

UNIQUENESS OF BESSEL MODELS: THE ARCHIMEDEAN CASE

DIHUA JIANG, BINYONG SUN AND CHEN-BO ZHU

Abstract. In the archimedean case, we prove uniqueness of Bessel models for general linear groups, unitary groups and orthogonal groups.

1 Introduction

Let G be one of the classical Lie groups

$$\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C}), \mathrm{U}(p, q), \mathrm{O}(p, q), \mathrm{O}_n(\mathbb{C}). \quad (1)$$

In order to consider Bessel models for G , we consider, for each non-negative integer r satisfying

$$n \geq 2r + 1, \quad p \geq r, \quad q \geq r + 1,$$

the r -th Bessel subgroup

$$S_r = N_{S_r} \rtimes G_0$$

of G , which is a semidirect product and which will be described explicitly in section 2.1. Here N_{S_r} is the unipotent radical of S_r , and G_0 is respectively identified with

$$\mathrm{GL}_{n-2r-1}(\mathbb{R}), \mathrm{GL}_{n-2r-1}(\mathbb{C}), \mathrm{U}(p-r, q-r-1), \mathrm{O}(p-r, q-r-1), \mathrm{O}_{n-2r-1}(\mathbb{C}). \quad (2)$$

Let χ_{S_r} be a generic character of S_r as defined in section 2.2. The main result of this paper is the following theorem, which is usually called the (archimedean) local uniqueness of Bessel models for G .

Theorem A. *Let G, G_0, S_r and χ_{S_r} be as above. For every irreducible representation π of G and π_0 of G_0 both in the class \mathcal{FH} , the inequality*

$$\dim \mathrm{Hom}_{S_r}(\pi \widehat{\otimes} \pi_0, \chi_{S_r}) \leq 1$$

holds.

We would like to make the following remarks on Theorem A. The symbol “ $\widehat{\otimes}$ ” stands for the completed projective tensor product of complete, locally convex topological vector spaces, and “ Hom_{S_r} ” stands for the space of continuous S_r -intertwining maps. Note that π_0 is viewed as a representation of S_r with the trivial N_{S_r} -action. As is quite common, we do not distinguish a representation with its underlying space.

Recall that a representation of G is said to be in the class \mathcal{FH} if it is Fréchet, smooth, of moderate growth, admissible and $Z(\mathfrak{g}_{\mathbb{C}})$ -finite. Here, and as usual, $Z(\mathfrak{g}_{\mathbb{C}})$ is the center of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G . Of course, the notion of representations in the class \mathcal{FH} fits all real reductive groups. The interested reader may consult [C] and [W, Ch. 11] for more details.

If $r = 0$, then $S_r = G_0$, and Theorem A is the multiplicity one theorem proved by Sun and Zhu in [SuZ] (and independently by Aizenbud and Gourevitch in [AG] for general linear groups). If G_0 is the trivial group, then Theorem A asserts uniqueness of Whittaker models for $GL_{2r+1}(\mathbb{R})$, $GL_{2r+1}(\mathbb{C})$, $U(r, r+1)$, $O(r, r+1)$ and $O_{2r+1}(\mathbb{C})$. See [Sh], [CHM] for local uniqueness of Whittaker models for quasi-split groups (or [JiSZ] for a quick proof). Hence the family of Bessel models interpolates between the Whittaker model (G_0 is trivial) and the spherical model ($r = 0$).

It is a basic problem in representation theory to establish various models with good properties. In particular, this has important applications to the classification of representations and to the theory of automorphic representations.

Whittaker models for representations of quasi-split reductive groups over complex, real and p -adic fields and their local uniqueness property are essential to the Langlands–Shahidi method [S] and the Rankin–Selberg method [Bu] to establish the Langlands conjecture on analytic properties of automorphic L -functions [GeS].

The notion of Bessel models originates from classical Bessel functions and it was first introduced by Novodvorski and Piatetski-Shapiro [NP] to study automorphic L -functions for $Sp(4)$. For orthogonal groups, the Bessel models are essential to establish analytic properties of automorphic L -functions as considered in [GiPSR]. The analogue for unitary groups is expected (see [BS], for example). More recently, Bessel models are used in the construction of automorphic descents from the general linear groups to certain classical groups [GiRS], as well as in the construction of local descents for supercuspidal representations of p -adic groups ([JiS], [So], [JiNQ1], and [JiNQ2]). Further applications of Bessel models to the theory of automorphic forms and automorphic L -functions are expected.

We remark that the local uniqueness of the Bessel models is one of the key properties, which makes applications of these models possible. An important purpose of this paper is to show that the archimedean local uniqueness of general Bessel models can be reduced to the uniqueness of the spherical models proved in [SuZ] (i.e. the $r = 0$ case). The key idea in this reduction is to construct an integral I_{μ} (equation (13) in section 3.3), where μ is a (non-zero) Bessel functional. We note that for p -adic fields, the reduction to the p -adic spherical models (proved in [AGRS]) is known by the work of Gan, Gross and Prasad [GGP]. The approach of this paper works for the p -adic local fields as well.

We now describe the contents and the organization of this paper. In section 2, we recall the general set-up of the Bessel models. In section 3, we outline our strategy, and give the proof of Theorem A, based on two propositions on the aforementioned integral I_{μ} (Propositions 3.3 and 3.4). This integral depends on a complex parameter s . Proposition 3.3 states that I_{μ} , when evaluated at a certain point of the

domain, is absolutely convergent and nonzero. On the other hand, Proposition 3.4 asserts that I_μ converges absolutely for all points of the domain when the real part of the parameter s is large, and it defines a G -invariant continuous linear functional on a representation of $G' \times G$ in the class \mathcal{FH} , where $G' \supset G$ is one of the spherical pairs considered in [SuZ]. The proof of Proposition 3.3 and Proposition 3.4 are given in sections 4 and 6, respectively. Section 5 is devoted to an explicit integral formula (Proposition 5.4), as a preparation for section 6.

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2 Bessel Subgroups and Generic Characters

2.1 Bessel subgroups. In order to describe the Bessel subgroups uniformly in all five cases, we introduce the following notation. Let \mathbb{K} be a \mathbb{R} -algebra, equipped with an involution τ . In this article, (\mathbb{K}, τ) is assumed to be one of the pairs

$$(\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}}), (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}}), (\mathbb{C}, -), (\mathbb{R}, 1_{\mathbb{R}}), (\mathbb{C}, 1_{\mathbb{C}}), \quad (3)$$

where $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are the maps which interchange the coordinates, “ $-$ ” is the complex conjugation, $1_{\mathbb{R}}$ and $1_{\mathbb{C}}$ are the identity maps.

Let E be a hermitian \mathbb{K} -module, namely it is a free \mathbb{K} -module of finite rank, equipped with a non-degenerate \mathbb{R} -bilinear map

$$\langle \cdot, \cdot \rangle_E : E \times E \rightarrow \mathbb{K}$$

satisfying

$$\langle u, v \rangle_E = \langle v, u \rangle_E^{\tau}, \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad a \in \mathbb{K}, \quad u, v \in E.$$

Denote by $G := U(E)$ the group of all \mathbb{K} -module automorphisms of E which preserve the form $\langle \cdot, \cdot \rangle_E$.

Assume that E is nonzero. Let $r \geq 0$ and

$$0 = X_0 \subset X_1 \subset \cdots \subset X_r \subset X_{r+1}$$

be a flag of E such that

- X_i is a free \mathbb{K} -submodule of E of rank i , $i = 0, 1, \dots, r, r + 1$;
- X_r is totally isotropic; and
- $X_{r+1} = X_r \oplus \mathbb{K}v'_0$ (orthogonal direct sum), with v'_0 a non-isotropic vector.

A group of the form

$$S_r := \{x \in G \mid (x - 1)X_{i+1} \subset X_i, \quad i = 0, 1, \dots, r\} \quad (4)$$

is called a r -th Bessel subgroup of G .

