INTRODUCTION

The problem of identifying $G$-invariant tempered distributions for the action of a classical group $G$ on a vector space is not new. In particular, the case of $G = O(p,q)$ acting on $\mathbb{R}^{p+q}$ was studied by many people, among them Methe, de Rham, Garding, Tengstrand, Gelfand and Shilov, in [Me], [dR], [Ga], [Te], [G-S], etc.. In this part of my thesis, I consider the following problem:

Let $G$ be a classical group of the following type:

$$O(p,q), U(p,q), Sp(p,q), Sp(2m,\mathbb{R}), O^*(2m),$$

$V$ be its standard module, namely,

$$\mathbb{R}^{p+q}, \mathbb{C}^{p+q}, \mathbb{H}^{p+q}, \mathbb{R}^{2m}, \mathbb{H}^{m}.$$  

The action of $G$ on $V$ induces an action of $G$ on $S^k(V^*)$, the space of tempered $k$ times distributions on $V^k = V \oplus \cdots \oplus V$, the direct sum of $k$ copies of $V$.

Our objective is to give a description of all the $G$-invariant distributions.

As it turns out, one can define a symplectic form $\langle , \rangle$ on $W = V^k \oplus V^k$ such that our original $V^k$ is a polarization, i.e., an isotropic subspace of half the dimension. Also, there exists a classical group $G'$ such that $G$ and $G'$ are mutual centralizers in $Sp = Sp(W)$, the isometry group of $\langle , \rangle$. In other words, $G$ and $G'$ form a reductive dual pair ([H1]).
We list below five corresponding dual pairs:

1) \((O(p, q), Sp(2k, \mathbb{R})) \subseteq Sp(2(p + q)k, \mathbb{R})\),

2) \((U(p, q), U(k, k)) \subseteq Sp(4(p + q)k, \mathbb{R})\),

3) \((Sp(p, q), O^*(4k)) \subseteq Sp(8(p + q)k, \mathbb{R})\),

4) \((Sp(2m, \mathbb{R}), O(k, k)) \subseteq Sp(4mk, \mathbb{R})\),

5) \((O^*(2m), Sp(k, k)) \subseteq Sp(8mk, \mathbb{R})\).

Since \(V^k\) is a polarization of the symplectic form, the oscillator representation of \(Sp\) (more precisely, the double cover \(\tilde{Sp}\) of \(Sp\)) has a Schrödinger realization in \(L^2(V^k)\) which induces an action of \(Sp\) on \(S^*(V^k)\), hereafter denoted by \(\omega\). Except for a character, it extends the \(G\)-action that we considered in the beginning.

Since \(G'\) commutes with \(G\), \(S^*(V^k)^G\), the space of \(G\)-invariant distributions is a \(G'\) module under \(\omega\). Except for \(G = Sp(p, q)\), where partial results are proved, our main theorem asserts the following two things:

a) \(S^*(V^k)^G = \text{closed span } \{\omega(g)\delta|g \in G'\}\), where \(\delta\) is the Dirac distribution at the origin of \(V^k\).

b) The multiplicity of \(\tau\) in \(S^*(V^k)^G\) is at most one, here \(\tau\) is any irreducible finite dimensional representation of \(K'\), a maximal compact subgroup of \(G'\).

In each case, we describe the set of \(\tau\) such that \(S^*(V^k)^G_\tau\) is multiplicity one.

We remark again that the case \(G = O(p, q), k = 1\), was solved by various people (Methee in Lorentzian case, Garding, Tengstrand, etc.), though the results were not stated in this form. Here, it is worthwhile to mention that these results can be used to give an elegant treatment for the fundamental solution of indefinite Laplacian ([dR], [H6]). For \(G = O(p, q)\) and arbitrary \(k\), the result is due to Kudla and Rallis ([KR]). Our method is different from theirs, and it is quite “canonical” from the point of view of invariant theory.

Some words about our approach are in order.

In Chapter 5, we prove that the multiplicity of \(G\)-invariant formal vectors of a
given $K'$-type is at most one. In the compact cases, this can be derived by properties of compact spherical pairs using see-saw dual pairs. All the $K'$-types which possibly have a $G$-invariant formal vector are also obtained. In the noncompact case, we use diamond dual pairs, an embedding property about them and a generalization of classical theory of spherical harmonics to prove a “reduction” theorem which, together with explicit descriptions of the harmonics with a given $K'$-type (most of these descriptions are due to [KV]) and the results in compact cases, implies the multiplicity one property in general and singles out all the $K'$-types having a possible $G$-invariant formal vector.

Chapter 6 deals with the existence of $G$-invariant distributions. That is probably the most interesting aspect of the present work. Our strategy is, for those “possible” $K'$-types, we prove that the projection of the Dirac distribution to such a $K'$-type $\tau$ is always nonzero, and that in turn is accomplished by explicitly computing the inner product in the Fock model between the Dirac distribution and the (simultaneous) lowest highest weight vectors of the dual pair $(M, K')$. This computation is made possible by a critical use of the famous Capelli-identity in classical invariant theory ([W1]). The author regrets being unable to carry out the computation for $G = Sp(p,q)$, which is why we have to exclude the “if” part of our main theorem for this case. Hopefully, it will be completed soon and appear elsewhere.
CHAPTER 4: PRELIMINARIES

§4.1 Notations and conventions

We shall denote by $I_p$, $I_q$, etc., the identity matrices of appropriate order, and

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$  

The transpose of a matrix $X$ is denoted by $X^t$.

Let $D$ be one of the three division algebras over $\mathbb{R}$, i.e., $D = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ as usual. Recall that as a real vector space, $\mathbb{H}$ has a standard basis consisting of the four elements $1, i, j, k$ with rules for multiplication:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$  

$D$ has a standard involution $\bar{\cdot}$, namely,

$$\bar{\cdot} = \begin{cases} \text{identity}, & D = \mathbb{R}. \\ \text{complex conjugation}, & D = \mathbb{C}. \\ \text{quaternionic conjugation}, & D = \mathbb{H}. \end{cases}$$  

Let $V = D^{p+q}_L$ be equipped with the following non-degenerate $\bar{\cdot}$-hermitian form $(\cdot, \cdot)_1$:

$$(z, w)_1 = z_1 w_1^h + z_2 w_2^h + \ldots + z_p w_p^h - z_{p+1} w_{p+1}^h - z_{p+2} w_{p+2}^h - \ldots - z_{p+q} w_{p+q}^h$$

where $z = \begin{pmatrix} z_1 \\ \vdots \\ z_{p+q} \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_{p+q} \end{pmatrix} \in V$.

Here, the subscript $L$ means that $V$ is regarded as a left vector space over $D$.

Also, we recall that a form being $\bar{\cdot}$-hermitian means

(i) $(\lambda z, w) = \lambda (z, w), \quad (z, \lambda w) = (z, w) \lambda^h, \quad \lambda \in D$,

(ii) $(w, z) = (z, w)^h$. 

45
Let $G$ be the isometry group of $(\cdot, \cdot)_1$, i.e.,

(4.1.1) \quad G = O(p, q), \quad \text{when } D = \mathbb{R}.

(4.1.2) \quad G = U(p, q), \quad \text{when } D = \mathbb{C}.

(4.1.3) \quad G = Sp(p, q), \quad \text{when } D = \mathbb{H}.

We also introduce two other series of classical groups. We use the same notations for convenience.

Let $V = \mathbb{R}^{2m}$ be equipped with the following symplectic form $(\cdot, \cdot)_1$:

$$(x, y)_1 = x_1 y_{m+1} - y_1 x_{m+1} + x_2 y_{m+2} - y_2 x_{m+2} + \ldots + x_m y_{2m} - y_m x_{2m}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_{2m} \end{pmatrix} \in \mathbb{R}^{2m}$.

Let $G$ be its isometry group, i.e.,

(4.1.4) \quad G = Sp(2m, \mathbb{R}).

Next, let $V = \mathbb{H}^{m}$ be the left vector space over \(\mathbb{H}\) equipped with the following \(\mathbb{H}\)-skew-hermitian form $(\cdot, \cdot)_1$:

$$(h, h')_1 = h_1 h'_{1}^{*} + h_2 h'_{2}^{*} + \ldots + h_m h'_{m}^{*}$$

where $h = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}$, $h' = \begin{pmatrix} h'_1 \\ \vdots \\ h'_m \end{pmatrix} \in \mathbb{H}^m$.

Let $G$ be its isometry group.

We identify $\mathbb{H}^{m}$ with $\mathbb{C}^{2m}$ by the rule:

$\mathbb{H}^{m} \ni h = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} \longleftrightarrow z = \begin{pmatrix} z_1 \\ \vdots \\ z_{2m} \end{pmatrix} \in \mathbb{C}^{2m},$

if $h_l = z_l + \varepsilon_{m+l} j$, $1 \leq l \leq m$.

Then, it is easy to see that $G$ is isomorphic to the subgroup of $GL(V, \mathbb{C})$ preserving the following two forms:

$$z_1^2 + z_2^2 + \ldots + z_{2m}^2$$
and
\[ z_1 \bar{z}_{m+1} - z_{m+1} \bar{z}_1 + z_2 \bar{z}_{m+2} - z_{m+2} \bar{z}_2 + \ldots + z_m \bar{z}_{2m} - z_{2m} \bar{z}_m \]

where \( z = \begin{pmatrix} z_1 \\ \vdots \\ z_{2m} \end{pmatrix} \in \mathbb{C}^{2m} \).

In other words,

(4.1.5) \[ G = O^*(2m). \]

See [He1] for this description of \( O^*(2m) \).

We always let \( G \) be one of the five series of classical groups just introduced, and let \( V \) be its standard module as above.

Let \( V^k = V \oplus \cdots \oplus V \), the direct sum of \( k \) copies of \( V \), \( \mathcal{S}(V^k) \) be the Schwarz space of rapidly decreasing functions on \( V^k \), \( \mathcal{S}^*(V^k) \) be its continuous dual. In usual terminology, \( \mathcal{S}^*(V^k) \) is the space of tempered distributions on \( V^k \). See [Tr].

The action of \( G \) on \( V \) induces an action of \( G \) on \( \mathcal{S}^*(V^k) \) given by:

\[(g \cdot \phi)(f) = \phi(g^{-1} \cdot f), \quad \phi \in \mathcal{S}^*(V^k), \quad f \in \mathcal{S}(V^k),\]

where

\[(g \cdot f)(v_1, v_2, \ldots, v_k) = f(g^{-1}v_1, g^{-1}v_2, \ldots, g^{-1}v_k), \quad v_i \in V, \quad 1 \leq i \leq k.\]

Our objective is to describe all the \( G \)-invariant distributions.
§4.2 Reductive dual pairs

(4.2.1) **Definition**: Let \((W, <, >)\) be a real symplectic vector space. A pair of subgroups \((G, G')\) of the symplectic group \(Sp(W, <, >)\) is called a **reductive dual pair** if

i) \(G'\) is the centralizer of \(G\) in \(Sp(W)\) and vice versa.

ii) Both \(G\) and \(G'\) act (absolutely) reductively on \(W\).

We introduce reductive dual pairs for our five series of classical groups.

Let \(W = V^k \oplus V^k\).

Consider the following isomorphisms:

\[(4.2.2) \quad W \cong V \otimes D^k \oplus V \otimes D^k \cong V \otimes D^{2k}.\]

Here, \(D^k\) (resp. \(D^{2k}\)) is regarded as a right vector space by composing the standard left vector space structure of \(D^k\) (resp. \(D^{2k}\)) with the involution \(\cdot^t\).

\(D^k_L\) has a \(\cdot\)-skew-hermitian form \((,)_2\):

\[(4.2.3) \quad ((u, v), (u', v')) = uv'^t - v'u'^t,
\]

and a \(\cdot\)-hermitian form, again denoted by \((,)_2\):

\[(4.2.4) \quad ((u, v), (u', v')) = uv't + vu't,
\]

where \(u = (u_1, u_2, ..., u_k) \in D^k_L\), etc..

They are both \(\cdot\)-sesquilinear, namely,

\[(4.2.5) \quad \lambda((u, v), (u', v')) = ((u, v), (u', v'))(\lambda(u', v')) = ((u, v), (u', v'))\lambda^t, \quad \lambda \in D.
\]

For the first three series of classical groups, we take \((,)_2\) to be \(\cdot\)-skew-hermitian, and for the last two, we take \((,)_2\) to be \(\cdot\)-hermitian.

Let \(\nu\) be the reduced trace map from \(D\) to \(\mathbb{R}\): \(\nu(\lambda) = \frac{1}{2}(\lambda + \lambda^t), \quad \lambda \in D\).

It satisfies

\[(4.2.6) \quad \nu(ab) = \nu(ba), \quad \nu(a^k) = \nu(a).
\]
Define \( < , > = \nu(( , )_1 \otimes ( , )_2) \), i.e.,

\[
(4.2.7) \quad < z \otimes z', w \otimes w' > = \nu((z, w)_1 (z', w')_2), \quad z, w \in V, \ z', w' \in D^{2k}.
\]

(4.2.8) **Lemma:** \(< , >\) is a well-defined real symplectic form on \( W \).

**Proof:** Let \( z, w \in V, \ z', w' \in D^{2k}, \lambda \in D \), then

\[
< \lambda z \otimes z', w \otimes w' > = \nu((\lambda z, w)_1 (z', w')_2) \\
= \nu(\lambda (z, w)_1 (z', w')_2) = \nu((z, w)_1 (z', w')_2 \lambda) \\
= \nu((z, w)_1 (\lambda^i z', w')_2) = < z \otimes \lambda^i z', w \otimes w' >.
\]

A similar relation holds in the second variable. Therefore, it is well defined.

Also, we can compute

\[
< z \otimes z', z \otimes z' > = \nu((z, z)_1 (z', z')_2) \\
= \nu((z', z')_2 (z, z)_1) = -\nu((z', z')_2 (z, z)_1) \\
= -\nu((z, z)_1 (z', z')_2) = - < z \otimes z', z \otimes z' >.
\]

Therefore, \(< z \otimes z', z \otimes z' > = 0\), i.e., \(< , >\) is symplectic. Q.E.D.

We denote the corresponding symplectic group by \( Sp \).

Notice that our original \( V^k \) is a **polarization** of the symplectic form, i.e., an isotropic subspace of half the dimension.

Let \( G' \) be the isometry group of \(( , )_2\).

(4.2.9) **Proposition:**

\[
G' \cong \begin{cases} 
  Sp(2k, \mathbb{R}), & \text{if } G = O(p, q) \\
  U(k, k), & \text{if } G = U(p, q) \\
  O^*(4k), & \text{if } G = Sp(p, q) \\
  O(k, k), & \text{if } G = Sp(2m, \mathbb{R}) \\
  Sp(k, k), & \text{if } G = O^*(2m).
\end{cases}
\]

**Proof:** Obvious.

(4.2.10) **Proposition:** \( G \) and \( G' \) form a reductive dual pair in \( Sp \).

For proof of proposition 4.2.10 and discussions about reductive dual pairs, see [H1], [H2], [H3], [H7].

52
§4.3 The oscillator representation

In this section, we review the basics of the oscillator representation \( \omega \) and then we specify to our five relevant cases. Differential operators representing \( \omega(g') \) are given in each case in terms of the coordinates, although we do not need them for the later development.

Let \((W, <>)\) be a real symplectic vector space with a nondegenerate symplectic form \(<,>\). We define the associated Heisenberg group \( H = H(W) \) as follows:

\[
(4.3.1) \quad H = W \oplus \mathbb{R}
\]

with the group law

\[
(w_1, t_1) \times (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2} < w_1, w_2 >)
\]

where \( w_1, w_2 \in W \), and \( t_1, t_2 \in \mathbb{R} \).

The symplectic group \( Sp(W) \) acts on \( H \) by

\[
g \cdot (w, t) = (gw, t).
\]

Let \( \chi \) be the unitary character of \( \mathbb{R} \) given by

\[
\chi : t \mapsto e^{it}.
\]

The following result is usually called Stone-Von Neumann Theorem.

\[
(4.3.2) \textbf{Theorem}: \text{ There is only one equivalence class of irreducible unitary representation } \rho \text{ of } H \text{ such that } \rho |_{\mathbb{R}} = \chi.
\]

Since \( Sp(W) \) acts as automorphisms of \( H \) leaving \( \mathbb{R} \) fixed, the above theorem implies that there exists a projective unitary representation \( \omega \) of \( Sp(W) \) on the space of \( \rho \) such that

\[
(4.3.3) \quad \omega(g) \rho(h) \omega(g)^{-1} = \rho(g(h)).
\]
Let $\tilde{Sp}(W)$ be the unique nontrivial two-fold cover of $Sp(W)$, it is a classical result of Shale ([Sh]) and Weil ([We]) that $\omega$ actually lifts to a unitary representation of $\tilde{Sp}(W)$. We call this representation the oscillator representation of $\tilde{Sp}(W)$.

(4.3.4) Convention: We identify $\tilde{Sp}(W)$ as a set with $Sp(W) \times \mathbb{Z}_2$. The projection from $\tilde{Sp}(W)$ to $Sp(W)$ is denoted by $\tilde{g} \mapsto g$. For any subgroup $B \subseteq Sp(W)$, the preimage of $B$ under the projection $\tilde{g} \mapsto g$ is denoted by $\tilde{B}$.

(4.3.5) Remark: For our later purpose of describing the invariant distributions, it is not very crucial to distinguish $\omega$ as a linear representation of $\tilde{Sp}(W)$ or as a projective representation of $Sp(W)$, although we shall try to do so.

Let $(X,Y)$ be a complete polarization of $W$, i.e.,

i) $W = X \oplus Y$,

ii) $X, Y$ are maximal isotropic with respect to $<,>$. 

The unique irreducible unitary representation $\rho$ of $H$ with central character $\chi$ can then be realized in $L^2(X)$ and the action of $H$ is given by

(4.3.6) $[\rho(x,y,t)f](x') = e^{i(t+<x',x'-\frac{1}{2}x>)}f(x'-x), \quad (x,y,t) \in H, \; x' \in X.$

It is usually refered to as the Schrödinger realization.

To describe the corresponding oscillator representation $\omega$, we let $P(Y)$ (resp. $P(X)$) be the subgroup of $Sp$ leaving $Y$ (resp. $X$) stable, $N(Y)$ be the subgroup of $P(Y)$ which acts as identity on $Y$, then

(4.3.7) $P(Y) = M \ltimes N(Y),$

where $M = P(X) \cap P(Y)$.

Let $\sigma$ be the following element of $Sp(W)$:

$\sigma : (x,y) \rightarrow (y,-x), \; x \in X, \; y \in Y.$

Then $P(Y)$ and $\sigma$ generate $Sp(W)$.  

54
Take \( \{ e_i, f_i \}_{i=1}^l \) to be a standard symplectic basis for the symplectic form \( \langle \cdot, \cdot \rangle \), i.e.,

\[
\langle e_i, e_j \rangle = 0, \quad \langle f_i, f_j \rangle = 0, \quad \langle e_i, f_j \rangle = \delta_{ij}
\]
such that \( X = \text{span of } \{ e_i \} \), \( Y = \text{span of } \{ f_i \} \).

