

EIGENDISTRIBUTIONS FOR ORTHOGONAL GROUPS AND REPRESENTATIONS OF SYMPLECTIC GROUPS

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ABSTRACT. We consider the action of $H = O(p, q)$ on the matrix space $M_{p+q, n}(\mathbb{R})$. We study a certain orbit \mathcal{O} of H in the null cone $\mathcal{N} \subseteq M_{p+q, n}(\mathbb{R})$ which supports an eigendistribution $d\nu_{\mathcal{O}}$ for H . Using some identities of Capelli type developed in the Appendix, we determine the structure of $\tilde{G} = Sp(2n, \mathbb{R})^{\sim}$ -cyclic module generated by $d\nu_{\mathcal{O}}$ under the oscillator representation of $\tilde{\mathbb{G}}$ (the metaplectic cover of $\mathbb{G} = Sp(2n(p+q), \mathbb{R})$). Applications to local theta correspondence and a generalized Huygens' Principle are given.

1. INTRODUCTION AND MAIN RESULT

Let $V = \mathbb{R}^{p+q}$ be equipped with the standard non-degenerate symmetric form of signature (p, q) , and let $H = O(p, q)$ be its isometry group.

For each natural number n , let V^n be the direct sum of n copies of V . We may identify V^n with $M_{p+q, n}(\mathbb{R})$, the space of real matrices of order $(p+q) \times n$. Then the induced action of H on $V^n \simeq M_{p+q, n}(\mathbb{R})$ is given by matrix multiplication on the left.

Recall that we have the reductive dual pair [9]

$$(H, G) = (O(p, q), Sp(2n, \mathbb{R})) \subseteq \mathbb{G} = Sp(2N, \mathbb{R}),$$

where $N = n(p+q)$. Consider the non-trivial double cover $\tilde{\mathbb{G}} = Mp(2N, \mathbb{R})$ of \mathbb{G} , called the metaplectic cover. For a subgroup E of \mathbb{G} , we denote the pullback of E in $\tilde{\mathbb{G}}$ by \tilde{E} .

Let Ω be the oscillator representation of $\tilde{\mathbb{G}}$ as described in [5]. The representation Ω may be realized in $L^2(M_{p+q, n}(\mathbb{R}))$. As usual, we normalize such an oscillator representation so that it will factor through the standard linear action of H on $L^2(M_{p+q, n}(\mathbb{R}))$, namely we have

$$(\Omega(h)f)(v) = f(h^{-1}v), \quad h \in H, v \in M_{p+q, n}(\mathbb{R}).$$

Let $\mathcal{S}(M_{p+q, n}(\mathbb{R}))$ and $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))$ be the space of Schwartz functions and the space of tempered distributions on $M_{p+q, n}(\mathbb{R})$, respectively. It is known that Ω preserves $\mathcal{S}(M_{p+q, n}(\mathbb{R}))$ inside $L^2(M_{p+q, n}(\mathbb{R}))$. By duality,

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we also get an action of \tilde{G} on $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$. We denote the action of \tilde{G} on $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$ again by Ω , and the dualized action on $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$ by Ω^* .

Let χ be a character of H . Note that there are four of them if $pq \neq 0$. Let

$$\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)} = \{\Phi \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R})) \mid \Omega^*(h)\Phi = \chi(h)\Phi, h \in H\}$$

denote the space of χ -eigendistributions for H .

Suppose that we are given an H -orbit \mathcal{O} in $M_{p+q,n}(\mathbb{R})$, and suppose that \mathcal{O} carries a signed measure $d\nu_{\mathcal{O}}$ such that

$$d\nu_{\mathcal{O}} \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}.$$

Since G commutes with H , we see that $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ is stable under the action of \tilde{G} by Ω^* . We consider the following \tilde{G} -cyclic submodule of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$:

$$\Omega^*(d\nu_{\mathcal{O}}) = \overline{\langle \Omega^*(\tilde{G})d\nu_{\mathcal{O}} \rangle},$$

where $\overline{\langle D \rangle}$ denotes closure of the span of a set D in the standard Frechet topology of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$. We are interested in the structure of this \tilde{G} -cyclic submodule, and in particular whether it is irreducible. Recall that according to [13], $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ always has a unique irreducible \tilde{G} -submodule.

Example: Take $\mathcal{O} = \{0\}$, the origin of $M_{p+q,n}(\mathbb{R})$, and $d\nu_{\mathcal{O}} = \delta$, the Dirac measure at the origin, then a theorem of Kudla and Rallis [18] asserts that

$$\Omega^*(\delta) = \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H,$$

the space of H -invariant tempered distributions. See also [31] for a similar statement in a more general context.

We introduce the null cone:

$$\mathcal{N} = \{v \in M_{p+q,n}(\mathbb{R}) \mid v^t I_{p,q} v = 0 \quad (\text{the } n \times n \text{ zero matrix})\},$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Clearly \mathcal{N} is stable under H .

We now assume that $p, q \geq n$. Then Witt's Extension Theorem [16] implies that the regular part of the null cone defined by

$$\mathcal{N}_n = \{v \in \mathcal{N} \mid \text{rank}(v) = n\}$$

consists of a single H -orbit. Note that the isotropic group of a point in \mathcal{N}_n is unimodular. Therefore we see [3] that \mathcal{N}_n carries an H -invariant Radon measure, denoted by $d\mu_n$. In fact $d\mu_n$ defines a tempered distribution on $M_{p+q,n}(\mathbb{R})$ (see §4 for details).

Let $K \simeq U(n)$ be a maximal compact subgroup of G . Denote

$$\mathbf{1}_n = \underbrace{(1, \dots, 1)}_n.$$

Theorem 1.1. *Assume that $p, q \geq n$. Then*

- (a) $\Omega^*(d\mu_n)$ is the unique irreducible \tilde{G} -submodule of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$.
- (b) $\Omega^*(d\mu_n)$ is \tilde{K} multiplicity-free with explicitly specified \tilde{K} -types (in Theorem 4.7). In particular,
- (c) $\Omega^*(d\mu_n)$ is finite dimensional if and only if $p, q \equiv n + 1 \pmod{2}$. In this case it is the irreducible finite dimensional representation of \tilde{G} (or rather G) with the highest weight

$$\left(\frac{p+q}{2} - (n+1)\right)\mathbf{1}_n.$$

Now consider the case when $p \geq q = n$.

We can break the regular part of the null cone into two parts:

$$\mathcal{N}_q = \mathcal{N}_q^+ \cup \mathcal{N}_q^-,$$

where

$$\mathcal{N}_q^+ = \left\{ v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}_q \mid \det y > 0 \right\}, \quad \mathcal{N}_q^- = \left\{ v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}_q \mid \det y < 0 \right\}.$$

Let $H^+ = O^+(p, q)$ be the subgroup of $H = O(p, q)$ stabilizing \mathcal{N}_q^+ (or \mathcal{N}_q^-). It is a subgroup of index two. This defines a character of $O(p, q)$ of order two, denoted by ϵ . It satisfies

$$\epsilon|_{O(p)} = \text{trivial}, \quad \epsilon|_{O(q)} = \text{determinant}.$$

Alternatively we may define $O^+(p, q)$ to be the kernel of this character.

Fix $\tau = \begin{pmatrix} I_p & 0 \\ 0 & \tau_q \end{pmatrix}$, where $\tau_q \in O(q) - SO(q)$. Then

$$O(p, q) = O^+(p, q) \cup O^+(p, q)\tau, \quad \text{and} \quad \tau(\mathcal{N}_q^\pm) = \mathcal{N}_q^\mp.$$

\mathcal{N}_q^\pm are then homogeneous spaces for $O^+(p, q)$. Again from general result on homogeneous spaces [3], we know that \mathcal{N}_q^\pm admit $O^+(p, q)$ -invariant Radon measures which we fix as $d\mu_q^\pm$ so that

$$d\mu_q = d\mu_q^+ + d\mu_q^-, \quad \tau d\mu_q^\pm = d\mu_q^\mp.$$

Remark 1.2. *When $p > q$, we can show that \mathcal{N}_q^\pm are in fact homogeneous spaces for $SO_e(p, q) = O^+(p, q) \cap SO(p, q)$, the connected component at the identity of $O(p, q)$. Thus in this case, both \mathcal{N}_q^\pm are connected. When $p = q$, \mathcal{N}_q breaks into four connected components. Each of the four components of \mathcal{N}_q is a homogeneous space for $SO_e(q, q)$.*

Set

$$d\nu_q = d\mu_q^+ - d\mu_q^- \in \mathcal{S}^*(M_{p+q,q}(\mathbb{R}))^{(H;\epsilon)}.$$

Theorem 1.3. *Assume that $p \geq q = n$. Then*

- (a) $\Omega^*(d\nu_q)$ is the unique irreducible \tilde{G} -submodule of $\mathcal{S}^*(M_{p+q,q}(\mathbb{R}))^{(H;\epsilon)}$.

- (b) $\Omega^*(d\nu_q)$ is \widetilde{K} multiplicity-free with explicitly specified \widetilde{K} -types (in Theorem 4.10). In particular,
- (c) $\Omega^*(d\nu_q)$ is finite dimensional if and only if $p > q$ and $p \equiv q \pmod{2}$. In this case it is the irreducible finite dimensional representation of \widetilde{G} (or rather G) with the highest weight

$$\left(\frac{p-q}{2} - 1\right)\mathbf{1}_q.$$

Recall that an irreducible admissible representation ρ of \widetilde{H} and an irreducible admissible representation π of \widetilde{G} are said to correspond to each other under the local theta correspondence if there exists a non-zero $\widetilde{H} \times \widetilde{G}$ -intertwining operator

$$\Omega \mapsto \rho \otimes \pi.$$

We write $\theta(\rho) = \pi$ or $\theta(\pi) = \rho$, and call π the theta lift of ρ , and vice versa. According to [13], this defines a one to one correspondence between certain subsets of irreducible admissible representations of \widetilde{H} and \widetilde{G} .

It is immediately clear that the unique irreducible \widetilde{G} -submodule in Theorems 1.1 and 1.3 is nothing but the dual of the theta lift of the character χ (trivial character in the first case, and ϵ in the second case). Moreover, part (c) of Theorems 1.1 and 1.3 implies that a (large) collection of irreducible finite dimensional representations of H and irreducible finite dimensional representations of G correspond under the local theta correspondence when certain parity conditions are met. We make this explicit.

For $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ are integers, let ρ_λ be the irreducible finite dimensional representation of $O(p+q, \mathbb{C})$ (and therefore of $O(p, q)$) parameterized by

$$(\lambda_1, \dots, \lambda_n, \underbrace{0, \dots, 0}_{[\frac{p+q}{2} - n]}; 1).$$

See [19]. Also let π_λ be the irreducible finite dimensional representation of $Sp(2n, \mathbb{C})$ (and therefore of $Sp(2n, \mathbb{R})$) with the highest weight λ .

Then we have

Theorem 1.4. *Consider the dual pair $(O(p, q), Sp(2n, \mathbb{R}))$ with $p, q \geq n$.*

- (a) *If $p, q \equiv n + 1 \pmod{2}$, then we have*

$$\theta(\rho_\lambda) = \pi_{\lambda + (\frac{p+q}{2} - (n+1))\mathbf{1}_n},$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ are integers.

- (b) *If $p > q = n$, and $p \equiv q \pmod{2}$, then we have*

$$\theta(\epsilon \otimes \rho_\lambda) = \pi_{\lambda + (\frac{p-q}{2} - 1)\mathbf{1}_q},$$

where $\lambda = (\lambda_1, \dots, \lambda_q)$, $\lambda_1 \geq \dots \geq \lambda_q \geq 0$ are integers.

Here is another application of our results. Recall that Huygens' Principle on \mathbb{R}^p says that when the space dimension p is odd and > 1 , waves propagate

on spherical shells. It was observed by the first-named author [10, 14] that Huygens' Principle may be interpreted in terms of representation theory. To be exact, for $p > q = n = 1$, part (c) of Theorem 1.3 is equivalent to Huygens' Principle. Our result for $p > q = n$ implies that solutions of a certain system of PDE's which are naturally associated to $O(p, q)$ (as opposed to the Lorentz group $O(p, 1)$ in the classical case) satisfy a generalization of Huygens' Principle (see Theorem 5.8). We note that Helgason proved a generalization of Huygen's Principle in a very different direction [7].

Here is the outline of our approach and some words on the organization of this paper. The first point is that \widetilde{K} -types of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ are rather restricted. In fact $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ is \widetilde{K} multiplicity-free, and all possible \widetilde{K} -types can be specified. Furthermore we find a criteria for a given $S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ (with an additional assumption) to generate a particular \widetilde{K} -type (in terms of the inner product of S with a certain \widetilde{K} highest weight vector). This is done in §2. The second point is that the $d\nu_{\mathcal{O}}$ in question transforms according to a character of \widetilde{P} under the oscillator representation, where $P = MN$ ($M \simeq GL(n, \mathbb{R})$) is the Siegel parabolic subgroup of G . In essence this means that $d\nu_{\mathcal{O}}$ is a $GL(n, \mathbb{R})^{\sim}$ -homogeneous distribution supported on the null cone \mathcal{N} . Thus it satisfies a set of differential equations which takes a particularly nice form in the Fock model of the oscillator representation. It turns out that the solutions of these differential equations satisfy some identities of Capelli type (see the Appendix). This in turn allows us to compute the inner product of $d\nu_{\mathcal{O}}$ with the \widetilde{K} highest weight vectors mentioned above, thus determining the \widetilde{K} -types of the \widetilde{G} -cyclic submodule $\Omega^*(d\nu_{\mathcal{O}}) \subseteq \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ (§3). Further it gives us the image of the Schwartz space $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$ under a natural \widetilde{G} -morphism into certain degenerate principal series representations of \widetilde{G} . The results of Johnson, Lee, Orsted-Zhang, Sahi, independently [17, 20, 23, 26] on degenerate principal series can then be applied to determine the \widetilde{G} -structure of $\Omega^*(d\nu_{\mathcal{O}})$. We do this in §4. In §5, we discuss applications of part of our results to local theta correspondence and a generalized Huygens' Principle.

