A GENERAL FORM OF GELFAND-KAZHDAN CRITERION

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ABSTRACT. We formalize the notion of matrix coefficients for distributional vectors in a representation of a real reductive group, which consist of generalized functions on the group. As an application, we state and prove a Gelfand-Kazhdan criterion for a real reductive group in very general settings.

1. TEMPERED GENERALIZED FUNCTIONS AND REPRESENTATIONS IN THE CLASS $\mathcal{FH}$

In this section, we review some basic terminologies in representation theory, which are necessary for this article. The two main ones are tempered generalized functions and representations in the class $\mathcal{FH}$. We refer the readers to [W88, W92] as general references.

Let G be a real reductive Lie group, by which we mean that

(a) the Lie algebra $\mathfrak{g}$ of G is reductive;
(b) G has finitely many connected components; and
(c) the connected Lie subgroup of G with Lie algebra $[\mathfrak{g},\mathfrak{g}]$ has a finite center.

We say that a (complex valued) function f on G is of moderate growth if there is a continuous group homomorphism

$$\rho: \ G \to \mathrm{GL}_n(\mathbb{C}), \ \text{for some } n \geq 1,$$

such that

$$|f(x)| \leq \mathrm{tr}(\overline{\rho(x)}^t\rho(x)) + \mathrm{tr}(\overline{\rho(x^{-1})}^t\rho(x^{-1})), \quad x \in G.$$

Here "$^-$" stands for the complex conjugation, and "$\,^t$" the transpose, of a matrix. A smooth function $f \in \mathrm{C}^\infty(G)$ is said to be tempered if Xf has moderate growth for all X in the universal enveloping algebra

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$\mathrm{U}(\mathfrak{g}_\mathbb{C})$. Here and as usual, $\mathfrak{g}_\mathbb{C}$ is the complexification of $\mathfrak{g}$, and $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ is identified with the space of all left invariant differential operators on G. Denote by $\mathrm{C}^\xi(G)$ the space of all tempered functions on G.

A smooth function $f \in \mathrm{C}^\infty(G)$ is called Schwartz if

$$|f|_{X,\phi} := \sup_{x\in G} \phi(x)\,|(Xf)(x)| < \infty$$

for all $X \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$, and all positive functions ϕ on G of moderate growth. Denote by $\mathrm{C}^\varsigma(G)$ the space of Schwartz functions on G, which is a nuclear Fréchet space under the seminorms $\{|\cdot|_{X,\phi}\}$. (See [Tr67] or [Ta95] for the notion as well as basic properties of nuclear Fréchet spaces.) We define the nuclear Fréchet space $\mathrm{D}^\varsigma(G)$ of Schwartz densities on G similarly. Fix a Haar measure dg on G, then the map

$$\begin{array}{rcl} \mathrm{C}^\varsigma(G) & \rightarrow & \mathrm{D}^\varsigma(G) \\ f & \mapsto & f\,dg \end{array}$$

is a topological linear isomorphism. We define a tempered generalized function on G to be a continuous linear functional on $\mathrm{D}^\varsigma(G)$. Denote by $\mathrm{C}^{-\xi}(G)$ the space of all tempered generalized functions on G, equipped with the strong dual topology. This topology coincides with the topology of uniform convergence on compact subsets of $\mathrm{D}^\varsigma(G)$, due to the fact that every bounded subset of a nuclear Fréchet space is relatively compact. Note that $\mathrm{C}^\xi(G)$ is canonically identified with a dense subspace of $\mathrm{C}^{-\xi}(G)$:

$$\mathrm{C}^\xi(G) \hookrightarrow \mathrm{C}^{-\xi}(G).$$

Remark 1.1. *In* [W92], *the space* $\mathrm{C}^\varsigma(G)$ *is denoted by* $\mathcal{S}(G)$ *and is called the space of rapidly decreasing functions on* G*. Note that* $\mathrm{C}^\varsigma(G)$ *(or* $\mathcal{S}(G)$*) is different from Harish-Chandra's Schwartz space of* G*, which is traditionally denoted by* $\mathcal{C}(G)$*.*

By a representation of G, or just a representation when G is understood, we mean a continuous linear action of G on a complete, locally convex, Hausdorff, complex topological vector space. When no confusion is possible, we do not distinguish a representation with its underlying space. Let V be a representation. It is said to be smooth if the action map $G \times V \to V$ is smooth as a map of infinite dimensional manifolds. The notion of smooth maps in infinite dimensional setting may be found in [GN09], for example.

