Understanding symmetries and their consequences

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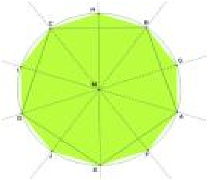
Outline

What is a group of symmetries?

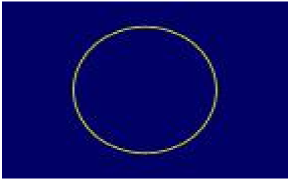
Example 1: finite group of symmetries



$r_{\frac{2k\pi}{n}}$: rotation by $\frac{2k\pi}{n}$ degrees, $k = 0, 1, 2, ..., n-1$.
s: reflection (along x-axis).
Group of symmetries of the regular n-gon:

$$D_n = \{r_{\frac{2k\pi}{n}}, sr_{\frac{2k\pi}{n}} \mid 0 \leq k \leq n-1\}.$$

Example 2: continuous group of symmetries
Let $n \to \infty$ in Example 1:



Group of symmetries of the circle:

$$O(2) = \text{distance preserving linear transformations of } \mathbb{R}^2.$$

Characterizing properties:

- e (identity transformation)
- Two symmetries can be composed (operated one after another) to obtain another symmetry: $x \circ y$
- A symmetry has an inverse which is also a symmetry: x^{-1}
- Symmetry compositions are “associative": $(x \circ y) \circ z = x \circ (y \circ z)$

finite groups $\approx$ symmetries of finite objects

continuous groups $=$ Lie groups
$\approx$ symmetries of continuous objects

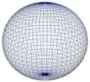
More examples:

(A) In $\mathbb{R}^n$: the group of linear transformations preserving the distance

$$x_1^2 + \cdots + x_n^2,$$

denoted by $O(n)$. This is also the group of symmetries of the $(n-1)$-dimensional sphere in $\mathbb{R}^n$:

$$S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$$



More generally: let $p + q = n$, group of linear transformations preserving the “pseudo distance"

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2,$$

denoted by $O(p, q)$.

(B) In $\mathbb{R}^2$: the group of linear transformations preserving area form $(xy' - yx')$.

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathbb{R}, ad - bc = 1 \right\}$$

More generally:

$$\mathbb{R}^{2n} = \underbrace{\mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2}_{n}$$

with the sum of area forms on $\mathbb{R}^2$.
Group of linear transformations preserving this sum of area forms, denoted by $Sp(2n,\mathbb{R})$.

$O(p, q)$ and $Sp(2n, \mathbb{R})$ are my favorite groups!

(classical groups)

Objects with symmetries

X possesses a certain symmetry G

means that for each $g \in G$, there is an invertible transformation

$$\rho_g : \underset{\circlearrowleft}{X}$$

and various ρ_g's compose just as they do in the group G.
Mathematical notation:

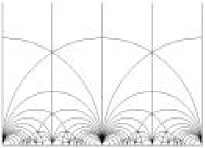
$$G \curvearrowright X$$

We say

G **acts on** X.

Thus rotation group T acts on the circle and $O(3)$ acts on the two dimensional sphere.

More examples:
(i) $SL(2,\mathbb{R})$ acts on the upper half plane by fractional linear transformations:



$$g \circ z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(ii) $SL(2,\mathbb{R})$ acts on two dimensional unit disk and on the circle (the boundary of the disk).

iii) In general there are lots of (closed) subgroups H of G:

$$G \curvearrowright G/H$$

(Homogeneous spaces for G)

(i) and (ii) are particular examples of homogeneous spaces for $SL(2,\mathbb{R})$.

(iv) Take $G = GL(n, \mathbb{R})$ (the group of $n \times n$ invertible real matrices). For $H = O(n)$,

$$GL(n, \mathbb{R})/O(n) = \text{equivalent classes of positive definite symmetric matrices}$$

For $H = B_n$ (the group of invertible upper triangular matrices),

$$GL(n, \mathbb{R})/B_n = \text{Flag variety}$$

Upshot: There are lots of exciting spaces with all kinds of continuous symmetries!

How to understand a given symmetry group

Basic idea: You can tell a lot about the real nature of G by knowing what possible places G can appear (as a group of transformations), namely through its actions.

