DUAL PAIR CORRESPONDENCE OF REPRESENTATIONS

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1. Constructing representations from those of subgroups

In the study of Lie (or continuous) groups, the class of reductive Lie groups occupies a special place. This class contains groups of interest to physics and many areas of mathematics such as geometry and number theory. Essentially they are the closed subgroups of complex matrices that are closed under conjugate transpose, and as such are "balanced" in some sense (unlike the group of invertible upper trianglar matrices). The group of all invertible matrices (the general linear group), the rotation groups, the Lorentz group and the various isometry groups of forms provide examples. They are called reductive for the following reason: any finite dimensional representation of such a group is completely reducible, namely it can be decomposed as the direct sum of irreducible representations.

The largest area of research in representation theory has been on representations of reductive groups. It is in this area that the most fascinating and useful results are discovered. A fundamental task here is to invent ways of constructing new representations, especially unitary representations. Those are the representations which admit positive definite invariant forms, and are the most relevant for physics and for problems in harmonic analysis and number theory. For compact groups, all unitary represntations were parameterized long time ago mainly through the work of Cartan and Weyl. Unfortunately if the reductive group is not compact, its unitary representations are either 1-dimensional or can only be found in infinite-dimensional spaces.

Two important methods of constructing representations are real parabolic induction and cohomological induction. Both start from representations of their subgroups and yield unitary representations under appropriate conditions. In the first case, the representation is realized in certain space of functions on the group (or geometrically space of sections of certain homogeneous vector bundles over a generalized flag variety G/P). Such representations can be analyzed via the method of real analysis. Gelfand (and his school) and Mackey laid the foundation for the subject. In the second case, the representation is in certain space of cohomology associated to homogeneous holomorphic vector bundles over a non-compact complex variety G/L (more precisely the derived functor analog of such objects). Harish-Chandra, Schmid, Zuckerman and Vogan are some of the main contributors for the theory.

2. Constructing representations from those of dual partners

Another powerful way of construct representations is through dual pair correspondence. Recall that two subgroups (G, G') of a group S is called a dual pair if

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G is the centralizer of G' in S, and vise versa. This notion was introduced by Howe for the case of S the symplectic group (the group of isometries of a non-degenerate skew symmetric form, necessarily in a even dimensional vector space).

The symplectic group has a truely distinguished unitary representation, with many names in the literature. It is called the oscillator representation, or metaplectic representation, harmonic representation, Weil (or Segal-Shale-Weil) representation. It arose in Weil's group theretical treatment of θ-series (in number theory) and it plays an important role in quantum mechanics as well. Its existence is (best) anticipated by the Stone-von Neumann Theorem, which says in a sense that the only "reasonable" operators satisfying the famous Heisenberg commutation relations

$$[X_i, Y_j] = \delta_{ij} I, \qquad 1 \leq i, j \leq N,$$

are the momentum operators $p_i = \frac{\partial}{\partial x_i}$ and the position operators $q_i = x_i$ (multiplication by coordinates). Since the symplectic group is the symmetry group of the canonical commutation relations (CCR), the uniqueness of CCR implies that for each element of the symplectic group, there must be some unitary operator linking the original CCR to the transformed one. This gives rise to the oscillator representation of (the double cover) of the symplectic group. It is the smallest non-trivial unitary representation for this group. The present author likes to think of this distinguished representation as the "mother" of all representations for a simple reason: the symplectic group, being the symmetry group of a skew-symmetric form, ia an "odd" symmetry, or a "Yin" symmetry. The basic symmetry pattern (or representation) of "Yin" is undoubtly "mother".

The real significance of the above setting derives from Howe's remarkable discovery that when you restrict the oscillator representation to a reductive dual pair, its spectrum will produce a one-to-one correspondence between (certain) representations of G and G'. A little bit more precisely, if a representation π of G is a quotient of the oscillator representation Ω (in a sense in a shadow of the "mother"), then there is a unique representation π' of G' (with the same property of being in a shadow of the "mother") such that

$$\pi \otimes \pi' \text{ is a } G \times G' \text{ quotient of } \Omega,$$

and vise versa. This correspondence of representations of G and G' is called dual pair correspondence. Another remarkable fact is that such a correspondence not only exists for Lie groups (which are groups over real numbers), it also exists for other local (p-adic) fields as well as global (number) fields. For local fields, we get a correspondence of representations (of Lie or p-adic groups), and for global fields, it anticipates a correspondence of automorphic forms on G and G'. This is where the theory of θ-series fits in. For this reason, this correspondence is also called (local) theta correspondence.

Now let me touch on the basic construction of dual pairs. I will just give a typical example: take a vector space V equiped with a non-degenerate symmetric form (think of this as a "Yang" object) and another one W equiped with a non-degenerate skew-symmetric form (think of this as a "Yin" object), then form the tensor product space $V \otimes W$. The corresponding tensor product of the two forms gives rise to a non-degenerate skew-symmetric form on $V \otimes W$. You may think of this in terms of

$$1 \times (-1) = (-1).$$

Let $G = G(V)$ be the isometry of the first form (a pseudo-rotation group), and $G' = G'(W)$ be the isometry of the second form (a symplectic group), then (G, G') forms a reductive dual pair inside the larger symplectic group, which is the isometry group of larger symplectic space $V \otimes W$. In view of this construction, the dual pair correspondence of representations of G and G' is therefore in some sense the correspondence of symmetry patterns of "Yin" and "Yang"!

The most important reason for studying such correspondence is that it provides one of the really few ways of constructing new representations and automorphic forms. In fact starting from very simple-minded representations of G (such as the 1-dimensional trivial representation), one can already produce very exotic representations of G' this way.

The basic problem here is to describe this correspondence of representations and to investigate its applications to other areas of representation theory and the theory of automorphic forms. Since its birth in the late seventies, it has been a very active area of study in representation theory. Much progress has been made starting from the fundamental work of Howe. Despite this, it is still mysterious in many ways. It is anticipated that understanding of this correspondence will yield greater understanding of the problem of unitary dual (determination of all irreducible unitary representations of a given reductive group), which is an even bigger mystery.

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