To be more explicit, we fix a totally isotropic free \mathbb{K} -submodule Y_r of

$$v_0^{\perp} := \{v \in E \mid \langle v, v'_0 \rangle_E = 0\}$$

of rank r so that the pairing

$$\langle \cdot, \cdot \rangle_E : X_r \times Y_r \rightarrow \mathbb{K}$$

is non-degenerate. Write

$$E_0 := v_0^\perp \cap (X_r \oplus Y_r)^\perp.$$

Then E is decomposed into an orthogonal sum of three submodules:

$$E = (X_r \oplus Y_r) \oplus E_0 \oplus \mathbb{K}v_0'. \quad (5)$$

According to the five cases of (\mathbb{K}, τ) in (3), G is one of the groups in (1). By scaling the form $\langle \cdot, \cdot \rangle_E$, we assume that

$$\langle v_0', v_0' \rangle_E = -1,$$

then $G_0 := U(E_0)$ is one of the groups in (2). The Bessel subgroup S_r is then a semidirect product

$$S_r = N_{S_r} \rtimes G_0, \quad (6)$$

where N_{S_r} is the unipotent radical of S_r .

2.2 Generic characters. Write

$$L_i := \text{Hom}_{\mathbb{K}}(X_{i+1}/X_i, X_i/X_{i-1}), \quad i = 1, 2, \dots, r,$$

which is a free \mathbb{K} -module of rank 1. For any $x \in S_r$, $x - 1$ obviously induces an element of L_i , which is denoted by $[x - 1]_i$. Denote by $[x]_0$ the projection of x to G_0 . It is elementary to check that the map

$$\begin{aligned} \eta_r : S_r &\rightarrow C_r := G_0 \times L_1 \times L_2 \times \cdots \times L_r, \\ x &\mapsto ([x]_0, [x - 1]_1, [x - 1]_2, \dots, [x - 1]_r), \end{aligned} \quad (7)$$

is a surjective homomorphism, and every character on S_r descends to one on C_r . A character on S_r is said to be generic if its descent to C_r has nontrivial restriction to every nonzero \mathbb{K} -submodule of L_i , $i = 1, 2, \dots, r$.

3 The Strategy, and Proof of Theorem A

3.1 The group G' . Introduce

$$E' := E \oplus \mathbb{K}v',$$

with v' a free generator. View it as a hermitian \mathbb{K} -module under the form $\langle \cdot, \cdot \rangle_{E'}$ so that

$$\langle \cdot, \cdot \rangle_{E'}|_{E \times E} = \langle \cdot, \cdot \rangle_E, \quad \langle E, v' \rangle_{E'} = 0 \quad \text{and} \quad \langle v', v' \rangle_{E'} = 1.$$

Then E' is the orthogonal sum of two submodules:

$$E' = (X'_{r+1} \oplus Y'_{r+1}) \oplus E_0,$$

where

$$X'_{r+1} := X_r \oplus \mathbb{K}(v_0' + v') \quad \text{and} \quad Y'_{r+1} := Y_r \oplus \mathbb{K}(v_0' - v')$$

are totally isotropic submodules.

Write $G' := \mathrm{U}(E')$, which contains G as the subgroup fixing v' . Denote by P'_{r+1} the parabolic subgroup of G' preserving X'_{r+1} , and by P_r the parabolic subgroup of G preserving X_r . As usual, we have

$$P'_{r+1} = N_{P'_{r+1}} \rtimes (G_0 \times \mathrm{GL}_{r+1}) \subset G' \quad \text{and} \quad (8)$$

$$P_r = N_{P_r} \rtimes (G'_0 \times \mathrm{GL}_r) \subset G, \quad (9)$$

where $N_{P'_{r+1}}$ and N_{P_r} are the unipotent radicals of P'_{r+1} and P_r , respectively,

$$\mathrm{GL}_{r+1} := \mathrm{GL}_{\mathbb{K}}(X'_{r+1}) \supset \mathrm{GL}_r := \mathrm{GL}_{\mathbb{K}}(X_r),$$

and

$$G'_0 := \mathrm{U}(E'_0) \supset G_0, \quad \text{with } E'_0 := E_0 \oplus \mathbb{K}v'_0.$$

Write

$$N_{r+1} = \{x \in \mathrm{GL}_{r+1} \mid (x-1)X'_{r+1} \subset X_r, (x-1)X_i \subset X_{i-1}, i = 1, 2, \dots, r\},$$

and

$$N_r = \{x \in \mathrm{GL}_r \mid (x-1)X_i \subset X_{i-1}, i = 1, 2, \dots, r\},$$

which are maximal unipotent subgroups of GL_{r+1} and GL_r , respectively.

We now describe other salient features of the Bessel group S_r . It is a subgroup of P_r :

$$S_r = N_{P_r} \rtimes (G_0 \times N_r) \subset P_r = N_{P_r} \rtimes (G'_0 \times \mathrm{GL}_r). \quad (10)$$

Although P_r is not a subgroup of P'_{r+1} , we have that $S_r \subset P'_{r+1}$ and the quotient map $P'_{r+1} \rightarrow G_0 \times \mathrm{GL}_{r+1}$ induces a surjective homomorphism

$$\tilde{\eta}_r : S_r \twoheadrightarrow G_0 \times N_{r+1}. \quad (11)$$

It is elementary to check that every character on S_r descends to one on $G_0 \times N_{r+1}$, and it is generic if and only if its descent to $G_0 \times N_{r+1}$ has generic restriction to N_{r+1} , in the usual sense.

Let χ_{S_r} be a generic character of S_r , as in Theorem A. Write

$$\chi_{S_r} = (\chi_{G_0} \otimes \psi_{r+1}) \circ \tilde{\eta}_r, \quad (12)$$

where χ_{G_0} is a character on G_0 , and ψ_{r+1} is a generic character on N_{r+1} . Throughout this article, we always assume that ψ_{r+1} is unitary. Otherwise the Hom space in Theorem A is trivial, due to the moderate growth condition on the representation π .

3.2 Induced representations of G' . Let π_0 and σ be irreducible representations of G_0 and GL_{r+1} in the class \mathcal{FH} , respectively. Write

$$\rho := \pi_0 \widehat{\otimes} \sigma,$$

which is a representation of $G_0 \times \mathrm{GL}_{r+1}$ in the class \mathcal{FH} .

Put

$$d_{\mathbb{K}} := \begin{cases} 1, & \text{if } \mathbb{K} \text{ is a field,} \\ 2, & \text{otherwise,} \end{cases}$$

and

$$\mathbb{K}_+^{\times} = \begin{cases} \mathbb{R}_+^{\times}, & \text{if } d_{\mathbb{K}} = 1, \\ \mathbb{R}_+^{\times} \times \mathbb{R}_+^{\times}, & \text{otherwise.} \end{cases}$$

Denote by

$$|\cdot| : \mathbb{K}^\times \rightarrow \mathbb{K}_+^\times$$

the map of taking componentwise absolute values. For all $a \in \mathbb{K}_+^\times$ and $s \in \mathbb{C}^{d_{\mathbb{K}}}$, put

$$a^s := a_1^{s_1} a_2^{s_2} \in \mathbb{C}^\times, \quad \text{if } d_{\mathbb{K}} = 2, \quad a = (a_1, a_2), \quad s = (s_1, s_2).$$

If $d_{\mathbb{K}} = 1$, $a^s \in \mathbb{C}^\times$ retains the usual meaning.

We now define certain representations of G' in the class \mathcal{FH} which are induced from the parabolic subgroup P'_{r+1} . For every $s \in \mathbb{C}^{d_{\mathbb{K}}}$, denote by π'_s the space of all smooth functions $f : G' \rightarrow \rho$ such that

$$f(n'gm x) = \chi_{G_0}(g)^{-1} |\det(m)|^s \rho(gm)(f(x)),$$

for all $n' \in N_{P'_{r+1}}$, $g \in G_0$, $m \in \mathrm{GL}_{r+1}$, $x \in G'$. (We introduce the factor $\chi_{G_0}(g)^{-1}$ for convenience only.)