We then identify \( W = X \oplus Y \) with \( R^l \oplus R^l \) by the rule:

\[
w \mapsto \begin{pmatrix} x \\ y \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_l \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_l \end{pmatrix} \in R^l \quad \text{if} \quad w = \sum_{i=1}^{l} (x_i e_i + y_i f_i).
\]

Under this identification, the symplectic form is given by

\[
\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \rangle = \begin{pmatrix} x \\ y \end{pmatrix}^t J_l \begin{pmatrix} x' \\ y' \end{pmatrix} = \sum_{i=1}^{l} (x_i y'_i - y_i x'_i).
\]

We have

\[
M = \{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \mid g \in GL(l, \mathbb{R}) \},
\]

\[
N(Y) = \{ \begin{pmatrix} I_l & 0 \\ s & I_l \end{pmatrix} \mid s = s' \in M_l(\mathbb{R}) \},
\]

\[
\sigma = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix},
\]

here, matrices act by left multiplication.

The oscillator representation \( \omega \) of \( \tilde{Sp}(2l, \mathbb{R}) \) can then be described as follows:

\[
\omega((g, \epsilon) f(x) = \gamma(g, \epsilon) \lvert \det g \rvert^{-\frac{1}{2}} f(g^{-1} x), \quad \tilde{g} = (g, \epsilon) \in \tilde{GL}(l, \mathbb{R}), \quad x \in \mathbb{R}^l,
\]

\[
\omega\left( \begin{pmatrix} I_l & 0 \\ s & I_l \end{pmatrix} \right) f(x) = e^{-\frac{1}{2} s^t x s} f(x),
\]

\[
\omega(\sigma) f(x) = \gamma_0 \left( \frac{1}{2\pi} \right)^{\frac{l}{2}} \int_{\mathbb{R}^l} e^{ix^t x'} f(x') dx', \quad \gamma_0 \in \mathbb{C}, \quad \lvert \gamma_0 \rvert = 1,
\]

where \( \gamma : \tilde{GL}(l, \mathbb{R}) \rightarrow \mathbb{C}^{\times} \), is a character of \( \tilde{GL}(l, \mathbb{R}) \) and its values are all eighth roots of unity.

(4.3.10) **Remark:** Let \( \mathfrak{sp} \) denote the Lie algebra of \( Sp \), then \( \omega(\mathfrak{sp}) \) consists of the differential operators of total degree 2, i.e.,

\[
x_i x_j, \quad \frac{\partial^2}{\partial x_i \partial x_j}, \quad \frac{1}{2} \left(x_i \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} x_i \right)
\]
where \( \{x_i\} \)'s denote the coordinates of the first factor of the complete polarization \( W \cong \mathbb{R}^l \oplus \mathbb{R}^l \).

Recall that in §4.2, we started with a classical group \( G \) acting on the standard module \( V \), associated to it, we defined a symplectic form \( \langle , \rangle \) on \( W = V^k \oplus V^k \) for which \( V^k \) is a polarization.

The oscillator representation \( \omega \) of \( Sp(W) \) is realized in \( L^2(V^k) \).

Since \( G \) preserves the complete polarization \( (V^k, V^k) \), i.e., \( G \subseteq M = P(X) \cap P(Y) \) in the notation of this section, (4.3.9) implies

\[
\omega(\tilde{g})f(v_1, v_2, \ldots, v_k) = \gamma(\tilde{g})((detg)^{(k)}|^{-\frac{1}{2}} f(g^{-1}v_1, g^{-1}v_2, \ldots, g^{-1}v_k),
\]

where \( \tilde{g} \in \tilde{G}, v_i \in V, 1 \leq i \leq k \), and \( (detg)^{(k)} \) is the determinant of \( g \) as a real linear transformation of \( V^k \).

We see from above that, except for a character of \( \tilde{G} \), the oscillator representation \( \omega \) is an extension of the action of \( G \) on \( L^2(V^k) \) induced from the given linear action of \( G \) on \( V \).

The dual statement holds true for the action on the space of tempered distributions \( S^*(V^k) \).

(4.3.11) Notation: We shall use \( \omega^t \) to denote the twist of \( \omega \) by the character of \( \tilde{G} \) for which \( \omega^t|_G \) factors through the linear action of \( G \) on \( V^k \).

(4.3.12) Remark: Recall \( (G, G') \) form a dual pair in \( Sp(W) \), \( W = V^k \oplus V^k \). Let \( g \) (resp. \( g' \)) be the Lie algebra of \( G \) (resp. \( G' \)), in the Schrödinger model of the oscillator representation, \( \omega(\mathfrak{g}') \) can be given explicitly by using concrete realizations case by case. See Appendix 1. Also, \( \omega(\mathfrak{g}') \) has the decomposition \( \omega(\mathfrak{g}') = \omega(\mathfrak{g}')^{(2,0)} \oplus \omega(\mathfrak{g}')^{(1,1)} \oplus \omega(\mathfrak{g}')^{(0,2)} \), where \( \omega(\mathfrak{g}')^{(a,b)} \) consists of differential operators in \( \omega(\mathfrak{g}') \) such that the total degree in \( x_i \) is \( a \) and the total degree in \( \frac{\partial}{\partial x_i} \) is \( b \).
Case 1): $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$.

$$
\omega(g')^{(2,0)} = \text{span of } \left\{ \sum_{t=1}^{p} x_{ti}x_{tj} - \sum_{t=p+1}^{p+q} x_{ti}x_{tj}, 1 \leq i, j \leq k \right\}
$$

(4.3.13)

$$
\omega(g')^{(1,1)} = \text{span of } \left\{ \sum_{t=1}^{p+q} (x_{ti} \frac{\partial}{\partial x_{tj}} + \frac{\delta_{ij}}{2}), 1 \leq i, j \leq k \right\}
$$

$$
\omega(g')^{(0,2)} = \text{span of } \left\{ \sum_{t=1}^{p} \frac{\partial^2}{\partial x_{ti}\partial x_{tj}} - \sum_{t=p+1}^{p+q} \frac{\partial^2}{\partial x_{ti}\partial x_{tj}}, 1 \leq i, j \leq k \right\}
$$

where $(x_{ij})_{1 \leq i \leq p, 1 \leq j \leq k}$ are the real coordinates of $V^k \cong \mathbb{R}^{p+q,k}$.

Case 2): $G = U(p, q)$, $G' = U(k, k)$.

$$
\omega(g')^{(2,0)} = \text{span of } \left\{ \sum_{t=1}^{p} z_{ti}\bar{z}_{tj} - \sum_{t=p+1}^{p+q} z_{ti}\bar{z}_{tj}, 1 \leq i, j \leq k \right\}
$$

(4.3.14)

$$
\omega(g')^{(1,1)} = \text{span of } \left\{ \sum_{t=1}^{p+q} (z_{ti} \frac{\partial}{\partial z_{tj}} + \frac{\delta_{ij}}{2}), \sum_{t=1}^{p+q} (\bar{z}_{ti} \frac{\partial}{\partial \bar{z}_{tj}} + \frac{\delta_{ij}}{2}), 1 \leq i, j \leq k \right\}
$$

$$
\omega(g')^{(0,2)} = \text{span of } \left\{ \sum_{t=1}^{p} \frac{\partial^2}{\partial z_{ti}\partial \bar{z}_{tj}} - \sum_{t=p+1}^{p+q} \frac{\partial^2}{\partial z_{ti}\partial \bar{z}_{tj}}, 1 \leq i, j \leq k \right\}
$$

where $(z_{ij})_{1 \leq i \leq p+q, 1 \leq j \leq k}$ are the complex coordinates of $V^k \cong \mathbb{C}^{p+q,k}$.
Case 3): $G = Sp(p, q), G' = O^*(4k)$.

\[
\omega(g')^{(2,0)} = \text{span of } \left\{ \sum_{t=1}^{2p} z_{ti} \bar{z}_{tj} - \sum_{t=2p+1}^{2p+2q} z_{ti} \bar{z}_{tj}, \ 1 \leq i, j \leq k \right\},
\]

\[
\sum_{t=1}^{p} (z_{ti} \bar{z}_{p+t,j} - z_{p+t,i} \bar{z}_{tj}) - \sum_{t=2p+1}^{2p+q} (z_{t,i} \bar{z}_{q+t,j} - z_{q+t,i} \bar{z}_{t,j}),
\]

\[
\sum_{t=1}^{p} (\bar{z}_{t,i} z_{p+t,j} - \bar{z}_{p+t,i} \bar{z}_{t,j}) - \sum_{t=2p+1}^{2p+q} (\bar{z}_{t,i} z_{q+t,j} - \bar{z}_{q+t,i} \bar{z}_{t,j}), \ 1 \leq i < j \leq k \}
\]

(4.3.15)

\[
\omega(g')^{(1,1)} = \text{span of } \left\{ \sum_{t=1}^{2p+2q} (z_{ti} \frac{\partial}{\partial z_{tj}} + \frac{\delta_{ij}}{2}), \ \sum_{t=1}^{2p+2q} (\bar{z}_{ti} \frac{\partial}{\partial \bar{z}_{tj}} + \frac{\delta_{ij}}{2}) \right\},
\]

\[
\sum_{t=1}^{p} \left( z_{t,i} \frac{\partial}{\partial z_{p+t,j}} - z_{p+t,i} \frac{\partial}{\partial z_{t,j}} \right) + \sum_{t=2p+1}^{2p+q} \left( z_{t,i} \frac{\partial}{\partial z_{q+t,j}} - z_{q+t,i} \frac{\partial}{\partial z_{t,j}} \right),
\]

\[
\sum_{t=1}^{p} \left( \bar{z}_{t,i} \frac{\partial}{\partial \bar{z}_{p+t,j}} - \bar{z}_{p+t,i} \frac{\partial}{\partial \bar{z}_{t,j}} \right) + \sum_{t=2p+1}^{2p+q} \left( \bar{z}_{t,i} \frac{\partial}{\partial \bar{z}_{q+t,j}} - \bar{z}_{q+t,i} \frac{\partial}{\partial \bar{z}_{t,j}} \right), \ 1 \leq i, j \leq k \}
\]

\[
\omega(g')^{(0,2)} = \text{span of } \left\{ \sum_{t=1}^{2p} \frac{\partial^2}{\partial z_{ti} \partial \bar{z}_{tj}}, \ \sum_{t=2p+1}^{2p+2q} \frac{\partial^2}{\partial z_{ti} \partial \bar{z}_{tj}}, \ 1 \leq i, j \leq k \right\},
\]

\[
\sum_{t=1}^{p} \left( \frac{\partial^2}{\partial z_{t,i} \partial z_{p+t,j}} - \frac{\partial^2}{\partial z_{p+t,i} \partial z_{t,j}} \right) + \sum_{t=2p+1}^{2p+q} \left( \frac{\partial^2}{\partial z_{t,i} \partial z_{q+t,j}} - \frac{\partial^2}{\partial z_{q+t,i} \partial z_{t,j}} \right),
\]

\[
\sum_{t=1}^{p} \left( \frac{\partial^2}{\partial \bar{z}_{t,i} \partial \bar{z}_{p+t,j}} - \frac{\partial^2}{\partial \bar{z}_{p+t,i} \partial \bar{z}_{t,j}} \right) + \sum_{t=2p+1}^{2p+q} \left( \frac{\partial^2}{\partial \bar{z}_{t,i} \partial \bar{z}_{q+t,j}} - \frac{\partial^2}{\partial \bar{z}_{q+t,i} \partial \bar{z}_{t,j}} \right), \ 1 \leq i < j \leq k \}
\]

where $(z_{ij})_{1 \leq i \leq 2p+2q, 1 \leq j \leq k}$ are the complex coordinates of $V^k \cong \mathbb{H}^{p+q,k} \cong \mathbb{C}^{2p+2q,k}$ induced by $\mathbb{H} = \mathbb{C} \oplus \mathbb{Cj}$.

Case 4): $G = Sp(2m, \mathbb{R}), G' = O(k, k)$.

\[
\omega(g')^{(2,0)} = \text{span of } \left\{ \sum_{t=1}^{m} (x_{t,i} x_{m+t,j} - x_{m+t,i} x_{t,j}), \ 1 \leq i < j \leq k \right\}
\]

(4.3.16)

\[
\omega(g')^{(1,1)} = \text{span of } \left\{ \sum_{t=1}^{m} (x_{ti} \frac{\partial}{\partial x_{tj}} + \frac{\delta_{ij}}{2}), \ 1 \leq i, j \leq k \right\}
\]

\[
\omega(g')^{(0,2)} = \text{span of } \left\{ \sum_{t=1}^{m} \left( \frac{\partial^2}{\partial x_{t,i} \partial x_{m+t,j}} - \frac{\partial^2}{\partial x_{m+t,i} \partial x_{t,j}} \right), \ 1 \leq i < j \leq k \right\}
\]

where $(x_{ij})_{1 \leq i \leq 2m, 1 \leq j \leq k}$ are the real coordinates of $V^k \cong \mathbb{R}^{2m,k}$. 

58
Case 5): \( G = O^*(2m), \ G' = Sp(k, k) \).

\[
\omega(g')^{(2,0)} = \text{span of } \left\{ \sum_{t=1}^{2m} z_{ti}z_{tj}, \sum_{t=1}^{2m} \bar{z}_{ti}\bar{z}_{tj}, \right. \\
\left. \sum_{t=1}^{m} (z_{t,i}z_{m+t,j} - z_{m+t,i}z_{t,j}), \sum_{t=1}^{m} (\bar{z}_{t,i}\bar{z}_{m+t,j} - \bar{z}_{m+t,i}\bar{z}_{t,j}), \ 1 \leq i, j \leq k \right\}
\]

(4.3.17)

\[
\omega(g')^{(1,1)} = \text{span of } \left\{ \sum_{t=1}^{2m} (z_{ti} \frac{\partial}{\partial z_{tj}} + \frac{\delta_{ij}}{2}), \sum_{t=1}^{2m} (\bar{z}_{ti} \frac{\partial}{\partial \bar{z}_{tj}} + \frac{\delta_{ij}}{2}), \right. \\
\left. \sum_{t=1}^{m} (z_{t,i} \frac{\partial}{\partial z_{m+t,j}} - z_{m+t,i} \frac{\partial}{\partial z_{t,j}}), \sum_{t=1}^{m} (\bar{z}_{t,i} \frac{\partial}{\partial \bar{z}_{m+t,j}} - \bar{z}_{m+t,i} \frac{\partial}{\partial \bar{z}_{t,j}}), \ 1 \leq i, j \leq k \right\}
\]

\[
\omega(g')^{(0,2)} = \text{span of } \left\{ \sum_{t=1}^{2m} \frac{\partial^2}{\partial z_{ti}\partial z_{tj}}, \sum_{t=1}^{2m} \frac{\partial^2}{\partial \bar{z}_{ti}\partial \bar{z}_{tj}}, \right. \\
\left. \sum_{t=1}^{m} \left( \frac{\partial^2}{\partial z_{t,i}\partial z_{m+t,j}} - \frac{\partial^2}{\partial z_{m+t,i}\partial z_{t,j}} \right), \sum_{t=1}^{m} \left( \frac{\partial^2}{\partial \bar{z}_{t,i}\partial \bar{z}_{m+t,j}} - \frac{\partial^2}{\partial \bar{z}_{m+t,i}\partial \bar{z}_{t,j}} \right), \ 1 \leq i, j \leq k \right\}
\]

where \((z_{ij})_{1 \leq i \leq 2m, 1 \leq j \leq k}\) are the complex coordinates of \(V^k \cong \mathbb{H}^m, k \cong \mathbb{C}^{2m}, k\) induced by \(\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j\).
§4.4 Statement of main result

We now state our main theorem.

For $G = Sp(p, q)$, we shall just claim the “only if” part of b).

For $G = Sp(2m, \mathbb{R})$, our notations and parametrizations for irreducible representations of $O(k)$ are the same as in [KV]. All the other notations are as in previous sections.

(4.4.1) Theorem:

a) $S^* (V^k)^G = \text{closed span } \{ \omega^s(q) \delta | q \in \hat{G}' \}$, where $\delta$ is the Dirac distribution at the origin of $V^k$.

b) The multiplicity of $\tau$ in $S^* (V^k)^G$ is at most one, here $\tau$ is any irreducible finite dimensional representation of $\hat{K}'$. It is equal to one if and only if the following conditions are satisfied:

$(1)$ $G = O(p, q)$, $G' = Sp(2k, \mathbb{R})$, $\tau \in \hat{U}(k)$, the highest weight of $\tau$ is

\[
\left( \frac{p-q}{2}, \frac{p-q}{2}, \frac{p-q}{2} \right) + (a_1, a_2, ..., a_t, 0, ..., 0, -b_s, ..., -b_2, -b_1),
\]

$a_1 \geq a_2 \geq ... \geq a_t > 0$, $b_1 \geq b_2 \geq ... \geq b_s > 0$, $t \leq \min(k, p)$, $s \leq \min(k, q)$,

$a_i, b_j$ are all even integers.

$(2)$ $G = U(p, q)$, $G' = U(k, k)$, $\tau = \tau_1 \otimes \tau_2$, $\tau_1, \tau_2 \in \hat{U}(k)$, $\tau_2 \cong \tau_1^*$, and the highest weight of $\tau_1$ is

\[
\left( \frac{p-q}{2}, \frac{p-q}{2}, \frac{p-q}{2} \right) + (a_1, a_2, ..., a_t, 0, ..., 0, -b_s, ..., -b_2, -b_1),
\]

$a_1 \geq a_2 \geq ... \geq a_t > 0$, $b_1 \geq b_2 \geq ... \geq b_s > 0$, $t \leq \min(k, p)$, $s \leq \min(k, q)$.