Denote by $\mathrm{C}(G;V)$ the space of V-valued continuous functions on G. It is a complete locally convex space under the topology of uniform convergence on compact sets. Similarly, denote by $\mathrm{C}^\infty(G;V)$ the (complete locally convex) space of smooth V-valued functions, with the usual smooth topology. For any $v \in V$, define $c_v \in \mathrm{C}(G;V)$ by

$$c_v(g) := gv.$$

The vector v is called smooth if $c_v \in \mathrm{C}^\infty(G;V)$. Denote by V_∞ the space of all smooth vectors in V, which clearly is stable under G. Note that the linear map

$$\begin{array}{rcl} V_\infty & \to & \mathrm{C}^\infty(G;V) \\ v & \mapsto & c_v \end{array}$$

is injective with closed image. Equip V_∞ with the subspace topology of $\mathrm{C}^\infty(G;V)$, then it becomes a smooth representation of G, which is called the smoothing of V. If V is smooth, then $V_\infty = V$ as a representation of G. In this case, its differential is defined to be the continuous $\mathrm{U}(\mathfrak{g}_\mathbb{C})$ action given by

$$Xv = (Xc_v)(1), \quad X \in \mathrm{U}(\mathfrak{g}_\mathbb{C}),\ v \in V,$$

where 1 is the identity element of G.

The representation V is said to be $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$ finite if a finite codimensional ideal of $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$ annihilates V_∞, where $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$ is the center of $\mathrm{U}(\mathfrak{g}_\mathbb{C})$. It is said to be admissible if every irreducible representation of a maximal compact subgroup K of G has finite multiplicity in V. A representation of G which is both admissible and $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$ finite is called a Harish-Chandra representation.

The representation V is said to be of moderate growth if for every continuous seminorm $|\ \ |_\mu$ on V, there is a positive function ϕ on G of moderate growth, and a continuous seminorm $|\ \ |_\nu$ on V such that

$$|gv|_\mu \leq \phi(g)|v|_\nu, \quad \text{for all } g \in G,\ v \in V.$$

The representation V is said to be in the class $\mathcal{FH}$ if the space V is Fréchet, and the representation is smooth and of moderate growth, and Harish-Chandra. The category of all such V is denoted by the same symbol $\mathcal{FH}$ (the morphisms being G-intertwining continuous linear maps). The strong dual of a representation in the class $\mathcal{FH}$ is again a representation which is smooth, of moderate growth, and Harish-Chandra. A representation which is isomorphic to such a strong dual is said to be in the class $\mathcal{DH}$. By the Casselman-Wallach globalization theorem, both the category $\mathcal{FH}$ and $\mathcal{DH}$ are equivalent to the category $\mathcal{H}$ of admissible finitely generated $(\mathfrak{g}_\mathbb{C}, K)$-modules ([C89], [W92, Chapter 11]). All representation spaces in $\mathcal{FH}$ are automatically nuclear Fréchet, and all morphisms in $\mathcal{FH}$ and $\mathcal{DH}$ are automatically topological homomorphisms with closed image. (Recall that in general, a linear map $\lambda : E \to F$ of topological vector spaces is called a topological homomorphism if the induced linear isomorphism $E/\mathrm{Ker}(\lambda) \to \mathrm{Im}(\lambda)$ is a topological isomorphism, where $E/\mathrm{Ker}(\lambda)$ is equipped with the quotient topology of E, and the image $\mathrm{Im}(\lambda)$ is equipped with the subspace topology of F.)