By using Langlands classification and the result of Speth–Vogan [SpV, Th. 1.1], we have

PROPOSITION 3.1. *The representation π'_s is irreducible except for a measure zero set of $s \in \mathbb{C}^{d_{\mathbb{K}}}$.*

3.3 The integral I_μ . Recall that ψ_{r+1} is the generic unitary character of N_{r+1} as in (12). Assume that the representation σ of GL_{r+1} is ψ_{r+1}^{-1} -generic, namely there exists a nonzero continuous linear functional

$$\lambda : \sigma \rightarrow \mathbb{C}$$

such that

$$\lambda(\sigma(m)u) = \psi_{r+1}(m)^{-1} \lambda(u), \quad m \in N_{r+1}, \quad u \in \sigma.$$

We fix one such λ . Define a continuous linear map Λ by the formula

$$\begin{aligned} \Lambda : \rho = \pi_0 \widehat{\otimes} \sigma &\rightarrow \pi_0 \\ u \otimes v &\mapsto \lambda(v)u. \end{aligned}$$

Let π be an irreducible representation of G in the class \mathcal{FH} , as in Theorem A, and let

$$\mu : \pi \times \pi_0 \rightarrow \mathbb{C}$$

be a Bessel functional, namely a continuous bilinear map which corresponds to an element of

$$\mathrm{Hom}_{S_r}(\pi \widehat{\otimes} \pi_0, \chi_{S_r}).$$

LEMMA 3.2. *For every $s \in \mathbb{C}^{d_{\mathbb{K}}}$, $u \in \pi$ and $f \in \pi'_s$, the smooth function*

$$g \mapsto \mu(\pi(g)u, \Lambda(f(g)))$$

on G is left invariant under S_r .

Proof. Let $x \in G$ and $b \in S_r \subset P'_{r+1}$. Write

$$b = n'gm, \quad n' \in N_{P'_{r+1}}, \quad g \in G_0, \quad m \in N_{r+1}.$$

Then

$$\Lambda(f(bx)) = \chi_{G_0}(g)^{-1} \Lambda(\rho(gm)(f(x)))$$

$$\begin{aligned}
&= \chi_{G_0}(g)^{-1} \psi_{r+1}(m)^{-1} \pi_0(g)(\Lambda(f(x))) \\
&= \chi_{S_r}(b)^{-1} \pi_0(g)(\Lambda(f(x))),
\end{aligned}$$

and therefore,

$$\begin{aligned}
\mu(\pi(bx)u, \Lambda(f(bx))) &= \chi_{S_r}(b)^{-1} \mu(\pi(b)\pi(x)u, \pi_0(g)(\Lambda(f(x)))) \\
&= \mu(\pi(x)u, \Lambda(f(x))).
\end{aligned}$$

The last equality holds as b maps to g under the quotient map $S_r \rightarrow G_0$, and π_0 is viewed as a representation of S_r via inflation. \square

Write

$$I_\mu(f, u) := \int_{S_r \backslash G} \mu(\pi(g)u, \Lambda(f(g))) dg, \quad f \in \pi'_s, \quad u \in \pi, \quad (13)$$

where dg is a right G -invariant positive measure on $S_r \backslash G$. It is clear that

$$I_\mu(\pi'_s(g)f, \pi(g)u) = I_\mu(f, u)$$

for all $g \in G$ whenever the integrals converge absolutely.

3.4 Proof of Theorem A. We shall postpone the proof of the following proposition to section 4.

PROPOSITION 3.3. *If $\mu \neq 0$, then there is an element $f_\rho \in \pi'_s$ and a vector $u_\pi \in \pi$ such that the integral $I_\mu(f_\rho, u_\pi)$ converges absolutely, and yields a nonzero number.*

Denote by

$$\operatorname{Re} : \mathbb{C}^{d_{\mathbb{K}}} \rightarrow \mathbb{R}^{d_{\mathbb{K}}}$$

the map of taking real parts componentwise. If $a \in \mathbb{R}^{d_{\mathbb{K}}}$ and $c \in \mathbb{R}$, by writing $a > c$, we mean that each component of a is $> c$.

The proof of the following proposition will be given in section 6 after preparation in section 5.

PROPOSITION 3.4. *There is a real number c_μ such that, for all $s \in \mathbb{C}^{d_{\mathbb{K}}}$ with $\operatorname{Re}(s) > c_\mu$, the integral $I_\mu(f, u)$ converges absolutely for all $f \in \pi'_s$ and all $u \in \pi$, and I_μ defines a continuous linear functional on $\pi'_s \widehat{\otimes} \pi$.*

We now complete the proof of Theorem A. We are given π, π_0 and a generic character χ_{S_r} of S_r as in equation (12). As noted there, we may assume that the generic character ψ_{r+1} of N_{r+1} is unitary. As is well known, there exists an irreducible representation σ of GL_{r+1} in the class \mathcal{FH} which is ψ_{r+1}^{-1} -generic (it follows from [CHM, Th.9.1], for example). For each $\mu \in \operatorname{Hom}_{S_r}(\pi \widehat{\otimes} \pi_0, \chi_{S_r})$, we may therefore define the integral I_μ , as in section 3.3.

Let F_r be a finite-dimensional subspace of $\operatorname{Hom}_{S_r}(\pi \widehat{\otimes} \pi_0, \chi_{S_r})$. By Proposition 3.4, there exists a real number c_{F_r} such that for all $\mu \in F_r$ and all $s \in \mathbb{C}^{d_{\mathbb{K}}}$ with $\operatorname{Re}(s) > c_{F_r}$, the integral $I_\mu(f, u)$ converges absolutely for all $f \in \pi'_s$ and all $u \in \pi$, and defines a continuous linear functional on $\pi'_s \widehat{\otimes} \pi$.

By Proposition 3.1, we may choose one s with $\operatorname{Re}(s) > c_{F_r}$ and π'_s irreducible. By Proposition 3.3, we have a linear embedding

$$F_r \hookrightarrow \operatorname{Hom}_G(\pi'_s \widehat{\otimes} \pi, \mathbb{C}), \quad \mu \mapsto I_\mu.$$

The latter space is at most one dimensional by [SuZ, Th. A], and so is F_r . This proves Theorem A. \square

4 Proof of Proposition 3.3

We continue with the notation of the last section and assume that $\mu \neq 0$. By absorbing the concerned characters into the representations π_0 and σ , we may and we will assume in this section that $\chi_{G_0} = 1$ and $s = 0 \in \mathbb{C}^{d_{\mathbb{K}}}$.

Let \bar{N}_{P_r} be the unipotent subgroup of G which is normalized by $G'_0 \times \mathrm{GL}_r$ so that $(G'_0 \times \mathrm{GL}_r)\bar{N}_{P_r}$ is a parabolic subgroup opposite to P_r . Then

$$P_r\bar{N}_{P_r} \text{ is open in } G.$$

Recall that the Bessel group S_r is a subgroup of P_r :

$$S_r = N_{P_r} \rtimes (G_0 \times N_r) \subset P_r = N_{P_r} \rtimes (G'_0 \times \mathrm{GL}_r).$$

We shall need to integrate over $S_r \backslash G$, and thus over the following product space:

$$N_r \backslash \mathrm{GL}_r \times (G_0 \backslash G'_0) \times \bar{N}_{P_r}.$$

4.1 A nonvanishing lemma on $G_0 \backslash G'_0$.

LEMMA 4.1. *There is a vector $u_\pi \in \pi$ and a smooth function $f_{\pi_0} : G'_0 \rightarrow \pi_0$, compactly supported modulo G_0 such that*

$$f_{\pi_0}(gg') = \pi_0(g)f_{\pi_0}(g'), \quad g \in G_0, \quad g' \in G'_0,$$

and

$$\int_{G_0 \backslash G'_0} \mu(\pi(g')u_\pi, f_{\pi_0}(g')) dg' \neq 0. \quad (14)$$

Proof. Pick $u_\pi \in \pi$ and $u_{\pi_0} \in \pi_0$ so that

$$\mu(u_\pi, u_{\pi_0}) = 1.$$

Let A' be a submanifold of G'_0 such that the multiplication map $G_0 \times A' \rightarrow G'_0$ is an open embedding, and

$$\mathrm{Re}(\mu(\pi(a)u_\pi, u_{\pi_0})) > 0, \quad a \in A'.$$

Let ϕ_0 be a compactly supported non-negative and nonzero smooth function on A' .