Finally we remark that the results and techniques of the current paper extend substantially those of [31]. Nevertheless, it remains an open problem to determine the \widetilde{G} -structure of H -eigendistributions completely.

2. \widetilde{K} -TYPES OF H -EIGENDISTRIBUTIONS

In this section, we examine the \widetilde{K} -types of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$, where χ is a character of H . We first deal with the case: $\chi = \mathbf{1}$, the trivial character of H .

We fix some notations. A (finite) sequence of non-negative integers $\{\lambda_i\}_{i \geq 1}$ is called dominant if $\lambda_1 \geq \lambda_2 \geq \dots$, and we denote its totality by Λ^+ . We also denote the subset consisting of those $\{\lambda_i\}_{i \geq 1} \in \Lambda^+$ with all λ_i 's even

(resp., odd) by Λ_e^+ (resp., Λ_o^+). For a compact group L , let \widehat{L} denote the set of equivalent classes of irreducible unitary representations of L .

Theorem 2.1. ([31]) $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$ is \widetilde{K} -multiplicity one. Further $\tau \in \widehat{\widetilde{K}}$ is a \widetilde{K} -type in $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$ if and only if the highest weight of τ^* is of the form

$$\frac{p-q}{2}\mathbf{1}_n + (\lambda_1, \dots, \lambda_k, 0, \dots, 0, -\mu_1, \dots, -\mu_l),$$

where $\{\lambda_i\}_{i=1}^k, \{\mu_j\}_{j=1}^l \in \Lambda_e^+$, and $k \leq p, l \leq q, k+l \leq n$.

We shall work in the formalism of $(\mathfrak{g}, \widetilde{K})$ modules.

Let $\mathbb{U} = U(N) \subseteq \mathbb{G} = Sp(2N, \mathbb{R})$ ($N = n(p+q)$) be a maximal compact subgroup of \mathbb{G} . Then under the Fock model of Ω [4], the space of $\widetilde{\mathbb{U}}$ -finite vectors of Ω is isomorphic to $\mathcal{P} = \mathcal{P}[M_{p+q,n}(\mathbb{C})]$, the space of polynomial functions on $M_{p+q,n}(\mathbb{C})$.

Let $\Omega^{-\infty}$ be the space of formal vectors of Ω [31]. Roughly speaking, $\Omega^{-\infty}$ is the space of formal power series on $M_{p+q,n}(\mathbb{C})$. It is a $(\text{Lie}(\mathbb{G})_{\mathbb{C}}, \widetilde{\mathbb{U}})$ -module, where $\text{Lie}(\mathbb{G})_{\mathbb{C}}$ is the complexified Lie algebra of \mathbb{G} . Further we have the inclusion of $(\text{Lie}(\mathbb{G})_{\mathbb{C}}, \widetilde{\mathbb{U}})$ -modules:

$$\mathcal{P} \subset \mathcal{S}^*(M_{p+q,n}(\mathbb{R})) \subset \Omega^{-\infty}.$$

We note that the action of $(\text{Lie}(\mathbb{G})_{\mathbb{C}}, \widetilde{\mathbb{U}})$ on $\Omega^{-\infty}$ is the extension of its action on $\mathcal{P} \subseteq \mathcal{S}(M_{p+q,n}(\mathbb{R}))$, while the action on $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$ is dual to the action on $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$. Thus on the level of $\widetilde{\mathbb{U}}$ -isotypic components, we have

$$\mathcal{P}_{\sigma} \subset \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))_{\sigma^*} \subset \Omega_{\sigma}^{-\infty},$$

for any $\sigma \in \widehat{\widetilde{\mathbb{U}}}$. Here and after, a subscript denotes the appropriate isotypic component.

We also note that Ω carries a (pre)unitary structure \langle, \rangle on \mathcal{P} , and so induces a natural pairing between $\Omega^{-\infty}$ and \mathcal{P} . Under our identification of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$ as a subspace of $\Omega^{-\infty}$, we have

$$S(\phi) = \langle S, \phi \rangle, \quad S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R})), \phi \in \mathcal{P},$$

where S on the right hand side is viewed as an element of $\Omega^{-\infty}$.

Let $\mathcal{H}(K) \subseteq \mathcal{P}$ be the space of K -harmonics [13]. For $\tau \in R(\widetilde{K}; \Omega^{-\infty})$, the set of \widetilde{K} -types which occur in $\Omega^{-\infty}$, define the projection map

$$p : (\Omega^{-\infty})_{\tau} \mapsto \mathcal{H}(K)_{\tau}$$

by taking an element $v \in (\Omega^{-\infty})_{\tau}$ to the lowest degree component of v .

Recall that $L = O(p) \times O(q)$ is a maximal compact subgroup of $H = O(p, q)$. Clearly the map p defined above is L -equivariant.

Denote the derived (\mathfrak{h}, L) module of a character χ of H by the same symbol, where \mathfrak{h} is the Lie algebra of H . Let $(\Omega^{-\infty})^{(\mathfrak{h}, L; \chi)}$ be the χ -eigenspace

of $\Omega^{-\infty}$ for (\mathfrak{h}, L) . Denote

$$\chi_1 = \chi|_{O(p)}, \quad \chi_2 = \chi|_{O(q)}.$$

Let $\mathcal{H}(K)^{(L; \chi_1 \otimes \chi_2)}$ be the $\chi_1 \otimes \chi_2$ -eigenspace of $\mathcal{H}(K)$ for L .

An argument similar to the one in [31] (for the case $\chi = \mathbf{1}$) yields the following

Lemma 2.2. *For any $\tau \in R(\widetilde{K}; \Omega^{-\infty})$, the induced map*

$$p: (\Omega^{-\infty})_{\tau}^{(\mathfrak{h}, L; \chi)} \mapsto \mathcal{H}(K)_{\tau}^{(L; \chi_1 \otimes \chi_2)}$$

is injective.

From the work of Kashiwara-Vergne ([19], see also [13]), we know that $\mathcal{H}(K)_{\tau}$ is irreducible under $\widetilde{M} \times \widetilde{K}$, where $M \simeq U(p) \times U(q)$, and so as $\widetilde{M} \times \widetilde{K}$ -modules, we have $\mathcal{H}(K)_{\tau} \simeq \rho(\tau) \otimes \tau$, where $\rho(\tau)$ is described in [19]. Thus $\mathcal{H}(K)_{\tau}^{(L; \chi_1 \otimes \chi_2)} \neq 0$ if and only if $\rho(\tau)^{(L; \chi_1 \otimes \chi_2)} \neq 0$. In view of the inclusion $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)} \subset (\Omega^{-\infty})^{(\mathfrak{h}, L; \chi)}$, and the fact that (M, L) is a symmetric pair, we arrive at the following result on the \widetilde{K} -structure of $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$.

Set $\Lambda_{\chi_1}^+ = \Lambda_e^+$ or Λ_o^+ depending on whether χ_1 is the trivial or the determinant character. Similarly for $\Lambda_{\chi_2}^+$.

Theorem 2.3. *$\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$ is \widetilde{K} -multiplicity one. Further $\tau \in \widehat{\widetilde{K}}$ is a \widetilde{K} -type in $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$ only if the highest weight of τ^* is of the form*

$$\frac{p-q}{2} \mathbf{1}_n + (\lambda_1, \dots, \lambda_k, 0, \dots, 0, -\mu_1, \dots, -\mu_l),$$

where $\{\lambda_i\}_{i=1}^k \in \Lambda_{\chi_1}^+$, $\{\mu_j\}_{j=1}^l \in \Lambda_{\chi_2}^+$, and $k \leq p$, $l \leq q$, $k+l \leq n$, and with the following additional requirement:

$$k = p, \text{ if } \chi_1 = \det, \quad \text{and} \quad l = q, \text{ if } \chi_2 = \det.$$

Remark 2.4. *Our result asserts that only certain \widetilde{K} -types can possibly occur in $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$. It does not assert that they actually occur. For the case $\chi = \mathbf{1}$, all of them do occur. See Theorem 2.1.*

We give two corollaries. The first one is well-known (See [24] for example).

Corollary 2.5. *$\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \det)}$ is non-zero only if $p+q \leq n$.*

Remark 2.6. *It turns out that $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \det)}$ is non-zero if and only if $p+q \leq n$, and $\tau \in \widehat{\widetilde{K}}$ is a \widetilde{K} -type in $\mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \det)}$ if and only if the highest weight of τ^* is of the form*

$$\frac{p-q}{2} \mathbf{1}_n + (\lambda_1, \dots, \lambda_p, 0, \dots, 0, -\mu_q, \dots, -\mu_1),$$

where $\{\lambda_i\}_{i=1}^p, \{\mu_j\}_{j=1}^q \in \Lambda_o^+$. See [21].

Corollary 2.7. *Assume that $p \geq q = n$. Then $\tau \in \widehat{K}$ is a \widetilde{K} -type in $\mathcal{S}^*(M_{p+q,q}(\mathbb{R}))^{(H;\epsilon)}$ only if the highest weight of τ^* is of the form*

$$\frac{p-q}{2}\mathbf{1}_q + (-\mu_q, \dots, -\mu_1),$$

where $\{\mu_j\}_{j=1}^q \in \Lambda_o^+$.

We proceed to describe a \widetilde{K} -highest weight vector in $\mathcal{H}(K)_\tau^{(L;\chi_1 \otimes \chi_2)}$ explicitly. We refer to [13] for some of the facts below.

Let

$$z = (z_{ij})_{p \times n} \in M_{p,n}(\mathbb{C}), \quad w = (w_{ij})_{q \times n} \in M_{q,n}(\mathbb{C})$$

be the complex coordinates of the Fock model of Ω .

For $1 \leq i, j \leq n$, let

$$\begin{aligned} r_{ij}(z) &= \sum_{k=1}^p z_{ki} z_{kj}, & r_{ij}(w) &= \sum_{k=1}^q w_{ki} w_{kj}, \\ \Delta_{ij}(z) &= \sum_{k=1}^p \frac{\partial^2}{\partial z_{ki} \partial z_{kj}}, & \Delta_{ij}(w) &= \sum_{k=1}^q \frac{\partial^2}{\partial w_{ki} \partial w_{kj}}, \\ E_{ij}(z) &= \sum_{k=1}^p z_{ki} \frac{\partial}{\partial z_{kj}}, & E_{ij}(w) &= \sum_{k=1}^q w_{ki} \frac{\partial}{\partial w_{kj}}. \end{aligned}$$

Further we let

$$\begin{aligned} U_{ij} &= E_{ij}(z) - E_{ji}(w) + \frac{p-q}{2} \delta_{ij}, \\ A_{ij} &= \Delta_{ij}(w) - r_{ij}(z), \\ B_{ij} &= r_{ij}(w) - \Delta_{ij}(z). \end{aligned}$$

Recall that $K = U(n)$ is a maximal compact subgroup of $G = Sp(2n, \mathbb{R})$. In the Fock model, the complexified action of $\widetilde{K}_{\mathbb{C}} = GL(n, \mathbb{C})^\sim$ is given by

$$(g \cdot f)(z, w) = (\det g)^{\frac{p-q}{2}} f(zg, wg^{t^{-1}}), \quad g \in GL(n, \mathbb{C})^\sim, \quad f \in \mathcal{P}[M_{p+q,n}(\mathbb{C})].$$

Recall also that we may choose the Harish-Chandra decomposition of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G :

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

such that

$$\begin{aligned} \Omega(\mathfrak{k}_{\mathbb{C}}) &= \text{span} \{U_{ij} | 1 \leq i, j \leq n\}, \\ \Omega(\mathfrak{p}^+) &= \text{span} \{A_{ij} | 1 \leq i \leq j \leq n\}, \\ \Omega(\mathfrak{p}^-) &= \text{span} \{B_{ij} | 1 \leq i \leq j \leq n\}. \end{aligned}$$

Here $\mathfrak{k}_{\mathbb{C}}$ is the complexification of \mathfrak{k} , the Lie algebra of $K = U(n)$.

For $1 \leq k \leq \min(p, n)$, we define the operator

$$D_{+k} = \det \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \cdots & \cdots & \cdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix}.$$

Note that this definition is unambiguous since A_{ij} 's commute among themselves.

