4), then $(x^*, q^*)$ is an optimal solution of (6).

(ii) if $(p^*, q^*)$ is optimal to (6), then $(p, p^*, q^*)$ with any $p$ satisfying $B(p) = p^*$ and $(p, q^*) \in G$ is optimal to (4).

(iii) they have the same optimal objective values.

**Proof.** Let $v_1$ and $v_2$ be the optimal objective values of (4) and (6) respectively.

If $(p^*, x^*, q^*)$ is an optimal solution of (4), then $x^* = B(p^*)$. Under the condition in the theorem, $(p^*, q^*) \in G$ implies $(x^*, q^*) \in G$. By Lemma 3 (i), $x^* \geq 0$. Thus, $(p, q) = (x^*, q^*)$ is a feasible solution of (6). By Lemma 7,

$$v_1 = (q^*)^T p^* - c(q^*) = (q^*)^T x^* - c(q^*) \leq v_2. \quad (7)$$

If $(p^*, q^*)$ is optimal to (6), then for any $p$ satisfying $B(p) = p^*$ and $(p, q^*) \in G$, $p^*$ is the solution of $LCP(p)$, thus $0 \leq b - Ap^* \perp p - p^* \geq 0$. Hence it is easy to see that $(p, p^*, q^*)$ is feasible to (4). By Lemma 7,

$$v_2 = (q^*)^T p^* - c(q^*) = (q^*)^T p - c(q^*) \leq v_1. \quad (8)$$

Combining (7) and (8), we have $v_1 = v_2$ and the equality holds in (7) and (8). Therefore, $(x^*, q^*)$ is an optimal solution to problem (6) and $(p, p^*, q^*)$ is an optimal solution to problem (4). □

For the case where the constraint $(p, q) \in G$ is absent in (4), we can say $G = R^N \times R^N$. In this case, the condition that $(p, q) \in G$ implies $(B(p), q) \in G$ for $p \geq 0$ certainly holds. Therefore, as a special case of Theorem 10, we have
Corollary 11 Under Assumption 1,

\[
\max_{(p,x,q)} q^T p - c(q) \\
\text{s.t. } 0 \leq q \leq b - Ax \\
e^T q \leq C \\
0 \leq b - Ax \perp p - x \geq 0 \\
p \geq 0.
\]

and

\[
\max_{(p,q)} q^T p - c(q) \\
\text{s.t. } 0 \leq q \leq b - Ap \\
e^T q \leq C \\
p \geq 0.
\]

are equivalent in the sense specified in Theorem 10.

The model (10) is a basic pricing model. We will show that it has a unique solution under certain conditions.

Lemma 12 Under Assumption 1 and \( C > 0 \), the solution \((\bar{p}, \bar{q})\) of problem (10) satisfies \( \bar{q} = b - A\bar{p} \).

Proof. Recall that we denote \( d(p) = b - Ap \). Suppose \( \bar{q}_i = d_i(\bar{p}) \) for some \( i \in \{1, \ldots, N\} \). That is, \( \bar{q}_i < d_i(\bar{p}) \).

Case 1: \( \bar{q}_i > 0 \). Then we can always find \( \Delta_i > 0 \) such that \( \bar{q}_i = d_i(\bar{p}_i + \Delta_i, \bar{p}_{-i}) \).

By setting \( p_i = \bar{p}_i + \Delta_i \), with other prices unchanged, a higher revenue will be obtained, which contradicts that \((\bar{p}, \bar{q})\) is the optimal solution to (10).
Case 2: $\bar{q}_i = 0$. Then again we can find $\delta_i > 0$ such that $\bar{q}_i = d_i(\bar{p}_i + \delta_i, \bar{p}_{-i})$. Now since $C > 0$, there must exist a $j$ such that $\bar{q}_j > 0$. Since $\bar{q}_j \leq d_j(\bar{p})$ and $\gamma_{i-j} > 0$, we have $\bar{q}_j < d_j(\bar{p}_i + \delta_i, \bar{p}_{-i})$. Thus we can find $\delta_j > 0$ such that $\bar{q}_j = d_j(\bar{p}_i + \delta_i, \bar{p}_j + \delta_j, \bar{p}_{-i,j})$.

Setting $p_i = \bar{p}_i + \delta_i$ and $p_j = \bar{p}_j + \delta_j$, with other prices unchanged, a higher revenue will be obtained, contradicting that $(\bar{p}, \bar{q})$ is optimal.

Hence we must have $\bar{q} = b - A\bar{p}$.

By the above lemma, if a seller uses this pricing model, he should supply in accordance with the market demands.

**Theorem 13** Under Assumption 1 and $C > 0$, there exists an optimal solution to problem (10). Furthermore, if $A + A^T$ is positive definite and $c(q)$ is convex, then the optimal solution is unique.

**Proof.** Since in problem (10), the objective function is continuous and the feasible constraint set is nonempty and compact, by Weierstrass theorem, there exists an optimal solution.

In addition, under Assumption 1, the optimal solution of (10), $(\bar{p}, \bar{q})$ satisfies $\bar{q} = b - A\bar{p}$ by Lemma 12. Hence problem (10) can be transformed to the following quadratic programming problem (which can be solved using many available optimization packages designed for such problems, like those found in Gill, Murray and Wright (1991) and Vanderbei (1999))

$$
\max_p \ -0.5p^T(A + A^T)p + b^Tp - c(b - Ap)
$$

s.t. $Ap \leq b$

$$
-e^TAp \leq C - e^Tb
$$

$p \geq 0.$

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By the hypothesis on $A + A^T$ and $c(q)$, the objective function is strictly concave and the constraints are linear. Thus the optimal solution of the problem is unique.

For example, if $A$ satisfies Assumption 1 and is strictly row dominant, then $A + A^T$ is positive definite (see McKenzie (1960)).

4 Conclusion

In this paper, we have formulated a demand function for multi-market segments and multiple prices (MSMP) using a linear complementarity problem (LCP). Based on this, we introduced a pricing model which is an LCP constrained optimization problem. We presented some properties of this model. Under certain conditions, we have also shown that the complementarity constrained pricing model can be simplified to a linearly constrained optimization problem. Such a simplification is possible for many other optimization problems involving MSMP. This is an important research topic in applications of the MSMP demand function. On the other hand, there are also many instances in which such simplifications can lead to wrong models, as shown by the example in the previous section. In these situations, the LCP constraints inherited from the MSMP demand function are unavoidable and remain as a core structure in the models.

Some future research directions:

- Our model of MSMP demand function need not be a unique one. It will be interesting to discover different models fitting different markets.

- Nonlinear demand functions may be necessary for some markets. Formulation of general or specific nonlinear MSMP demand functions is also an important direction.

- With an increasing number of available models of MSMP demand function, the
investigation of their applications is certainly a rich area for researchers and practitioners.

References