Put

$$f_{\pi_0}(g') := \begin{cases} \phi_0(a)\pi_0(g)u_{\pi_0}, & \text{if } g' = ga \in G_0A', \\ 0, & \text{otherwise,} \end{cases}$$

which clearly fulfills all the desired requirements. \square

4.2 Whittaker functions on GL_r . Fix u_π and f_{π_0} as in Lemma 4.1. Set

$$\Phi(m, \bar{n}) := \int_{G_0 \backslash G'_0} \mu(\pi(mg'\bar{n})u_\pi, f_{\pi_0}(g')) dg',$$

which is a smooth function on $\mathrm{GL}_r \times \bar{N}_{P_r}$. It is nonzero since $\Phi(1, 1) \neq 0$ by (14).

Note that μ is χ_{S_r} -equivariant, the representation π_0 of S_r has trivial restriction to N_r , and χ_{S_r} and ψ_{r+1} have the same restriction to N_r . Therefore, we have

$$\Phi(lm, \bar{n}) = \psi_{r+1}(l)\Phi(m, \bar{n}), \quad l \in N_r, \quad m \in \mathrm{GL}_r, \quad \bar{n} \in \bar{N}_{P_r}.$$

Let W_r be a smooth function on GL_r with compact support modulo N_r such that

$$W_r(lm) = \psi_{r+1}(l)^{-1}W_r(m), \quad l \in N_r, \quad m \in \mathrm{GL}_r. \quad (15)$$

The following lemma is due to Jacquet and Shalika ([JS, §3], see also [Co, §4]).

LEMMA 4.2. *For every W_r as above, there is a vector $u_\sigma \in \sigma$ such that*

$$W_r(m) = \lambda(\sigma(m)u_\sigma), \quad m \in \mathrm{GL}_r.$$

Let $\phi_{\bar{N}}$ be a smooth function on \bar{N}_{P_r} with compact support. Pick W_r and $\phi_{\bar{N}}$ appropriately so that

$$\int_{(N_r \backslash \mathrm{GL}_r) \times \bar{N}_{P_r}} \delta_{P_r}^{-1}(m) \Phi(m, \bar{n}) W_r(m) \phi_{\bar{N}}(\bar{n}) dm d\bar{n} \neq 0. \quad (16)$$

Here and as usual, we denote by

$$\delta_H : h \mapsto |\det(\mathrm{Ad}_h)|$$

the modular character of a Lie group H .

4.3 The construction of f_ρ . Note that

$$P'_{r+1} \cap G = N_{P_r} \rtimes (G_0 \times \mathrm{GL}_r). \quad (17)$$

By counting the dimensions of the concerned Lie groups, we check that the multiplication map

$$P'_{r+1} \times G \rightarrow G' \quad \text{is a submersion.} \quad (18)$$

From (17) and (18), we see that the multiplication map

$$\iota_{G'} : (N_{P'_{r+1}} \rtimes \mathrm{GL}_{r+1}) \times (G'_0 \rtimes \bar{N}_{P_r}) \rightarrow G'$$

is an open embedding.

Put

$$f_\rho(x) := \begin{cases} \phi_{\bar{N}}(\bar{n}) f_{\pi_0}(g') \otimes (\sigma(m)u_\sigma), & \text{if } x = \iota_{G'}(n', m, g', \bar{n}), \\ 0, & \text{if } x \text{ is not in the image of } \iota_{G'}, \end{cases}$$

where u_σ is as in Lemma 4.2. Then $f_\rho \in \pi'_s$ (recall that s is assumed to be 0).

Finally, we have that

$$\begin{aligned} I_\mu(f_\rho, u_\pi) &= \int_{S_r \backslash G} \mu(\pi(x)u_\pi, \Lambda(f_\rho(x))) dx \\ &= \int_{(N_r \backslash \mathrm{GL}_r) \times (G_0 \backslash G'_0) \times \bar{N}_{P_r}} \mu(\pi(mg'\bar{n})u_\pi, \Lambda(f_\rho(mg'\bar{n}))) \delta_{P_r}^{-1}(m) dm dg' d\bar{n} \\ &= \int_{(N_r \backslash \mathrm{GL}_r) \times (G_0 \backslash G'_0) \times \bar{N}_{P_r}} \delta_{P_r}^{-1}(m) \lambda(\sigma(m)u_\sigma) \phi_{\bar{N}}(\bar{n}) \\ &\quad \cdot \mu(\pi(mg'\bar{n})u_\pi, f_{\pi_0}(g')) dm dg' d\bar{n} \\ &= \int_{(N_r \backslash \mathrm{GL}_r) \times \bar{N}_{P_r}} \delta_{P_r}^{-1}(m) \Phi(m, \bar{n}) W_r(m) \phi_{\bar{N}}(\bar{n}) dm d\bar{n}, \end{aligned}$$

which converges to a nonzero number by (16). This finishes the proof of Proposition 3.3.

5 Another Integral Formula on $S_r \backslash G$

This section and the next section are devoted to a proof of Proposition 3.4 in the case when $E'_0 := E_0 \oplus \mathbb{K}v'_0$ is isotropic, i.e. when E'_0 has a torsion-free isotropic vector. The anisotropic case is simpler and is left to the reader. We first develop some generalities in the following two subsections.

5.1 Commuting positive forms. Let F be a free \mathbb{K} -module of finite rank. By a positive form on F , we mean a \mathbb{R} -bilinear map

$$[\cdot, \cdot]_F : F \times F \rightarrow \mathbb{K}$$

satisfying

$$[u, v]_F = \overline{[v, u]_F}, \quad [au, v]_F = a[u, v]_F, \quad a \in \mathbb{K}, \quad u, v \in F.$$

and

$$[u, u]_F \in \mathbb{K}_+^\times \quad \text{for all torsion-free } u \in F.$$

Here $\bar{a} \in \mathbb{K}$ denotes the componentwise complex conjugation of a , for $a \in \mathbb{K}$.

Now further assume that F is a hermitian \mathbb{K} -module, i.e. a non-degenerate hermitian form $\langle \cdot, \cdot \rangle_F$ (with respect to τ) on F is given. We say that the positive form $[\cdot, \cdot]_F$ is commuting (with respect to $\langle \cdot, \cdot \rangle_F$) if

$$\theta_F^2 = 1,$$

where $\theta_F : F \rightarrow F$ is the \mathbb{R} -linear map specified by

$$[u, v]_F = \langle u, \theta_F v \rangle_F, \quad u, v \in F. \quad (19)$$

The following lemma is elementary.

LEMMA 5.1. *Up to the action of $U(F)$, there exists a unique commuting positive form on F .*

Proof. We check the case of complex orthogonal groups, and leave other cases to the reader. So assume that $(\mathbb{K}, \tau) = (\mathbb{C}, 1_{\mathbb{C}})$. Then

$$[\cdot, \cdot]_F \mapsto \text{the eigenspace of } \theta_F \text{ of eigenvalue } 1$$

defines a $U(F)$ -equivariant bijection:

$$\begin{aligned} & \{\text{commuting positive forms on } F\} \\ \leftrightarrow & \{\text{real forms } F_0 \text{ of } F \text{ so that } \langle \cdot, \cdot \rangle_{F_0 \times F_0} \text{ is real valued and positive definite}\}. \end{aligned}$$

The assertion follows immediately. \square

5.2 A Jacobian. Now fix a commuting positive form $[\cdot, \cdot]_F$, and denote by $K(F)$ its stabilizer in $U(F)$ (which is also the centralizer of θ_F in $U(F)$). Then $K(F)$ is a maximal compact subgroup of $U(F)$. Write

$$F = F_+ \oplus F_-,$$

where F_+ and F_- are eigenspaces of θ_F of eigenvalues 1 and -1 , respectively.