Remark 1.2. *Our terminology follows that of Wallach in* [W92]*, where presumably $\mathcal{F}$ refers to Fréchet, $\mathcal{H}$ refers to Harish-Chandra, and $\mathcal{FH}$ hints at Fréchet globalization of Harish-Chandra modules. Note that Casselman calls a representation in the class $\mathcal{FH}$ a Harish-Chandra representation (*[C89]*, page 394), while some other authors spell out all conditions of the class $\mathcal{FH}$, without attaching a name.*

2. Statement of results

Let U^∞, V^∞ be a pair of representations of G in the class $\mathcal{FH}$ which are contragredient to each other, i.e., we are given a G-invariant nondegenerate continuous bilinear map

$$\langle \, , \, \rangle : U^\infty \times V^\infty \to \mathbb{C}. \tag{1}$$

Denote by $U^{-\infty}$ the strong dual of V^∞, which is a representation in the class $\mathcal{DH}$ containing U^∞ as a dense subspace. Similarly, denote by $V^{-\infty}$ the strong dual of U^∞. For any $u \in U^\infty$, $v \in V^\infty$, the (usual) matrix coefficient $c_{u\otimes v}$ is defined by

$$c_{u\otimes v}(g) := \langle gu, v\rangle, \quad g \in G. \tag{2}$$

By the moderate growth conditions of U^∞ and V^∞, one easily checks that a matrix coefficient $c_{u\otimes v}$ is a tempered function on G.

The following theorem, which defines the notion of matrix coefficients for distributional vectors, is in a sense well-known. See the work of Shilika ([S74, Section 3]) in the context of unitary representations, and the work of Kostant ([K78, Section 6.1]) or Yamashita ([Y88, Section 2.3]) in the context of Hilbert representations. With the benefit of the Casselman-Wallach theorem, it is of interest and most natural to state the result in the context of representations in the class $\mathcal{FH}$. This is also partially justified by the increasing use of this class of representations, due to the recent progress in restriction problems for classical groups. One purpose of this note is to provide a detailed proof of this result.

Theorem 2.1. *Let G be a real reductive group. Denote by $\mathrm{C}^\xi(G)$ (resp. $\mathrm{C}^{-\xi}(G)$) the space of all tempered functions (resp. tempered generalized functions) on G. Let (U^∞, V^∞) be a pair of representations of G in the class $\mathcal{FH}$ which are contragredient to each other. Then the matrix* *coefficient map*

$$(3) \qquad \begin{array}{rcl} U^\infty \times V^\infty & \to & \mathrm{C}^\xi(G), \\ (u,v) & \mapsto & c_{u\otimes v} \end{array}$$

extends to a continuous bilinear map

$$U^{-\infty} \times V^{-\infty} \to \mathrm{C}^{-\xi}(G),$$

and the induced $G \times G$ intertwining continuous linear map

$$(4) \qquad c : U^{-\infty}\widehat{\otimes}V^{-\infty} \to \mathrm{C}^{-\xi}(G)$$

is a topological homomorphism with closed image.

Here "$\widehat{\otimes}$" stands for the completed projective tensor product of Hausdorff locally convex topological vector spaces. In our case, this coincides with the completed inductive tensor product as the spaces involved are nuclear. Recall again that a linear map $\lambda : E \to F$ of topological vector spaces is called a topological homomorphism if the induced linear isomorphism $E/\mathrm{Ker}(\lambda) \to \mathrm{Im}(\lambda)$ is a topological isomorphism, where $E/\mathrm{Ker}(\lambda)$ is equipped with the quotient topology of E, and the image $\mathrm{Im}(\lambda)$ is equipped with the subspace topology of F. The action of $G \times G$ on $\mathrm{C}^{-\xi}(G)$ is obtained by continuously extending its action on $\mathrm{C}^\xi(G)$:

$$((g_1, g_2)f)(x) := f(g_2^{-1}xg_1).$$

Remark 2.2. *(A) Denote by t_{U_∞} the paring $V^\infty \times U^\infty \to \mathbb{C}$, and view it as an element of $U^{-\infty}\widehat{\otimes}V^{-\infty}$. Then $c(t_{U_\infty}) \in \mathrm{C}^{-\xi}(G)$ is the character of the representation U^∞.*

(B) Let $U_1^\infty, U_2^\infty, \cdots, U_k^\infty$ be pairwise inequivalent irreducible representations of G in the class $\mathcal{FH}$. Let V_i^∞ be a representation of G in the class $\mathcal{FH}$ which is contragredient to U_i^∞, $i = 1, 2, \cdots, k$. Then the second assertion of Theorem 2.1 implies that the sum

$$\bigoplus_{i=1}^k U_i^{-\infty}\widehat{\otimes}V_i^{-\infty} \to \mathrm{C}^{-\xi}(G)$$

of the matrix coefficient maps is a topological embedding with closed image.