$(3)$ $G = Sp(p, q)$, $G' = O^*(4k)$, $\tau \in \hat{U}(2k)$, the highest weight of $\tau$ is

\[
(p-q, p-q, ..., p-q) + (a_1, a_2, a_2, ..., a_t, a_t, 0, ..., 0, -b_s, -b_s, ..., -b_2, -b_2, -b_1, -b_1),
\]

$a_1 \geq a_2 \geq ... \geq a_t > 0$, $b_1 \geq b_2 \geq ... \geq b_s > 0$, $t \leq \min(k, p)$, $s \leq \min(k, q)$.
(4) \( G = Sp(2m, \mathbb{R}), G' = O(k, k), \tau = \tau_1 \otimes \tau_2, \tau_1, \tau_2 \in \hat{O}(k), \tau_2 \cong \tau_1^*, \) and the highest weight of \( \tau_1 \) is:

If \( k \) is odd, \( k = 2r + 1 \),
\[
\left( \frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon), \quad \epsilon = (-1)^{a_1+a_2+\ldots+a_t},
\]
\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad t \leq \min(m, r),
\]
or
\[
\left( \frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon), \quad \epsilon = (-1)^{a_1+a_2+\ldots+a_t+1},
\]
\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad 2r + 1 - m \leq t \leq \min(m, r).
\]

If \( k \) is even, \( k = 2r \),
\[
\left( \frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0)_+, \quad \epsilon = (-1)^{a_1+a_2+\ldots+a_t},
\]
\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad t \leq \min(m, r),
\]
or
\[
\left( \frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0)_-, \quad \epsilon = (-1)^{a_1+a_2+\ldots+a_t},
\]
\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad 2r - m \leq t \leq \min(m, r).
\]

(5) \( G = O^*(2m), G' = Sp(k, k), \tau = \tau_1 \otimes \tau_2, \tau_1, \tau_2 \in \hat{Sp}(k), \tau_2 \cong \tau_1^*, \) and the highest weight of \( \tau_1 \) is
\[
\left( \frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2} \right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),
\]
\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad t \leq \min(m, k).
\]

**Remark:** In case (1), (2) (resp. case (4), (5)), when \( p - q \) (resp. \( m \)) is even, the representations \( \tau, \tau_1 \) etc. are actual representations of \( U(k) \) (resp. \( O(k), Sp(k) \)), instead of \( \hat{U}(k) \) (resp. \( \hat{O}(k), \hat{Sp}(k) \)).

We shall prove the “only if” part of b) in Chapter 5. The theorem then follows by showing that the projection of the Dirac distribution to the possible \( \bar{K}' \)-types prescribed above is not zero. That is done in Chapter 6.
CHAPTER 5: MULTIPLICITY ONE PROPERTY OF INVARIANT FORMAL VECTORS

§5.1 Review of structure of dual pairs in the Fock model

We introduce the Fock model of the oscillator representation. In this model, we then review the results of [H2] about the structure of dual pairs, among them the Duality Correspondence Theorem when one member of the pair is compact, the see-saw and diamond dual pairs.

The essentially unique irreducible unitary representation of the Heisenberg group $H = H(W)$ with central character $\chi$ (see §4.3 for notations) has another realization which we shall describe now.

Let $\{e_j, f_j\}_{j=1}^l$ be a standard symplectic basis for the symplectic form $<,>$, and $\{x_j, y_j\}$ be the coordinates of a typical vector $w \in W$ under this basis, again as in §4.3.

Introduce a complex structure on $W$ by setting

$$z_j = x_j + iy_j, \quad 1 \leq j \leq l.$$  

Let

$$z = (z_1, ..., z_l) = (x_1 + iy_1, ..., x_l + iy_l)$$

with

$$\bar{z} = (\bar{z}_1, ..., \bar{z}_l) = (x_1 - iy_1, ..., x_l - iy_l).$$

Set

$$z \cdot z' = \sum_{j=1}^l z_j \bar{z}_j,'$$

so that

$$|z|^2 = z \cdot \bar{z}.$$  

(5.1.1) **Theorem** (see [Ba] [Ca]): The essentially unique unitary representation of
$H$ with $\chi$ as central character may be realized in the Hilbert space

$$\mathcal{F} = \{ f(z) \text{ holomorphic on } W : \left( \frac{i}{2\sqrt{\pi}} \right)^l \int_W |f(z)|^2 e^{-|z|^2} dz_1 d\bar{z}_1 \ldots dz_l d\bar{z}_l < \infty \}$$

with the inner product

$$(f_1, f_2) = \left( \frac{i}{2\sqrt{\pi}} \right)^l \int_W f_1(z) \overline{f_2(z)} e^{-|z|^2} dz_1 d\bar{z}_1 \ldots dz_l d\bar{z}_l, \quad f_1, f_2 \in \mathcal{F}. $$

The action, denoted by $\nu$, of $H$ on $\mathcal{F}$ is given by

$$\nu(z)f(z') = e^{-\frac{|z|^2}{8}} e^{z' \cdot \bar{z}} f(z' - z), \quad z \in W \subseteq H.$$ 

We call $\nu$ the Fock model.

One of the original sources for the following facts is [Ba]. A nice account can also be found in [Fo].

(5.1.2) **Remark:** In the Fock model, we have

$$\left( \frac{\partial}{\partial z_j} \right)^* = z_j, \quad 1 \leq j \leq l,$$

where $A^*$ is the adjoint operator of $A$ acting on the Hilbert space $\mathcal{F}$.

(5.1.3) **Remark:** If we set $X = \text{span of } \{ e_j \}$, $Y = \text{span of } \{ f_j \}$, the Schrödinger representation $\rho$ of $H$ with central character $\chi$ is realized in $L^2(X)$. The isomorphism between $\rho$ and $\nu$ can be established by requiring

$$L^2(X) \ni e^{-\frac{1}{8} \sum_{i=1}^l x_i^2} \mapsto 1 \in \mathcal{F}$$

and

$$x_j \rightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z_j} + z_j \right),$$

$$\frac{\partial}{\partial x_j} \rightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z_j} - z_j \right).$$

(5.1.4)

Corresponding to the Fock model $\nu$, we have a representation $\omega_\nu$ of $Sp(W)$ satisfying

$$\omega_\nu(\tilde{g}) \nu(h) \omega_\nu(\tilde{g})^{-1} = \nu(g(h)),$$

(5.1.5)
where \( h \in H = H(W) \), \( \tilde{g} \in \tilde{Sp}(W) \). \( \omega_\nu \) is called the Fock model of the oscillator representation. We shall use the notation \( \omega \) instead of \( \omega_\nu \) for convenience.

(5.1.6) **Remark:** \( U_l \) is the isometry group of \( z \cdot \tilde{z} \), is a maximal compact subgroup of \( Sp(W) \). \( \tilde{U}_l \) is identified with \( \{ (g, \delta) | g \in U_l, \delta^2 = detg \} \). Under the Fock model, \( \tilde{U}_l \) acts in a simple manner:

\[
\omega(g, \delta)f(z') = \delta^{-1}f(g^{-1}z'), \quad \tilde{g} = (g, \delta) \in \tilde{U}_l.
\]

We now review some results we will need from the theory of reductive dual pairs ([H2]). We shall state everything in the Fock model \( \omega \) of the oscillator representation of \( \tilde{Sp} \) without sometimes explicitly mentioning \( \omega \).

Consider a reductive dual pair \( (G, G') \subseteq Sp \).

A maximal compact subgroup of \( Sp \) is \( U = U_l \), the unitary group in \( l \) variables. In the Fock model, the space of \( \tilde{U} \)-finite vectors in \( \omega \) is isomorphic to \( \mathcal{P} = \mathcal{P}(\mathbb{C}^l) \), the space of polynomials on \( \mathbb{C}^l \).

Using this identification, we have

\[
(5.1.7) \quad \omega(\mathfrak{sp}_C) = \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)},
\]

\[
\mathfrak{sp}^{(1,1)} = \text{span of } \{ \frac{1}{2}(z_i \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} z_i) \},
\]

\[
\mathfrak{sp}^{(2,0)} = \text{span of } \{ z_i z_j \},
\]

\[
\mathfrak{sp}^{(0,2)} = \text{span of } \{ \frac{\partial^2}{\partial z_i \partial z_j} \}.
\]

Here and after, a subscript \( C \) denotes complexification.

Let

\[
(5.1.8) \quad \mathfrak{sp} = u \oplus q
\]

be the Cartan decomposition of \( \mathfrak{sp} \) with this choice of \( U \). We have

\[
(5.1.9) \quad \omega(u_C) = \mathfrak{sp}^{(1,1)}, \quad \omega(q_C) = \mathfrak{sp}^{(2,0)} \oplus \mathfrak{sp}^{(0,2)}.
\]
The structure of a dual pair \((M, K') \subseteq \text{Sp}, K'\) compact.

Assume \(K' \subseteq U = U_l\). Write

\[
(5.1.10) \quad \mathfrak{m}_C = \mathfrak{m}^{(1,1)} \oplus \mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)}
\]

with \(\mathfrak{m}^{(i,j)} = \mathfrak{m}_C \cap \mathfrak{sp}^{(i,j)}\).

We define the space of generalized \(K'\)-harmonics.

\[
(5.1.11) \quad \mathcal{H}(K') = \{ \mathcal{P} \in \mathcal{P} : X \cdot \mathcal{P} = 0 \text{ for all } X \in \mathfrak{m}^{(0,2)} \}.
\]

Let

\[
(5.1.12) \quad \mathcal{P} = \sum_{\tau \in R(K', \omega)} \mathcal{P}_\tau
\]

be the isotypic decomposition of \(\mathcal{P}\) as a \(K'\)-module, where \(R(K', \omega)\) is the set of the \(K'\)-types occurring in the oscillator representation \(\omega\).

The following theorem is a generalization of the classical theory of spherical harmonics, but sometimes I shall also refer it as the Duality Correspondence Theorem.

\[
(5.1.13) \quad \textbf{Theorem: (see } [H2], \text{ p.542)}
\]

Let \((M, K')\) be a reductive dual pair in \(\text{Sp}, K'\) compact. Then

a) The joint action of \(\mathfrak{m} \times K'\) on \(\mathcal{P}_\tau\) is irreducible for each \(\tau \in R(K', \omega)\).

b) \(\mathcal{H}(K')_\tau = \mathcal{P}_\tau \cap \mathcal{H}(K')\) consists precisely of the space of polynomials of lowest degree in \(\mathcal{P}_\tau\).

c) One has \(\mathcal{P}_\tau = \mathcal{U}(\mathfrak{m}^{(2,0)}) \cdot \mathcal{H}(K')_\tau\), where \(\mathcal{U}(\mathfrak{m}^{(2,0)})\) is the universal enveloping algebra of \(\mathfrak{m}^{(2,0)}\).

d) The group \(K'\) and the Lie algebra \(\mathfrak{m}^{(1,1)}\) generate mutual commutants on \(\mathcal{H}(K')\). Equivalently, each \(\mathcal{H}(K')_\tau\) is irreducible under the joint action of \(\mathfrak{m}^{(1,1)} \times K'\), and if we write \(\mathcal{H}(K')_\tau \cong \Omega(\tau) \otimes \tau\) for \(\tau \in R(K', \omega)\), then \(\tau\) determines \(\Omega(\tau)\) and vice versa, so that \(\tau \mapsto \Omega(\tau)\) is an injection from \(R(K', \omega)\) into \(R(\mathfrak{m}^{(1,1)}, \omega)\).

See-saw and diamond dual pairs

Let \((G, G') \subseteq \text{Sp}\) be a reductive dual pair. We may assume \(G\) and \(G'\) are embedded in \(\text{Sp}\) in such a way that the Cartan decomposition (5.1.8) of \(\mathfrak{sp}\) also
induces Cartan decompositions of $g$ and $g'$. Thus,

$$g = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = u \cap g, \quad \mathfrak{p} = q \cap g,$$

$$g' = \mathfrak{k}' \oplus \mathfrak{p}', \quad \mathfrak{k}' = u \cap g', \quad \mathfrak{p}' = q \cap g'.$$

(5.1.14) **Fact:** $\mathfrak{k}$ and $\mathfrak{k}'$ are themselves members of reductive dual pairs $(\mathfrak{k}, \mathfrak{m}')$ and $(\mathfrak{m}, \mathfrak{m}')$ ([H2], p.539).

$$\begin{array}{cc}
\mathfrak{k} & \mathfrak{k}' \\
\cap & \cap \\
m & \cap m'
\end{array}$$

The pairs of Lie algebras placed in the opposite position of the above diagram are reductive dual pairs. Therefore, they are called see-saw dual pairs.

We have the following inclusions:

$$\begin{array}{cc}
\mathfrak{k} & \mathfrak{k}' \\
\cap & \cap \\
g & \cap g'
\end{array}$$

(5.1.15)

$$\begin{array}{cc}
m & m'
\cap & \cap
\mathfrak{g} & \mathfrak{g}'
\end{array}$$

Since $\mathfrak{k} \subseteq u, \text{ad}\mathfrak{k}$ preserves the decomposition (5.1.7). Since $\mathfrak{m}'$ is the full centralizer of $\mathfrak{k}$ in $\mathfrak{sp}$, we have

(5.1.16)

$$m'_C = m'^{(1,1)} \oplus m'^{(2,0)} \oplus m'^{(0,2)}, \quad m'^{(i,j)} = m'_C \cap \mathfrak{sp}^{(i,j)},$$

$$m_C = m^{(1,1)} \oplus m^{(2,0)} \oplus m^{(0,2)}, \quad m^{(i,j)} = m_C \cap \mathfrak{sp}^{(i,j)}.$$

Clearly, $m'^{(1,1)}$ is the complexification of the Lie algebra of the maximal compact subgroup of $M'$.

Let

$$m_0^{(1,1)} = m^{(1,1)} \cap \mathfrak{sp}, \quad m'^{(1,1)} = m'^{(1,1)} \cap \mathfrak{sp}.$$  

(5.1.17) **Fact:** $(m_0^{(1,1)}, m'^{(1,1)})$ is a reductive dual pair in $\mathfrak{sp}$ ([H2], p.540).
Thus we can expand (5.1.15) to

\[ (5.1.18) \]

The pairs of Lie algebras similarly placed in the two diamonds are reductive dual pairs. Therefore, they are called diamond dual pairs.

Lastly, we have the following embedding property

\[ (5.1.19) \textbf{Fact} ([H2], p.540): \]

\[ m^{(2,0)} \oplus m^{(0,2)} = p_C \oplus m^{(2,0)} = m^{(2,0)} \oplus p_C. \]

List of diamond dual pairs

We write down the two diamonds of reductive dual pairs in our five cases.

(Case 1:)

\[ G = O(p, q), \quad G' = Sp(2k, \mathbb{R}). \]

\[ U(p) \times U(q) \quad \bigcirc \quad U(k) \times U(k) \]

\[ O(p, q) \quad \bigcirc \quad Sp(2k, \mathbb{R}) \times Sp(2k, \mathbb{R}) \quad U(k) \]

(Case 2:)

\[ G = U(p, q), \quad G' = U(k, k). \]

\[ U(p) \times U(q) \times U(p) \times U(q) \quad U(k) \times U(k) \times U(k) \times U(k) \]

\[ U(p, q) \times U(p, q) \quad U(k, k) \times U(k, k) \quad U(k) \times U(k) \]

(Case 3:)

\[ G = Sp(p, q), \quad G' = O^*(4k). \]
(5.1.22) \[ \begin{align*} U(2p) & \times U(2q) \\ \cup & \\ \cap & \\ Sp(p) \times Sp(q) \end{align*} \]

(5.1.22) \[ \begin{align*} O^*(4k) & \times O^*(4k) \\ \cup & \\ U(2k) \end{align*} \]

(Case 4:) \[ G = Sp(2m, \mathbb{R}), \quad G' = O(k, k). \]

(5.1.23) \[ \begin{align*} U(m) & \times U(m) \\ \cup & \\ \cap & \\ Sp(2m, \mathbb{R}) \times Sp(2m, \mathbb{R}) \end{align*} \]

(5.1.23) \[ \begin{align*} U(k, k) & \quad O(k) \times O(k) \\ \cup & \\ O(k, k) \end{align*} \]

(Case 5:) \[ G = O^*(2m), \quad G' = Sp(k, k). \]

(5.1.24) \[ \begin{align*} U(m) & \quad O^*(2m) \times O^*(2m) \\ \cup & \\ O^*(2m) \end{align*} \]

(5.1.24) \[ \begin{align*} U(2k) & \quad Sp(k) \times Sp(k) \\ \cup & \\ Sp(k, k) \end{align*} \]
§5.2 Multiplicity one property: Compact cases

In this section, we consider three dual pairs \((G, G') \subseteq Sp\) introduced in §4.2 with \(G = O(p), \ U(p), \ Sp(p)\). We prove that the \(K'\)-types of the space of \(G\)-invariant formal vectors have multiplicity at most one. By writing down simultaneous highest (or lowest) weight vectors of a dual pair \((M, K')\) and therefore explicitly describing "the duality correspondence" in the Fock model, we also determine which \(K'\)-types occur. The respective results are summarized in Theorems (5.2.32), (5.2.44) and (5.2.50).

We shall use the same notation as in Chapter 4.

(5.2.1) **Definition**: Let \(\omega\) be the oscillator representation of \(\tilde{Sp} = \tilde{Sp}(2l, \mathbb{R})\), \(\mathcal{P}\) be the space of \(\tilde{U} = \tilde{U}(l)\)-finite vectors of \(\omega\). The space of formal vectors of \(\omega\), denoted by \(\omega^{-\infty}\), is the space of formal linear combinations of a basis of \(\mathcal{P}\).

(5.2.2) **Remark**: In the Fock model, \(\mathcal{P} \cong \mathcal{P}_l\), the space of polynomials in \(l\) variables. Therefore, \(\omega^{-\infty}\) is isomorphic to the space of formal power series in \(l\) variables.

(5.2.3) **Remark**: In the Schrödinger model, the oscillator representation of \(\tilde{Sp} = \tilde{Sp}(W)\) is realized in \(L^2(X)\), where \(X\) is a factor of the complete polarization \(W = X \ominus Y\). \(\omega^{-\infty}\) contains as a proper subspace the space of tempered distributions on \(X\) for the following reason: The normalized Hermite functions \(\{\phi_{k_1, k_2, \ldots, k_l}\}\) correspond to the monomials \(\left\{ \frac{1}{(\sqrt{\pi})^{\frac{l}{2}}(k_1!k_2!\ldots k_l!)^{\frac{1}{2}}} z_1^{k_1} z_2^{k_2} \ldots z_l^{k_l} \right\}\) under the isomorphism in (5.1.3). Since these monomials form a basis of \(\tilde{U}\)-finite vectors in the Fock model, we see that the Hermite functions form a basis of \(\tilde{U}\)-finite vectors in the Schrödinger model. Since they also form an orthonormal basis of \(L^2(X)\) and since \(L^2(X)\) is dense in \(\mathcal{S}^\ast(X)\), we see that tempered distributions are formal vectors. In fact the following is true:

\[\sum \lambda_{k_1, k_2, \ldots, k_l} \phi_{k_1, k_2, \ldots, k_l}\]

is a tempered distribution if and only if \(\lambda_{k_1, k_2, \ldots, k_l}\)'s have at most polynomial growth.
We finally note that everything in this section remains true if we replace $P$ by $\omega^{-\infty}$, since we work in $P$ and all the groups we deal with are compact.

Take a reductive dual pair $(G, G') \subseteq Sp$, $G$ compact.

From §5.1, we know there exists a reductive group $M \supseteq G$ such that

\[(5.2.4) \quad (M, K') \subseteq Sp\]

is a reductive dual pair.

Therefore, if we take $\tau \in \hat{K}'$, an irreducible finite dimensional representation of $\hat{K}'$, the $\tau$-isotypic component of $P$, denoted by $P_\tau$, is of the form:

\[(5.2.5) \quad P_\tau \cong \sigma(\tau) \otimes \tau\]

where $\sigma(\tau) \in \hat{M}$. See Theorem (5.1.13).