With the preparation of the commuting positive forms, we set

$$S_F := \{u + v \mid u \in F_+, v \in F_-, [u, u]_F = [v, v]_F = 1, [u, v]_F = 0\}.$$

Assume that F is isotropic, i.e. there is a torsion-free vector of F which is isotropic with respect to $\langle \cdot, \cdot \rangle_F$. This is the case of concern. Then S_F is nonempty. It is easy to check that $K(F)$ acts transitively on S_F . According to [Shi], S_F is in fact a Nash-manifold. Furthermore, it is a Riemannian manifold with the restriction of the metric

$$\frac{1}{\dim_{\mathbb{R}} \mathbb{K}} \operatorname{tr}_{\mathbb{K}/\mathbb{R}} [\cdot, \cdot]_F.$$

Write

$$\Gamma_{F,-1} := \{u \in F \mid \langle u, u \rangle_F = -1\},$$

which is a Nash-manifold. It is also a pseudo-Riemannian manifold with the restriction of the metric

$$\frac{1}{\dim_{\mathbb{R}} \mathbb{K}} \operatorname{tr}_{\mathbb{K}/\mathbb{R}} \langle \cdot, \cdot \rangle_F.$$

Equip \mathbb{R}_+^{\times} with the invariant Riemannian metric so that the tangent vector $t \frac{d}{dt}$ at $t \in \mathbb{R}_+^{\times}$ has length 1. As a product of one or two copies of \mathbb{R}_+^{\times} , \mathbb{K}_+^{\times} is again a Riemannian manifold.

Define a map

$$\begin{aligned} \eta_F : S_F \times \mathbb{K}_+^{\times} &\rightarrow \Gamma_{F,-1}, \\ (w, t) &\mapsto \frac{t-t^{-\tau}}{2}u + \frac{t+t^{-\tau}}{2}v, \end{aligned}$$

where

$$w = u + v, \quad u \in F_+, \quad v \in F_-.$$

Note that the domain and the range of the smooth map η_F have the same real dimension. Denote by J_{η_F} the Jacobian of η_F (with respect to the metrics defined above), which is a non-negative continuous function on $S_F \times \mathbb{K}_+^{\times}$. Since η_F and all the involved metrics are semi-algebraic, J_{η_F} is also semi-algebraic (see [Shi] for the notion of semi-algebraic maps and Nash maps). Note that $K(F)$ acts transitively on S_F (and trivially on \mathbb{K}_+^{\times}), η_F and all the involved metrics are $K(F)$ -equivariant. Therefore, there is a non-negative continuous semi-algebraic function J_F on \mathbb{K}_+^{\times} such that

$$J_{\eta_F}(w, t) = J_F(t), \quad w \in S_F, \quad t \in \mathbb{K}_+^{\times}.$$

Denote $C(X)$ the space of continuous functions on any (topological) space X .

LEMMA 5.2. *For $\phi \in C(\Gamma_{F,-1})$, one has that*

$$\int_{\Gamma_{F,-1}} \phi(x) dx = \frac{1}{2} \int_{\mathbb{K}_+^{\times}} J_F(t) \int_{S_F} \phi(\eta_F(w, t)) dw d^{\times} t,$$

where dx , dw and $d^{\times} t$ are the volume forms associated to the respective metrics.

Proof. For every $t \in \mathbb{K}_+^{\times}$, write

$$\langle t \rangle := \begin{cases} t, & \text{if } d_{\mathbb{K}} = 1, \\ t_1 t_2, & \text{if } d_{\mathbb{K}} = 2 \text{ and } t = (t_1, t_2). \end{cases} \quad (20)$$

Write

$$\Gamma_{F,-1}(1) = \{u \in \Gamma_{F,-1} \mid \langle [u, u]_F \rangle = 1\},$$

which is a closed submanifold of $\Gamma_{F,-1}$ of measure zero. One checks case by case that η_F induces diffeomorphisms from both

$$S_F \times \{t \in \mathbb{K}_+^\times \mid \langle t \rangle > 1\} \quad \text{and} \quad S_F \times \{t \in \mathbb{K}_+^\times \mid \langle t \rangle < 1\}$$

onto $\Gamma_{F,-1} \setminus \Gamma_{F,-1}(1)$. The lemma then follows. \square

5.3 A preliminary integral formula on $G_0 \backslash G'_0$. We recall the notation of section 2. The isotropic condition on $E'_0 = E_0 \oplus \mathbb{K}v'_0$ (which we assume as in the beginning of this section) ensures that there is a vector $v_0 \in E_0$ such that

$$\langle v_0, v_0 \rangle = 1.$$

Denote by Z_0 its orthogonal complement in E_0 . Then E is an orthogonal sum of four submodules:

$$E = (X_r \oplus Y_r) \oplus Z_0 \oplus \mathbb{K}v_0 \oplus \mathbb{K}v'_0. \quad (21)$$

Fix a commuting positive form $[\cdot, \cdot]_E$ on E so that (21) is an orthogonal sum of five submodules with respect to $[\cdot, \cdot]_E$. Recall that

$$G := U(E), \quad G'_0 := U(E'_0), \quad G_0 = U(E_0).$$

Put

$$K := K(E), \quad K'_0 := K(E'_0), \quad K_0 := K(E_0).$$

For every $t \in \mathbb{K}_+^\times$, denote by $g_t \in G'_0$ the element which is specified by

$$\begin{cases} g_t(v_0 + v'_0) = t(v_0 + v'_0), \\ g_t(v_0 - v'_0) = t^{-\tau}(v_0 - v'_0), \quad \text{and} \\ g_t|_{X_r \oplus Y_r \oplus Z_0} = \text{the identity map.} \end{cases} \quad (22)$$

We use the results of the last two subsections to prove the following lemma.

LEMMA 5.3. *For $\phi \in C(G_0 \backslash G'_0)$, we have*

$$\int_{G_0 \backslash G'_0} \phi(x) dx = \int_{\mathbb{K}_+^\times} J_{E'_0}(t) \int_{K'_0} \phi(g_t k) dk d^\times t,$$

where dk is the normalized haar measure on K'_0 , and dx is a suitably normalized G'_0 -invariant positive measure on $G_0 \backslash G'_0$.

Proof. By first integrating over K'_0 , we just need to show that

$$\int_{G_0 \backslash G'_0} \phi(x) dx = \int_{\mathbb{K}_+^\times} J_{E'_0}(t) \phi(g_t) d^\times t, \quad \phi \in C(G_0 \backslash G'_0 / K'_0). \quad (23)$$

We identify $G_0 \backslash G'_0$ with $\Gamma_{E'_0, -1}$ by the map $g \mapsto g^{-1}v'_0$. Note that $v_0 + v'_0 \in S_{E'_0}$ and $G_0 g_t$ is identified with $\eta_{E'_0}(v_0 + v'_0, t^{-1})$. The measure dx is identified with a constant C multiple of the metric measure dy on $\Gamma_{E'_0, -1}$.

Let

$$\phi \in C(G_0 \backslash G'_0 / K'_0) = C(K'_0 \backslash \Gamma_{E'_0, -1}).$$

Then the function $\phi(\eta_{E'_0}(w, t^{-1}))$ is independent of $w \in S_{E'_0}$. Also note that

$$J_{E'_0}(t) = J_{E'_0}(t^{-1}), \quad t \in \mathbb{K}_+^\times.$$

Therefore by Lemma 5.2, we have

$$\begin{aligned}
\int_{G_0 \backslash G'_0} \phi(x) dx &= C \int_{\Gamma_{E'_0, -1}} \phi(y) dy \\
&= \frac{1}{2} C \int_{\mathbb{K}_+^\times} J_{E'_0}(t) \int_{S_{E'_0}} \phi(\eta_{E'_0}(w, t)) dw d^\times t \\
&= \frac{1}{2} C \int_{\mathbb{K}_+^\times} J_{E'_0}(t^{-1}) \int_{S_{E'_0}} \phi(\eta_{E'_0}(w, t^{-1})) dw d^\times t \\
&= \frac{1}{2} C \int_{\mathbb{K}_+^\times} J_{E'_0}(t) \int_{S_{E'_0}} \phi(\eta_{E'_0}(v_0 + v'_0, t^{-1})) dw d^\times t \\
&= \frac{1}{2} C \int_{\mathbb{K}_+^\times} J_{E'_0}(t) \phi(gt) \int_{S_{E'_0}} 1 dw d^\times t.
\end{aligned}$$

We finish the proof by putting

$$C := 2 \left(\int_{S_{E'_0}} 1 dw \right)^{-1}. \quad \square$$

5.4 The integral formula on $S_r \backslash G$. Denote by B_r the Borel subgroup of GL_r stabilizing the flag

$$0 = X_0 \subset X_1 \subset \cdots \subset X_r.$$

For every $\mathbf{t} = (t_1, t_2, \dots, t_r) \in (\mathbb{K}_+^\times)^r$, denote by $a_{\mathbf{t}}$ the element of GL_r whose restriction to

$$\{v \in X_i \mid [v, X_{i-1}]_E = 0\}$$

is the scalar multiplication by t_i , for $i = 1, 2, \dots, r$.