- Out of all actions, linear actions (by invertible linear transformations) are by far the simplest.
- All actions can be converted in some sense to linear actions by the following scheme:

If

$$G \curvearrowright X,$$

let

$$C(X) = \text{ space of functions on } X.$$

(think $C(X)$ as the full collection of observables on X)

Then G acts on $C(X)$ by:

$$(g \cdot F)(x) = F(g^{-1} \cdot x), \quad g \in G.$$

representations = linear actions

Big advantages:

- Representations can be superimposed (direct sum).
- Representations can multiplied (tensor product).
- Elements in G are represented by invertible linear transformations and so can be analyzed in great detail.
- The family of invertible linear transformations must behave (according to the symmetry law of G), thus greatly limiting their possibilities.

Take $T = \{e^{i\theta}\}$, the circle group. For each integer k,

$$e^{i\theta} \mapsto \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$$

is a representation of the circle group by 2×2 invertible real matrices.

If you like complex numbers,

$$e^{i\theta} \mapsto e^{ik\theta}.$$

Call this (1-d) representation χ_k.

- Any representation ρ of T on a vector space V is in a suitable sense a direct sum of χ_k with multiplicities:

$$\rho = \sum_k m_k \chi_k,$$

where m_k are non-negative integers. Consequently

- for any vector $v \in V$,

$$v = \sum_k v_k, \quad \rho_\theta(v_k) = e^{ik\theta} v_k.$$

(v_k is a simultaneous eigenvector for ρ_θ).

- All representations of the circle group thus enjoy “Fourier Series" expansion. For functions on the circle:

$$C(T) = \sum_k \chi_k.$$

Remarks:

- Linear transformations having simultaneous eigenspace decomposition commute with each other.
- One therefore should not expect such such simultaneous eigenspace decompositions for groups which are not “commutative".

Example: $G = SO(3)$, the group of rotations in $\mathbb{R}^3$. It has a natural representation on $\mathbb{R}^3$.
Claim: apart from the zero space and $\mathbb{R}^3$, there are no other subspaces (1-d or 2-d) which are preserved by all rotations.

Appropriate analog of simultaneous eigenstates:

irreducible representations

- For each non-negative integer j, there is a unique irreducible representation π_j of $SO(3)$ with dimension $2j+1$.

  (j is called the spin)

- They exhaust all irreducible representations of $SO(3)$.

  (if you like, elementary particles with $SO(3)$ symmetry)

- (Recall) $SO(3) \curvearrowright S^2$:

$$C(S^2) = \sum_j \pi_j.$$

Example: $G = SL(2, \mathbb{R})$. It is not commutative, and worse it is not compact. This means there are directions in $SL(2, \mathbb{R})$ which can go to infinity:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad a \to \infty.$$

Irreducible unitary (think “physically relevant") representations of $SL(2, \mathbb{R})$ are either

- the trivial one dimensional representation, or are
- infinite-dimensional!

Where are they?

(A) In space of functions on $\mathbb{R}$:

$$(g \cdot F)(x) = |cx+d|^{-(1+\nu i)} F(\frac{ax+b}{cx+d}), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(B) In certain space of holomorphic functions on the upper half plane:

$$(g \cdot F)(z) = (cz+d)^{-k} F(\frac{az+b}{cz+d}), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Fundamental tasks:

- Find all possible irreducible representations of a given G (done by Langlands)
- Find all possible irreducible unitary representations of a given G (only done for some Lie groups of low rank, like $SL(2, \mathbb{R})$)
- For all naturally occurring representations, carry out the harmonic synthesis

One of the things which concern me

All familiar

$$X = M_{m,n}(\mathbb{R}).$$

Fix p, q such that $p + q = m$. Then

$$O(p,q) \curvearrowright M_{m,n}(\mathbb{R}) \ \text{(by matrix multiplication from left), and so}$$

$$O(p,q) \curvearrowright C(M_{m,n}(\mathbb{R})).$$

Observation: The function space has a far larger symmetry than the action of $O(p,q)$.

Hidden Symmetries!

(by certain action of $Sp(2n, \mathbb{R})$)

Now look for possible irreducible representations of $O(p, q)$ and $Sp(2n, \mathbb{R})$ which can be obtained as an image (think “shadow") of $C(M_{m,n}(\mathbb{R}))$:

$$C(M_{m,n}(\mathbb{R})) \mapsto \pi \otimes \pi'.$$

(Howe): There is a **1-1 correspondence** of such representations of $O(p, q)$ and $Sp(2n, \mathbb{R})$!

Now the game began:

What is the nature of this pairing of representations?

Related questions:

- Can one produce many interesting representations (unknown by previous methods) through this formalism?
- What can you do with them (applications)?

My talk ends

Thank you!