We use a superscript $G$ to denote the space of $G$-invariants.

Then it is clear that

\[(5.2.6) \quad P_\tau^G \cong \sigma(\tau)^G \otimes \tau.\]

(5.2.7) Definition:

\[
M(\omega^G_\tau) \overset{\text{def}}{=} \text{Multiplicity of } \tau \text{ in } P^G \overset{\text{def}}{=} \dim \text{Hom}_{\hat{K}'}(\tau, P^G).
\]

(5.2.8) Proposition: $M(\omega^G_\tau) = \dim \sigma(\tau)^G$.

Proof: Obvious from (5.2.6).

We list the see-saw dual pairs

\[
(5.2.9) \quad \begin{array}{c|c}
G & K' \\
\hline
\cap & \cap \\
M & G'
\end{array}
\]

70
for our three relevant cases. See §5.1 or [H2].

\[(5.2.10)\]

\[
\begin{array}{c|c}
O(p) & U(k) \\
\hline
U(p) & Sp(2k, \mathbb{R}) \\
\end{array}
\]

\[(5.2.11)\]

\[
\begin{array}{c|c}
U(p) & U(k) \times U(k) \\
\hline
U(p) \times U(p) & U(k, k) \\
\end{array}
\]

\[(5.2.12)\]

\[
\begin{array}{c|c}
Sp(p) & U(2k) \\
\hline
U(2p) & O^*(4k) \\
\end{array}
\]

Notice that all \(M\)'s are compact.

\[(5.2.13)\] **Definition**: A pair of compact groups \(G_1 \subseteq G_2\) is called a **spherical pair** if the following is satisfied:

For \(\pi \in \hat{G}_2\), the trivial representation of \(G_1\) occurs in \(\pi|_{G_1}\) at most once, i.e.,

\[
\dim \pi^{G_1} \leq 1.
\]

\[(5.2.14)\] **Definition**: A pair of compact groups \(G_1 \subseteq G_2\) is called a **symmetric pair** if \(G_1\) is the group of the fixed points of an involutive automorphism.

We have the following classical result. See [Wa] for its proof.

\[(5.2.15)\] **Theorem** (Gelfand): A symmetric pair is a spherical pair.

We see easily that the following three pairs are all symmetric pairs with \(\theta\) as respective involutive automorphisms.

\[(5.2.16)\]

\[
O(p) \subseteq U(p), \quad \theta(X) = \overline{X}
\]
(5.2.17) \[ U(p) \subseteq U(p) \times U(p), \quad \theta((X,Y)) = (Y,X) \]
(diagonal imbedding)

(5.2.18) \[ Sp(p) \subseteq U(2p), \quad \theta(X) = J_p X J_p^{-1}, \text{ where } J_p = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}. \]

Combining all the above, we have

(5.2.19) **Theorem:** \[ M(\omega^G_\tau) \leq 1, \text{ for any } \tau \in \hat{K}'. \]

In order to describe the set of \( \tau \)'s with \( M(\omega^G_\tau) = 1 \), we shall state below some results from the theory of models of representations for the classical groups or from classical invariant theory. We refer the reader to [BGG], [H1].

As usual, we parametrize the irreducible finite dimensional representations of \( U(p) \) by their highest weights, which are in turn given by a \( p \)-tuple of integers \( a_1, a_2, ..., a_p \) satisfying

(5.2.20) \[ a_1 \geq a_2 \geq ... \geq a_p. \]

(5.2.21) **Proposition** (see [BGG], [H1]): An irreducible finite dimensional representation of \( U(p) \) has an \( O(p) \)-invariant if and only if its highest weight is of the form:

\[ (a_1, a_2, ..., a_p), \quad a_i \in 2\mathbb{Z}, \quad 1 \leq i \leq p. \]

Moreover, the dimension of \( O(p) \) invariants is one.

In other words, the Young diagram of that representation has even number of cells in each row.

Now we consider the spherical pair \( (Sp(p), U(2p)) \).

(5.2.22) **Proposition** (see [BGG], [H1]): An irreducible finite dimensional representation of \( U(2p) \) has a \( Sp(p) \)-invariant if and only if its highest weight is of the form:

\[ (a_1, a_1, a_2, a_2, ..., a_p, a_p). \]

Moreover, the dimension of \( Sp(p) \) invariants is one.
In other words, the Young diagram of that representation has even number of cells in each column.

As for the spherical pair \((U(p), U(p) \times U(p))\), we have

\[(5.2.23) \textbf{Proposition}: \text{ Let } U(p) \hookrightarrow U(p) \times U(p) \text{ be the diagonal imbedding and } \pi_1 \otimes \pi_2 \text{ be an irreducible finite dimensional representation of } U(p) \times U(p). \text{ Then } \pi_1 \otimes \pi_2 \text{ has a } U(p)\text{-invariant if and only if } \pi_2 \text{ is isomorphic to the contragradient of } \pi_1. \text{ Moreover, the dimension of } U(p)\text{ invariants is one.}
\]

\textbf{Proof:} Use } \pi_1 \otimes \pi_2 \cong \text{End}_{U(p)}(\pi_1^*, \pi_2) \text{ and Schur's Lemma.}

\[(5.2.24) \textbf{Remark}: \text{ All the above are also consequences of a general result of Helgason about finite dimensional spherical representations. See } [\text{He2}].
\]

We now describe } \sigma(\tau) \in \hat{M} \text{ for } \tau \in \hat{K}'.

\[(\text{Case 1}) \quad (O(p), Sp(2k)) \subseteq Sp(2l, \mathbb{R}), \quad l = pk.
\]

We have

\[(5.2.25) \quad M = U(p), \quad K' = U(k).
\]

In the Fock model of } \omega, \text{ the } \tilde{U}(l)\text{-finite vectors in } \omega \text{ form a space isomorphic to } P_{p,k}, \text{ the space of polynomials on } \mathbb{C}^{p,k}.

Moreover, if we write the coordinates in terms of a } p \times k \text{ matrix:

\[ Z = \begin{pmatrix}
z_{11} & z_{12} & \cdots & z_{1k} \\
\vdots & \ddots & \ddots & \vdots \\
z_{p1} & z_{p2} & \cdots & z_{pk}
\end{pmatrix},
\]

\text{then } \tilde{U}(p) \times \tilde{U}(k) \text{ acts by:}

\[(5.2.26) \quad (\tilde{A}, \tilde{B}) \circ Z = a^k b^{-p} AZB^{-1},
\]

\text{where } \tilde{A} = (A, a) \in \tilde{U}(p) \text{ with } a^2 = \text{det} A, \quad \tilde{B} = (B, b) \in \tilde{U}(k) \text{ with } b^2 = \text{det} B. \text{ See Remark (5.1.6).}
(5.2.27) **Convention:** From now on, we shall write a formula like (5.2.26) simply as 
\[ (\tilde{A}, \tilde{B}) \circ Z = (\det A)^{\frac{1}{2}} (\det B)^{-\frac{1}{2}} A Z B^{-1}. \]
These formulas should be interpreted similarly as in (5.2.26).

(5.2.28) **Notation:** We denote by \( h^+_k, h^-_k \) the upper and lower triangular Borel subalgebras of \( \mathfrak{gl}(k, \mathbb{C}) \), i.e., the Lie algebras of the upper and lower triangular matrices of order \( k \).

With respect to \( h^-_p \times h^+_k \), the simultaneous highest weight vectors of \( \tilde{U}(k) \) and \( \tilde{U}(p) \) are of the following form:

\[
d_1^{a_1} d_2^{a_2} ... d_t^{a_t},
\]
where
\[
d_i = \det \left( \begin{array}{ccc}
z_{11} & z_{12} & \cdots & z_{1i} \\
z_{21} & z_{22} & \cdots & z_{2i} \\
\vdots & \vdots & \ddots & \vdots \\
z_{ii} & z_{i2} & \cdots & z_{ii}
\end{array} \right),
\]

\( a_1, a_2, ..., a_t \) are non-negative integers, \( t \leq \min(p, k) \).

The corresponding weights are:

(5.2.30) \( \tilde{U}(p) : \left( \frac{k}{2}, \frac{k}{2}, ..., \frac{k}{2} \right) - (a_1 + a_2 + ... + a_t, a_2 + ... + a_t, ..., a_t, 0, 0, ..., 0) \),

(5.2.31) \( \tilde{U}(k) : \left( \frac{p}{2}, \frac{p}{2}, ..., \frac{p}{2} \right) + (a_1 + a_2 + ... + a_t, a_2 + ... + a_t, ..., a_t, 0, 0, ..., 0) \).

Combining Theorem (5.2.19), Proposition (5.2.21), (5.2.30), (5.2.31) and using the fact that \( (\chi \otimes \omega)|_{\tilde{U}(p)} \) factors through the original linear action of \( O(p) \) on \( V^k \) where \( \chi \) is the character of \( \tilde{U}(p) : A \mapsto (\det A)^{-\frac{k}{2}} \) (see [KR]), we obtain

(5.2.32) **Theorem:** Let \( (G, G') = (O(p), Sp(2k, \mathbb{R})) \) be the reductive dual pair in \( Sp(2pk, \mathbb{R}) \), \( \omega \) be the oscillator representation of \( \tilde{Sp}(2pk, \mathbb{R}) \). Then \( M((\omega^\tau)^{G'}) \leq 1 \), for \( \tau \in \tilde{U}(k) \). Moreover, \( M((\omega^\tau)^{G'}) = 1 \) if and only if \( \tau \) has highest weight
\[
(\frac{p}{2}, \frac{p}{2}, ..., \frac{p}{2}) + (c_1, c_2, ..., c_t, 0, 0, ..., 0),
\]

74
$c_1 \geq c_2 \geq ... \geq c_t > 0$, all even integers, $t \leq \min(p, k)$.

(Case 2) \hspace{1cm} (U(p), U(k, k)) \subseteq Sp(2l, \mathbb{R}), \hspace{0.5cm} l = 2pk.

We have

\begin{equation}
M = U(p) \times U(p), \hspace{0.5cm} K' = U(k) \times U(k).
\end{equation}

In the Fock model of $\omega$, the $\bar{U}(l)$-finite vectors in $\omega$ form a space isomorphic to $\mathcal{P}_{p, k, \bar{p}, \bar{k}}$, the space of polynomials on

\begin{equation}
(Q, \bar{Q}) = \begin{pmatrix}
q_{11} & q_{12} & ... & q_{1k} & \bar{q}_{11} & \bar{q}_{12} & ... & \bar{q}_{1k} \\
q_{21} & q_{22} & ... & q_{2k} & \bar{q}_{21} & \bar{q}_{22} & ... & \bar{q}_{2k} \\
... & ... & ... & ... & ... & ... & ... & ... \\
q_{p1} & q_{p2} & ... & q_{pk} & \bar{q}_{p1} & \bar{q}_{p2} & ... & \bar{q}_{pk}
\end{pmatrix}.
\end{equation}

The action of $\bar{U}(p) \times \bar{U}(p) \times \bar{U}(k) \times \bar{U}(k)$ is given by

\begin{equation}
((\bar{A}, \bar{B}), (\bar{C}, \bar{D})) \circ (Q, \bar{Q}) = ((\det A)\frac{\bar{b}}{\bar{a}}(\det C)^{-\frac{\bar{a}}{\bar{b}}}AQC^{-1}, (\det B)^{-\frac{\bar{b}}{\bar{a}}}(\det D)^{\frac{\bar{a}}{\bar{b}}B^{-1}D})
\end{equation}

where $(\bar{A}, \bar{B}) \in \bar{U}(p) \times \bar{U}(p)$, $(\bar{C}, \bar{D}) \in \bar{U}(k) \times \bar{U}(k)$.

With respect to $b^-_p \times b^+_p \times b^-_k \times b^+_k$, the simultaneous highest weight vectors are of the following form:

\begin{equation}
a_1^{a_1}a_2^{a_2}...a_t^{a_t}b_1^{b_1}b_2^{b_2}...b_t^{b_t},
\end{equation}

where

\begin{equation}
d_i = \det \begin{pmatrix}
q_{11} & q_{12} & ... & q_{1i} \\
q_{21} & q_{22} & ... & q_{2i} \\
... & ... & ... & ... \\
q_{i1} & q_{i2} & ... & q_{ii}
\end{pmatrix}, \hspace{0.5cm} \bar{d}_i = \det \begin{pmatrix}
\bar{q}_{11} & \bar{q}_{12} & ... & \bar{q}_{1i} \\
\bar{q}_{21} & \bar{q}_{22} & ... & \bar{q}_{2i} \\
... & ... & ... & ... \\
\bar{q}_{i1} & \bar{q}_{i2} & ... & \bar{q}_{ii}
\end{pmatrix},
\end{equation}

$a_i, b_i$ are non-negative integers, $t \leq \min(p, k)$.

The respective weights are, for $\bar{U}(p) \times \bar{U}(p)$,

\begin{equation}
-(\begin{pmatrix}
k \big/ 2 \\
k \big/ 2 \\
k \big/ 2
\end{pmatrix}) - (a_1 + a_2 + ... + a_t, a_2 + ... + a_t, 0, 0, ..., 0),
\end{equation}

75
\[(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) + (b_1 + b_2 + \ldots + b_t, b_2 + \ldots + b_t, \ldots, b_t, 0, 0, \ldots, 0)].\]

The corresponding representation of \(\hat{U}(p) \times \hat{U}(p)\) is denoted by \(\pi_1 \otimes \pi_2\). For \(U(k) \times U(k)\),

\[(5.2.39) \quad [\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}] + (a_1 + a_2 + \ldots + a_t, a_2 + \ldots + a_t, \ldots, a_t, 0, 0, \ldots, 0),\]
\[-(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}) - (b_1 + b_2 + \ldots + b_t, b_2 + \ldots + b_t, \ldots, b_t, 0, 0, \ldots, 0)].\]

The corresponding representation of \(\hat{U}(k) \times \hat{U}(k)\) is denoted by \(\tau = \tau_1 \otimes \tau_2\).

In this section’s notation, \(\pi_1 \otimes \pi_2 = \sigma(\tau)\).

Let \(\chi = \chi_1 \otimes \chi_2\) be the following character of \(\hat{U}(p) \times \hat{U}(p)\):

\[(5.2.40) \quad \chi : (A, B) \mapsto (det A)^{-\frac{p}{2}} (det B)^{\frac{p}{2}},\]

then \((\chi \otimes \omega)|_{\hat{U}(p) \times \hat{U}(p)}\) factors through the original linear action of \(U(p)\) on \(V^k\).

We know that

\[(5.2.41) \quad M((\omega^4)^{G_\tau}) = 1 \iff (\chi^{-1} \otimes \sigma(\tau))|_{U(p)} \neq 0.\]

This is equivalent to

\(\chi_2^{-1} \otimes \pi_2 \cong (\chi_1^{-1} \otimes \pi_1)^*\) by Theorem (5.2.23).

In turn, it is equivalent to

\[(5.2.42) \quad a_i = b_i, \quad 1 \leq i \leq t.\]

In other words,

\[(5.2.43) \quad \tau_2 \cong \tau_1^*\]

\[(5.2.44) \text{Theorem: Let } (G, G') = (U(p), U(k,k)) \text{ be the reductive dual pair in } Sp(4pk, \mathbb{R}), \omega \text{ be the oscillator representation of } \hat{Sp}(4pk, \mathbb{R}). \text{ Then } M((\omega^4)^{G_\tau}) \leq 1, \text{ for } \tau = \tau_1 \otimes \tau_2 \in \hat{U}(k) \times \hat{U}(k). \text{ Moreover, } M((\omega^4)^{G_\tau}) = 1 \text{ if and only if } \tau_2 \cong \tau_1^*, \text{ and } \tau_1 \text{ has highest weight}\]
\[(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2}) + (c_1, c_2, \ldots, c_t, 0, 0, \ldots, 0),\]

76
\( \geq c_2 \geq \ldots \geq c_t > 0, \ t \leq \min(p,k). \)

(Case 3) \( (Sp(p), O^*(4k)) \subseteq Sp(2l, \mathbb{R}), \ l = 4pk. \)

We have

\[(5.2.45) \quad M = U(2p), \quad K' = U(2k). \]

In this case, the oscillator representation \( \omega \) of \( \hat{Sp}(2l, \mathbb{R}) \) factors through \( Sp(2l, \mathbb{R}) \).

In the Fock model of \( \omega \), the \( U(l) \)-finite vectors in \( \omega \) form a space isomorphic to \( P_{2p,2k} \), the space of polynomials on

\[(5.2.46) \quad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,2k-1} & x_{1,2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{p,1} & x_{p,2} & \ldots & x_{p,2k-1} & x_{p,2k} \\ y_{1,1} & y_{1,2} & \ldots & y_{1,2k-1} & y_{1,2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{p,1} & y_{p,2} & \ldots & y_{p,2k-1} & y_{p,2k} \end{pmatrix}. \]

The action of \( U(2p) \times U(2k) \) is given by

\[(5.2.47) \quad (A, B) \circ \begin{pmatrix} X \\ Y \end{pmatrix} = (\det A)^k (\det B)^{-p} A \begin{pmatrix} X \\ Y \end{pmatrix} B^{-1}, \]

where \( A \in U(2p), \ B \in U(2k). \)

Let \( b_{2p}^- \) be the following Borel subalgebra of \( gl(2p, \mathbb{C}) \):

\[\begin{pmatrix} b_p^- & \beta \\ b_p^- & b_p^- \end{pmatrix}, \ \beta \text{ is strictly lower triangular}.\]

With respect to \( b_{2p}^- \times b_{2k}^+ \), the simultaneous highest weight vectors of the above action are of the following form:

\[(5.2.48) \quad d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_{2l-1}^{\alpha_{2l-1}} d_{2l}^{\alpha_{2l}}, \]

where

\[(5.2.49) \quad d_{2i} = \det \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,2i-1} & x_{1,2i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i,1} & x_{i,2} & \ldots & x_{i,2i-1} & x_{i,2i} \\ y_{1,1} & y_{1,2} & \ldots & y_{1,2i-1} & y_{1,2i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{i,1} & y_{i,2} & \ldots & y_{i,2i-1} & y_{i,2i} \end{pmatrix}, \]

77
\[
d_{2i+1} = \det \begin{pmatrix}
  x_{1,1} & x_{1,2} & \ldots & x_{1,2i-1} & x_{1,2i} & x_{1,2i+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  x_{i,1} & x_{i,2} & \ldots & x_{i,2i-1} & x_{i,2i} & x_{i,2i+1} \\
  x_{i+1,1} & x_{i+1,2} & \ldots & x_{i+1,2i-1} & x_{i+1,2i} & x_{i+1,2i+1} \\
  y_{1,1} & y_{1,2} & \ldots & y_{1,2i-1} & y_{1,2i} & y_{1,2i+1} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  y_{i,1} & y_{i,2} & \ldots & y_{i,2i-1} & y_{i,2i} & y_{i,2i+1}
\end{pmatrix},
\]

\(a_i\)'s are non-negative integers, \(t \leq \min(p, k)\).