PROPOSITION 5.4. *For $\phi \in C(S_r \backslash G)$, one has that*

$$\int_{S_r \backslash G} \phi(g) dg = \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times \times K} \phi(a_{\mathbf{t}} g t k) \delta_{P_r}^{-1}(a_{\mathbf{t}}) \delta_{B_r}^{-1}(a_{\mathbf{t}}) J_{E'_0}(t) d^\times \mathbf{t} d^\times t dk,$$

where dg is a suitably normalized right G -invariant measure on $S_r \backslash G$.

Proof. Write $K_r = K \cap \mathrm{GL}_r$. Then we have

$$\begin{aligned}
\int_{S_r \backslash G} \phi(g) dg &= \int_{(N_{P_r} N_r G_0) \backslash (N_{P_r} \mathrm{GL}_r G'_0 K)} \phi(g) dg \\
&= \int_{(N_r \backslash \mathrm{GL}_r) \times (G_0 \backslash G'_0) \times K} \phi(m g' k) \delta_{P_r}^{-1}(m) dm dg' dk \\
&= \int_{(\mathbb{K}_+^\times)^r \times K_r \times (G_0 \backslash G'_0) \times K} \phi(a_{\mathbf{t}} l g' k) \delta_{P_r}^{-1}(a_{\mathbf{t}}) \delta_{B_r}^{-1}(a_{\mathbf{t}}) d^\times \mathbf{t} dl dg' dk \\
&= \int_{(\mathbb{K}_+^\times)^r \times (G_0 \backslash G'_0) \times K} \phi(a_{\mathbf{t}} g' k) \delta_{P_r}^{-1}(a_{\mathbf{t}}) \delta_{B_r}^{-1}(a_{\mathbf{t}}) d^\times \mathbf{t} dg' dk \\
&\quad (l \in K_r \subset \mathrm{GL}_r \text{ commutes with } g' \in G'_0)
\end{aligned}$$

$$\begin{aligned}
&= \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times \times K'_0 \times K} \phi(a_t g_t l k) \delta_{P_r}^{-1}(a_t) \delta_{B_r}^{-1}(a_t) J_{E'_0}(t) d^\times \mathbf{t} d^\times t dl dk \\
&\quad \text{(by Lemma 5.3)} \\
&= \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times \times K} \phi(a_t g_t k) \delta_{P_r}^{-1}(a_t) \delta_{B_r}^{-1}(a_t) J_{E'_0}(t) d^\times \mathbf{t} d^\times t dk. \quad \square
\end{aligned}$$

6 Proof of Proposition 3.4

6.1 An Iwasawa decomposition. Recall that we have a hermitian \mathbb{K} -module

$$E' = X_r \oplus Y_r \oplus Z_0 \oplus \mathbb{K}v_0 \oplus \mathbb{K}v'_0 \oplus \mathbb{K}v'. \quad (24)$$

Equip it with the commuting positive form $[\ , \]_{E'}$ which extends $[\ , \]_E$ and makes v' and E perpendicular. Also recall that $G' := U(E')$. Put $K' := K(E')$.

Write

$$E_3 = \mathbb{K}v_0 \oplus \mathbb{K}v'_0 \oplus \mathbb{K}v', \quad (25)$$

and denote by N_{E_3} the unipotent radical of the Borel subgroup of $U(E_3)$ stabilizing the line $\mathbb{K}(v'_0 + v')$. For every $t \in \mathbb{K}_+^\times$, denote by $b_t \in U(E_3)$ the element specified by

$$\begin{cases} b_t(v_0) = v_0, \\ b_t(v'_0 + v') = t(v'_0 + v'), \\ b_t(v'_0 - v') = t^{-\tau}(v'_0 - v'). \end{cases} \quad (26)$$

For every $t \in \{t \in \mathbb{K}_+^\times \mid tt^\tau = 1\}$, denote by $c_t \in U(E_3)$ the element specified by

$$\begin{cases} c_t(v_0) = tv_0, \\ c_t(v'_0) = v'_0, \\ c_t(v') = v'. \end{cases}$$

Recall the element $g_t \in G'_0 \subset G'$ in (22). Note that it also stays in $U(E_3)$. By Iwasawa decomposition, we write

$$g_t = c_{t'} n_t b_{t'} k_t, \quad n_t \in N_{E_3}, \quad k_t \in K(E_3). \quad (27)$$

Then both t' and t'' are Nash functions of t .

LEMMA 6.1. *One has that*

$$t' = 2(t^{-2} + t^{2\tau} + 2)^{-1/2}.$$

Proof. Note that v_0, v'_0, v' is an orthonormal basis of E_3 with respect to $[\ , \]_{E'}$. We have that

$$\begin{aligned}
&[g_t^{-1}(v'_0 + v'), g_t^{-1}(v'_0 + v')]_{E'} \\
&= [k_t^{-1} b_{t'}^{-1} n_t^{-1} c_{t''}^{-1}(v'_0 + v'), k_t^{-1} b_{t'}^{-1} n_t^{-1} c_{t''}^{-1}(v'_0 + v')]_{E'} \\
&= [t'^{-1}(v'_0 + v'), t'^{-1}(v'_0 + v')]_{E'} \\
&= 2t'^{-2}.
\end{aligned}$$

On the other hand,

$$g_t^{-1}(v'_0 + v') = \frac{t^{-1} - t^\tau}{2} v_0 + \frac{t^{-1} + t^\tau}{2} v'_0 + v',$$

and

$$\begin{aligned} [g_t^{-1}(v'_0 + v'), g_t^{-1}(v'_0 + v')]_{E'} &= \left(\frac{t^{-1} - t^\tau}{2} \right)^2 + \left(\frac{t^{-1} + t^\tau}{2} \right)^2 + 1 \\ &= \frac{t^{-2} + t^{2\tau} + 2}{2}. \end{aligned}$$

Therefore the lemma follows. \square

6.2 Majorization of Whittaker functions. We define a norm function on G' by

$$\|g\| := \max \{ (\langle [gu, gu]_{E'} \rangle)^{1/2} \mid u \in E', \langle [u, u]_{E'} \rangle = 1 \}, \quad g \in G',$$

where $\langle \cdot \rangle$ is as in (20).

For every $\tilde{\mathbf{t}} = (t_1, t_2, \dots, t_r, t_{r+1}) \in (\mathbb{K}_+^\times)^{r+1}$, write

$$\xi(\tilde{\mathbf{t}}) = \begin{cases} \prod_{i=1}^r \left(1 + \frac{t_i}{t_{i+1}}\right), & \text{if } d_{\mathbb{K}} = 1, \\ \prod_{i=1}^r \left(1 + \frac{t_{i,1}}{t_{i+1,1}}\right) \times \prod_{i=1}^r \left(1 + \frac{t_{i,2}}{t_{i+1,2}}\right), & \text{if } d_{\mathbb{K}} = 2 \text{ and } t_i = (t_{i,1}, t_{i,2}). \end{cases} \quad (28)$$

Write

$$a_{\tilde{\mathbf{t}}} = a_{\mathbf{t}} b_{t_{r+1}} \in \mathrm{GL}_{r+1}, \quad \text{with } \mathbf{t} = (t_1, t_2, \dots, t_r).$$

Recall that $a_{\mathbf{t}}$ is defined in section 5.4 and b_t is defined in (26).