A second purpose of this note (and additional reason for writing down a proof of Theorem 2.1) is to prove the following generalized form of the Gelfand-Kazhdan criterion. For applications towards uniqueness of certain degenerate Whittaker models, it is highly desirable to have the most general form of the Gelfand-Kazhdan criterion. We refer the reader to [JSZ08] for one such application.

Theorem 2.3. *Let S_1 and S_2 be two closed subgroups of G, with continuous characters*

$$\chi_i : S_i \to \mathbb{C}^\times, \quad i = 1, 2.$$

(a) *Assume that there is a continuous anti-automorphism τ of G such that for every $f \in \mathrm{C}^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g}_\mathbb{C})^G$, the conditions*

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply that

$$f(x^\tau) = f(x).$$

Then for any pair of irreducible representations (U^∞, V^∞) of G in the class $\mathcal{FH}$ which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \leq 1.$$

The equality holds only if $U^\infty_{\check{\tau}}$ and V^∞ are equivalent, where $U^\infty_{\check{\tau}}$ is the representation obtained from U^∞ by (pre)composing with the automorphism $g \mapsto \tau(g^{-1})$.

(b) *Assume that for every $f \in \mathrm{C}^{-\xi}(G)$ which is an eigenvector of $\mathrm{U}(\mathfrak{g})^G$, the conditions*

$$f(sx) = \chi_1(s) f(x), \quad s \in S_1,$$

and

$$f(xs) = \chi_2(s)^{-1} f(x), \quad s \in S_2$$

imply that

$$f = 0.$$

Then for any pair of irreducible representations (U^∞, V^∞) of G in the class $\mathcal{FH}$ which are contragredient to each other, one has that

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) = 0.$$

Here and as usual, $\mathrm{U}(\mathfrak{g}_\mathbb{C})^G$ is identified with the space of bi-invariant differential operators on G, $\mathbb{C}_{\chi_i}$ is the one dimensional representation of S_i given by the character χ_i, and "Hom_{S_i}" stands for continuous S_i homomorphisms. The equalities in the theorem are to be understood as equalities of generalized functions. For example, $f(sx)$ denotes the left translation of f by s^{-1}. Similar notations apply throughout this article.

Remark 2.4. *(A) In concrete applications, the data* $S_1, S_2, \chi_1, \chi_2, \tau$ *are interconnected, and various considerations often imply that the two terms in the product* $\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2})$ *are equal, and so each of them is at most 1. This is how Gelfand-Kazhdan criterion (Part (a) of Theorem 2.3) is applied to prove the (local) uniqueness of certain models in the theory of automorphic forms. Part (b) serves a similar purpose in showing the disjointness of certain periods.*

(B) The original Gelfand-Kazhdan criterion is in [GK71] *(for the non-archimedean case), and their idea has been very influential ever since. Various versions for real reductive groups have appeared in the literature, including* [S74] *and* [K78] *(for the study of Whittaker models, but both implicitly). Later works which state some versions of Gelfand-Kazhdan criterion explicitly include that of H. Yamashita (*[Y88, Theorem 2.10]*), and of A. Aizenbud, D. Gourevitch and E. Sayag (*[AGS, Section 2]*). The papers* [S74] *and* [Y88] *have been particularly instructive for the current article.*