Combining Theorem (5.2.19), Proposition (5.2.22) and using an argument analogous to that giving Theorem (5.2.32), we obtain

(5.2.50) **Theorem:** Let \((G, G') = (Sp(p), O^*(4k))\) be the reductive dual pair in \(Sp(8pk, \mathbb{R})\), \(\omega\) be the oscillator representation of \(Sp(8pk, \mathbb{R})\). Then \(M((\omega^4)^G) \leq 1\), for \(\tau \in \hat{U}(2k)\). Moreover, \(M((\omega^4)^G) = 1\) if and only if \(\tau\) has highest weight

\[(p, p, \ldots, p) + (c_1, c_1, c_2, \ldots, c_t, c_t, 0, 0, \ldots, 0),\]

\[c_1 \geq c_2 \geq \ldots \geq c_t > 0, \ t \leq \min(p, k).\]
§5.3 Multiplicity one property: Noncompact cases

In this section, we prove the Reduction Theorem (5.3.4) which implies the $\hat{K}'$-type multiplicity (at most) one property of $G$-invariant formal vectors in general. It also implies, for example, that a holomorphic representation of $U(p, q)$ can have at most one $O(p, q)$ invariant. The possible $\hat{K}'$-types are then determined by using results of [KV] about pluriharmonics (with one exception). The above named results describe the correspondence $\tau \rightarrow \Omega(\tau)$ in Theorem (5.1.13) by writing down simultaneous highest (or lowest) weight vectors in the space of pluriharmonics.

Consider five reductive dual pairs $(G, G') \subseteq Sp = Sp(2l, \mathbb{R})$ as defined in Chapter 4.

We shall work in $\mathcal{P} \cong \mathcal{P}_l$, the space of $\tilde{U} = \tilde{U}(l)$-finite vectors of the oscillator representation $\omega$ of $\tilde{S}p = \tilde{S}p(2l, \mathbb{R})$. We shall also use the standard formalism of $(\mathfrak{g}, \hat{K}')$ modules. Because of the algebraic nature of this formalism, we note that everything in this section remains true if we replace $\mathcal{P}$ by $\omega^{-\infty}$, the space of formal vectors of $\omega$. See the previous section for its definition and a brief discussion of it.

Recall the diamonds of reductive dual pairs (§5.1):

\begin{equation}
\begin{array}{c}
m_0^{(1,1)} \\
\bigcirc \bigcirc \bigcirc
\end{array}
\begin{array}{c}
m_0^{(1,1)}' \\
\bigcirc \bigcirc \bigcirc
\end{array}
\begin{array}{c}
m \\
\bigcirc \bigcirc \bigcirc
\end{array}
\begin{array}{c}
m' \\
\bigcirc \bigcirc \bigcirc
\end{array}
\begin{array}{c}
\mathfrak{g} \\
\bigcirc \bigcirc \bigcirc
\end{array}
\begin{array}{c}
\mathfrak{g}' \\
\bigcirc \bigcirc \bigcirc
\end{array}
\end{equation}

Recall also that $m_0^{(1,1)}$ (resp. $m_0^{(1,1)}'$) is the Lie algebra of the maximal compact subgroup of $M$ (resp. $M'$).

Let $\tau \in R(\hat{K}'; G, \omega)$, the set of the $\hat{K}'$-types occurring in the oscillator representation $\omega$ associated to the dual pair $(G, G') \subseteq Sp$. We have the following isotypic decomposition of $\mathcal{P}$,

\begin{equation}
\mathcal{P} = \sum_{\tau \in R(\hat{K}'; G, \omega)} \mathcal{P}_\tau.
\end{equation}
Let \( v \in \mathcal{P}_\tau \), and

\[
(5.3.3) \quad v = v_N + v_{N+1} + v_{N+2} + \cdots
\]

be the decomposition of \( v \) into homogeneous components (with respect to the notion of degree in \( \mathcal{P} \)). \( v_N \) is of the lowest degree \( N \), \( N \in \mathbb{Z}^+ \), and \( \deg v_{N+i} = N + i \), \( i \geq 0 \).

We denote by \( P : v \mapsto v_N \), for \( v \in \mathcal{P}_\tau \). The image of \( \mathcal{P}_\tau \) under \( P \) is \( \mathcal{H}(K')_\tau \), the \( K' \)-harmonics with the given \( K' \)-type \( \tau \). See Theorem (5.1.13).

Clearly if \( v \) is \((\mathfrak{g},K)\)-invariant, \( v_N \) is \( K \)-invariant.

Let us prove the following "reduction theorem":

(5.3.4) **Theorem:** \( P : \mathcal{P}^{(\mathfrak{g},K)}_\tau \longrightarrow \mathcal{H}(K')_\tau \) is injective for \( \tau \in R(\tilde{K}'; G, \omega) \).

**Proof:** Suppose \( v \in \mathcal{P}^{(\mathfrak{g},K)}_\tau \) and \( v_N = 0 \).

Fix a Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) as in §5.1, then we have

\[
(5.3.5) \quad m^{(2,0)} \oplus m^{(0,2)} = p_C \oplus m^{(0,2)} = m^{(2,0)} \oplus p_C, \quad (\text{by } 5.1.19).
\]

Let \( X \in p_C \). By (5.3.5), we can write (uniquely)

\[
X = L_X + R_X, \quad L_X \in m^{(0,2)}, \quad R_X \in m^{(2,0)}.
\]

It follows again by (5.3.5) that

\[
\{L_X\}_{X \in p_C} \quad \text{(resp. } \{R_X\}_{X \in p_C}\text{)} \text{ spans } m^{(0,2)} \quad \text{(resp. } m^{(2,0)}\text{)}.
\]

Now since \( p_C \cdot v = 0 \) by the invariance assumption, we obtain

\[
(5.3.6) \quad (L_X + R_X)v = 0, \quad \text{i.e.,}
\]

\[
L_Xv_{N+1} + R_Xv_{N+1} + L_Xv_{N+2} + R_Xv_{N+2} + \cdots = 0.
\]

Since \( L_X \) lowers the degree by 2, \( R_X \) raises the degree by 2, (5.3.6) implies

\[
(5.3.7) \quad L_Xv_{N+1} = 0.
\]

Since \( \{L_X\}_{X \in p_C} \) spans \( m^{(0,2)} \), (5.3.7) says that \( v_{N+1} \) is \( K' \)-harmonic, and therefore by Theorem (5.1.13)(b), it is of the lowest degree \( N \) which is impossible unless \( v_{N+1} = 0 \).
Continuing this way, we obtain \( v_{N+i} = 0, \ i \geq 1. \)

So \( v = 0. \) Q.E.D.

Now since \( \mathcal{H}(K')_\tau \) is irreducible as a \( M_0^{(1,1)} \times K' \) module (Theorem 5.1.13), we can write

\[
H(K')_\tau \cong \Omega(\tau) \otimes \tau
\]

(5.3.8)

where \( \Omega(\tau) \in \hat{M}_0^{(1,1)}. \) By looking at the list of diamond dual pairs in §5.1, we see that the pairs \( K \subseteq M_0^{(1,1)} \) are compact spherical pairs, all of which are product of compact spherical pairs that we considered in §5.2. So the multiplicity of \( \tau \) in \( \mathcal{H}(K')^K \) is at most one.

Combining the above with our reduction theorem, we have

(5.3.9) **Proposition:** The multiplicity of \( \tau \) in \( \mathcal{P}^{(\mathfrak{g},K)} \) is at most one, here \( \tau \) is any irreducible representation of \( \hat{K'} \), a maximal compact subgroup of \( \check{G}' \).

In turn, this implies (notations are as in Chapter 4)

(5.3.10) **Theorem:** The multiplicity of \( \tau \) in \( \mathcal{S}(V^k)^G \) is at most one for any irreducible representation \( \tau \) of \( \hat{K}' \).

By Theorem (5.1.13), the \( \tau \)-isotypic component, \( \mathcal{P}_\tau \), is of the form

\[
\mathcal{P}_\tau = \sigma(\tau) \otimes \tau,
\]

where \( \sigma(\tau) \in \hat{M}. \) Thus \( \sigma(\tau)^{\mathfrak{g},K} \) is at most one dimensional.

For example, if we take \( G = O(p,q) \), then \( M = U(p,q). \) Since any holomorphic representation of \( U(p,q) \) can be realized as \( \sigma(\tau) \) with some choices of \( k \) and \( \tau \) (see [EHW]), our reduction theorem implies the following:

(5.3.11) **Corollary:** A holomorphic representation of \( U(p,q) \) has an \( O(p,q) \)-invariant only if its lowest \( U(p) \times U(q) \)-type has an \( O(p) \times O(q) \)-invariant, and therefore only if its lowest highest weight is "even".

In order to determine the possible \( \hat{K}' \)-types \( \tau \) such that \( \mathcal{S}(V^k)^G_\tau \neq 0 \), we need to first determine \( \mathcal{H}(K')_\tau. \)
Let us observe the following:

Since $m^{(2,0)} = sp^{(2,0)} \cap m_C$ by definition and since $(m_C, t')$ form a dual pair, we see that $m^{(2,0)}$ consists precisely of $t'$-invariant differential operators of second order in $\mathcal{P}$. Therefore, $\Omega = \mathcal{H}(K')$ is the space of polynomials in $\mathcal{P}$ annihilated by all the $t'$-invariant differential operators of second order. In the analysis literature, $\Omega$ is usually referred to as the space of pluriharmonics (c.f. 5.3.12 below).

Let us quote the explicit results of [KV] (I-III below).

1) Let $\Omega$ be the space of polynomials $f$ on $M_{p,k}(\mathbb{C}) \times M_{q,k}(\mathbb{C})$ such that

\begin{equation}
(\Delta_{i,j}f)(x, y) = 0, \quad 1 \leq i \leq p, 1 \leq j \leq q
\end{equation}

where $\Delta_{i,j} = \sum_{p=1}^{k} \frac{\partial^2}{\partial x_{ip} \partial y_{jp}}, \quad x = (x_{ij}) \in M_{p,k}, \quad y = (y_{ij}) \in M_{q,k}$.

$\Omega$ is usually referred to as the space of pluriharmonics.

Let $Gl(p, \mathbb{C}) \times Gl(q, \mathbb{C}) \times Gl(k, \mathbb{C})$ act on $M_{p,k}(\mathbb{C}) \times M_{q,k}(\mathbb{C})$ by

\begin{equation}
(g_1, g_2, c) \cdot (x, y) = (g_1 x c^{-1}, g_2 y c^t)
\end{equation}

The system (5.3.12) is invariant under this action, so $\Omega$ is a representation of $Gl(p, \mathbb{C}) \times Gl(q, \mathbb{C}) \times Gl(k, \mathbb{C})$.

Let

\begin{equation}
\Omega = \bigoplus_{\tau \in \Sigma \subseteq Gl(k, \mathbb{C})} \Omega(\tau) \otimes \tau
\end{equation}

be the isotypic decomposition of the above action.

In the following, we shall parametrize the irreducible representations of general linear groups by their highest weights with respect to the upper triangular Borel subalgebras.

(5.3.15) **Theorem** ([KV]):

a) $\Sigma = \{\tau = (a_1, a_2, \ldots, a_t, 0, \ldots, 0, -b_s, -b_{s-1}, \ldots, -b_1)\}$

\[ a_1 \geq a_2 \geq \ldots \geq a_t > 0, \quad b_1 \geq b_2 \geq \ldots \geq b_s > 0, \quad t \leq \min(k, p), \quad s \leq \min(k, q). \]
b) If \( \tau \in \Sigma \),
\[
\tau = (a_1, a_2, \ldots, a_t, 0, \ldots, 0, -b_s, -b_{s-1}, \ldots, -b_1),
\]
the representation \( \Omega(\tau) \) is
\[
(0, \ldots, 0, -a_t, -a_{t-1}, \ldots, -a_1) \otimes (b_1, b_2, \ldots, b_s, \ldots, 0, \ldots, 0).
\]

II) Let
\[
O(2r + 1, \mathbb{C}) = \{ g \in GL(2r + 1, \mathbb{C}) | g^t J g = J \}
\]
where \( J = \begin{pmatrix} 0 & I_r & 0 \\ I_r & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

Let \( \Omega \) be the space of polynomials \( f \) on \( M_{m,2r+1}(\mathbb{C}) \) such that
\[
(\Delta_{i,j} f)(x, y, t) = 0, \quad 1 \leq i \leq m, 1 \leq j \leq m
\]
where
\[
\Delta_{i,j} = \sum_{\nu=1}^{r} \left( \frac{\partial^2}{\partial x_{i\nu} \partial y_{j\nu}} + \frac{\partial^2}{\partial x_{j\nu} \partial y_{i\nu}} \right) + \frac{\partial^2}{\partial t_i \partial t_j},
\]
x = (x_{ij}), y = (y_{ij}) \in M_{m,r}(\mathbb{C}), \ t = (t_i) \in M_{m,1}(\mathbb{C}).

\( GL(m, \mathbb{C}) \times O(2r + 1, \mathbb{C}) \) acts on \( M_{m,2r+1}(\mathbb{C}) \) by
\[
(g, c) \cdot (x, y, t) = g(x, y, t) c^{-1}.
\]
The system (5.3.16) is invariant under this action, so \( \Omega \) is a representation of \( GL(m, \mathbb{C}) \times O(2r + 1, \mathbb{C}) \).

Let
\[
\Omega = \bigoplus_{\tau \in \Sigma \subseteq O(2r+1, \mathbb{C})} \Omega(\tau) \otimes \tau
\]
be the isotypic decomposition of the above action.

Let \( b^+(\mathfrak{so}(2r + 1)) \) be the following Borel subalgebra of \( \mathfrak{so}(2r + 1) \):
\[
\begin{pmatrix}
\alpha & \beta & \delta \\
0 & -\alpha^t & 0 \\
0 & \delta^t & 0
\end{pmatrix}, \ \ \text{\( \alpha \) upper triangular, \( \beta \) skew symmetric.}
\]
Since $O(2r + 1, \mathbb{C}) \cong SO(2r + 1) \times \{\pm 1\}$, the irreducible representations of $O(2r + 1, \mathbb{C})$ are parametrized by the highest weights with respect to $\mathfrak{h}^+(\mathfrak{so}(2r + 1))$ and a number $\epsilon = 1$ or $-1$.

(5.3.19) **Theorem** ([KV]): The set $\Sigma$ and the correspondence $\tau \to \Omega(\tau)$ are described by the following:

If \( \tau = (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon) \), \( \epsilon = (-1)^{a_1+a_2+\cdots+a_t} \),

\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ t \leq \min(m, r),
\]

then

\[
\Omega(\tau) = (0, \ldots, 0, -a_t, -a_{t-1}, \ldots, -a_1).
\]

If \( \tau = (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon) \), \( \epsilon = (-1)^{a_1+a_2+\cdots+a_t+1} \),

\[
a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ 2r + 1 - m \leq t \leq \min(m, r),
\]

then

\[
\Omega(\tau) = (0, \ldots, 0, -1, \ldots, -1, -a_t, -a_{t-1}, \ldots, -a_1).
\]

III) Let

\[
O(2r, \mathbb{C}) = \{ g \in \text{GL}(2r, \mathbb{C}) | g^t J g = J \}
\]

where \( J = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \).

Let $\Omega$ be the space of polynomials $f$ on $M_{m,2r+1}(\mathbb{C})$ such that

(5.3.20) \( (\Delta_{i,j} f)(x, y) = 0, \ 1 \leq i \leq m, 1 \leq j \leq m \)

where

\[
\Delta_{i,j} = \sum_{\nu=1}^{r} \left( \frac{\partial^2}{\partial x_{ij} \partial y_{j\nu}} + \frac{\partial^2}{\partial x_{j\nu} \partial y_{i\nu}} \right), \ x = (x_{ij}), \ y = (y_{ij}) \in M_{m,r}(\mathbb{C}).
\]

$\text{GL}(m, \mathbb{C}) \times O(2r, \mathbb{C})$ acts on $M_{m,2r}(\mathbb{C})$ by

(5.3.21) \( (g, c) \cdot (x, y) = g(x, y)c^{-1} \).
The system (5.3.20) is invariant under this action, so \( \Omega \) is a representation of \( \text{Gl}(m, \mathbb{C}) \times O(2r, \mathbb{C}) \).

Let

\[
\Omega = \bigoplus_{\tau \in \Sigma \subseteq O(2r, \mathbb{C})} \Omega(\tau) \otimes \tau
\]

be the isotypic decomposition of the above action.

We have \( O(2r) \cong SO(2r) \times \{ \pm 1 \} \).

The irreducible representations of \( SO(2r, \mathbb{C}) \) are parametrized by the highest weights \((\lambda_1, \lambda_2, ..., \lambda_r)\) with respect to \( \mathfrak{b}^+ (\mathfrak{so}(2r))\), \( \lambda_1 \geq \lambda_2 \geq ... \geq |\lambda_r| > 0 \). If \( \lambda_r \neq 0 \), we obtain an irreducible representation of \( O(2r, \mathbb{C}) \) by inducing \((\lambda_1, \lambda_2, ..., \lambda_r)_+ \). This induced representation will be denoted by \((\lambda_1, \lambda_2, ..., \lambda_r)_+ \). If \( \lambda_r = 0 \), then \((\lambda_1, \lambda_2, ..., \lambda_r)\) can be extended to \( O(2r, \mathbb{C}) \), and we denote by \((\lambda_1, \lambda_2, ..., \lambda_r)_\pm \) the two possible extensions.

(5.3.23) Theorem ([KV]): The set \( \Sigma \) and the correspondence \( \tau \rightarrow \Omega(\tau) \) are described by the following:

if \( \tau = (a_1, a_2, ..., a_t, 0, ..., 0)_+ \),

\[ a_1 \geq a_2 \geq ... \geq a_t > 0, \ t \leq \min(m, r), \]

then

\[ \Omega(\tau) = (0, ..., 0, -a_t, -a_{t-1}, ..., -a_1). \]

if \( \tau = (a_1, a_2, ..., a_t, 0, ..., 0)_- \),

\[ a_1 \geq a_2 \geq ... \geq a_t > 0, \ 2r - m \leq t \leq \min(m, r), \]

then

\[ \Omega(\tau) = (0, ..., 0, \underbrace{-1, ..., -1, -a_t, -a_{t-1}, ..., -a_1}_{2r-t}). \]

One other description of pluriharmonics which is not explicitly given in [KV] is the following:
III) Let

$$Sp(2k,\mathbb{C}) = \{ g \in Gl(2k,\mathbb{C}) | g^T J g = J \}$$

where $J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$.