Following [J, Prop. 3.1], we have

LEMMA 6.2. *Let notation be as in section 3.2. Let c_ρ be a positive number, $|\cdot|_{\pi_0}$ a continuous seminorm on π_0 , and $|\cdot|_{\rho,0}$ a continuous seminorm on ρ . Assume that*

$$|\Lambda(\rho(g)u)|_{\pi_0} \leq \|g\|^{c_\rho} |u|_{\rho,0}, \quad g \in G_0 \times \mathrm{GL}_{r+1}, \quad u \in \rho.$$

Then for any positive integer N , there is a continuous seminorm $|\cdot|_{\rho,N}$ on ρ such that

$$|\Lambda(\rho(a_{\tilde{\mathbf{t}}})u)|_{\pi_0} \leq \xi(\tilde{\mathbf{t}})^{-N} \|a_{\tilde{\mathbf{t}}}\|^{c_\rho} |u|_{\rho,N}, \quad \tilde{\mathbf{t}} \in (\mathbb{K}_+^\times)^{r+1}, \quad u \in \rho.$$

Proof. To ease the notation, we assume that $d_{\mathbb{K}} = 1$. The other case is proved in the same way. For every $i = 1, 2, \dots, r$, let Y_i be a vector in the Lie algebra of GL_{r+1} so that

$$\mathrm{Ad}_{a_{\tilde{\mathbf{t}}}} Y_i = \frac{t_i}{t_{i+1}} Y_i, \quad \tilde{\mathbf{t}} = (t_1, t_2, \dots, t_{r+1}),$$

and

$$m_i := -\psi_{r+1}(Y_i) \neq 0.$$

Here ψ_{r+1} stands for the differential of the same named character. Similar notation will be used for the differentials of representations.

For every sequence $\mathbf{N} = (N_1, N_2, \dots, N_r)$ of non-negative integers, write

$$\tilde{\mathbf{t}}^{(\mathbf{N})} := \prod_{i=1}^r (t_i/t_{i+1})^{N_i}, \quad \tilde{\mathbf{t}} = (t_1, t_2, \dots, t_{r+1}) \in (\mathbb{K}_+^\times)^{r+1}.$$

Also write

$$Y^{\mathbf{N}} = Y_1^{N_1} Y_2^{N_2} \dots Y_r^{N_r},$$

which is an element in the universal enveloping algebra of the Lie algebra of GL_{r+1} .

Then

$$\begin{aligned}\Lambda(\rho(a_{\tilde{\mathbf{t}}})\rho(Y^{\mathbf{N}})u) &= \Lambda(\rho(\text{Ad}_{a_{\tilde{\mathbf{t}}}}Y^{\mathbf{N}})\rho(a_{\tilde{\mathbf{t}}})u) \\ &= \tilde{\mathbf{t}}^{(\mathbf{N})}\Lambda(\rho(Y^{\mathbf{N}})\rho(a_{\tilde{\mathbf{t}}})u) \\ &= \left(\prod_{i=1}^r m_i^{N_i}\right)\tilde{\mathbf{t}}^{(\mathbf{N})}\Lambda(\rho(a_{\tilde{\mathbf{t}}})u).\end{aligned}$$

Therefore,

$$\tilde{\mathbf{t}}^{(\mathbf{N})}\left|\Lambda(\rho(a_{\tilde{\mathbf{t}}})u)\right|_{\pi_0} \leq |m|^{-\mathbf{N}}\|a_{\tilde{\mathbf{t}}}\|^{c_\rho}\left|\rho(Y^{\mathbf{N}})u\right|_{\rho,0}, \quad (29)$$

where

$$|m|^{-\mathbf{N}} := \prod_{i=1}^r |m_i|^{-N_i}.$$

Given the positive integer N , write

$$\xi(\tilde{\mathbf{t}})^N = \sum_{\mathbf{N}} a_{\mathbf{N}}\tilde{\mathbf{t}}^{(\mathbf{N})},$$

where $a_{\mathbf{N}}$'s are non-negative integers. In view of (29), we finish the proof by setting

$$|u|_{\rho,N} := \sum_{\mathbf{N}} a_{\mathbf{N}}|m|^{-\mathbf{N}}\left|\rho(Y^{\mathbf{N}})u\right|_{\rho,0}. \quad \square$$

6.3 Convergence of an integral.

LEMMA 6.3. *For any non-negative continuous semi-algebraic function J on $(\mathbb{K}_+^\times)^{r+1}$, there is a positive number c_J with the following property: for every $s \in \mathbb{R}^{d_{\mathbb{K}}}$ with $s > c_J$, there is a positive integer N such that*

$$\int_{(\mathbb{K}_+^\times)^{r+1}} (t_1 t_2 \cdots t_r t'_{r+1})^s \xi(t_1, t_2, \dots, t_r, t'_{r+1})^{-N} J(\tilde{\mathbf{t}}) d^\times \tilde{\mathbf{t}} < \infty, \quad (30)$$

where

$$\tilde{\mathbf{t}} = (t_1, t_2, \dots, t_r, t_{r+1}), \quad t'_{r+1} = 2(t_{r+1}^{-2} + t_{r+1}^{2r} + 2)^{-1/2},$$

and ξ is defined in (28).

Proof. To ease the notation, we again assume that $d_{\mathbb{K}} = 1$. Note that the change of variable

$$\tilde{\mathbf{t}} \mapsto \tilde{\alpha} := \left(\alpha_1 = \frac{t_1}{t_2}, \alpha_2 = \frac{t_2}{t_3}, \dots, \alpha_{r-1} = \frac{t_{r-1}}{t_r}, \alpha_r = \frac{t_r}{t'_{r+1}}, t_{r+1}\right)$$

is a measure preserving Nash isomorphism from $(\mathbb{K}_+^\times)^{r+1}$ onto itself. So J is also a continuous semi-algebraic function of $\tilde{\alpha}$. It is well known that every continuous semi-algebraic function (on a closed semi-algebraic subset of a finite-dimensional real vector space) is of polynomial growth [BoCR, Prop.2.6.2]. (This is the reason that we work in the semi-algebraic setting in this article.) Therefore, there is a positive number c'_J such that

$$J(\tilde{\alpha}) \leq \left(\prod_{j=1}^r (\alpha_j + \alpha_j^{-1})^{c'_J}\right) \times (t_{r+1} + t_{r+1}^{-1})^{c'_J}, \quad \tilde{\mathbf{t}} \in (\mathbb{K}_+^\times)^{r+1}.$$

Take a positive number c_J , large enough so that

$$\int_{\mathbb{K}_+^\times} t'_{r+1} {}^{(r+1)c_J} (t_{r+1} + t_{r+1}^{-1})^{c_J} d^\times t_{r+1} < \infty.$$

and

$$\int_0^1 \alpha_j^{jc_J} (\alpha_j + \alpha_j^{-1})^{c_J} d^\times \alpha_j < \infty, \quad j = 1, 2, \dots, r.$$

The integral (30) is equal to

$$\begin{aligned} & \int_{(\mathbb{K}_+^\times)^{r+1}} \alpha_1^s \alpha_2^{2s} \cdots \alpha_r^{rs} t'_{r+1} {}^{(r+1)s} \prod_{j=1}^r (1 + \alpha_j)^{-N} J(\tilde{\alpha}) d^\times \tilde{\alpha} \\ & \leq \left(\prod_{j=1}^r \int_{\mathbb{K}_+^\times} \frac{\alpha_j^{js} (\alpha_j + \alpha_j^{-1})^{c_J}}{(1 + \alpha_j)^N} d^\times \alpha_j \right) \times \int_{\mathbb{K}_+^\times} t'_{r+1} {}^{(r+1)s} (t_{r+1} + t_{r+1}^{-1})^{c_J} d^\times t_{r+1}. \end{aligned}$$

Now it is clear that the above integral converges when $s > c_J$ and N is large enough so that

$$\int_1^\infty \frac{\alpha_j^{js} (\alpha_j + \alpha_j^{-1})^{c_J}}{(1 + \alpha_j)^N} d^\times \alpha_j < \infty, \quad j = 1, 2, \dots, r. \quad \square$$