For $1 \leq l \leq \min(q, n)$, we similarly define the operator

$$D_{-l} = \det \begin{pmatrix} B_{n-l+1, n-l+1} & \cdots & B_{n-l+1, n} \\ \cdots & \cdots & \cdots \\ B_{n, n-l+1} & \cdots & B_{n, n} \end{pmatrix}.$$

We also define the following polynomials q_{χ_1}, q_{χ_2} :

$$q_{\chi_1}(z) = \begin{cases} 1, & \text{if } \chi_1 = 1, \\ \det_p(z), & \text{if } \chi_1 = \det, p \leq n, \end{cases}$$

$$q_{\chi_2}(w) = \begin{cases} 1, & \text{if } \chi_2 = 1, \\ \det_q(w), & \text{if } \chi_2 = \det, q \leq n, \end{cases}$$

where

$$\det_p(z) = \det \begin{pmatrix} z_{11} & \cdots & z_{1p} \\ \cdots & \cdots & \cdots \\ z_{p1} & \cdots & z_{pp} \end{pmatrix},$$

$$\det_q(w) = \det \begin{pmatrix} w_{n-q+1, n-q+1} & \cdots & w_{n-q+1, n} \\ \cdots & \cdots & \cdots \\ w_{n, n-q+1} & \cdots & w_{n, n} \end{pmatrix}.$$

For $1 \leq k \leq p, 1 \leq l \leq q$ with $k+l \leq n$, and sets of non-negative integers $\{\alpha_i\}_{1 \leq i \leq k}, \{\beta_j\}_{1 \leq j \leq l}$, define the polynomials

(2.8)

$$D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) = (D_{+1}^{\alpha_1} \cdots D_{+k}^{\alpha_k} D_{-1}^{\beta_1} \cdots D_{-l}^{\beta_l})(q_{\chi_1} q_{\chi_2}).$$

When $\chi_1 = \det$ and $p \leq n$, we may require $k = p$ by making some of the α_i 's to be zero. Similarly for χ_2 . We note that D_i 's and D_{-j} 's commute for $i+j \leq n$.

It is easy to see that (up to the a sign, or $(-1)^{\alpha_1+2\alpha_2+\cdots+k\alpha_k}$ to be exact)

$$D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) = d_1(z)^{\alpha_1} \cdots d_k(z)^{\alpha_k} q_{\chi_1}(z) d_1(w)^{\beta_1} \cdots d_l(w)^{\beta_l} q_{\chi_2}(w),$$

where

$$d_k(z) = \det \begin{pmatrix} r_{11}(z) & \cdots & r_{1k}(z) \\ \cdots & \cdots & \cdots \\ r_{k1}(z) & \cdots & r_{kk}(z) \end{pmatrix},$$

and

$$d_l(w) = \det \begin{pmatrix} r_{n-l+1, n-l+1}(w) & \cdots & r_{n-l+1, n}(w) \\ \cdots & \cdots & \cdots \\ r_{n, n-l+1}(w) & \cdots & r_{n, n}(w) \end{pmatrix}.$$

In particular, $D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ transforms according to $\chi_1 \otimes \chi_2$ under the action of $L = O(p) \times O(q)$.

Note that in the case we are considering, the space of K -harmonics is given by

$$\mathcal{H}(K) = \{f \in \mathcal{P} \mid \sum_{k=1}^n \frac{\partial^2 f}{\partial z_{ik} \partial w_{jk}} = 0, \forall 1 \leq i \leq p, 1 \leq j \leq q\}.$$

Set

$$\begin{aligned} a_i &= \sum_{i \leq s \leq k} \alpha_s, & 1 \leq i \leq k, \\ b_j &= \sum_{j \leq t \leq l} \beta_t, & 1 \leq j \leq l, \end{aligned}$$

and

$$\epsilon_{\chi_1} = \begin{cases} 0, & \text{if } \chi_1 = \mathbf{1}, \\ 1, & \text{if } \chi_1 = \det. \end{cases}$$

Similarly we define ϵ_{χ_2} .

Proposition 2.9. *We have*

$$D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \in \mathcal{H}(K)^{(L; \chi_1 \otimes \chi_2)},$$

and it is a $\widetilde{K}_{\mathbb{C}}$ highest weight vector with the highest weight

$$\frac{p-q}{2} \mathbf{1}_n + (2a_1 + \epsilon_{\chi_1}, \dots, 2a_k + \epsilon_{\chi_1}, 0, \dots, 0, -2b_l - \epsilon_{\chi_2}, \dots, -2b_1 - \epsilon_{\chi_2}).$$

Consequently we have

$$\begin{aligned} U_{ij}(D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)) &= 0, & 1 \leq i < j \leq n, \\ U_{ii}(D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)) &= \lambda_i D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l), \end{aligned}$$

where

$$\lambda_i = \begin{cases} \frac{p-q}{2} + 2a_i + \epsilon_{\chi_1}, & 1 \leq i \leq k, \\ \frac{p-q}{2}, & k+1 \leq i \leq n-l, \\ \frac{p-q}{2} - 2b_{n-i+1} - \epsilon_{\chi_2}, & n-l+1 \leq i \leq n. \end{cases}$$

Proof. The fact that $D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ is K -harmonic is immediate. It is also clear that $D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ is invariant under the upper triangular matrices with ones in the diagonal, and transforms according to the given character of the diagonal subgroup of $\widetilde{K}_{\mathbb{C}} = GL(n, \mathbb{C})^\sim$. For the rest, we just have to observe that the derived action of e_{ij} is given by U_{ij} , where e_{ij} is the $n \times n$ matrix with one at the (i, j) entry and zeroes elsewhere. \square

We now prove a criteria for the non-vanishing of \widetilde{K} -types for a given distribution $S \in \mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$. Recall that P is the Siegel parabolic subgroup of G (see §3 for an explicit description).

Theorem 2.10. *Suppose that $S \in \mathcal{S}^*(M_{p+q, n}(\mathbb{R}))^{(H; \chi)}$ and S transforms according to a character of \widetilde{Q} , where $Q = P \cap K \simeq O(n) \subseteq K$. Let $\tau \in \widetilde{K}$ be such that the highest weight of τ^* is given by*

$$\frac{p-q}{2} \mathbf{1}_n + (2a_1 + \epsilon_{\chi_1}, \dots, 2a_k + \epsilon_{\chi_1}, 0, \dots, 0, -2b_l - \epsilon_{\chi_2}, \dots, -2b_1 - \epsilon_{\chi_2}),$$

where $a_i = \sum_{i \leq s \leq k} \alpha_s$, $b_j = \sum_{j \leq t \leq l} \beta_t$, and $\alpha_s, \beta_t \in \mathbb{Z}^+$. Then the following three conditions are equivalent:

- (a) τ occurs in the space $\overline{\Omega^*(\widetilde{K})S}$;
- (b) $S_\tau \neq 0$, where S_τ is the τ -isotypic component of S ;

(c) *the inner product*

$$\langle S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle \neq 0.$$

Proof. Clearly (a) is equivalent to (b).

By Lemma 2.2, $S_\tau \neq 0$ if and only if $p(S_\tau) \neq 0$. Here $p(S_\tau) \in \mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$ is the image of S_τ under the projection map p .

Note that $\mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$ is irreducible as a \widetilde{K} -module. See the discussions leading to Theorem 2.3. Denote $v_\tau = D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$, which is a \widetilde{K} -highest weight vector in $\mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$. We can write

$$p(S_\tau) = \lambda_\tau v_\tau + N_\tau,$$

where $\lambda_\tau \in \mathbb{C}$, and N_τ is a sum of \widetilde{A} -weight vectors in $\mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$ with \widetilde{A} -weights strictly less than that of v_τ . Here A is a maximal torus of K .

We shall show that $p(S_\tau) \neq 0$ if and only if $\lambda_\tau \neq 0$.

By choosing a Borel subgroup B of $K_{\mathbb{C}} = GL(n, \mathbb{C})$ appropriately, we may assume [28] that

$$\mathfrak{k}_{\mathbb{C}} = \bar{\mathfrak{b}} + \mathfrak{q},$$

where \mathfrak{q} is the complexified Lie algebra of Q , and $\bar{\mathfrak{b}}$ is the Borel subalgebra of $\mathfrak{k}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$ opposite to \mathfrak{b} .

By the Poincaré-Birkhoff-Witt theorem, we have

$$\mathcal{U}(\mathfrak{k}_{\mathbb{C}}) = \mathcal{U}(\bar{\mathfrak{b}})\mathcal{U}(\mathfrak{q}).$$

Suppose that $\lambda_\tau = 0$. Then $p(S_\tau) = N_\tau$. Since S transforms according to a character of \widetilde{Q} , $p(S_\tau)$ transforms under \widetilde{Q} in the same way. Thus we have

$$\mathcal{U}(\mathfrak{k}_{\mathbb{C}})p(S_\tau) = \mathcal{U}(\bar{\mathfrak{b}})\mathcal{U}(\mathfrak{q})p(S_\tau) \subseteq \mathcal{U}(\bar{\mathfrak{b}})p(S_\tau) = \mathcal{U}(\bar{\mathfrak{b}})N_\tau.$$

Now each vector in $\mathcal{U}(\bar{\mathfrak{b}})N_\tau$ is a sum of \widetilde{A} weight vectors with weights strictly less than that of v_τ , we see that $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})p(S_\tau) \neq \mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$, a contradiction to the fact that $\mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$ is irreducible as a \widetilde{K} -module.

We thus conclude that $p(S_\tau) \neq 0$ is the same thing as $\lambda_\tau \neq 0$, and the latter is equivalent to the statement that the inner product of S with v_τ is not zero. \square

Remark 2.11. *The argument above was first used in [30] in the context of multiplicity-free actions.*

3. \widetilde{K} -TYPES OF \widetilde{P} -EIGENDISTRIBUTIONS

Let P be the Siegel parabolic subgroup of $G = Sp(2n, \mathbb{R})$. We have $P \simeq MN$, where

$$M = \{m(a) = \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}, a \in GL(n, \mathbb{R})\},$$

$$N = \{n(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, b = b^t \in M_{n,n}(\mathbb{R})\}.$$

Let χ_0 be the character of

$$\widetilde{M} = \{(m(a), \epsilon) \mid a \in GL(n, \mathbb{R}), \epsilon = \pm 1\}$$

given by:

$$\chi_0(m(a), \epsilon) = \epsilon \cdot \begin{cases} 1, & \text{if } \det(a) > 0, \\ i, & \text{if } \det(a) < 0. \end{cases}$$

Note that χ_0 is of order 4.

We then have [5]

$$\begin{aligned} [\Omega(m(a), \epsilon)f](v) &= \chi_0(m(a), \epsilon)^\alpha |\det a|^{\frac{p+q}{2}} f(va), \quad a \in GL(n, \mathbb{R}), \\ [\Omega(n(b))](v) &= e^{\frac{i}{2} \operatorname{tr}(v^t I_{p,q} v b)} f(v), \quad f \in L^2(M_{p+q,n}(\mathbb{R})), v \in M_{p+q,n}(\mathbb{R}). \end{aligned}$$

Here $\alpha \equiv p - q \pmod{4}$, and tr denotes the trace of a square matrix.

We write a typical element $v \in M_{p+q,n}(\mathbb{R})$ as

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x = (x_{ij})_{p \times n} \in M_{p,n}(\mathbb{R}), y = (y_{ij})_{q \times n} \in M_{q,n}(\mathbb{R}).$$

For $1 \leq i, j \leq n$, let

$$\begin{aligned} r_{ij} &= (v^t I_{p,q} v)_{ij} = \sum_{k=1}^p x_{ki} x_{kj} - \sum_{k=1}^q y_{ki} y_{kj}, \\ E_{ij} &= \sum_{k=1}^p x_{ki} \frac{\partial}{\partial x_{kj}} + \sum_{k=1}^q y_{ki} \frac{\partial}{\partial y_{kj}} + \frac{p+q}{2} \delta_{ij}. \end{aligned}$$

For a character ξ of \widetilde{P} , let

$$\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(\widetilde{P}; \xi)} = \{S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R})) \mid \Omega^*(\tilde{p})S = \xi(\tilde{p})S, \tilde{p} \in \widetilde{P}\}$$

denote the space of ξ -eigendistributions for \widetilde{P} .

Suppose that we are given a tempered distribution

$$S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(\widetilde{P}; \xi_0 \otimes |\cdot|^\nu)},$$

where ξ_0 is a character of \widetilde{P} of order at most four, ν is a complex number, and $|\cdot|^\nu$ is the character of \widetilde{P} so that $(m(a), \epsilon) \mapsto |\det a|^\nu$. By differentiation, S satisfies the following system of equations:

$$\begin{cases} E_{ij} S = \nu \delta_{ij} S, \\ r_{ij} S = 0, \end{cases}$$

where $1 \leq i, j \leq n$. Here and after, δ_{ij} denotes the usual Kronecker symbol. More generally we may consider a formal vector $S \in \Omega^{-\infty}$ satisfying the same system of equations.

It turns out that the above system is easier to work with in the Fock model. Recall that there is an isomorphism between the Schrodinger model and the Fock model. This correspondence is such that

$$\begin{aligned} x_{ij} &\rightarrow \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z_{ij}} + z_{ij} \right), & \frac{\partial}{\partial x_{ij}} &\rightarrow \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z_{ij}} - z_{ij} \right), \\ y_{ij} &\rightarrow \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial w_{ij}} + w_{ij} \right), & \frac{\partial}{\partial y_{ij}} &\rightarrow \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial w_{ij}} - w_{ij} \right). \end{aligned}$$

See [2]. For the ease of notation, we shall use equality instead of the arrow sign to indicate the same operator in the two models.

Recall from §2 the notations U_{ij} , A_{ij} , B_{ij} , and $r_{ij}(z)$, $\Delta_{ij}(z)$, $E_{ij}(Z)$ etc..