Let $\Omega$ be the space of polynomials $f$ on $M_{m,2k}(\mathbb{C})$ such that

$$\Delta_{i,j} f(x,y) = 0, \quad 1 \leq i < j \leq m$$

where

$$\Delta_{i,j} = \sum_{\nu=1}^{k} \left( \frac{\partial^2}{\partial x_{i\nu} \partial y_{j\nu}} - \frac{\partial^2}{\partial x_{j\nu} \partial y_{i\nu}} \right), \quad x = (x_{ij}), \quad y = (y_{ij}) \in M_{m,k}(\mathbb{C}).$$

$Gl(m,\mathbb{C}) \times Sp(2k,\mathbb{C})$ acts on $M_{m,2k}(\mathbb{C})$ by

$$(g,c) \cdot (x,y) = g(x,y)c^{-1}.$$  

(5.3.25)

The system (5.3.24) is invariant under this action, so $\Omega$ is a representation of $Gl(m,\mathbb{C}) \times Sp(2k,\mathbb{C})$.

Let

$$\Omega = \bigoplus_{\tau \in \Sigma \subseteq Sp(2k,\mathbb{C})} \Omega(\tau) \otimes \tau$$

be the isotypic decomposition of the above action.

Let $b^{+}(sp(2k))$ be the following Borel subalgebra of $sp(2k,\mathbb{C})$:

$$\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha^t \end{pmatrix}, \quad \alpha \text{ upper triangular, } \beta \text{ symmetric.}$$

We parametrize the irreducible representations of $Sp(2k,\mathbb{C})$ by their highest weights with respect to $b^{+}(sp(2k))$, namely, we write $\tau = (m_1,m_2,...,m_k)$ with $m_1 \geq m_2 \geq ... \geq m_k \geq 0$, if the highest weight of $\tau$ is $\sum m_i t_i$, where $t_i$'s are the coordinates of the following Cartan subalgebra:

$$\mathfrak{h}(sp(2k)) = \begin{pmatrix} t_1 \\ \vdots \\ t_k \\ -t_1 \\ \vdots \\ -t_k \end{pmatrix}.$$
Following [KV], we prove

\((5.3.27)\) Theorem ([KV]): The set \(\Sigma\) and the correspondence \(\tau \rightarrow \Omega(\tau)\) are described by the following:

If \(\tau = (a_1, a_2, ..., a_t, 0, ..., 0),\)

\[a_1 \geq a_2 \geq ... \geq a_t > 0, \ t \leq \min(m, k),\]

then

\[\Omega(\tau) = (0, ..., 0, -a_t, -a_{t-1}, ..., -a_1).\]

Actually, we only need to repeat the proofs as for other cases in [KV]. For the sake of completeness, we give such a proof.

Proof: For the moment, we take \(b^-_m \times b^+(\mathfrak{sp}(2k))\) for our Borel subalgebra of \(Gl(m, \mathbb{C}) \times Sp(2k, \mathbb{C}).\)

Obviously, the function of the following form is pluriharmonic and is a highest weight vector with respect to \(b^-_m \times b^+(\mathfrak{sp}(2k)),\)

\((5.3.28)\)

\[\Delta_1(x)^{\alpha_1} \Delta_2(x)^{\alpha_2} ... \Delta_j(x)^{\alpha_j}, \ 0 \leq j \leq m, k,\]

where

\[\Delta_j(x) = \det \begin{pmatrix} x_{11} & ... & x_{1j} \\ ... & ... & ... \\ x_{j1} & ... & x_{jj} \end{pmatrix}.\]

We claim that all the highest weight vectors in \(\Omega\) are of this form.

First, let us assume \(k \leq m\) and let \(\tau\) be any irreducible representation of \(Sp(2k, \mathbb{C})\) parametrized by its highest weight with respect to \(b^+(\mathfrak{sp}(2k)),\)

\[\tau = (m_1, m_2, ..., m_t, 0, ..., 0), \ m_1 \geq m_2 \geq ... \geq m_t > 0, \ t \leq k,\]

then the function

\[\Delta_1(x)^{m_1-m_2} \ldots \Delta_{t-1}(x)^{m_{t-1}-m_t} \Delta_t(x)^{m_t}\]

is a simultaneous highest weight vector with the \(Sp(2k, \mathbb{C})\)-type \(\tau.\)
Now the point is:

Since \((Gl(m, \mathbb{C}), Sp(2k, \mathbb{C}))\) form a dual pair in \(\Omega\) (see Theorem 5.1.13), we see that such a simultaneous highest weight vector is unique up to a scalar.

Therefore, our assertion is established in this case and \(\Sigma = \hat{Sp}(2k, \mathbb{C})\).

If \(m, k\) are arbitrary, we take \(N \geq m, k\).

For any \(f \in \Omega\), we define

\[
\hat{f}(\hat{x}, \hat{y}) = f(x, y), \quad \hat{x} = \begin{pmatrix} x \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} y \end{pmatrix} \in M_{N,k}(\mathbb{C}).
\]

If \(f\) is pluriharmonic and is a highest weight vector of \(Gl(m, \mathbb{C}) \times Sp(2k, \mathbb{C})\), then \(\hat{f}\) is pluriharmonic and is a highest weight vector of \(Gl(N, \mathbb{C}) \times Sp(2k, \mathbb{C})\). Hence, \(\hat{f}\) is of the form (5.3.28), which implies that so is \(f\).

If we parametrize an irreducible representation of \(Gl(2m, \mathbb{C})\) by its highest weight with respect to \(b^+_m\), the result then follows.

Combining Reduction Theorem (5.3.4), the explicit descriptions of pluriharmonics (I–III) and Propositions (5.2.21), (5.2.22), (5.2.23) about spherical pairs \(O(p) \subseteq U(p), U(p) \subseteq U(p) \times U(p), Sp(p) \subseteq U(2p)\), we conclude (see §5.2 for a parallel proof in compact cases)

(5.3.29) **Multiplicity One Theorem:** For any \(\tau \in \hat{K}'\), the multiplicity of \(\tau\) in \(S^*(V^k)^G\) is at most one. Moreover, \(S^*(V^k)^G \neq 0\) only if \(\Omega(\tau)^K \neq 0\), where \(\Omega(\tau) \in \hat{M}_0^{(1,1)}\) and \(H(K')\tau \cong \Omega(\tau) \otimes \tau\). Therefore, \(\tau\) has to satisfy the following conditions:

1. \(G = O(p, q), G' = Sp(2k, \mathbb{R}), \tau \in \hat{U}(k),\) the highest weight of \(\tau\) is

\[
\left( \frac{p-q}{2}, \frac{p-q}{2}, ..., \frac{p-q}{2} \right) + (a_1, a_2, ..., a_t, 0, ..., 0, -b_s, ..., -b_2, -b_1),
\]

\[a_1 \geq a_2 \geq ... \geq a_t > 0, \quad b_1 \geq b_2 \geq ... \geq b_s > 0, \quad t \leq \min(k, p), \quad s \leq \min(k, q),
\]

\(a_i, b_j\) are all even integers.

2. \(G = U(p, q), G' = U(k, k), \tau = \tau_1 \otimes \tau_2, \tau_1, \tau_2 \in \hat{U}(k), \tau_2 \cong \tau_1^*,\) and the highest weight of \(\tau_1\) is

\[
\left( \frac{p-q}{2}, \frac{p-q}{2}, ..., \frac{p-q}{2} \right) + (a_1, a_2, ..., a_t, 0, ..., 0, -b_s, ..., -b_2, -b_1),
\]

88
\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ b_1 \geq b_2 \geq \ldots \geq b_s > 0, \ t \leq \min(k, p), \ s \leq \min(k, q).\)

(3) \(G = Sp(p, q), \ G' = O^*(4k), \ \tau \in \hat{U}(2k),\) the highest weight of \(\tau\) is

\((p-q, p-q, \ldots, p-q) + (a_1, a_1, a_2, a_2, \ldots, a_t, a_t, 0, \ldots, 0, -b_s, -b_s, \ldots, -b_2, -b_2, -b_1, -b_1),\)

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ b_1 \geq b_2 \geq \ldots \geq b_s > 0, \ t \leq \min(k, p), \ s \leq \min(k, q).\)

(4) \(G = Sp(2m, \mathbb{R}), \ G' = O(k, k), \ \tau = \tau_1 \otimes \tau_2, \ \tau_1, \tau_2 \in \hat{O}(k), \ \tau_2 \cong \tau_1^*,\) and the highest weight of \(\tau_1\) is:

If \(k\) is odd, \(k = 2r + 1,\)

\[
\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon), \quad \epsilon = (-1)^{a_1 + a_2 + \ldots + a_t},
\]

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ t \leq \min(m, r),\)

or

\[
\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0; \epsilon), \quad \epsilon = (-1)^{a_1 + a_2 + \ldots + a_t + 1},
\]

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ 2r + 1 - m \leq t \leq \min(m, r).\)

If \(k\) is even, \(k = 2r,\)

\[
\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),
\]

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ t \leq \min(m, r),\)

or

\[
\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),
\]

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ 2r - m \leq t \leq \min(m, r).\)

(5) \(G = O^*(2m), \ G' = Sp(k, k), \ \tau = \tau_1 \otimes \tau_2, \ \tau_1, \tau_2 \in \hat{Sp}(k), \ \tau_2 \cong \tau_1^*,\) and the highest weight of \(\tau_1\) is

\[
\left(\frac{m}{2}, \frac{m}{2}, \ldots, \frac{m}{2}\right) + (a_1, a_2, \ldots, a_t, 0, \ldots, 0),
\]

\(a_1 \geq a_2 \geq \ldots \geq a_t > 0, \ t \leq \min(m, k).\)
CHAPTER 6: EXISTENCE OF INVARIANT DISTRIBUTIONS

§6.1 Compact cases: Inner product formulas

In this section, we shall exhibit $G$-invariant distributions with appropriate $\tilde{K}'$-types by showing that the projections of the Dirac distribution onto those $\tilde{K}'$-types are nonzero. In turn, this is accomplished by explicitly computing the inner product of the Dirac distribution with the highest weight vectors of those $\tilde{K}'$-types. During the process, we invoke the Capelli-identity in classical invariant theory ([W1]). Everything is done within the Fock model. The explicit description of these Fock models is given in §5.2, from (5.2.25) on.

(Case 1) \( (O(p), Sp(2k, \mathbb{R})) \subseteq Sp(2l, \mathbb{R}), l = pk. \)

Since the isomorphism of the Schrödinger model with the Fock model is such that

\[
\begin{align*}
   x_{ij} & \to \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z_{ij}} + z_{ij} \right), \\
   \frac{\partial}{\partial x_{ij}} & \to \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial z_{ij}} - z_{ij} \right),
\end{align*}
\]

(6.1.1)

where \((x_{ij})_{1 \leq i \leq p, 1 \leq j \leq k}\) are the coordinates of \(V^k = \mathbb{R}^{p,k}\), and since the Dirac distribution at the origin of \(V^k\) satisfies

\[ x_{ij} \delta = 0, \]

it must have the form

\[
\delta = e^{-\frac{x^2}{4}} \overset{\text{def.}}{=} e^{-\frac{\sum_{i,j} s_{ij}^2}{2}} \quad \text{(up to a scalar)}
\]

(6.1.2)
in the Fock model.

(6.1.3) **Definition:** Given a matrix \( \{A_{ij}, 1 \leq i, j \leq n\} \) of noncommuting variables, then

\[
det(A_{ij}) \overset{\text{def.}}{=} \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1)1} A_{\sigma(2)2} \ldots A_{\sigma(n)n}.
\]

Let

\[
\partial_n = \det \left( \begin{array}{ccc}
\frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{12}} & \frac{\partial}{\partial z_{1n}} \\
\frac{\partial}{\partial z_{21}} & \frac{\partial}{\partial z_{22}} & \frac{\partial}{\partial z_{2n}} \\
\frac{\partial}{\partial z_{n1}} & \frac{\partial}{\partial z_{n2}} & \frac{\partial}{\partial z_{nn}}
\end{array} \right),
\]

then

\[
\partial_n \delta = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \frac{\partial}{\partial z_{\sigma(1)1}} \frac{\partial}{\partial z_{\sigma(2)2}} \ldots \frac{\partial}{\partial z_{\sigma(n)n}} \delta
\]

\[
= (-1)^n \sum_{\sigma \in S_n} \text{sgn}(\sigma) z_{\sigma(1)1} z_{\sigma(2)2} \ldots z_{\sigma(n)n} \delta = (-1)^n d_n \delta.
\]

Let us now recall the Capelli-identity ([W1] [H1]). In our context, it asserts

(6.1.6) \[
d_t \partial_t = \det(E_{ij}^t + \delta_{ij}(t-i))_{1 \leq i \leq t, 1 \leq j \leq t}
\]

\[
= \det \left( \begin{array}{ccc}
E_{11}^t + (t - 1) & E_{12}^t & \ldots & E_{1t}^t \\
E_{21}^t & E_{22}^t + (t - 2) & \ldots & E_{2t}^t \\
\vdots & \vdots & \ddots & \vdots \\
E_{t1}^t & E_{t2}^t & \ldots & E_{tt}^t
\end{array} \right),
\]

where

\[
E_{ij}^t = \sum_{\mu=1}^{t} z_{\mu i} \frac{\partial}{\partial z_{\mu j}}.
\]

(6.1.7) **Lemma:** \( E_{ij}^t(d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t}) = 0, \quad 1 \leq i < j \leq t. \)

Proof: \( Gl(t, \mathbb{C}) \) acts on \( Z_t = \left( \begin{array}{c}
\cdots \\
\cdots \\
Z_{t1} \\
\cdots \\
Z_{tt}
\end{array} \right) \) by right translation with the derived action of \( gl(t, \mathbb{C}) \) given by

\[
\epsilon_{ij}^t \rightarrow E_{ij}^t = \sum_{\mu=1}^{t} z_{\mu i} \frac{\partial}{\partial z_{\mu j}},
\]

where \( \epsilon_{ij}^t \) is the \( t \times t \) matrix with one at the \((i, j)\) entry and zero's elsewhere.

\( d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t} \) is obviously invariant under the upper triangular matrices with 1's in the diagonal, therefore the lemma follows.
Applying the Capelli-identity to $d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t$, we get

\[
d_t \partial_t (d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t)
= det(E_{ij}^t + \delta_{ij}(t-i)) d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t \quad \text{(by 6.1.6)}
= \Pi_{i=1}^t (E_{ii}^t + (t-i)) d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t \quad \text{(by 6.1.7)}
= B(a_1, a_2, ..., a_t) d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t
\]

where

\[(6.1.8) \quad B(a_1, a_2, ..., a_t) = \prod_{i=1}^t \{ \sum_{s \geq i} [a_s + (t-i)] \}.
\]

In other words,

\[(6.1.9) \quad \partial_t (d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t) = B(a_1, a_2, ..., a_t) d^{a_1}_1 d^{a_2}_2 ... d^{a_t-1}_t.
\]

Since in the Fock model, we have \((\frac{\partial}{\partial z_{ij}})^* = z_{ij}\) (see Remark 5.1.2). So,

\[(6.1.10) \quad d^*_n = \partial_n.
\]

We compute the following inner product:

\[
(\delta, d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t)
= (\delta, d_t \cdot d^{a_1}_1 d^{a_2}_2 ... d^{a_t-1}_t)
= (\partial_t \delta, d^{a_1}_1 d^{a_2}_2 ... d^{a_t-1}_t) \quad \text{(by 6.1.10)}
= (-1)^t (d_t \delta, d^{a_1}_1 d^{a_2}_2 ... d^{a_t-1}_t) \quad \text{(by 6.1.5)}
= (-1)^t (\delta, \partial_t (d^{a_1}_1 d^{a_2}_2 ... d^{a_t-1}_t)) \quad \text{(by 6.1.10)}
= (-1)^t B(a_1, a_2, ..., a_t - 1)(\delta, d^{a_1}_1 d^{a_2}_2 ... d^{a_t-2}_t). \quad \text{(by 6.1.9)}
\]

Therefore by the above induction formula, we have

\[(6.1.11) \textbf{Proposition:} \quad (\delta, d^{a_1}_1 d^{a_2}_2 ... d^{a_t}_t) \neq 0\]
if and only if $a_i \in 2\mathbb{Z}^+$, $1 \leq i \leq t$. Moreover,

$$(6.1.12) \quad (\delta, a_1^{a_1} a_2^{a_2} \cdots a_t^{a_t}) = \prod_{1 \leq i \leq t} \prod_{1 \leq c_i \leq a_i - 1} (-1)^i B(a_1, a_2, \ldots, a_{i-1}, c_i)(\delta, 1)$$

if $a_1, a_2, \ldots, a_t$ are all even.

Two examples:

We shall look at two special situations where we can find explicit projections of the Dirac distribution onto various $K'$-types.

**Example 1:** $(O(p), SL_2(\mathbb{R})) \subseteq Sp(2p, \mathbb{R})$.

We have

$$\delta = \exp \left( -\sum_{j=1}^{p} \frac{z_j^2}{2} \right)$$

where $\{z_i\}_{1 \leq i \leq p}$ are the complex coordinates under the Fock model.

One easily decomposes $\delta$ into $U(p) \times U(1)$ constituents, this is exactly the “Taylor expansion” of $\delta$.

$$\delta = \sum_{t=0}^{\infty} \frac{(-\sum_{j=1}^{p} z_j^2)^t}{2^t \cdot t!}.$$ 

$\frac{(-\sum_{j=1}^{p} z_j^2)^t}{2^t \cdot t!}$ is the projection of the Dirac distribution $\delta$ to the $U(p) \times U(1)$ type with the simultaneous highest weight vector $z_1^{2t}$. Its $U(1)$-weight is $2t$. These are the only $U(1)$-weights which have nontrivial $O(p)$-invariant distributions on $V = \mathbb{R}^p$.