3. Proof of Theorem 2.1

Let $(U, \langle\, , \,\rangle_U)$ be a Hilbert space which carries a continuous representation of G so that its smoothing coincides with U^∞. Existence of such a representation is well known, and it follows from Casselman's embedding theorem together with the Casselman-Wallach globalization theorem ([W92, Chapter 11]). Denote by V the strong dual of U, which carries a representation of G. (Its topology is given by the inner product

$$\langle \bar{u_1}, \bar{u_2} \rangle_V := \langle u_2, u_1 \rangle_U, \quad u_1, u_2 \in U,$$

where $\bar{u_i} \in V$ is the linear functional $\langle \cdot\, , u_i \rangle_U$ on U.) Note that the smoothing of V coincides with V^∞. Recall, as is well-known, that the three pairs U and V, U^∞ and $V^{-\infty}$, and $U^{-\infty}$ and V^∞, are strong duals of each other as representations of G. For $u \in U$, $v \in V$, set

$$|u|_U := \sqrt{\langle u, u \rangle_U} \quad \text{and} \quad |v|_V := \sqrt{\langle v, v \rangle_V}.$$

Lemma 3.1. *There is a continuous seminorm* $|\cdot|_G$ *on* $\mathrm{D}^\varsigma(G)$ *such that*

$$\int_G |f(g)|\, |gu|_U \, dg \leq |\omega|_G \, |u|_U, \quad \omega = f\, dg \in \mathrm{D}^\varsigma(G),\ u \in U.$$

Proof. This is well known, and follows easily from the facts that

(a) U is (automatically) of moderate growth ([W92, Lemma 2.A.2.2]), and

(b) there is a positive continuous function ϕ on G of moderate growth so that $1/\phi$ is integrable. See [W92, Lemma 2.A.2.4].

$\square$

By Lemma 3.1, for any $\omega \in \mathrm{D}^{\varsigma}(G)$ and $u \in U$, the integral (in the sense of Riemann)

$$\omega u := \int_G \omega(g)\, gu \tag{5}$$

converges absolutely, and thus defines a vector in U. Furthermore, the bilinear map

$$\begin{array}{rcl} \mathrm{D}^{\varsigma}(G) \times U & \rightarrow & U, \\ (\omega, u) & \mapsto & \omega u \end{array} \tag{6}$$

is continuous.

Lemma 3.2. *For $\omega \in \mathrm{D}^{\varsigma}(G)$ and $u \in U$, we have $\omega u \in U^{\infty}$.*

Proof. Denote by L the representation of G on $\mathrm{D}^{\varsigma}(G)$ by left translations. Thus for $g \in G$ and $\omega \in \mathrm{D}^{\varsigma}(G)$, $L_g(\omega)$ is the push forward of ω via the map

$$\begin{array}{rcl} G & \rightarrow & G, \\ x & \mapsto & gx. \end{array}$$

It is routine to check that

$$L : \; G \times \mathrm{D}^{\varsigma}(G) \rightarrow \mathrm{D}^{\varsigma}(G)$$

is a smooth representation. For $X \in \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$, denote by

$$L_X : \; \mathrm{D}^{\varsigma}(G) \rightarrow \mathrm{D}^{\varsigma}(G)$$

its differential. Trivially we have

$$c_{\omega u}(g) = (L_g(\omega))u, \quad g \in G. \tag{7}$$

This implies that $c_{\omega u} \in \mathrm{C}^{\infty}(G; U)$, namely $\omega u \in U^{\infty}$. $\square$

The following two lemmas are refinements of [S74, Proposition 3.2].

Lemma 3.3. *The bilinear map*

$$\begin{array}{rrcl} \Phi_U : & \mathrm{D}^{\varsigma}(G) \times U & \rightarrow & U^{\infty}, \\ & (\omega, u) & \mapsto & \omega u. \end{array}$$

is continuous.

Proof. By the defining topology on U^{∞}, we need to show that the map

$$\begin{array}{rcl} \mathrm{D}^{\varsigma}(G) \times U & \rightarrow & \mathrm{C}^{\infty}(G; U), \\ (\omega, u) & \mapsto & c_{\omega u}. \end{array}$$

is continuous. In view of the topology on $\mathrm{C}^\infty(G;U)$, this is equivalent to showing that the bilinear map

$$\begin{array}{rcl} \mathrm{D}^\varsigma(G)\times U & \to & \mathrm{C}(G;U), \\ (\omega,u) & \mapsto & X(c_{\omega u}), \end{array}$$

is continuous for all $X\in \mathrm{U}(\mathfrak{g}_\mathbb{C})$. This is clearly true by observing that

$$X(c_{\omega u}) = c_{(L_X(\omega))u}, \quad \omega\in \mathrm{D}^\varsigma(G),\ u\in U.$$