Proposition 3.1. *The system of differential equations*

$$\begin{cases} E_{ij}S = \nu\delta_{ij}S, \\ r_{ij}S = 0 \end{cases}$$

in the Schrodinger model is equivalent to the system of differential equations

$$\begin{cases} B_{ij}S = (U_{ij} - \nu\delta_{ij})S, \\ A_{ji}S = (U_{ji} + \nu\delta_{ji})S \end{cases}$$

in the Fock model.

Proof. We compute

$$\begin{aligned} x_{ki}\frac{\partial}{\partial x_{kj}} &= \frac{1}{2}\left(\frac{\partial}{\partial z_{ki}} + z_{ki}\right)\left(\frac{\partial}{\partial z_{kj}} - z_{kj}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial z_{ki}\partial z_{kj}} + z_{ki}\frac{\partial}{\partial z_{kj}} - \frac{\partial}{\partial z_{ki}}z_{kj} - z_{ki}z_{kj}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial z_{ki}\partial z_{kj}} + z_{ki}\frac{\partial}{\partial z_{kj}} - z_{kj}\frac{\partial}{\partial z_{ki}} - z_{ki}z_{kj} - \delta_{ij}\right), \end{aligned}$$

and

$$\begin{aligned} x_{ki}x_{kj} &= \frac{1}{2}\left(\frac{\partial}{\partial z_{ki}} + z_{ki}\right)\left(\frac{\partial}{\partial z_{kj}} + z_{kj}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial z_{ki}\partial z_{kj}} + z_{ki}\frac{\partial}{\partial z_{kj}} + \frac{\partial}{\partial z_{ki}}z_{kj} + z_{ki}z_{kj}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial z_{ki}\partial z_{kj}} + z_{ki}\frac{\partial}{\partial z_{kj}} + z_{kj}\frac{\partial}{\partial z_{ki}} + z_{ki}z_{kj} + \delta_{ij}\right). \end{aligned}$$

Thus in the Fock model, the equation $E_{ij}S = \nu\delta_{ij}S$ becomes

$$(3.2) \quad (\Delta_{ij}(z) + E_{ij}(z) - E_{ji}(z) - r_{ij}(z) + \Delta_{ij}(w) + E_{ij}(w) - E_{ji}(w) - r_{ij}(w))S = 2\nu\delta_{ij}S,$$

and the equation $r_{ij}S = 0$ becomes

$$(3.3) \quad -\Delta_{ij}(w) - E_{ij}(w) - E_{ji}(w) - r_{ij}(w) + (p - q)\delta_{ij})S = 0.$$

Adding and subtracting the Equations (3.2) and (3.3), we get

$$\begin{cases} (\Delta_{ij}(z) + E_{ij}(z) - E_{ji}(w) - r_{ij}(w) + \frac{p-q}{2}\delta_{ij})S = \nu\delta_{ij}S, \\ (\Delta_{ij}(w) + E_{ij}(w) - E_{ji}(z) - r_{ij}(z) - \frac{p-q}{2}\delta_{ij})S = \nu\delta_{ij}S. \end{cases}$$

The result follows. \square

Note that (by a straightforward computation)

$$\begin{aligned} [E_{ij}(z), r_{kl}(z)] &= \delta_{jk}r_{il}(z) + \delta_{jl}r_{ki}(z), \\ [E_{ij}(z), \Delta_{kl}(z)] &= -(\delta_{ik}\Delta_{jl}(z) + \delta_{il}\Delta_{kj}(z)). \end{aligned}$$

Exactly the same commutation relations hold if we replace z by w . This easily implies the following

Lemma 3.4. *The differential operators $\{U_{ij}\}$, $\{A_{ij}\}$, $\{B_{ij}\}$ satisfy the following commutation relations:*

$$\begin{aligned} [U_{ij}, A_{kl}] &= \delta_{jk}A_{il} + \delta_{jl}A_{ki}, \\ [U_{ij}, B_{kl}] &= -(\delta_{ik}B_{jl} + \delta_{il}B_{kj}). \end{aligned}$$

Given a matrix $X = (X_{ij})_{n \times n}$, where $\{X_{ij}\}$ is a set of (not necessarily commuting) linear operators of a fixed vector space, we define its **column determinant**

$$(3.5) \quad \det(X) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) X_{\sigma(1)1} X_{\sigma(2)2} \cdots X_{\sigma(n)n},$$

and also its **row determinant**

$$(3.6) \quad \text{rdet}(X) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) X_{1\sigma(1)} X_{2\sigma(2)} \cdots X_{n\sigma(n)}.$$

Thus $\text{rdet}(X) = \det(X^t)$. Furthermore if $\{X_{ij}\}$ commute among themselves, then $\text{rdet}(X) = \det(X)$.

In the appendix of this paper, certain identities of Capelli type are proved. For the convenience of the reader, we include special cases of these identities below.

Theorem 3.7. Identities of Capelli type

- (a) *Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying $X_{ij} = X_{ji}$, and the commutation relations*

$$[E_{ij}, X_{kl}] = \epsilon(\delta_{jk}X_{il} + \delta_{jl}X_{ki}),$$

for some complex number ϵ . Suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i, j \leq n.$$

Then we have

$$\det(X_{ij})S = \det(E_{n+1-i, n+1-j} + \epsilon(n-i)\delta_{ij})S,$$

and

$$\det(X_{n+1-i, n+1-j})S = \det(E_{i,j} + \epsilon(n-i)\delta_{ij})S.$$

- (b) *Let $E = (E_{ij})_{n \times n}$, $Y = (Y_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying $Y_{ij} = Y_{ji}$, and the commutation relations*

$$[E_{ij}, Y_{kl}] = \epsilon(\delta_{ik}Y_{jl} + \delta_{il}Y_{kj}),$$

for some complex number ϵ . Suppose that $S \in V$ satisfies

$$Y_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i, j \leq n.$$

Then we have

$$\text{rdet}(Y_{ij})S = \text{rdet}(E_{n+1-i, n+1-j} + \epsilon(n-i)\delta_{ij})S,$$

and

$$\text{rdet}(Y_{n+1-i, n+1-j})S = \text{rdet}(E_{i,j} + \epsilon(n-i)\delta_{ij})S.$$

Recall the (pre)unitary structure \langle, \rangle of \mathcal{P} , which induces a pairing between $\Omega^{-\infty}$ and \mathcal{P} , still denoted by \langle, \rangle .

The main result of this section is the following

Theorem 3.8. *Suppose that $S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$ satisfies*

$$\begin{cases} E_{ij}S = \nu\delta_{ij}S, \\ r_{ij}S = 0, \end{cases}$$

where $1 \leq i, j \leq n$ and $\nu \in \mathbb{C}$. Let χ be a character of $H = O(p, q)$. For $1 \leq k \leq p$, $1 \leq l \leq q$ with $k + l \leq n$, and sets of non-negative integers $\{\alpha_i\}_{1 \leq i \leq k}$, $\{\beta_j\}_{1 \leq j \leq l}$, let $D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \in \mathcal{H}(K)_\tau^{(L; \chi_1 \otimes \chi_2)}$ be the \widetilde{K} highest weight vector defined in Equation (2.8). Denote

$$\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) = \langle S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle.$$

Then we have the recursive formula:

$$\begin{aligned} & \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \\ &= \lambda(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \prod_{i=1}^k (2a_i + \epsilon_{\chi_1} + \frac{p-q}{2} - (\nu + i + 1)) \\ &= \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \prod_{j=1}^l (-2b_j - \epsilon_{\chi_2} + \frac{p-q}{2} + (\nu + j + 1)). \end{aligned}$$

Here $a_i = \sum_{i \leq s \leq k} \alpha_s$ and $b_j = \sum_{j \leq t \leq l} \beta_t$.

Proof. With respect to the pairing \langle, \rangle , we have the adjoint relations:

$$\left(\frac{\partial}{\partial z_{ij}} \right)^* = z_{ij}, \quad \left(\frac{\partial}{\partial w_{ij}} \right)^* = w_{ij}.$$

See [2]. Thus we have

$$\begin{aligned} A_{ij}^* &= B_{ij}, & B_{ij}^* &= A_{ij}, \\ U_{ij}^* &= U_{ji}. \end{aligned}$$

We compute (using $A_{ij}^* = B_{ij}$):

$$\begin{aligned} & \langle S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle \\ &= \langle S, D_{+k} D_\chi(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \rangle \\ &= \langle \text{rdet}(B_{ij})_{k \times k} S, D_\chi(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \rangle. \end{aligned}$$

From Proposition 3.1 and Lemma 3.4, we have $B_{ij}S = (U_{ij} - \nu\delta_{ij})S$ and $[U_{ij}, B_{kl}] = -(\delta_{ik}B_{jl} + \delta_{il}B_{kj})$, and so by Theorem 3.7 (b), we have

$$\text{rdet}(B_{ij})_{k \times k} S = \text{rdet}(U_{k+1-i, k+1-j} - (\nu + k - i)\delta_{ij})_{k \times k} S.$$

In the expansion formula for the above row determinant, every term except the diagonal term starts with a U_{st} with $s > t$. Since $U_{st}^* = U_{ts}$, and U_{ts} annihilates any \widetilde{K} highest weight vector, we see that only the diagonal term

in the expansion of $\text{rdet}(U_{k+1-i, k+1-j} - (\nu + k - i)\delta_{ij})_{k \times k} S$ will contribute to the inner product

$$\langle \text{rdet}(U_{k+1-i, k+1-j} - (\nu + k - i)\delta_{ij})_{k \times k} S, D_\chi(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \rangle .$$

Thus from Proposition 2.9, we have

$$\begin{aligned} & \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \\ &= \langle S, \prod_{i=1}^k (U_{ii} - (\nu + i - 1)) D_\chi(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \rangle \\ &= \prod_{i=1}^k (2(a_i - 1) + \epsilon_{\chi_1} + \frac{p-q}{2} - (\nu + i - 1)) \lambda(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \\ &= \prod_{i=1}^k (2a_i + \epsilon_{\chi_1} + \frac{p-q}{2} - (\nu + i + 1)) \lambda(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l). \end{aligned}$$

We also compute $\langle S, D(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle$ (using $B_{ij}^* = A_{ij}$):

$$\begin{aligned} & \langle S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle \\ &= \langle S, D_{-l} D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \rangle \\ &= \langle \det(A_{n-l+i, n-l+j})_{l \times l} S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \rangle . \end{aligned}$$

From Proposition 3.1 and Lemma 3.4, we have $A_{ij} S = (U_{ij} + \nu \delta_{ij}) S$ and $[U_{ij}, A_{kl}] = \delta_{jk} A_{il} + \delta_{jl} A_{ki}$, and so by Theorem 3.7 (a), we have

$$\det(A_{n-l+i, n-l+j})_{l \times l} S = \det(U_{n-l+i, n-l+j} + (\nu + l - i)\delta_{ij})_{l \times l} S.$$

In the expansion formula for the above column determinant, every term except the diagonal term starts with a U_{st} with $s > t$. Since $U_{st}^* = U_{ts}$, and U_{ts} annihilates any \widetilde{K} highest weight vector, we see that only the diagonal term in the expansion of $\det(U_{n-l+i, n-l+j} + (\nu + l - i)\delta_{ij})_{l \times l} S$ will contribute to the inner product

$$\langle \det(U_{n-l+i, n-l+j} + (\nu + l - i)\delta_{ij})_{l \times l} S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \rangle .$$

Thus from Proposition 2.9, we have

$$\begin{aligned} & \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \\ &= \langle S, \prod_{j=1}^l (U_{n-l+j, n-l+j} + (\nu + l - j)) D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \rangle \\ &= \prod_{j=1}^l (-2(b_{l-j+1} - 1) - \epsilon_{\chi_2} + \frac{p-q}{2} + (\nu + l - j)) \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \\ &= \prod_{j=1}^l (-2b_j - \epsilon_{\chi_2} + \frac{p-q}{2} + (\nu + j + 1)) \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1). \end{aligned}$$

This completes the proof of the theorem. \square

Remark 3.9. *We comment on how the recursive formula of Theorem 3.8 may be applied. Note that in our notations, we have*

$$\begin{aligned}\lambda(\alpha_1, \dots, \alpha_{k-1}, 0; \beta_1, \dots, \beta_l) &= \lambda(\alpha_1, \dots, \alpha_{k-1}; \beta_1, \dots, \beta_l), \\ \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_{l-1}, 0) &= \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_{l-1}).\end{aligned}$$

We may reduce the computation of $\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ to that of $\lambda(0; 0)$ from the recursion. We also note the following: Suppose that $0 \neq S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(H;\chi)}$ satisfies the conditions of Theorem 2.10, then corresponding to one of the non-zero \widetilde{K} -types, certain $\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l)$ must be non-zero. Our recursion formula implies in particular that $\lambda(0; 0) \neq 0$.