In this case, we compute the inner product (see 5.1.1) in a straightforward way:

$$(\delta, z_1^{2t}) = \frac{(-\sum_{j=1}^{p} z_j^2)^t}{2^t \cdot t!} \bigg|_{z_1^{2t}} = \frac{(-1)^t (z_1^{2t}, z_1^{2t})}{2^t \cdot t!}$$

$$= \frac{(-1)^t (\sqrt{\pi})^p (2t)!}{2^t \cdot t!} = (-1)^t (\sqrt{\pi})^p 1 \cdot 3 \cdots (2t - 1)$$

(c.f. Formula 6.1.12). The $B$-function in this case is of course $B(a_t) = a_t$, since

$$\frac{\partial}{\partial z_1} z_1^{a_t} = a_t z_1^{a_t-1}.$$ 

In the Schrödinger picture, $sl_2(\mathbb{R})$ is generated by

$$\epsilon^+ = \frac{i}{2} \sum_{j=1}^{p} x_j^2, \quad \epsilon^- = \frac{i}{2} \sum_{j=1}^{p} \frac{\partial^2}{\partial x_j^2}, \quad \hbar = \sum_{j=1}^{p} (x_j \frac{\partial}{\partial x_j} + \frac{1}{2}),$$

93
with the usual commutation relation:

\[ [\mathfrak{h}, \epsilon^+] = 2\epsilon^+, \quad [\mathfrak{h}, \epsilon^-] = -2\epsilon^-, \quad [\epsilon^+, \epsilon^-] = \mathfrak{h}. \]

\( \mathfrak{sl}_2(\mathbb{R}) \) has another basis \( \mathfrak{e}, n^+, n^- \) with

\[ \mathfrak{e} = \epsilon^+ - \epsilon^-, \quad n^+ = \frac{1}{2}(\mathfrak{h} + i\epsilon^+ + i\epsilon^-), \quad n^- = \frac{1}{2}(\mathfrak{h} - i\epsilon^+ - i\epsilon^-). \]

In particular, \( n^+ = \frac{1}{2} \sum_{j=1}^{p} (x_j \frac{\partial}{\partial x_j} + \frac{1}{2} - \frac{1}{2} x_j^2 - \frac{1}{2} \frac{\partial^2}{\partial x_j^2}) = -\frac{1}{4} \sum_{j=1}^{p} (x_j - \frac{\partial}{\partial x_j})^2. \)

Going to the Fock picture,

\[ x_j \rightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} + x_j \right) \]

\[ \frac{\partial}{\partial x_j} \rightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} - x_j \right), \]

so that \( x_j - \frac{\partial}{\partial x_j} \rightarrow \sqrt{2} x_j, \) i.e., \( n^+ \rightarrow -\sum_{j=1}^{p} x_j^2. \)

Let \( v_0 = 1, \) \( v_t = (n^+)^t v_0, \) \( t \geq 1. \)

We have

\[ \delta = \sum_{t=0}^{\infty} \frac{v_t}{t!}. \]

Similarly, using the characterization of the Lebesgue measure \( L \) on \( \mathbb{R}^p, \) \( \frac{\partial}{\partial x_j} L = 0, \)

we have

\[ L = \frac{\sum_{j=1}^{p} x_j^2}{2} = \sum_{t=0}^{\infty} (-1)^t \frac{(v_t)}{t!}. \]

Howe [H6] obtained the above two formulas in a different way.

**Example 2.** An explicit formula of invariant distributions: \((O(2), Sp(4, \mathbb{R})) \subset Sp(8, \mathbb{R})\) case.

The above means that we are trying to determine explicitly the \( O_2 \)-invariant distributions on \( V \oplus V, \) where \( V = \mathbb{R}^2. \)

We have

\[ \delta = exp(\sum_{1 \leq i, j \leq 2} \frac{z_{ij}^2}{2}) = \sum_{t=0}^{\infty} \frac{\left( \sum_{1 \leq i, j \leq 2} \frac{z_{ij}^2}{2} \right)^t}{t!}. \]
\(U(2) \times U(2)\) acts on \(Z = \left( \begin{array}{cc} z_{11} & z_{12} \\ z_{21} & z_{22} \end{array} \right)\) by left and right multiplication, and we want to decompose \((\frac{r^2}{2})^t / t! = (\sum_{i \leq \frac{t}{2}} z_{ii}^2)^t / t!\) into \(U(2) \times U(2)\) constituents.

Let

\[
\left( \frac{r^2}{2} \right)^t / t! = f_0^t \det^t + f_1^t \det^{t-1} + \ldots + f_i^t \det^{t-i} + \ldots + f_{t-1}^t \det + f_t^t
\]

where \(\det = \det \left( \begin{array}{cc} z_{11} & z_{12} \\ z_{21} & z_{22} \end{array} \right)\), \(f_i^t\) belongs to the \(U(2) \times U(2)\) irreducible constituent with simultaneous highest weight \(z_{11}^{2i}\), i.e.,

\[
\deg f_i^t = 2i, \quad \partial_{\det}(f_i^t) = 0,
\]

with \(\partial_{\det} = \det \left( \begin{array}{ccc} \frac{\partial}{\partial z_{11}} & \frac{\partial}{\partial z_{12}} & 0 \\ \frac{\partial}{\partial z_{21}} & \frac{\partial}{\partial z_{22}} & 0 \\ 0 & 0 & \frac{\partial}{\partial \det} \end{array} \right) = \frac{\partial}{\partial z_{11}} \frac{\partial}{\partial z_{22}} - \frac{\partial}{\partial z_{21}} \frac{\partial}{\partial z_{12}}\).

We compute

\[
\partial_{\det} \left( \frac{r^2}{2} \right)^t / t! = \det \cdot \frac{\left( \frac{r^2}{2} \right)^{t-2}}{(t-2)!},
\]

\[
\partial_{\det}(f_i^t \det^{t-i}) = (t + i + 1)(t - i) f_i^t \det^{t-i-1},
\]

so we obtain

\[
f_i^t = \begin{cases} \frac{1}{(t+i+1)(t-i)} f_i^{t-2}, & \text{if } t - i \text{ is even, } t - i \geq 2, \\ 0, & \text{if } t - i \text{ is odd.} \end{cases}
\]

Thus,

\[
\delta = \sum_{t=0}^{\infty} \sum_{i=0}^{t} f_i^t \det^{t-i} = \sum_{i=0}^{\infty} f_i^t 0 F_1(i + \frac{3}{2}, \frac{\det^2}{4}),
\]

where

\[
0 F_1(\lambda, x) = \sum_{n=0}^{\infty} \frac{1}{\lambda(\lambda + 1) \cdots (\lambda + n - 1)} \frac{x^n}{n!}.
\]

Notice that Formula (\(\star\)) determines \(f_i^t\) uniquely, for one can equate the homogeneous components of \(\delta = \sum_{t=0}^{\infty} \frac{(r^2)^t}{t!}\) with those of \(\sum_{i=0}^{\infty} f_i^t 0 F_1(i + \frac{3}{2}, \frac{\det^2}{4})\).

(Case 2) \(\left( U(p), U(k, k) \right) \subseteq Sp(2l, \mathbb{R}), \ l = 2pk.\)

95
We shall go through computations for case 2 in parallel fashion without commenting on the justification, which is similar to case 1.

Since the isomorphism of the Schrödinger model with the Fock model is such that

\begin{equation}
\begin{align*}
 z_{ij} & \longrightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q_{ij}} + q_{ij}, \right), \quad \frac{\partial}{\partial z_{ij}} & \longrightarrow \frac{1}{2\sqrt{2}} \left( \frac{\partial}{\partial q_{ij}} - \bar{q}_{ij}, \right), \\
\bar{z}_{ij} & \longrightarrow \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \bar{q}_{ij}} + \bar{q}_{ij}, \right), \quad \frac{\partial}{\partial \bar{z}_{ij}} & \longrightarrow \frac{1}{2\sqrt{2}} \left( \frac{\partial}{\partial \bar{q}_{ij}} - q_{ij}, \right),
\end{align*}
\end{equation}

where \((z_{ij})_{1 \leq i \leq p, 1 \leq j \leq k}\) are the coordinates of \(V^k \cong \mathbb{C}^{n_k}\), and since the Dirac distribution at the origin of \(V^k\) satisfies

\[ z_{ij} \cdot \delta = \bar{z}_{ij} \cdot \delta = 0, \]

it must have the form

\begin{equation}
\delta = \exp \left( - \frac{\sum_{ij} |q_{ij}|^2}{2} \right) \overset{\text{def.}}{=} \exp \left( - \frac{\sum_{ij} q_{ij} \bar{q}_{ij}}{2} \right) \quad \text{(up to a scalar)}
\end{equation}

in the Fock model.

Let

\begin{equation}
\partial_n = \det \begin{pmatrix}
\frac{\partial}{\partial q_{11}} & \frac{\partial}{\partial q_{12}} & \cdots & \frac{\partial}{\partial q_{1n}} \\
\frac{\partial}{\partial q_{21}} & \frac{\partial}{\partial q_{22}} & \cdots & \frac{\partial}{\partial q_{2n}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial}{\partial q_{n1}} & \frac{\partial}{\partial q_{n2}} & \cdots & \frac{\partial}{\partial q_{nn}}
\end{pmatrix}
\end{equation}

and \(\bar{\partial}_n\) by replacing \(q_{ij}\) by \(\bar{q}_{ij}\).

Then we have

\begin{equation}
\partial_n \delta = \left( -\frac{1}{2} \right)^n \bar{d}_n \delta, \quad \bar{\partial}_n \delta = \left( -\frac{1}{2} \right)^n d_n \delta,
\end{equation}

by a computation analogous to that giving (6.1.5).

In the Fock model, we have \(\left( \frac{\partial}{\partial q_{ij}} \right)^* = \frac{1}{2} \bar{q}_{ij}\), \(\left( \frac{\partial}{\partial \bar{q}_{ij}} \right)^* = \frac{1}{2} q_{ij}\) (This is so essentially because \(z_{ij}\)'s are complex variables). Therefore,

\begin{equation}
\left( \partial_n \right)^* = \frac{1}{2^n} d_n, \quad \left( \bar{\partial}_n \right)^* = \frac{1}{2^n} \bar{d}_n.
\end{equation}
We compute the following inner product:

\[
(\delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t}) \\
= (\delta, d_t \cdot d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t-1} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t}) \\
= (2^t \delta_t \delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t-1} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t}) \quad \text{(by 6.1.17)}
\]

\[
= (-1)^t (\delta_t \delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t-1} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t}) \quad \text{(by 6.1.16)}
\]

\[
= (-1)^t (\delta_t, 2^t \delta_t (d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t-1} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t})) \quad \text{(by 6.1.17)}
\]

\[
= (-2)^t B(b_1, b_2, \ldots, b_t)(\delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t-1} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t-1}). \quad \text{(by 6.1.9)}
\]

Therefore by the above induction formula, we have

\[(6.1.18) \text{ Proposition:} \]

\[
(\delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t}) \neq 0
\]

if and only if \(a_i = b_i, \ 1 \leq i \leq t\). Moreover,

\[(6.1.19) \quad (\delta, d_1^{a_1} d_2^{a_2} \ldots d_t^{a_t} \overline{\delta}_1^{b_1} \overline{d}_2^{b_2} \ldots \overline{d}_t^{b_t})
= \prod_{1 \leq i \leq t} \prod_{1 \leq i \leq t} (-2)^{c_1+2c_2+\ldots+t c_t} B(c_1, c_2, \ldots, c_t)(\delta, 1).
\]

(Case 3) \( (Sp(k), O^*(4k) \subseteq Sp(2l, \mathbb{R}), \ l = 4pk). \)

The author is unable to obtain the inner product formulas for this case in general.

Still it is interesting just to see how the Dirac distribution looks under the Fock model. Therefore, the following paragraph is included.

Since the isomorphism of the Schrödinger model with the Fock model is such that

\[(6.1.20)
\]

\[
\begin{align*}
& z_{i,j} \rightarrow \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_{i,2j}} + x_{i,2j-1}), \quad \frac{\partial}{\partial z_{i,j}} \rightarrow \frac{1}{2\sqrt{2}}(2\frac{\partial}{\partial x_{i,2j-1}} - y_{i,2j}), \\
& \overline{z}_{i,j} \rightarrow \frac{1}{\sqrt{2}}(2\frac{\partial}{\partial x_{i,2j-1}} + y_{i,2j}), \quad \frac{\partial}{\partial \overline{z}_{i,j}} \rightarrow \frac{1}{2\sqrt{2}}(2\frac{\partial}{\partial y_{i,2j-1}} - x_{i,2j-1}), \\
& z_{p+i,j} \rightarrow \frac{1}{\sqrt{2}}(-2\frac{\partial}{\partial x_{i,2j}} + y_{i,2j-1}), \quad \frac{\partial}{\partial z_{p+i,j}} \rightarrow \frac{1}{2\sqrt{2}}(2\frac{\partial}{\partial y_{i,2j-1}} + x_{i,2j}), \\
& \overline{z}_{p+i,j} \rightarrow \frac{1}{\sqrt{2}}(2\frac{\partial}{\partial y_{i,2j-1}} - x_{i,2j}), \quad \frac{\partial}{\partial \overline{z}_{p+i,j}} \rightarrow \frac{1}{2\sqrt{2}}(2\frac{\partial}{\partial x_{i,2j}} + y_{i,2j-1}),
\end{align*}
\]

97
where \((z_{i,j})_{1 \leq i \leq 2p, 1 \leq j \leq k}\) are the complex coordinates of \(V^k \cong \mathbb{H}^{p,k}\) as in (4.3.15), and since the Dirac distribution at the origin of \(V^k\) satisfies

\[z_{i,j} \cdot \delta = z_{i,j} \cdot \delta = 0,\]

it must have the form

\[
\delta = \exp\left(-\frac{\sum_{1 \leq i \leq p, 1 \leq j \leq k} \text{det} \left( \begin{array}{cc} x_{i,2j-1} & x_{i,2j} \\ y_{i,2j-1} & y_{i,2j} \end{array} \right)}{2} \right)
\]

(6.1.21)

\[
def \delta = \exp\left(-\frac{\sum_{1 \leq i \leq p, 1 \leq j \leq k} (x_{i,2j-1}y_{i,2j} - y_{i,2j-1}x_{i,2j})}{2}\right) \text{ (up to a scalar)}
\]

in the Fock model.

TO BE CONTINUED IN THE FUTURE.
§6.2 Noncompact cases

In this section, we show that, in the noncompact cases, the existence of $G$-invariant distributions with certain $\tilde{K}'$-types is a consequence of the explicit descriptions of pluriharmonics, the inner product formulas in the compact cases together with the functorial properties of the oscillator representation and the Dirac distribution.

Let $R(\tilde{K}';G,\omega)_0$ be the set of $\tau \in R(\tilde{K}';G,\omega)$ such that $\mathcal{H}(K')_r^K \neq 0$. All the highest weights of $\tau \in R(\tilde{K}';G,\omega)_0$ are listed in Multiplicity One Theorem (5.3.29).

(6.2.1) **Theorem:** The Dirac distribution at the origin of $V^k$ has a nonzero projection to the $\tau$-type for any $\tau \in R(\tilde{K}';G,\omega)_0$.

**Proof:**

First, let us assume $G = O(p,q), U(p,q), Sp(p,q)$.

Recall the two diamonds of reductive dual pairs:

\[
\begin{array}{cccc}
& M_0^{(1,1)} & & M_0^{(1,1)} \\
\cap & K & \cap & M' \\
\cup & G & \cup & G'
\end{array}
\]

\[
\begin{array}{cccc}
& M_0^{(1,1)} & & M_0^{(1,1)} \\
\cap & K' & \cap & M
\end{array}
\]

Notice that in these three cases, we have $K = G_1 \times G_2$ with

$$G_1 = O(p), U(p), Sp(p), \quad G_2 = O(q), U(q), Sp(q),$$

$M_0^{(1,1)} = M_1 \times M_2$ with

$$M_1 = U(p) \times U(p), U(2p), \quad M_2 = U(q) \times U(q), U(2q),$$

and $M_0^{(1,1)} = M_1' \times M_2'$ with

$$M_1' = U(k), U(k) \times U(k), U(2k), \quad M_2' = U(k), U(k) \times U(k), U(2k).$$
By the functorial properties of the oscillator representation (see [H7], for example), we have
\[
\omega|_{M_0^{(1,1)} \times M_0^{(1,1)}} \cong \omega_1 \otimes \omega_2,
\]
where \(\omega_i\) is the oscillator presentation associated to the dual pair \((M_i, M'_i)\), \(i = 1, 2\).

Let us observe the following fact about pluriharmonics:
\(\mathcal{H}(K')\) is irreducible under the joint action of \(M_0^{(1,1)} \times K'\), and all the simultaneous highest weight vectors of \(\mathcal{H}(K')\) under this action are products of two simultaneous highest weight vectors for the dual pairs \((M_i, M'_i)\), \(i = 1, 2\). See the list of diamond dual pairs, Theorem (5.3.15) or directly from [KV].

Let \(v_\tau \in \mathcal{H}(K')\) be such a simultaneous highest weight vector for \(M_0^{(1,1)} \times K'\),
\[
v_\tau^0 = v_{\tau_1}^0 \otimes v_{\tau_2}^0,
\]
where \(v_{\tau_i}\) is a simultaneous highest weight vector for the dual pair \((M_i, M'_i)\), \(\tau_i \in \tilde{M}'_i\), \(i = 1, 2\).

Let \(\delta_i\) be the Dirac distribution at the origin of \(V_i^k\), the direct sum of \(k\)-copies of the standard module for \(G_i\), \(i = 1, 2\).

Since \(\tau \in R(K'; G, \omega)_0\), we have \((\delta_i, v_{\tau_i}) \neq 0\) by the inner product formulas in compact cases. See Propositions (6.1.11), (6.1.18).

Now the Dirac distribution \(\delta\) at the origin of \(V^k\) can be expressed as
\[
\delta = \delta_1 \otimes \delta_2,
\]
we have
\[
(\delta, v_\tau) = (\delta_1, v_{\tau_1})(\delta_2, v_{\tau_2}) \neq 0.
\]

Secondly, let us assume \(G = Sp(2m, \mathbb{R}), O^*(2m)\).

We again look at the two diamonds of reductive dual pairs. Specifically, we have
\[
U(m) \times U(m) \quad \quad U(k) \times U(k)
\]
For the sake of convenience, we take \( G = Sp(2m, \mathbb{C}) \). The same proof works for \( G = O^*(2m) \), as long as we make some obvious adjustments as to what are the corresponding groups in this case.

By the functoriality of the oscillator representation, we have

\[
\mathcal{H}(K') = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad \mathcal{H}_2 \cong \mathcal{H}_1^*,
\]

where \( \mathcal{H}_1 \) is the space of pluriharmonics for \( O(k, \mathbb{C}) \) in \( M_{m,k}(\mathbb{C}) \). It is irreducible under the joint action of \( Gl(m, \mathbb{C}) \times O(k, \mathbb{C}) \).

We also observe the following fact about \( \mathcal{H}_1 \):

All the simultaneous highest weight vectors of \( Gl(m, \mathbb{C}) \times O(k, \mathbb{C}) \) in \( \mathcal{H}_1 \) are simultaneous highest weight vectors of \( Gl(m, \mathbb{C}) \times Gl(k, \mathbb{C}) \). See Theorem (5.3.19) and (5.3.23) or directly from [KV].

Therefore for \( \tau \in R(\tilde{K}'; G, \omega)_0 \), the projection of the Dirac distribution at the origin of \( V^k \) to \( \mathcal{H}(K')_\tau \) is nonzero by the inner product formulas for the case of dual pairs \((U(m), U(k, k))\). Q.E.D.
APPENDIX 1: EXPLICIT REALIZATIONS
OF THE OSCILLATOR REPRESENTATIONS

1. \((O_p, Sp(2k, \mathbb{R})) \subseteq Sp(2pk, \mathbb{R})\).

Let \(\mathbb{R}^{2k}\) be equipped with the symplectic form \(\langle \cdot, \cdot \rangle\):
\[
\langle (x, y), (x', y') \rangle = xy'^t - yx'^t = (x, y)\sigma_k \left( \begin{array}{l} y' \\ x' \end{array} \right),
\]
where
\[
\sigma_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}, \quad x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k.
\]

The isometry group of \(\langle \cdot, \cdot \rangle\), \(Sp(2k, \mathbb{R})\), is generated by the following elements:
\[
\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in GL(k, \mathbb{R}), \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}, b = b^t \in M_{k, k}(\mathbb{R}), \text{ and } \sigma_k,
\]
here, matrices act by right multiplication, \(g \cdot (x, y) = (x, y)g^{-1}\).