□

For any $\omega\in \mathrm{D}^\varsigma(G)$, denote by $\omega^\vee$ its push forward via the map

$$\begin{array}{rcl} G & \to & G, \\ g & \mapsto & g^{-1}. \end{array}$$

Applying Lemma 3.3 to V, we get a continuous bilinear map

$$\begin{array}{rrcl} \Phi_V: & \mathrm{D}^\varsigma(G)\times V & \to & V^\infty, \\ & (\omega,v) & \mapsto & \omega v. \end{array}$$

Now for any $\omega\in \mathrm{D}^\varsigma(G)$, we define the continuous linear map

$$\begin{array}{rcl} U^{-\infty} & \to & U, \\ u & \mapsto & \omega u \end{array}$$

to be the transpose of

$$\begin{array}{rcl} V & \to & V^\infty, \\ v & \mapsto & \omega^\vee v, \end{array}$$

i.e.,

$$\langle \omega u, v\rangle = \langle u, \omega^\vee v\rangle, \quad u\in U^{-\infty},\ v\in V. \tag{8}$$

Lemma 3.4. *The bilinear map*

$$\begin{array}{rrcl} \Phi_V^\vee: & \mathrm{D}^\varsigma(G)\times U^{-\infty} & \to & U, \\ & (\omega,u) & \mapsto & \omega u \end{array} \tag{9}$$

is continuous and extends (6).

Proof. It is routine to check that (9) extends (6). Since $\mathrm{D}^\varsigma(G)$ is nuclear, we only need to show that (9) is separably continuous ([Tr67]). We already know that (9) is continuous in the second variable.

Fix $u\in U^{-\infty}$, then the continuity of the bilinear map

$$\begin{array}{rcl} \theta_u: \mathrm{D}^\varsigma(G)\times V & \to & \mathbb{C}, \\ (\omega,v) & \mapsto & \langle u, \omega^\vee v\rangle. \end{array}$$

clearly implies the continuity of the map

$$\begin{array}{rcl} \mathrm{D}^\varsigma(G) & \to & U, \\ \omega & \mapsto & \omega u = \theta_u(\omega,\,\cdot\,). \end{array}$$

□

Lemma 3.5. *The image of* $\Phi_V^\vee$ *is contained in* U^∞*, and the induced bilinear map*

$$\begin{array}{rrcl} \Phi_V^\vee: & \mathrm{D}^\varsigma(G) \times U^{-\infty} & \to & U^\infty, \\ & (\omega, u) & \mapsto & \omega u \end{array} \tag{10}$$

is continuous.

Proof. By chasing the definition of ωu, we see that the equality (7) still holds for all $\omega \in \mathrm{D}^\varsigma(G)$ and $u \in U^{-\infty}$. Again, this implies that $\omega u \in U^\infty$.

The proof for the continuity is similar to that of Lemma 3.3. We need to prove that the map

$$\begin{array}{rcl} \mathrm{D}^\varsigma(G) \times U^{-\infty} & \to & \mathrm{C}^\infty(G; U), \\ (\omega, u) & \mapsto & c_{\omega u}. \end{array}$$

is continuous. It is the same as that the bilinear map

$$\begin{array}{rcl} \mathrm{D}^\varsigma(G) \times U^{-\infty} & \to & \mathrm{C}(G; U), \\ (\omega, u) & \mapsto & X(c_{\omega u}), \end{array}$$

is continuous for all $X \in \mathrm{U}(\mathfrak{g}_\mathbb{C})$. This is again true by checking that

$$X(c_{\omega u}) = c_{(L_X(\omega))u}, \quad \omega \in \mathrm{D}^\varsigma(G),\ u \in U^{-\infty}.$$

□

Now define the (distributional) matrix coefficient map by

$$\begin{array}{l} c: U^{-\infty} \times V^{-\infty} \to \mathrm{C}^{-\xi}(G), \\ c_{u\otimes v}(\omega) := \langle \omega u, v\rangle, \quad \omega \in \mathrm{D}^\varsigma(G). \end{array} \tag{11}$$