We give an immediate application of our inner product formula. For $S = \delta$, the Dirac measure at the origin of $M_{p+q,n}(\mathbb{R})$, we have $\nu = -\frac{p+q}{2}$. In the notations of Theorem 3.8, we note that for $i \leq k \leq p$,

$$2a_i + \frac{p-q}{2} - (\nu + i + 1) = 2a_i + p - i - 1 > 0, \quad \text{if } a_i > 0,$$

and for $j \leq l \leq q$,

$$-2b_j + \frac{p-q}{2} + (\nu + i + 1) = -2b_j - q + j + 1 < 0, \quad \text{if } b_j > 0.$$

This implies that

$$\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) = \langle S, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle \neq 0.$$

In view of Theorem 2.10, we have the following

Corollary 3.10. *The Dirac measure δ has a non-zero projection to every \widetilde{K} -type τ listed in Theorem 2.1.*

Remark 3.11. *The above statement was proved in [31] by using the original Capelli identity. Together with Theorem 2.1, it implies the result of [18] alluded to in the Introduction, namely*

$$\Omega^*(\delta) = \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H.$$

4. INVARIANT MEASURES AND DEGENERATE PRINCIPAL SERIES

Recall the regular part of the null cone

$$\mathcal{N}_n = \{v \in \mathcal{N} \mid \text{rank}(v) = n\},$$

where $p, q \geq n$. Recall also the invariant measure $d\mu_n$ on \mathcal{N}_n from the Introduction.

First we assume that $p + q > 2n$. We shall see that the measure $d\mu_n$ can be extended to a tempered distribution I on $M_{p+q,n}(\mathbb{R})$ as follows.

Let $\text{Sym}_n^2(\mathbb{R})$ be the space of $n \times n$ real symmetric matrices. Note that for any $v \in M_{p+q,n}(\mathbb{R})$, $v^t I_{p,q} v \in \text{Sym}_n^2(\mathbb{R})$.

For $f \in \mathcal{S}(M_{p+q,n}(\mathbb{R}))$ and a neighborhood \mathcal{U} of the $n \times n$ zero matrix in $\text{Sym}_n^2(\mathbb{R})$, consider the integral

$$I(f)\mathcal{U} = \int_{v^t I_{p,q} v \in \mathcal{U}} f(v) dv.$$

For $v = \begin{pmatrix} x \\ y \end{pmatrix} \in M_{p+q,n}(\mathbb{R})$, write (except for a set of measure zero)

$$x = AR^{\frac{1}{2}},$$

where A is an element of the Stiefel manifold defined by

$$S^{p,n} = \{x \in M_{p,n}(\mathbb{R}) \mid x^t x = I_n\} \simeq O(p)/O(p-n),$$

and $R \in P_n$, the space of $n \times n$ positive definite real symmetric matrices. Similarly we write

$$y = BS^{\frac{1}{2}}, \quad B \in S^{q,n}, \quad S \in P_n.$$

Then up to constants, we have [8]

$$dx = (\det R)^{\frac{p-n-1}{2}} dR dA, \quad dy = (\det S)^{\frac{q-n-1}{2}} dS dB,$$

where dA is the $O(p)$ -invariant metric on $S^{p,n}$, and likewise for dB . Thus we have

$$I(f)\mathcal{U} = \int_{A,B,R-S} f \left(\begin{pmatrix} AR^{\frac{1}{2}} \\ BS^{\frac{1}{2}} \end{pmatrix} \right) (\det R)^{\frac{p-n-1}{2}} (\det S)^{\frac{q-n-1}{2}} dR dA dS dB,$$

where the integral is over the set $A \in S^{p,n}$, $B \in S^{q,n}$, $R-S \in \mathcal{U}$. By making the change of variables: $U = R-S$, we see that

$$I(f)\mathcal{U} = \int_{A,B,U} f \left(\begin{pmatrix} AR^{\frac{1}{2}} \\ B(R-U)^{\frac{1}{2}} \end{pmatrix} \right) (\det R)^{\frac{p-n-1}{2}} \det(R-U)^{\frac{q-n-1}{2}} dR dA dU dB.$$

Therefore we see that the following limit is well-defined:

$$(4.1) \quad I(f) = \lim_{\mathcal{U} \rightarrow 0} \frac{1}{\text{vol}(\mathcal{U})} I(f)\mathcal{U},$$

where $\text{vol}(\mathcal{U})$ is the volume of \mathcal{U} with respect to the standard Euclidean metric in $\text{Sym}_n^2(\mathbb{R})$. Furthermore we have

$$I(f) = \int_{A \in S^{p,n}, B \in S^{q,n}, R \in P_n} f \left(\begin{pmatrix} AR^{\frac{1}{2}} \\ BR^{\frac{1}{2}} \end{pmatrix} \right) (\det R)^{\frac{p+q-2n-2}{2}} dR dA dB.$$

Since we are assuming $p+q > 2n$, we see that the above defines a tempered distribution I . Clearly I is $H = O(p,q)$ -invariant, and is supported on \mathcal{N}_n .

Recall the Siegel parabolic subgroup $P \simeq MN$. For $\widetilde{m}_a \in \widetilde{M}$, we have

$$\Omega(\widetilde{m}_a) = \chi_0(\widetilde{m}_a)^\alpha |\det a|^{\frac{p+q}{2}} \lambda(a), \quad a \in GL(n, \mathbb{R}),$$

where λ denotes the (unnormalized) action of $GL(n, \mathbb{R})$ on $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$:

$$(4.2) \quad (\lambda(a)\psi)(x, y) = \psi(xa, ya), \quad \psi \in \mathcal{S}(M_{p+q,n}(\mathbb{R})),$$

and $\alpha \equiv p - q \pmod{4}$. See §3.

A straightforward computation using Equation (4.1) yields

$$\begin{aligned}\lambda^*(a)I &= |\det a|^{p+q-(n+1)}I, & a \in GL(n, \mathbb{R}), \\ \Omega^*(n)I &= I, & n \in N.\end{aligned}$$

Therefore we have

$$\Omega^*(\tilde{p})I = \chi_0(\widetilde{m_a})^{-\alpha} |\det a|^{\frac{p+q}{2}-(n+1)}I, \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

Now assume that $p = q = n$. Let $S(M_{p+q,n}(\mathbb{R})) \subseteq \mathcal{S}(M_{p+q,n}(\mathbb{R}))$ be the subspace of Schwartz functions on $M_{p+q,n}(\mathbb{R})$ which correspond to polynomials in the Fock model of the oscillator representation Ω . Namely $f \in S(M_{p+q,n}(\mathbb{R}))$ if and only if f is of the form $p\phi_0$, where p is a polynomial and ϕ_0 is the so-called Gaussian given by

$$\phi_0(v) = e^{-\frac{\text{tr}(x^t x) + \text{tr}(y^t y)}{2}}, \quad v = \begin{pmatrix} x \\ y \end{pmatrix} \in M_{p+q,n}(\mathbb{R}).$$

For $f \in S(M_{p+q,n}(\mathbb{R}))$, and $s \in \mathbb{C}$, consider the integral

$$J_s(f) = \int_{A \in S^{p,n}, B \in S^{q,n}, R \in P_n} f \begin{pmatrix} AR^{\frac{1}{2}} \\ BR^{\frac{1}{2}} \end{pmatrix} (\det R)^s dR dA dB.$$

Note that when $f = p\phi_0$, we have

$$J_s(f) = \int_{A \in S^{p,n}, B \in S^{q,n}, R \in P_n} p \begin{pmatrix} AR^{\frac{1}{2}} \\ BR^{\frac{1}{2}} \end{pmatrix} e^{-R} (\det R)^s dR dA dB.$$

From the theory of Zeta integrals [6], we know that $J_s(f)$ is absolutely convergent if $\text{Re}(s) > -1$, and $J_s(f)$ can be analytically continued in the complex plane, except at $s = -\frac{n+2-j}{2} - \mathbb{Z}^+$ for $j = 1, 2, \dots, n$, where $J_s(f)$ has poles. Furthermore $J_s(f)$ has at most a simple pole at $s = -1$.

We define for $f \in S(M_{p+q,n}(\mathbb{R}))$

$$J(f) = \lim_{s \rightarrow -1} (s+1)J_s(f),$$

the residue of $J_s(f)$ at $s = -1$. Note that

$$J_s(\phi_0) = c \prod_{j=1}^n \Gamma(s + \frac{n+2-j}{2}),$$

where c is a non-zero number. See [8]. Thus $J_s(\phi_0)$ has a non-zero residue at $s = -1$. In particular $J \neq 0$. The association $f \mapsto J(f)$ for $f \in S(M_{p+q,n}(\mathbb{R}))$ therefore defines an element $J \in \Omega^{-\infty}$, the space of formal vectors of Ω . We clearly have

$$\Omega^*(\tilde{p})J = |\det a|^{-1}J, \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

We shall see in the latter part of this section that J is in fact a tempered distribution on $M_{p+q,n}(\mathbb{R})$.

We summarize the above discussions by the following

Proposition 4.3. *Assume that $p, q \geq n$. Then*

- (a) *If $p + q > 2n$, then $d\mu_n$ can be extended to a tempered distribution $I \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$ such that*

$$\Omega^*(\tilde{p})I = \chi_0(\widetilde{m_a})^{-\alpha} |\det a|^{\frac{p+q}{2} - (n+1)} I, \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

- (b) *If $p = q = n$, then $d\mu_n$ can be extended to a formal vector $J \in (\Omega^{-\infty})^{(h,L)}$ such that*

$$\Omega^*(\tilde{p})J = |\det a|^{-1} J, \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

In a similar way, we have

Proposition 4.4. *Assume that $p \geq q = n$. Then*

- (a) *If $p > q$, then $d\nu_q$ can be extended to a tempered distribution $I' \in \mathcal{S}^*(M_{p+q,q}(\mathbb{R}))^{(H;\epsilon)}$ such that*

$$\Omega^*(\tilde{p})I' = \chi_0(\widetilde{m_a})^{-\alpha+2} |\det a|^{\frac{p-q}{2} - 1} I', \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

- (b) *If $p = q$, then $d\nu_q$ can be extended to a formal vector $J' \in (\Omega^{-\infty})^{(h,L;\epsilon)}$ such that*

$$\Omega^*(\tilde{p})J' = \text{sgn}(\det a) |\det a|^{-1} J', \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

Remark 4.5. *Note that $\chi_0(\widetilde{m_a})^2 = \text{sgn}(\det a)$ for $a \in GL(n, \mathbb{R})$.*

To minimize notations, we shall use $d\mu_n$ and $d\nu_q$ to represent the corresponding tempered distributions (or formal vectors) given in Propositions 4.3 and 4.4.

We now define some degenerate principal series representations of \tilde{G} .

For $\sigma \in \mathbb{C}$ and $\alpha = 0, 1, 2, 3$, define the following character of \tilde{P} :

$$\chi_\alpha(\sigma)(\tilde{p}) = \chi_0(\widetilde{m_a})^\alpha |\det a|^\sigma, \quad \tilde{p} = \widetilde{m_a}n \in \tilde{P}.$$

Let $I_\alpha(\sigma)$ be the representation of \tilde{G} induced from the character $\chi_\alpha(\sigma)$ of \tilde{P} as follows:

$$\begin{aligned} I_\alpha(\sigma) &= \text{Ind}_{\tilde{P}}^{\tilde{G}}(\chi_\alpha(\sigma)) \\ &= \{f \in C^\infty(\tilde{G}) \mid f(\tilde{p}\tilde{g}) = \chi_\alpha(\sigma + \rho_n)(\tilde{p})f(\tilde{g}), \tilde{p} = \widetilde{m_a}n \in \tilde{P}\}, \end{aligned}$$

where $\rho_n = \frac{n+1}{2}$. The group \tilde{G} acts on $I_\alpha(\sigma)$ by right translation.

We will be concerned with the following situations:

$$(a) : p, q \geq n, \quad \text{and} \quad (b) : p \geq q = n.$$

Let $S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))$ be given by

$$S = \begin{cases} d\mu_n, & \text{in case (a),} \\ d\nu_q, & \text{in case (b).} \end{cases}$$

Then we have (from Propositions 4.3 and 4.4)

$$S \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{(\tilde{P}; \chi_\alpha(\sigma)^{-1})},$$

where

$$\sigma = -\nu, \begin{cases} \nu = \frac{p+q}{2} - (n+1), & \alpha \equiv p - q \pmod{4}, \text{ in case (a),} \\ \nu = \frac{p-q}{2} - 1, & \alpha \equiv p - q + 2 \pmod{4}, \text{ in case (b).} \end{cases}$$

For $\phi \in \mathcal{S}(M_{p+q,n}(\mathbb{R}))$, consider the function $\Phi_S(\phi)$ on \tilde{G} defined by:

$$(4.6) \quad \Phi_S(\phi)(\tilde{g}) = S(\Omega(\tilde{g})\phi), \quad \tilde{g} \in \tilde{G}.$$

Then $\Phi_S(\phi) \in I_\alpha(s)$, where $s = \sigma - \rho_n$. Thus we have defined a map

$$\Phi_S : \mathcal{S}(M_{p+q,n}(\mathbb{R})) \rightarrow I_\alpha(s).$$

This map is clearly \tilde{G} -equivariant.

Let n_0 (resp., n_1) be the maximal even integer (resp., odd integer) less than or equal to n . We are now in a position to prove the following result, which implies Theorem 1.1 in the Introduction.