6.4 End of proof of Proposition 3.4. Take a continuous seminorm $|\cdot|_{\pi,0}$ on π and a continuous seminorm $|\cdot|_{\pi_0}$ on π_0 such that

$$|\mu(u, v)| \leq |u|_{\pi,0} \cdot |v|_{\pi_0}, \quad u \in \pi, \quad v \in \pi_0. \quad (31)$$

Take a positive integer c_π and a continuous seminorm $|\cdot|_{\pi,1}$ on π such that

$$|\pi(g)u|_{\pi,0} \leq \|g\|^{c_\pi} |u|_{\pi,1}, \quad g \in G, \quad u \in \pi. \quad (32)$$

Take a positive integer c_ρ and a continuous seminorm $|\cdot|_{\rho,0}$ on ρ such that

$$|\Lambda(\rho(g)(u))|_{\pi_0} \leq \|g\|^{c_\rho} |u|_{\rho,0}, \quad g \in G_0 \times \mathrm{GL}_{r+1}, \quad u \in \rho. \quad (33)$$

Now assume that

$$g = a_{\mathbf{t}} g_{\mathbf{t}} k, \quad \mathbf{t} = (t_1, t_2, \dots, t_r) \in (\mathbb{K}_+^\times)^r, \quad t \in \mathbb{K}_+^\times, \quad k \in K.$$

Write

$$g_{\mathbf{t}} = c_{\mathbf{t}'} n_{\mathbf{t}} b_{\mathbf{t}'} k_{\mathbf{t}}$$

as in (27). Then for all $s \in \mathbb{C}^{d_{\mathbb{K}}}$, $u \in \pi$, $f \in \pi'_s$, we have

$$\begin{aligned} \mu(\pi(g)u, \Lambda(f(g))) &= \mu(\pi(c_{\mathbf{t}'} n_{\mathbf{t}} b_{\mathbf{t}'} a_{\mathbf{t}} k_{\mathbf{t}} k)u, \Lambda(f(c_{\mathbf{t}'} n_{\mathbf{t}} b_{\mathbf{t}'} a_{\mathbf{t}} k_{\mathbf{t}} k))) \\ & \quad (k_{\mathbf{t}} \in \mathrm{U}(E_3) \text{ commutes with } a_{\mathbf{t}} \in \mathrm{GL}_r) \\ &= \mu(\pi(n_{\mathbf{t}} b_{\mathbf{t}'} a_{\mathbf{t}} k_{\mathbf{t}} k)u, \Lambda(f(n_{\mathbf{t}} b_{\mathbf{t}'} a_{\mathbf{t}} k_{\mathbf{t}} k))) \\ & \quad (\text{By Lemma 3.2 and the fact that } c_{\mathbf{t}'} \in G_0 \subset S_r) \\ &= \mu(\pi(n_{\mathbf{t}} b_{\mathbf{t}'} a_{\mathbf{t}} k_{\mathbf{t}} k)u, (t_1 t_2 \cdots t_r t')^s \Lambda(\rho(a_{\mathbf{t}} b_{\mathbf{t}'}) f(k_{\mathbf{t}} k))) \\ & \quad (n_{\mathbf{t}} \in N_{P'_{r+1}} \text{ and } a_{\mathbf{t}} b_{\mathbf{t}'} \in \mathrm{GL}_{r+1}). \end{aligned}$$

Therefore, by (31), (32) and the fact that the norm function $\|\cdot\|$ on G' is right K' -invariant, we have

$$\begin{aligned}
& |\mu(\pi(g)u, \Lambda(f(g)))| \\
& \leq (t_1 t_2 \cdots t_r t')^{\operatorname{Re}(s)} \times \|n_t b_{t'} a_t\|^{c_\pi} \times |u|_{\pi,1} \times \left| \Lambda(\rho(a_t b_{t'}) f(k_t k)) \right|_{\pi_0}. \quad (34)
\end{aligned}$$

Let J be the (non-negative continuous semi-algebraic) function on $(\mathbb{K}_+^\times)^{r+1}$ defined by

$$J(\mathbf{t}, t) := \|n_t b_{t'} a_t\|^{c_\pi} \times \|a_t b_{t'}\|^{c_\rho} \times \delta_{P_r}^{-1}(a_t) \delta_{B_r}^{-1}(a_t) J_{E'_0}(t).$$

Let $c_\mu := c_J$ be as in Lemma 6.3, and assume that the real part $\operatorname{Re}(s) > c_\mu$. Let N be a large integer as in Lemma 6.3 so that

$$c_{s,N} := \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times} (t_1 t_2 \cdots t_r t')^{\operatorname{Re}(s)} \xi(\mathbf{t}, t')^{-N} J(\mathbf{t}, t) d^\times \mathbf{t} d^\times t < \infty.$$

Take a continuous seminorm $|\cdot|_{\rho,N}$ on ρ as in Lemma 6.2. Then we have

$$\left| \Lambda(\rho(a_t b_{t'}) f(k_t k)) \right|_{\pi_0} \leq \xi(\mathbf{t}, t')^{-N} \|a_t b_{t'}\|^{c_\rho} |f(k_t k)|_{\rho,N} \leq \xi(\mathbf{t}, t')^{-N} \|a_t b_{t'}\|^{c_\rho} |f|_{\pi'}, \quad (35)$$

where

$$|f|_{\pi'} := \max \{ |f(k')|_{\rho,N} \mid k' \in K' \},$$

which defines a continuous seminorm on π'_s . Combining (34) and (35), we get

$$\begin{aligned}
& |\mu(\pi(g)u, \Lambda(f(g)))| \\
& \leq |u|_{\pi,1} \times |f|_{\pi'} \times (t_1 t_2 \cdots t_r t')^{\operatorname{Re}(s)} \times \xi(\mathbf{t}, t')^{-N} \times \|n_t b_{t'} a_t\|^{c_\pi} \times \|a_t b_{t'}\|^{c_\rho}. \quad (36)
\end{aligned}$$

Then by Proposition 5.4 and (36), we have

$$\begin{aligned}
& \int_{S_r \setminus G} |\mu(\pi(g)u, \Lambda(f(g)))| dg \\
& = \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times \times K} |\mu(\pi(a_t g_t k)u, \Lambda(f(a_t g_t k)))| \delta_{P_r}^{-1}(a_t) \delta_{B_r}^{-1}(a_t) J_{E'_0}(t) d^\times \mathbf{t} d^\times t dk \\
& \leq |u|_{\pi,1} \times |f|_{\pi'} \times \int_{(\mathbb{K}_+^\times)^r \times \mathbb{K}_+^\times} (t_1 t_2 \cdots t_r t')^{\operatorname{Re}(s)} \xi(\mathbf{t}, t')^{-N} J(\mathbf{t}, t) d^\times \mathbf{t} d^\times t \\
& = c_{s,N} \times |u|_{\pi,1} \times |f|_{\pi'}.
\end{aligned}$$

Therefore, the integral $I_\mu(f, u)$ converges absolutely. Finally,

$$|I_\mu(f, u)| \leq \int_{S_r \setminus G} |\mu(\pi(g)u, \Lambda(f(g)))| dg \leq c_{s,N} \times |u|_{\pi,1} \times |f|_{\pi'},$$

which proves the continuity of I_μ . This finishes the proof of Proposition 3.4.

Added in Proof. Theorem A of the current paper remains valid for special orthogonal groups, in view of results of the article ‘‘A note on special orthogonal groups following Waldspurger’’, by Binyong Sun and Chen-Bo Zhu ([arXiv.org: http://arxiv.org/abs/1005.4484](http://arxiv.org/abs/1005.4484))

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DIHUA JIANG, School of Mathematics, University of Minnesota, 206 Church St. S.E., Minneapolis, MN 55455, USA
dhjiang@math.umn.edu

BINYONG SUN, Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China
sun@math.ac.cn

CHEN-BO ZHU, Department of Mathematics, National University of Singapore, 2 Science drive 2, Singapore 117543
matzhucb@nus.edu.sg

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