Lemma 3.6. *The matrix coefficient map* c *defined in (11) is continuous.*

Proof. As $U^{-\infty}$ is nuclear, again we only need to prove the separable continuity. First fix $u \in U^{-\infty}$, then the map

$$\begin{array}{rcl} V^{-\infty} & \to & \mathrm{C}^{-\xi}(G), \\ v & \mapsto & c_{u\otimes v}, \end{array}$$

is continuous since it is the transpose of the continuous linear map

$$\begin{array}{rcl} \mathrm{D}^\varsigma(G) & \to & U^\infty, \\ \omega & \mapsto & \omega u. \end{array}$$

Then fix $v \in V^{-\infty}$, since the bilinear map

$$\begin{array}{rrcl} \vartheta_v: & U^{-\infty} \times \mathrm{D}^\varsigma(G) & \to & \mathbb{C}, \\ & (u, \omega) & \mapsto & \langle \omega u, v\rangle, \end{array}$$

is continuous, it induces a continuous bilinear map

$$\begin{array}{rcl} U^{-\infty} & \to & \mathrm{C}^{-\xi}(G), \\ u & \mapsto & c_{u\otimes v} = \vartheta_v(u, \cdot). \end{array}$$

□

It is straightforward to check that (11) extends the usual matrix coefficient map (2). The proof of the first assertion of Theorem 2.1 is now complete.

To prove the second assertion of Theorem 2.1, we need two elementary lemmas.

Lemma 3.7. *Let $\lambda : E \to F$ be a G intertwining continuous linear map of representations of G. Assume that E is Fréchet, smooth, and of moderate growth, and F is in the class $\mathcal{FH}$. Then $E/\mathrm{Ker}(\lambda)$ is a representation of G in the class $\mathcal{FH}$, and λ is a topological homomorphism.*

As the proof is routine, we omit the details.

Lemma 3.8. *Let $\lambda : E \to F$ be a continuous linear map of nuclear Fréchet spaces. Equip the dual spaces E' and F' with the strong dual topologies. Then λ is a topological homomorphism if and only if its transpose $\lambda^t : F' \to E'$ is. When this is the case, both λ and λ^t have closed images.*

Proof. The first assertion is a special case of [B87, Section IV.2, Theorem 1]. (Recall that every bounded set in a nuclear Fréchet space is relatively compact.)

Now assume that λ is a topological homomorphism. Then as a Hausdorff quotient of a Fréchet space, $E/\mathrm{Ker}(\lambda)$ is complete, and so is $\mathrm{Im}(\lambda)$, which implies that $\mathrm{Im}(\lambda)$ is closed in F. By the Extension Theorem of continuous linear functionals, the image of λ^t consists of all elements in E' which vanish on $\mathrm{Ker}(\lambda)$. This is closed in E'. □

Recall that both $V^\infty \widehat{\otimes} U^\infty$ and $\mathrm{D}^\varsigma(G)$ are nuclear Fréchet spaces. In particular, they are both reflexive. The map c of (4) is the transpose of a $G \times G$ intertwining continuous linear map

$$c^t : \mathrm{D}^\varsigma(G) \to V^\infty \widehat{\otimes} U^\infty.$$

Lemma 3.7 for the group $G \times G$ implies that c^t is a topological homomorphism. Lemma 3.8 then implies that c is a topological homomorphism with closed image. This completes the proof of the second assertion of Theorem 2.1.

4. Proof of Theorem 2.3

The argument is standard (c.f. [GK71] or [S74]). We use the notation and the assumption of Theorem 2.3. As before, $U^{-\infty}$ is the strong dual of V^∞, and $V^{-\infty}$ is the strong dual of U^∞. Suppose that both $\mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1})$ and $\mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2})$ are non-zero. Pick

$$0 \neq u_0 \in \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \subset U^{-\infty}$$

and

$$0 \neq v_0 \in \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) \subset V^{-\infty}.$$