Theorem 4.7. *Assume that $p, q \geq n$. Then*

- (a) τ is a \tilde{K} -type in $\Omega^*(d\mu_n)$ if and only if the highest weight of τ^* is of the form

$$(\lambda_1, \dots, \lambda_n), \quad \lambda_i \equiv \frac{p-q}{2} \pmod{2},$$

and

$$\begin{cases} \lambda_1 \leq \frac{p+q}{2} - (n+1), \\ \lambda_{n_1} \geq n_1 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } n - q \equiv 1, p \equiv 0 \pmod{2},$$

$$\begin{cases} \lambda_2 \leq \frac{p+q}{2} - n, \\ \lambda_{n_0} \geq n_0 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } n - q \equiv 0, p \equiv 1 \pmod{2},$$

$$\begin{cases} \lambda_1 \leq \frac{p+q}{2} - (n+1), \\ \lambda_{n_0} \geq n_0 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } n - q \equiv 1, p \equiv 1 \pmod{2},$$

$$\begin{cases} \lambda_2 \leq \frac{p+q}{2} - n, \\ \lambda_{n_1} \geq n_1 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } n - q \equiv 0, p \equiv 0 \pmod{2}.$$

- (b) The image of $\Phi_{d\mu_n}$ in $I_\alpha(s)$ contains a \tilde{K} -type τ if and only if the highest weight of τ is of the form described above. Furthermore this image is irreducible and is the unique irreducible submodule of $I_\alpha(s)$.
- (c) $\Omega^*(d\mu_n)$ is irreducible, and is therefore the unique irreducible \tilde{G} -submodule of $\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$.

Proof. (a) Since $\Omega^*(d\mu_n) \subseteq \mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^H$, a \tilde{K} -type τ can occur in $\Omega^*(d\mu_n)$ only if the highest weight of τ^* is of the form

$$(4.8) \quad \begin{aligned} (\lambda_1, \dots, \lambda_n) &= \frac{p-q}{2} \mathbf{1}_n + (2a_1, \dots, 2a_k, 0, \dots, 0, -2b_l, \dots, -2b_1), \\ &= \frac{p-q}{2} \mathbf{1}_n + (\xi_1, \dots, \xi_n), \end{aligned}$$

where $a_i = \sum_{i \leq s \leq k} \alpha_s$, $b_j = \sum_{j \leq t \leq l} \beta_t$, and $\alpha_s, \beta_t \in \mathbb{Z}^+$. See Theorem 2.1.

Since $d\mu_n$ is a \widetilde{P} -eigendistribution, we have $\Omega^*(d\mu_n) = \overline{\langle \Omega^*(\widetilde{K})d\mu_n \rangle}$, and $d\mu_n$ transforms according to a character of \widetilde{Q} , where $Q = P \cap K \simeq O(n) \subseteq K$. By Theorem 2.10, a \widetilde{K} -type τ given in Equation (4.8) occurs in $\Omega^*(d\mu_n)$ if and only if the inner product

$$\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) = \langle d\mu_n, D_\chi(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \rangle \neq 0.$$

By Theorem 3.8, we have the recursive formula:

$$\begin{aligned} & \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \\ &= \lambda(\alpha_1, \dots, \alpha_k - 1; \beta_1, \dots, \beta_l) \prod_{i=1}^k \left(2a_i + \frac{p-q}{2} - (\nu + i + 1) \right) \\ &= \lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l - 1) \prod_{j=1}^l \left(-2b_j + \frac{p-q}{2} + (\nu + j + 1) \right), \end{aligned}$$

where (the normalized homogeneity degree) $\nu = \frac{p+q}{2} - (n+1)$. See Proposition 4.3.

Observe that $2a_i + \frac{p-q}{2} - (\nu + i + 1) = 2a_i - q + n - i$. Thus if $n - q \equiv 1 \pmod{2}$, then $2a_i - q + n - i \neq 0$ for even i , and in order for $\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \neq 0$, we must have $\xi_1 = 2a_1 \leq q - n - 1$. Similarly by examining $-2b_j + \frac{p-q}{2} + (\nu + j + 1)$, we see that if $p \equiv 0 \pmod{2}$, then in order for $\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \neq 0$, we must have $\xi_{n_1} \geq n_1 - p + 1$. Conversely since $a_1 \geq \dots \geq a_k$, $b_1 \geq \dots \geq b_l$, we see that if $\xi_1 \leq q - n - 1$ and $\xi_{n_1} \geq n_1 - p + 1$, then $\lambda(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_l) \neq 0$. We thus get the required inequality if $n - q \equiv 1$ and $p \equiv 0 \pmod{2}$. The other three cases are entirely similar.

(b) Note that for $\phi \in \mathcal{S}(M_{p+q,n}(\mathbb{R}))$ and for $\tilde{k} \in \widetilde{K}$, we have $\Phi_{d\mu_n}(\phi)(\tilde{k}) = (\Omega^*(\tilde{k}^{-1})d\mu_n)(\phi)$. Thus a \widetilde{K} -type τ occurs in the image of $\Phi_{d\mu_n}$ in $I_\alpha(s)$ if and only if τ^* occurs in $\langle \Omega^*(\widetilde{K})d\mu_n \rangle$, and is therefore given as in part (b).

Now the structure of $I_\alpha(s)$ is completely known [17, 20, 23, 26]. We note that their results cover the symplectic group, but the metaplectic case can be extended easily. In all the four cases (which correspond to the parity of $n + \alpha$ and the parity of $s + \rho_n + \frac{\alpha}{2}$), straightforward verifications show that for $p, q \geq n$, the corresponding $I_\alpha(s)$ has a unique irreducible submodule, and the \widetilde{K} -types of this submodule coincide with the \widetilde{K} -types of the image of $\Phi_{d\mu_n}$ described above. Since $I_\alpha(s)$ is \widetilde{K} -multiplicity free, part (b) follows.

Note that $\Omega^*(d\mu_n)$ is the dual of the image of $\Phi_{d\mu_n}$ in $I_\alpha(s)$. Now part (c) clearly follows from part (b). \square

Remark 4.9. *The recursive formula implies in particular that the “generalized Fourier coefficients” of the formal vector $d\mu_n$ (for $p = q = n$) grow at most polynomially, and therefore $d\mu_n$ actually defines a tempered distribution in this case. See [14, 31] for such a characterization of tempered distributions.*

In exactly the same way, we have the following result which implies Theorem 1.3 in the Introduction. Its proof will be omitted.

Theorem 4.10. *Assume that $p \geq q = n$. Then*

- (a) τ is a \widetilde{K} -type in $\Omega^*(d\nu_q)$ if and only if the highest weight of τ^* is of the form

$$(\lambda_1, \dots, \lambda_q), \quad \lambda_i \equiv \frac{p-q}{2} + 1 \pmod{2},$$

and

$$\begin{cases} \lambda_1 \leq \frac{p-q}{2} - 1, \\ \lambda_{q_1} \geq q_1 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } p > q, p \equiv 1 \pmod{2},$$

$$\begin{cases} \lambda_1 \leq \frac{p-q}{2} - 1, \\ \lambda_{q_0} \geq q_0 + 1 - \frac{p+q}{2}, \end{cases} \quad \text{if } p > q, p \equiv 0 \pmod{2},$$

$$\lambda_1 = \dots = \lambda_{q-2} = -1, \quad \text{if } p = q.$$

- (b) The image of $\Phi_{d\nu_q}$ in $I_\alpha(s)$ contains a \widetilde{K} -type τ if and only if the highest weight of τ is of the form described above. Furthermore when $p > q$, this image is the unique irreducible submodule of $I_\alpha(s)$, and when $p = q$, it is one of the two irreducible submodules of the corresponding $I_\alpha(s)$.
- (c) $\Omega^*(d\nu_q)$ is irreducible, and is therefore the unique irreducible \widetilde{G} -submodule of $\mathcal{S}^*(M_{p+q,q}(\mathbb{R}))^{(H;\epsilon)}$.

Remark 4.11. *When $p = q$, the Harish-Chandra module of $\Omega^*(d\nu_q)$ is isomorphic to that of the unitary highest weight module of highest weight $-\mathbf{1}_q$. Also the other irreducible submodule of $I_\alpha(s)$ is the image of $\Phi_{d\gamma_q}$, where $d\gamma_q \in \mathcal{S}^*(M_{2q,q}(\mathbb{R}))^{(H;\epsilon')}$, and $\epsilon' = \epsilon \otimes \det$.*

5. APPLICATIONS

We highlight the following results which are immediate consequences of Theorems 4.7 and 4.10. Both our applications in this section will be derived from these two finite-dimensionality assertions.

- (a) $p, q \geq n$, and $p, q \equiv n + 1 \pmod{2}$:

$$\overline{\langle \Omega^*(g)d\mu_n, g \in Sp(2n, \mathbb{R}) \rangle}$$

is finite dimensional, and carries an irreducible $Sp(2n, \mathbb{R})$ -module of highest weight $(\frac{p+q}{2} - (n+1))\mathbf{1}_n$.

- (b) $p > q = n$, and $p \equiv q \pmod{2}$:

$$\overline{\langle \Omega^*(g)d\nu_q, g \in Sp(2q, \mathbb{R}) \rangle}$$

is finite dimensional, and carries an irreducible $Sp(2q, \mathbb{R})$ -module of highest weight $(\frac{p-q}{2} - 1)\mathbf{1}_q$.

Denote

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{Z}^+)^n \mid \lambda_1 \geq \dots \geq \lambda_n \geq 0\}.$$

Recall that for $\lambda \in \Lambda_n^+$, ρ_λ is the irreducible finite dimensional representation of $O(p, q)$ parameterized by

$$(\lambda_1, \dots, \lambda_n, \underbrace{0, \dots, 0}_{[\frac{p+q}{2}-n]}; 1),$$

and π_λ is the irreducible finite dimensional representation of $Sp(2n, \mathbb{R})$ with the highest weight λ .

Let $\mathcal{P}(M_{p+q, n}(\mathbb{R}))$ be the algebra of polynomials on $M_{p+q, n}(\mathbb{R})$, where $H = O(p, q)$ acts by

$$(h \cdot f)(v) = f(h^{-1}v), \quad h \in H, f \in \mathcal{P}(M_{p+q, n}(\mathbb{R})), v \in M_{p+q, n}(\mathbb{R}).$$

Let $\mathcal{H}(M_{p+q, n}(\mathbb{R}))$ be the subspace of $O(p, q)$ -harmonics. It is well-known that ρ_λ^* can be realized in $\mathcal{H}(M_{p+q, n}(\mathbb{R}))$. See [19]. We let $\mathcal{H}(M_{p+q, n}(\mathbb{R}))_{\rho_\lambda^*}$ be its ρ_λ^* -isotypic component.

We prove the following result which implies Theorem 1.4 in the Introduction.

Theorem 5.1. Correspondence of finite dimensional representations

- (a) Assume that $p, q \geq n$, and $p, q \equiv n + 1 \pmod{2}$. For $\lambda \in \Lambda_n^+$, let b_λ be a non-zero polynomial in $\mathcal{H}(M_{p+q, n}(\mathbb{R}))_{\rho_\lambda}$, and define the $O(p, q) \times Sp(2n, \mathbb{R})$ -invariant subspace

$$F_\lambda = \overline{\langle \Omega^*(g)(b_\lambda d\mu_n), g \in O(p, q) \times Sp(2n, \mathbb{R}) \rangle}.$$

Then F_λ is finite dimensional and irreducible under the joint action of $O(p, q)$ and $Sp(2n, \mathbb{R})$, and as $O(p, q) \times Sp(2n, \mathbb{R})$ modules, we have

$$F_\lambda \simeq \rho_\lambda^* \otimes \pi_{\lambda + (\frac{p+q}{2} - (n+1))\mathbf{1}_n}^*.$$

In particular, ρ_λ and $\pi_{\lambda + (\frac{p+q}{2} - (n+1))\mathbf{1}_n}$ correspond to each other under the local theta correspondence.

- (b) Assume that $p > q$ and $p \equiv q \pmod{2}$. For $\lambda \in \Lambda_q^+$, let b_λ be a non-zero polynomial in $\mathcal{H}(M_{p+q, q}(\mathbb{R}))_{\rho_\lambda}$, and define the $O(p, q) \times Sp(2q, \mathbb{R})$ -invariant subspace

$$F_\lambda = \overline{\langle \Omega^*(g)(b_\lambda d\nu_q), g \in O(p, q) \times Sp(2q, \mathbb{R}) \rangle}.$$

Then F_λ is finite dimensional and irreducible under the joint action of $O(p, q)$ and $Sp(2q, \mathbb{R})$, and as $O(p, q) \times Sp(2q, \mathbb{R})$ modules, we have

$$F_\lambda \cong (\epsilon \otimes \rho_\lambda^*) \otimes \pi_{\lambda + (\frac{p-q}{2} - 1)\mathbf{1}_q}^*.$$

In particular, $(\epsilon \otimes \rho_\lambda)$ and $\pi_{\lambda + (\frac{p-q}{2} - 1)\mathbf{1}_q}$ correspond to each other under the local theta correspondence.

Proof. We shall prove part (b). Part (a) is similar.

Consider the following subspace of $\mathcal{P}(M_{p+q,q}(\mathbb{R}))$:

$$B_\lambda = \langle h \cdot b_\lambda, h \in O(p, q) \rangle.$$

$O(p, q)$ acts on B_λ by a multiple of ρ_λ^* .