The oscillator representation \(\omega_1\) acts on \(L^2(\mathbb{R}^k)\) by:
\[
[\omega_1 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](x) = (det a)^{\frac{k}{2}} f(xa),
\]
\[
[\omega_1 \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix} f](x) = e^{\frac{i}{2} \pi bx^t} f(x),
\]
\[
[\omega_1(\sigma)f](x) = (\frac{i}{2\pi})^{\frac{k}{2}} \int_{\mathbb{R}^k} e^{i x y^t} f(y) dy, \quad x \in \mathbb{R}^k = M_{1, k}(\mathbb{R}).
\]

The \(p\)-th tensor product of \(\omega_1, \omega_p\), acts on \(L^2(M_{p, k}(\mathbb{R}))\) by
\[
[\omega_p \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](x) = (det a)^{\frac{k}{2}} f(xa),
\]
\[
[\omega_p \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix} f](x) = e^{\frac{i}{2} \pi Tr(bx^t)} f(x)
\]
\[
[\omega(\sigma)f](x) = (\frac{i}{2\pi})^{\frac{pk}{2}} \int_{M_{p, k}(\mathbb{R})} e^{i Tr(xy^t)} f(y) dy, \quad x \in M_{p, k}(\mathbb{R}).
\]

Let \(O_p\) act on the left by
\[
(g \cdot f)(x) = f(g^{-1} x), \quad g \in O_p, x \in M_{p, k}(\mathbb{R}).
\]

102
It's easy to see that $O_p$ commutes with $\omega_p$.

2. $(U_p, U(k, k)) \subseteq Sp(4pk, \mathbb{R})$.

Let $\mathbb{C}^{2k}$ be equipped with the "hermitian-symplectic" form $< >$:

$$< (z, w), (z', w') > = z\bar{w}' - w\bar{z}' = (z, w)\sigma_k \begin{pmatrix} z' & \bar{w}' \\ \bar{z}' & w' \end{pmatrix},$$

where $\sigma_k = \begin{pmatrix} a & b \\ 0 & (a^{-1}) \end{pmatrix}, z = (z_1, z_2, ..., z_k), w = (w_1, w_2, ..., w_k) \in \mathbb{C}^k$.

The real part of $< >$ is a real symplectic form on $\mathbb{C}^k$ regarded as a real vector space.

The isometry group of $< >$, $U(k, k)$, is generated by the following elements:

$$\begin{pmatrix} a & 0 \\ 0 & (a^{-1}) \end{pmatrix}, a \in GL(k, \mathbb{C}), \begin{pmatrix} b & 0 \\ 0 & I_k \end{pmatrix}, b = \bar{b} \in M_{k, k}(\mathbb{C}), \text{and } \sigma_k,$$

here, matrices act by right multiplication, $g \cdot (z, w) = (z, w)g^{-1}$.

The oscillator representation $\omega_1$ acts on $L^2(\mathbb{C}^k)$ by

$$[\omega_1 \begin{pmatrix} a & 0 \\ 0 & (a^{-1}) \end{pmatrix} f](z) = det a f(za),$$

$$[\omega_1 \begin{pmatrix} b & 0 \\ 0 & I_k \end{pmatrix} f](z) = e^{i\frac{\text{Tr}(z\bar{w}')}{2}} f(z)$$

$$[\omega_1(\sigma)f](z) = \left(\frac{i}{2\pi}\right)^k \int_{\mathbb{C}^k} e^{i\text{Re}(z\bar{w}')} f(w) dw, \quad z \in \mathbb{C}^k = M_{1,k}(\mathbb{C}).$$

The $p$-th tensor product of $\omega_1, \omega_p$, acts on $L^2(M_{p,k}(\mathbb{C}))$ by

$$[\omega_p \begin{pmatrix} a & 0 \\ 0 & (a^{-1}) \end{pmatrix} f](z) = (det a)^p f(za)$$

$$[\omega_p \begin{pmatrix} b & 0 \\ 0 & I_k \end{pmatrix} f](z) = e^{i\frac{\text{Tr}(z\bar{w}')}{2}} f(z)$$

$$[\omega_p(\sigma)f](z) = \left(\frac{i}{2\pi}\right)^{pk} \int_{M_{p,k}(\mathbb{C})} e^{i\text{Re}(z\bar{w}')} f(w) dw, \quad z \in M_{p,k}(\mathbb{C}).$$

Let $U_p$ act on the left by

$$(g \cdot f)(z) = f(g^{-1}z), \quad g \in U_p, z \in M_{p,k}(\mathbb{C}).$$

It is easy to see that $U_p$ commutes with $\omega_p$.  

103
Let $\mathbb{H}^{2k}$ be equipped with the “quaternionic-symplectic” form $\langle \rangle$:

$$\langle (u, v), (u', v') \rangle = uu'^{qt} - vv'^{qt} = (u, v)\sigma_k \left( \begin{pmatrix} u'^{qt} \\ v'^{qt} \end{pmatrix} \right),$$

where $\sigma_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, $u = (u_1, u_2, \ldots, u_k)$, $v = (v_1, v_2, \ldots, v_k) \in \mathbb{H}^k$.

The real part of $\langle \rangle$ is a real symplectic form on $\mathbb{H}^k$ regarded as a real vector space.

The isometry group of $\langle \rangle$, $O^*(4k)$, is generated by the following elements:

$$\begin{pmatrix} a & 0 \\ 0 & (a^{qt})^{-1} \end{pmatrix}, a \in GL(k, \mathbb{H}), \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}, b = b^{qt} \in M_{k,k}(\mathbb{H})$$

and $\sigma_k$, here, matrices again act by right multiplication, $g \cdot (u, v) = (u, v)g^{-1}$.

The oscillator representation $\omega_1$ acts on $L^2(\mathbb{H}^k)$ by

$$[\omega_1 \begin{pmatrix} a & 0 \\ 0 & (a^{qt})^{-1} \end{pmatrix}] f(u) = |\det a|^2f(ua)$$

$$[\omega_1 \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}] f(u) = e^{i\frac{k\text{tr}(bu)}{2}}f(u)$$

$$[\omega_1(\sigma)]f(u) = \left( \frac{i}{2\pi} \right)^{2k} \int_{\mathbb{H}^k} e^{i\text{Re}(uv^{qt})}f(u)dv, \quad u \in \mathbb{H}^k = M_{1,k}(\mathbb{H}).$$

The $p$-th tensor product of $\omega_1, \omega_p$, acts on $L^2(M_{p,k}(\mathbb{H}))$ by

$$[\omega_p \begin{pmatrix} a & 0 \\ 0 & (a^{qt})^{-1} \end{pmatrix}] f(u) = |\det a|^{2p}f(ua)$$

$$[\omega_p \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}] f(u) = e^{i\frac{p\text{tr}(bu)}{2}}f(u)$$

$$[\omega_p(\sigma)]f(u) = \left( \frac{i}{2\pi} \right)^{2pk} \int_{M_{p,k}(\mathbb{H})} e^{i\text{Re}(uv^{qt})}f(v)dv, \quad u \in M_{p,k}(\mathbb{H}).$$

Let $Sp(p)$ act on the left by

$$(g \cdot f)(u) = f(g^{-1}u), \quad g \in Sp(p), u \in M_{p,k}(\mathbb{H}).$$

It is easy to see that $Sp(p)$ commutes with $\omega_p$.

4. $\langle Sp(2m, \mathbb{R}), O(k, k) \rangle \subseteq Sp(4mk, \mathbb{R})$. 

104
Let $\mathbb{R}^{2k}$ be equipped with the symmetric form $(\cdot, \cdot)$:

$$(x, y), (x', y') \mapsto xy'y' + yx'y' = (x, y)\tau_k \left( \frac{x'}{y'} \right),$$

where $\tau_k = \begin{pmatrix} I_k & 0 \\ 0 & I_k \end{pmatrix}$, $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$.

The isometry group of $(\cdot, \cdot)$, $O(k, k)$, is generated by the following elements:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in GL(k, \mathbb{R}), \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}, b = -b^t \in M_{k,k}(\mathbb{R})$$

and $\tau_k$.

Let $\mathbb{R}^{2,2k}$ be the direct sum of two copies of $\mathbb{R}^{2k}$ with a general vector written in the form: $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ where $x_i = (x_{1i}, x_{2i}, \ldots, x_{ki}), y_i = (y_{1i}, y_{2i}, \ldots, y_{ki}) \in \mathbb{R}^k$.

We define $<>$ to be the following symplectic form on $\mathbb{R}^{2,2k}$:

$$< \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}, \begin{pmatrix} x'_1 & y'_1 \\ x'_2 & y'_2 \end{pmatrix} > = ((x_1, y_1), (x'_1, y'_1)) - ((x_2, y_2), (x'_2, y'_2)).$$

Its isometry group is isomorphic to $Sp(4k, \mathbb{R})$. Clearly $O(k, k) \subseteq Sp(4k, \mathbb{R})$ and $\mathbb{R}^{2,k} = \{ \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, x_1, x_2 \in \mathbb{R}^k \}$ is a polarization of the symplectic form $< >$.

The oscillator representation $\omega_1$ of $Sp(4k, \mathbb{R})$, when restricted to $O(k, k)$, has the following form:

$$[\omega_1 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](x_1, x_2) = detaf((x_1, a))$$

$$[\omega_1 \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix} f](x_1, x_2) = e^{\frac{i}{2} (x_2 - x_1 b x_1)} f((x_1, x_2))$$

$$[\omega_1(\tau_k) f](x_1, x_2) = \left( \frac{i}{2\pi} \right)^k \int_{\mathbb{R}^k} e^{i(x_1 y_2 - x_2 y_1)} f(y_1, y_2) d(y_1, y_2)$$

The last two formulas are so because

$$< \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_1b \\ 0 & x_2b \end{pmatrix} > = x_1(x_2b) - x_2(x_1b) = x_2bx_1 - x_1bx_2$$

$$< \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_1 \\ 0 & y_2 \end{pmatrix} > = x_1y_2 - x_2y_1.$$

The $m$-th tensor product of $\omega_1, \omega_m$, acts on $L^2(\mathbb{R}^{2m,k})$ by

$$[\omega_m \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](x_1, x_2) = (detam f((x_1, a)), x_1, x_2 \in \mathbb{R}^{m,k}.$$
Let $Sp(2m, \mathbb{R})$ act on the left by

$$(g \cdot f)(x_1 \ x_2) = f(g^{-1}(x_1)),$$  
$g \in Sp(2m, \mathbb{R}), \ (x_1 \ x_2) \in \mathbb{R}^{2m,k}.$

It’s easy to see that $Sp(2m, \mathbb{R})$ commutes with $\omega_m$.

5. $(O^*(2m), Sp(k, k)) \subseteq Sp(8m, \mathbb{R})$

Let $\mathbb{H}^{2k} = \mathbb{H}^k \oplus \mathbb{H}^k$ be equipped the “$\frac{1}{2}$-hermitian” form $(\ ):$

$$((u, v), (u', v')) = uu'^{t2} + vv'^{t2} = (u, v)\tau_k \begin{pmatrix} u'^{t1} \\ v'^{t1} \end{pmatrix},$$

where $\tau_k = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix}$, $u = (u_1, u_2, ..., u_k)$, $v = (v_1, v_2, ..., v_k) \in \mathbb{H}^k$.

The isometry group of $(\ )$, $Sp(k, k)$, is generated by the following elements:

$$\begin{pmatrix} a & 0 \\ 0 & a^{t1} \end{pmatrix}, a \in GL(k, \mathbb{H}), \begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix}, b = -b^{t2} \in M_{k, k}(\mathbb{H}), \text{and } \tau_k.$$

Consider $j(\ )$. Since

$$[j(\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix})] = -(v' u'^{t1} + u' v'^{t1})j$$

and

$$Re[(v' u'^{t1} + u' v'^{t1})j] = Re[j(v' u'^{t1} + u' v'^{t1})],$$

we see that

$$Re(j(\ )) = Re(juv'^{t1} + jvu'^{t1})$$

is a real symplectic form, its isometry group is isomorphic to $Sp(8k, \mathbb{R})$.

Clearly $Sp(k, k) \subseteq Sp(8k, \mathbb{R})$.

The oscillator representation $\omega_1$ of $Sp(8k, \mathbb{R})$, when restricted to $Sp(k, k)$, has the following form:

$$[\omega_1(\begin{pmatrix} a & 0 \\ 0 & a^{t1} \end{pmatrix})]f(u) = |det\alpha|^2 f(u\alpha), \ u \in \mathbb{H}^k = M_{1, k}(\mathbb{H}),$$

$$[\omega_1(\begin{pmatrix} I_k & b \\ 0 & I_k \end{pmatrix})]f(u) = e^{iRe(juv'^{t1})} f(u) = e^{iRe(jvu'^{t1})} f(u),$$

$$[\omega_1(\tau_k)]f(u) = (\frac{i}{2\pi})^{2k} \int_{\mathbb{H}^k} e^{iRe(juv'^{t1})} f(v) dv.$$
The \( m \)-the tensor product of \( \omega_1, \omega_m \), acts on \( L^2(M_{m,k}(\mathbb{H})) \) by

\[
[\omega_m \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f](u) = |det a|^{2m} f(ua), \ u \in M_{m,k}(\mathbb{H}).
\]

\[
[\omega_m \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} f](u) = e^{-iRe Tr(juv^t)} f(u),
\]

\[
[\omega_m(\tau_k) f](u) = \left( \frac{i}{2\pi} \right)^{2km} \int_{M_{m,k}(\mathbb{H})} e^{iRe Tr(juv^t)} f(v) dv.
\]

Let \( O^*(2m) \) act on the left by

\[
(g \cdot f)(u) = f(g^{-1}u), \ g \in O^*(2m), u \in M_{m,k}(\mathbb{H}).
\]

It is easy to see that \( O^*(2m) \) commutes with \( \omega_m \).
APPENDIX 2: AN EXPLICIT FORMULA FOR $O(p, 1)$ INVARIANTS IN SOME $U(p, 1)$ MODULES OF VERMA TYPE

Fix a Harish-Chandra decomposition of $U(p, 1)$:

$$u(p, 1)_C = \mathfrak{t}_C \oplus p^+ \oplus p^-.$$ 

Let $\pi \in \hat{K}$, we then define a representation of $u(p, 1)$ in $\pi \otimes S(p^+)$ by requiring:

$$X^+ \cdot (v \otimes f(p^+)) = v \otimes X^+ f(p^+), \quad X^+ \in p^+, \quad v \in \pi, \quad f(p^+) \in S(p^+),$$

$$k \cdot (v \otimes f(p^+)) = kv \otimes \text{ad}k \cdot f(p^+), \quad k \in K,$$

$$X^- \cdot (v \otimes 1) = 0, \quad X^- \in p^-.$$ 

We know from Corollary (5.3.11) that in order for $\pi \otimes S(p^+)$ to have a $O(p, 1)$-invariant, the highest weight of $\pi$ has to be "even". We are interested in producing explicit formulas of these invariants.

We shall only deal with the case when $\pi$ is a linear character of $U(p) \times U(1)$. Namely,

$$\pi : (x, y) \rightarrow (dx)^m \cdot (dy)^n, \quad m, n \in 2\mathbb{Z}, \quad x \in U(p), \; y \in U(1).$$

We fix a $v \in \pi$, $v \neq 0$.

Set $X_i^+ = E_{i,p+1}, X_i^- = E_{p+1,i}, \; 1 \leq i \leq p$, where $E_{ij}$ is the $(p + 1) \times (p + 1)$ matrix having one in the $(i, j)$ entry and zero's elsewhere.

We have $p^+$ (resp. $p^-$) = span of $\{X_i^+\}_{1 \leq i \leq p}$ (resp. $\{X_i^-\}_{1 \leq i \leq p}$).

We introduce coordinates on $p^+$ by the rule

$$X_i^+ \hookrightarrow x_i, \quad 1 \leq i \leq p.$$
Proposition: \( v \otimes_0 F_1\left(\frac{m-n+1}{2}, \sum_{i} x_i^2\right) = \)

\[
v \otimes \left\{ 1 + \frac{1}{(m-n+1) \cdot 2 \left( \sum x_i^2 \right)} + \frac{1}{(m-n+1) \cdot 2 \cdot (m-n+3) \cdot 4 \left( \sum x_i^2 \right)^2} + \ldots + \frac{1}{(m-n+1) (m-n+1 + 1) \ldots (m-n+1 + l-1)} \left( \frac{\sum x_i^2}{4} \right)^l / l! + \ldots \right\}
\]

is an \( O(p, 1) \) invariant.

\( _0F_1(\alpha, x) \) is the standard \( \binom{\alpha}{1} \) hypergeometric function.

Sketch of Proof:

By using commutation relations of \( E_{ij} \)'s, the action of \( p^+, p^- \) on a monomial is found to be

\[
X_i^+(v \otimes x_1^{a_1} x_2^{a_2} \ldots x_i^{a_i} \ldots x_p^{a_p}) = v \otimes x_1^{a_1} x_2^{a_2} \ldots x_i^{a_i+1} \ldots x_p^{a_p},
\]

\[
X_i^-(v \otimes x_1^{a_1} x_2^{a_2} \ldots x_i^{a_i} \ldots x_p^{a_p}) = -a_i \left( \sum_{j=1}^{p} a_j + m - n - 1 \right) v \otimes x_1^{a_1} x_2^{a_2} \ldots x_{i-1}^{a_{i-1}} \ldots x_p^{a_p}
\]

\[
= -v \otimes \left[ \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{p} x_j \frac{\partial}{\partial x_j} + m - n - 1 \right) \right] x_1^{a_1} x_2^{a_2} \ldots x_{i-1}^{a_{i-1}} \ldots x_p^{a_p}.
\]

Suppose \( v \otimes f = v \otimes \sum a_{a_1, a_2, \ldots, a_p} x_1^{a_1} x_2^{a_2} \ldots x_p^{a_p} \) is \( O(p, 1) \)-invariant, then

\( v \) and \( f \) is \( O(p) \times O(1) \) invariant,

\( (X_i^+ + X_i^-)(v \otimes f) = 0, \ \forall 1 \leq i \leq p. \)

The first condition implies that \( f \) is of the form:

\[
f = \sum_{l=0}^{\infty} \lambda_l (x_1^2 + x_2^2 + \ldots + x_p^2)^l.
\]

There is a unique solution to the second invariance condition, and that is

\[
\lambda_{l+1} = \frac{1}{(2l + m - n + 1)(2l + 2)} \lambda_l, \ \ l \geq 0,
\]

thus we arrive at the desired formula.

Remark: When \( m = n, \ f =_0 F_1\left(\frac{1}{2}, \sum_{i} x_i^2\right) = \cosh(\sqrt{\sum x_i^2}). \)
BIBLIOGRAPHIES