Then the matrix coefficient $c_{u_0 \otimes v_0} \in \mathrm{C}^{-\xi}(G)$ satisfies the followings:

$$\begin{cases} \text{it is an eigenvector of } \mathrm{U}(\mathfrak{g}_{\mathbb{C}})^G, \text{ (by the irreducibility hypothesis)} \\ c_{u_0 \otimes v_0}(sx) = \chi_1(s)\, c_{u_0 \otimes v_0}(x), \quad s \in S_1, \text{ and} \\ c_{u_0 \otimes v_0}(xs) = \chi_2(s)^{-1}\, c_{u_0 \otimes v_0}(x), \quad s \in S_2. \end{cases}$$

By the assumption of the theorem, we have

$$c_{u_0 \otimes v_0}(x^\tau) = c_{u_0 \otimes v_0}(x). \tag{12}$$

Lemma 4.1. *Let $\omega \in \mathrm{D}^\varsigma(G)$. Denote by*

$$\tau_* : \mathrm{D}^\varsigma(G) \to \mathrm{D}^\varsigma(G)$$

the push forward map by τ. Then

$$\omega u_0 = 0 \quad \textit{if and only if} \quad (\tau_*(\omega))^\vee v_0 = 0.$$

Proof. As a consequence of (12), we have

$$c_{u_0 \otimes v_0}(\omega) = 0 \quad \text{if and only if} \quad c_{u_0 \otimes v_0}(\tau_*(\omega)) = 0. \tag{13}$$

By the irreducibility of U^∞, $\omega u_0 = 0$ if and only if

$$\langle g(\omega u_0), v_0 \rangle = 0 \quad \text{for all } g \in G,$$

i.e.,

$$\langle (L_g \omega) u_0, v_0 \rangle = 0 \quad \text{for all } g \in G.$$

By (13), this is equivalent to saying that

$$\langle (\tau_*(L_g \omega)) u_0, v_0 \rangle = 0 \quad \text{for all } g \in G.$$

Now the lemma follows from the following elementary identity and the irreducibility of V^∞:

$$\langle (\tau_*(L_g \omega)) u_0, v_0 \rangle = \langle (\tau_* \omega)(g^\tau u_0), v_0 \rangle = \langle (g^\tau u_0), (\tau_* \omega)^\vee\, v_0 \rangle.$$

□

Let

$$0 \neq u_0' \in \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) \subset U^{-\infty}$$

be another element. Applying Lemma 4.1 twice, we get that for all $\omega \in \mathrm{D}^\varsigma(G)$,

$$\omega u_0 = 0 \quad \text{if and only if} \quad \omega u_0' = 0.$$

Therefore the two continuous G homomorphisms

$$\Phi : \omega \mapsto \omega u_0, \quad \text{and} \quad \Phi' : \omega \mapsto \omega u_0',$$

from $\mathrm{D}^\varsigma(G)$ to U^∞ have the same kernel, say J. Here and as before, we view $\mathrm{D}^\varsigma(G)$ as a representation of G via left translations. Both Φ and Φ' induce nonzero injective G homomorphisms

$$\bar{\Phi}, \bar{\Phi}' : \mathrm{D}^\varsigma(G)/J \to U^\infty.$$

Lemma 3.7 says that $\mathrm{D}^\varsigma(G)/J$ is a representation of G in the class $\mathcal{FH}$. Recall that U^∞ is assumed to be irreducible. By Schur's lemma for the category $\mathcal{FH}$, $\bar{\Phi}'$ is a scalar multiple of $\bar{\Phi}$, which implies that u_0' is a scalar multiple of u_0. This proves that

$$\dim \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2}) = 1.$$

Similarly,

$$\dim \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1}) = 1.$$

To finish the proof of Part (a) of Theorem 2.3, we first note the identity $c_{u_0 \otimes v_0}(x^{-1}) = c_{v_0 \otimes u_0}(x)$ (as generalized functions). The equality of matrix coefficients in (12) will then imply that U^∞_τ and V^∞ are equivalent, in view of Remark 2.2, Part (B).

Part (b) of Theorem 2.3 is immediate as the matrix coefficient $c_{u_0 \otimes v_0}$ would have to be zero if there were nonzero $u_0 \in \mathrm{Hom}_{S_2}(V^\infty, \mathbb{C}_{\chi_2})$ and nonzero $v_0 \in \mathrm{Hom}_{S_1}(U^\infty, \mathbb{C}_{\chi_1})$.

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