Clearly

$$F_\lambda = \overline{\langle \Omega^*(g)(B_\lambda d\nu_q), g \in Sp(2q, \mathbb{R}) \rangle}.$$

Let $W(M_{p+q,q}(\mathbb{R}))$ be the Weyl algebra on the matrix space $M_{p+q,q}(\mathbb{R})$, i.e. the algebra of polynomial coefficient differential operators on $M_{p+q,q}(\mathbb{R})$. For $m \in \mathbb{Z}_{\geq 0}$, let $W_m(M_{p+q,q}(\mathbb{R}))$ be the subspace of $W(M_{p+q,q}(\mathbb{R}))$ consisting of those polynomial coefficient differential operators with total degree $\leq m$. We shall regard $\mathcal{P}(M_{p+q,q}(\mathbb{R}))$ as a subalgebra of the Weyl algebra $W(M_{p+q,q}(\mathbb{R}))$. Note that B_λ is a subspace of $W_m(M_{p+q,q}(\mathbb{R}))$ for some m .

Since

$$\Omega^*(g)W_m(M_{p+q,q}(\mathbb{R}))\Omega^*(g)^{-1} \subseteq W_m(M_{p+q,q}(\mathbb{R}))$$

for each $g \in Sp(2q, \mathbb{R})$ (see [11]), we see that the image of

$$g \mapsto \Omega^*(g)B_\lambda\Omega^*(g)^{-1}, \quad g \in Sp(2q, \mathbb{R})$$

is a finite dimensional space in the Weyl algebra $W(M_{p+q,q}(\mathbb{R}))$. Further since

$$\overline{\langle \Omega^*(g)d\nu_q, g \in Sp(2q, \mathbb{R}) \rangle}$$

is finite dimensional, we conclude that F_λ is finite dimensional.

$O(p, q)$ acts on F_λ by the representation $\epsilon \otimes \rho_\lambda^*$, and as $O(p, q) \times Sp(2q, \mathbb{R})$ -module, we can write

$$F_\lambda \cong (\epsilon \otimes \rho_\lambda^*) \otimes E_\lambda^*$$

where E_λ is a $Sp(2q, \mathbb{R})$ module.

Now according to [13], E_λ is quasi-simple, namely it has a fixed $Sp(2q, \mathbb{R})$ infinitesimal character, and further it has a unique irreducible $Sp(2q, \mathbb{R})$ submodule. From the duality correspondence of infinitesimal characters [25], we know that the infinitesimal character of E_λ is

$$\lambda + \left(\frac{p+q}{2} - 1, \frac{p+q}{2} - 2, \dots, \frac{p+q}{2} - q \right).$$

Thus E_λ is a multiple of $\pi_{\lambda + (\frac{p-q}{2}-1)\mathbf{1}_q}$, this multiple must be one for E_λ to have a unique irreducible submodule. \square

We shall present our second application: a generalization of Huygens' Principle.

Consider the Cauchy problem for the wave equation:

$$\begin{aligned} \sum_{k=1}^p \frac{\partial^2 \phi}{\partial x_k^2} &= \frac{\partial^2 \phi}{\partial y^2}, \\ \phi(x, y)|_{y=0} &= \phi_0(x), \\ (\partial_y \phi)(x, y)|_{y=0} &= \phi_1(x), \end{aligned}$$

where $x = (x_1, \dots, x_p) \in \mathbb{R}^p$, $y \in \mathbb{R}$, and ϕ_0, ϕ_1 are two given functions of Schwartz class.

Standard techniques involving the Fourier transform show that there exist distributions P_0 and P_1 on \mathbb{R}^{p+1} such that

$$\phi = P_0 *_x \phi_0 + P_1 *_x \phi_1,$$

where $*_x$ is convolution in the x variables. The distributions P_l ($l = 0, 1$) are called propagators. They have been computed classically and the following results are known.

(a) $\phi(x', y)$ depends only on the values of $\phi_0(x)$ and $\phi_1(x)$ for $\|x - x'\| \leq |y|$, where $\|x\| = (\sum_{k=1}^p x_k^2)^{\frac{1}{2}}$. In other words P_l is supported inside the light cone, i.e., in the set $\{(x, y) : \|x\|^2 - y^2 \leq 0\}$.

(b) If p is odd and > 1 , then $\phi(x', y)$ depends only on the values of $\phi_0(x)$ and $\phi_1(x)$ (and their derivatives) for $\|x - x'\| = |y|$. In other words P_l is supported on the light cone, i.e., in the set $\{(x, y) : \|x\|^2 - y^2 = 0\}$.

Part (b) is what is known as **Huygens' Principle**: when the space dimension p is odd and > 1 , waves propagate on spherical shells.

We shall now discuss a generalization of the above phenomenon.

Let p, q be two natural numbers with $p \geq q$. Consider

$$x = (x_{ij})_{p \times q} \in M_{p,q}(\mathbb{R}), \quad \text{and} \quad y = (y_{ij})_{q \times q} \in M_{q,q}(\mathbb{R}).$$

Given two Schwartz class functions ϕ_i ($i = 0, 1$) of x , we consider the system of partial differential equations

$$(5.2) \quad \sum_{k=1}^p \frac{\partial^2 \phi}{\partial x_{ki} \partial x_{kj}} = \sum_{k=1}^q \frac{\partial^2 \phi}{\partial y_{ki} \partial y_{kj}}, \quad 1 \leq i, j \leq q,$$

$$(5.3) \quad \phi(x, sy) = \phi(x, y), \quad s \in SO(q),$$

$$(5.4) \quad \phi(x, y)|_{y=0} = \phi_0(x),$$

$$(5.5) \quad (\partial_{\det y} \phi)(x, y)|_{y=0} = \phi_1(x).$$

Here $\partial_{\det y} = \det(\partial_{y_{ij}})_{q \times q}$.

We shall call Equation (5.2) the generalized wave equation, and if ϕ satisfies Equations (5.2-5.5), we say ϕ solves the generalized wave equation with initial data ϕ_0, ϕ_1 .

As in the case of the wave equation ($q = 1$), there exist distributions P_0 and P_1 on $M_{p+q,q}(\mathbb{R})$ such that

$$\phi = P_0 *_x \phi_0 + P_1 *_x \phi_1$$

where $*_x$ is convolution in the x variables. The distributions P_l are again called propagators.

Definition 5.6. *We say that the system of PDE's (5.2-5.5) satisfies **Generalized Huygens' Principle** if the propagators P_l ($l = 0, 1$) are supported*

on the null cone

$$\mathcal{N} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in M_{p+q,q}(\mathbb{R}) \mid x^t x = y^t y \right\}.$$

We recall the dual pair

$$(H, G) = (O(p, q), Sp(2q, \mathbb{R})) \subseteq \mathbb{G} = Sp(2(p+q)q, \mathbb{R}),$$

and the associated oscillator representation Ω of $\tilde{\mathbb{G}}$ on $L^2(M_{p+q,q}(\mathbb{R}))$. \mathbb{G} acts on $\mathcal{S}(M_{p+q,q}(\mathbb{R}))$ by restriction, and then on $\mathcal{S}^*(M_{p+q,q}(\mathbb{R}))$ by duality.

The following observation was made in [10] (for $q = 1$). See also the discussion at the end of this section.

Criterion. Generalized Huygens' Principle holds if and only if P_l generates a finite-dimensional module for $\tilde{G} = \tilde{Sp}(2q, \mathbb{R})$ under Ω^* .

When $p+q$ is odd, $\tilde{Sp}(2q, \mathbb{R})$ is the metaplectic cover of $Sp(2q, \mathbb{R})$, which has no faithful finite dimensional representations. This immediately implies the following

Proposition 5.7. *Generalized Huygens' Principle cannot hold when $p+q$ is odd.*

On the other hand, we have

Theorem 5.8. *When $p > q$ and $p+q$ even, P_l generates an irreducible finite dimensional $\tilde{Sp}(2q, \mathbb{R})$ module of highest weight $(\frac{p-q}{2} - l)\mathbf{1}_q$. In particular, Generalized Huygens' Principle holds.*

This will follow from part (b) of Theorem 5.1 and a number of propositions below. As the proof of these propositions follow very closely the case of $q = 1$ [14], we will be contented to just state the results.

Denote the Fourier transform with respect to x by F_x . Thus

$$F_x(\psi)(x, y) = (2\pi)^{-\frac{pq}{2}} \int_{M_{p,q}(\mathbb{R})} e^{-i \operatorname{tr}(x^t x')} \psi(x', y) dx'.$$

We include the following proposition for completeness. We fix a $\tau_q \in O(q)$ with $\det \tau_q = -1$, so that $O(q) = SO(q) \cup SO(q)\tau_q$. We also fix the invariant measure du on $O(q)$ so that it has total measure 2. We shall use the same symbol du for $du|_{SO(q)}$. Therefore du is also the invariant measure on $SO(q)$ with total measure 1.

Proposition 5.9. *The solution of system of Equations (5.2-5.5) is given uniquely by*

$$\phi(x, y) = (P_0(y) *_x \phi_0)(x) + (P_1(y) *_x \phi_1)(x),$$

where $*_x$ is convolution in the x variables, and for a fixed y , $P_l(y)$ is the distribution on $M_{p,q}(\mathbb{R})$ given by

$$P_l(y) = F_x^{-1}(Q_l(\cdot, y)), \quad l = 0, 1,$$

and

$$Q_0(x, y) = \frac{1}{2} \int_{u \in SO(q)} \{e^{i \operatorname{tr}((x^t x)^{\frac{1}{2}} u^t y)} + e^{i \operatorname{tr}((x^t x)^{\frac{1}{2}} (u \tau_q)^t y)}\} du,$$

$$Q_1(x, y) = \frac{1}{2} i^{-q} \det(x^t x)^{-\frac{1}{2}} \int_{u \in SO(q)} \{e^{i \operatorname{tr}((x^t x)^{\frac{1}{2}} u^t y)} - e^{i \operatorname{tr}((x^t x)^{\frac{1}{2}} (u \tau_q)^t y)}\} du.$$

Define the distributions P_l ($l = 0, 1$) on $M_{p+q, q}(\mathbb{R})$ by

$$P_l(\psi_1 \otimes \psi_2) = \int_{M_{q, q}(\mathbb{R})} P_l(y) (\psi_1) \psi_2(y) dy,$$

where $\psi_1 \in \mathcal{S}(M_{p, q}(\mathbb{R}))$, $\psi_2 \in \mathcal{S}(M_{p, q}(\mathbb{R}))$ are functions of Schwartz class on $M_{p, q}(\mathbb{R})$ and $M_{q, q}(\mathbb{R})$, respectively. We shall then write the solution of Equations (5.2-5.5) as

$$\phi = P_0 *_x \phi_0 + P_1 *_x \phi_1.$$

We shall investigate symmetries of the propagators by exploiting symmetries of generalized wave equation together with its initial data.

Define the Fourier transform on $\mathcal{S}(M_{p+q, q}(\mathbb{R}))$ with respect to the metric

$$\operatorname{tr}(v^t I_{p, q} v) = \operatorname{tr}(x^t x) - \operatorname{tr}(y^t y), \quad v = \begin{pmatrix} x \\ y \end{pmatrix} \in M_{p+q, q}(\mathbb{R}) :$$

(5.10)

$$F(\psi)(x, y) = (2\pi)^{-\frac{(p+q)q}{2}} \int_{M_{p+q, q}(\mathbb{R})} e^{-i(\operatorname{tr}(x^t x') - \operatorname{tr}(y^t y'))} \psi(x', y') dx' dy'.$$

Define also the Fourier transform of a tempered distribution T by

$$F(T)(\psi) = T(F^{-1}(\psi)), \quad \psi \in \mathcal{S}(M_{p+q, q}(\mathbb{R})).$$

Recall that we have the action of $H = O(p, q)$ on $\mathcal{S}^*(M_{p+q, q}(\mathbb{R}))$ via Ω^* . Recall also the (unnormalized) action λ of $GL(q, \mathbb{R})$ on $\mathcal{S}(M_{p+q, q}(\mathbb{R}))$. See Equation (4.2). By understanding the effects of $O(p) \otimes O(q)$ and $GL(q, \mathbb{R})$ actions on the generalized wave equation and its initial data, and then by taking the Fourier transform, we obtain

Proposition 5.11. *We have*

$$\begin{aligned} \Omega^*(A)P_0 &= P_0, \quad \Omega^*(A)P_1 = P_1, \quad A \in O(p), \\ \Omega^*(B)P_0 &= P_0, \quad \Omega^*(B)P_1 = (\det B)P_1, \quad B \in O(q), \\ \lambda^*(a)P_l &= (\det a)^l |\det a|^q P_l, \quad a \in GL(q, \mathbb{R}), \quad l = 0, 1, \end{aligned}$$

and so

$$\begin{aligned} \Omega^*(A)F(P_0) &= F(P_0), \quad \Omega^*(A)F(P_1) = F(P_1), \quad A \in O(p), \\ \Omega^*(B)F(P_0) &= F(P_0), \quad \Omega^*(B)F(P_1) = (\det B)F(P_1), \quad B \in O(q), \\ \lambda^*(a)F(P_l) &= (\det a)^{-l} |\det a|^p F(P_l), \quad a \in GL(q, \mathbb{R}), \quad l = 0, 1. \end{aligned}$$

Denote by $\mathfrak{sp}(2q, \mathbb{R})$ the Lie algebra of $Sp(2q, \mathbb{R})$, then we have

$$\Omega(\mathfrak{sp}(2q, \mathbb{R})) = \langle r_{ij}, \Delta_{ij}, E_{ij} | 1 \leq i, j \leq q \rangle,$$

where

$$\begin{aligned} r_{ij} &= (v^t I_{p,q} v)_{ij} = \sum_{k=1}^p x_{ki} x_{kj} - \sum_{k=1}^q y_{ki} y_{kj}, \\ E_{ij} &= \sum_{k=1}^p x_{ki} \frac{\partial}{\partial x_{kj}} + \sum_{k=1}^q y_{ki} \frac{\partial}{\partial y_{kj}} + \frac{p+q}{2} \delta_{ij}, \end{aligned}$$

as before, and

$$\Delta_{ij} = \sum_{k=1}^p \frac{\partial^2}{\partial x_{ki} \partial x_{kj}} - \sum_{k=1}^q \frac{\partial^2}{\partial y_{ki} \partial y_{kj}}.$$

Observe that P_l ($l = 0, 1$) satisfy the generalized wave equation, namely $\Delta_{ij} P_l = 0$. This may be seen from the constructive proof of Proposition 5.9. By taking the Fourier transform, we get

$$r_{ij} F(P_l) = 0, \quad 1 \leq i, j \leq q.$$

The above equation says that $F(P_l)$ is supported on the null cone \mathcal{N} . We note that the Jacobian matrix $J(r_{11}, r_{12}, \dots, r_{qq})$ at any point of \mathcal{N}_q has full rank $\frac{q(q+1)}{2}$. See [27] for this fact. Thus away from $\mathcal{N} - \mathcal{N}_q$, $F(P_l)$ factors through the restriction map to \mathcal{N}_q , and so may be regarded as a distribution on \mathcal{N}_q . Since $O(p) \times GL(q, \mathbb{R})$ acts on \mathcal{N}_q transitively, the transformation properties under $O(p)$ and $GL(q, \mathbb{R})$ in Proposition 5.11 determine $F(P_l)$ up to multiples. Thus we conclude

Proposition 5.12.

$$\begin{aligned} F(P_0) &= c_0 |\det y| d\mu_q = c_0 (\det y) d\nu_q, \\ F(P_1) &= c_1 d\nu_q = c_1 \operatorname{sgn}(\det y) d\mu_q, \end{aligned}$$

for some constants c_0 and c_1 .

Note that $\{r_{ij}\}_{1 \leq i, j \leq q}$ are functionally independent. Thus P_l is supported on the null cone if and only if for each i, j , we have $r_{ij}^{d_{ij}} P_l = 0$ for some natural number d_{ij} . Using the $GL(q, \mathbb{R})$ -homogeneity of P_l (Proposition 5.11), we deduce that

$$E_{ij} P_l = \left(\frac{p-q}{2} - l \right) \delta_{ij} P_l.$$

Since $\Delta_{ij} P_l = 0$ for $1 \leq i, j \leq q$, we see that Generalized Huygens' Principle holds for the system of PDE's (5.2-5.5) if and only if each P_l (or $F(P_l)$) generates a finite-dimensional $\Omega^*(\mathfrak{sp}(2q, \mathbb{R}))$ module. In turn this is true if and only if each P_l (or $F(P_l)$) generates a finite-dimensional $\widetilde{Sp}(2q, \mathbb{R})$ module under Ω^* . Note that [5] the Fourier transform F as defined in Equation (5.10) is an element of $\Omega(\widetilde{Sp}(2q, \mathbb{R}))$, or to be exact an element of $\Omega(\widetilde{K})$.

Now it is easy to see that Proposition 5.12 and Theorem 5.1, part (b) imply Theorem 5.8.

Appendix: Identities of Capelli type

In this appendix, we prove two kinds of identities of Capelli type for solutions of certain system of operator equations, which generalize the famous Capelli identity in classical invariant theory [29]. As demonstrated by Weyl, the Capelli identity plays a central role in deriving polynomial invariants and relative invariants of classical groups. The identities proven here should play similar roles in applications to problems concerning eigendistributions of classical groups. See §3 and in particular Theorem 3.8.

A1 The first identity: Replacing the last few columns of the determinant

We will be concerned with two matrices of linear operators $\{E_{ij}\}_{1 \leq i, j \leq n}$, $\{X_{ij}\}_{1 \leq i, j \leq n}$ acting on the same vector space which satisfy the commutation relations:

$$[E_{ij}, X_{kl}] = \epsilon \delta_{jk} X_{il}, \quad 1 \leq i, j, k, l \leq n,$$

where ϵ is a complex number. We will also have some element S in the vector space satisfying equations of the form

$$E_{ij}S = X_{ij}S, \quad 1 \leq i, j \leq n.$$

Remark A1.1. *In actual applications, the operators E_{ij} and X_{ij} are differential operators, and $\{E_{ij}\}_{1 \leq i, j \leq n}$ satisfy the commutation relations of the standard basis of the Lie algebra \mathfrak{gl}_n of GL_n , namely*

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

If that is the case, then $\epsilon = \pm 1$. In what follows, we do not assume any commutation relations among the E_{ij} 's.

Recall that for a matrix $A = (A_{ij})_{n \times n}$ of noncommuting variables, we have the column determinant $\det(A)$ and the row determinant $\text{rdet}(A)$, defined in Equations (3.5) and (3.6), respectively.

Theorem A1.2. *Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying the commutation relations*

$$[E_{ij}, X_{kl}] = \epsilon \delta_{jk} X_{il}.$$

For each $0 \leq r \leq n$, define a matrix P_r of order $n \times n$ as follows:

$$P_r = \begin{pmatrix} X_{11} & \cdots & X_{1r} & E_{1n} & \cdots & E_{1,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r1} & \cdots & X_{rr} & E_{rn} & \cdots & E_{r,r+1} \\ X_{n1} & \cdots & X_{nr} & E_{nn} + (n-r-1)\epsilon & \cdots & E_{n,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r+1,1} & \cdots & X_{r+1,r} & E_{r+1,n} & \cdots & E_{r+1,r+1} \end{pmatrix}.$$

Suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq n, r+1 \leq j \leq n.$$

Then we have

$$\det(X)S = \det(P_r)S.$$

Remark A1.3. When $r = 0$, we have the full “Capelli identity”:

$$\det \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \cdots & \cdots & \cdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} S = \det \begin{pmatrix} E_{nn} + (n-1)\epsilon & \cdots & E_{n1} \\ \cdots & \cdots & \cdots \\ E_{1n} & \cdots & E_{11} \end{pmatrix} S.$$

Proof. For $r = n$, it is trivial. Also for $r = n - 1$ from the definition of $\det(X)$ and the assumption $X_{in}S = E_{in}S$, we can clearly replace the last column of X in $\det(X)S$ by the last column of $E = (E_{ij})_{n \times n}$. Thus we have $\det(X)S = \det(P_{n-1})S$.

Assume that $\det(X)S = \det(P_r)S$ for certain $r \leq n - 1$. We shall compute $\det(P_r)S$. The idea is to examine the r th column of P_r , and try to interchange with columns on the right. Namely for each individual product term in the formula of $\det(P_r)S$, we try to interchange the r th element with $(r + 1)$ th element, and then $(r + 2)$ th element, and so on, until we have interchanged with the n th element. This can be done using the given commutation relations. It turns out that the extra term (corresponding to a permutation $\sigma \in S_n$) arising from application of commutation relations can be absorbed with another product term corresponding to a certain $\tilde{\sigma} \in S_n$ with the opposite sign, producing an extra ϵ in a diagonal term of the lower right corner of P_r .

We will be more precise in the following.

Notice that the r th column of P_r from top to bottom is $X_{1r}, \dots, X_{rr}, X_{nr}, \dots, X_{r+1,r}$.

Observe that X_{1r}, \dots, X_{rr} commute with every column on the right, namely with all the elements in the $(r + 1)$ th, ..., n th columns of P_r .

Observe also that X_{nr} commutes with every column on the right except the $(r + 1)$ th column, $X_{n-1,r}$ commutes with every column on the right except the $(r + 2)$ th column, and so on, and finally $X_{r+1,r}$ commutes with every column on the right except the n th column.

In other words, for $r + 1 \leq j \leq n$, X_{jr} commutes with every column on the right except the $(r + n - j + 1)$ th column of P_r , which consists of (from top to bottom) $E_{1j}, \dots, E_{rj}, E_{nj}, \dots, E_{jj} + (j - r - 1)\epsilon, \dots, E_{r+1,j}$. Further the non-zero commutation relations are

$$[X_{jr}, E_{kj} + (j - r - 1)\epsilon\delta_{kj}] = [X_{jr}, E_{kj}] = -[E_{kj}, X_{jr}] = -\epsilon X_{kr},$$

where $i + 1 \leq j \leq n$, $1 \leq k \leq n$.

Consider a term $T(\sigma)$ in $\det(P_r)$ corresponding to a permutation $\sigma \in S_n$. We write

$$T(\sigma) = a * * * b X_{jr} * * * (E_{kj} + (j - r - 1)\epsilon\delta_{kj}) * * * d,$$

where $\sigma(r) = j$ and term $E_{kj} + (j - r - 1)\epsilon\delta_{kj}$ is in the $(r + n - j + 1)$ th place, namely it is from the $(r + n - j + 1)$ th column of P_r .

We have

$$T(\sigma) = \begin{cases} a *** b *** (E_{kj} + (j - r - 1)\epsilon\delta_{kj}) *** dX_{jr}, & \text{if } 1 \leq j \leq r, \\ a *** b *** (E_{kj} + (j - r - 1)\epsilon\delta_{kj}) *** dX_{jr} \\ \quad - \epsilon a *** b *** X_{kr} *** d, & \text{if } r + 1 \leq j \leq n. \end{cases}$$

In the second case, we shall call the term $-\epsilon a *** b *** X_{kr} *** d$ the extra term corresponding to σ .

We now handle these extra terms. To fix ideas, we examine the case $j = n$, and $1 \leq k \leq r$. Then $r + n - j + 1 = r + 1$, so that E_{kn} is from the $(r + 1)$ th column. We have

$$\begin{aligned} T(\sigma) &= a *** bX_{nr}E_{kn} *** d \\ &= a *** bE_{kn} *** dX_{nr} - \epsilon a *** bX_{kr} *** d \\ &= a *** bE_{kn} *** dX_{nr} - \epsilon a *** b *** dX_{kr}. \end{aligned}$$

Here $\sigma(r) = n$ and $\sigma(r + 1) = k$. Hence

$$T(\sigma)S = a *** bE_{kn} *** dE_{nr}S - a *** b *** dE_{kr}S.$$

Now consider $\tilde{\sigma} \in S_n$ such that

$$\begin{aligned} \tilde{\sigma}(r) &= k, \quad \tilde{\sigma}(r + 1) = n, \\ \tilde{\sigma}|_{\{1, \dots, n\} - \{r, r+1\}} &= \sigma|_{\{1, \dots, n\} - \{r, r+1\}}. \end{aligned}$$

Clearly $\text{sgn}(\tilde{\sigma}) = -\text{sgn}(\sigma)$.

Corresponding to $\tilde{\sigma}$, we have

$$\begin{aligned} T(\tilde{\sigma})S &= a *** bX_{kr}(E_{nn} + (n - r - 1)\epsilon) *** dS \\ &= a *** b(E_{nn} + (n - r - 1)\epsilon) *** dX_{kr}S \\ &= a *** b(E_{nn} + (n - r - 1)\epsilon) *** dE_{kr}S. \end{aligned}$$

By combining the extra term corresponding to σ to the term corresponding to $\tilde{\sigma}$, we get

$$\begin{aligned} &\text{sgn}(\sigma)T(\sigma)S + \text{sgn}(\tilde{\sigma})T(\tilde{\sigma})S \\ &= \text{sgn}(\sigma)a *** bE_{kn} *** dE_{nr}S + \\ &\quad \text{sgn}(\tilde{\sigma})a *** b(E_{nn} + (n - r)\epsilon) *** dE_{kr}S. \end{aligned}$$

In general for $r + 1 \leq j \leq n$, we conclude similarly that the effect of interchanging X_{jr} with the columns on the right is accounted for through adding an ϵ to the $(n - j + 1) \times (n - j + 1)$ th diagonal term of the lower right corner of the matrix P_r .

We therefore get $\det(P_r)S = \det(P_{r-1})S$, if $X_{ir}S = E_{ir}S$ for $1 \leq i \leq n$.

□

A2 The second identity: Replacing the first few columns of the determinant

Theorem A2.1. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying the commutation relations

$$[E_{ij}, X_{kl}] = \epsilon \delta_{jk} X_{il}.$$

Fix $0 \leq r \leq n$, and suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq r.$$

Then we have

$$\begin{aligned} \det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdots & E_{1r} & X_{1,r+1} & \cdots & X_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r1} & \cdots & E_{rr} + (n-r)\epsilon & X_{r,r+1} & \cdots & X_{rn} \\ E_{r+1,1} & \cdots & E_{r+1,r} & X_{r+1,r+1} & \cdots & X_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{n1} & \cdots & E_{nr} & X_{n,r+1} & \cdots & X_{nn} \end{pmatrix} S \\ = \det \begin{pmatrix} X_{r+1,r+1} & \cdots & X_{r+1,n} & X_{r+1,r} & \cdots & X_{r+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n,r+1} & \cdots & X_{nn} & X_{nr} & \cdots & X_{n1} \\ X_{r,r+1} & \cdots & X_{rn} & X_{rr} & \cdots & X_{r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,r+1} & \cdots & X_{1n} & X_{1r} & \cdots & X_{11} \end{pmatrix} S. \end{aligned}$$

Remark A2.2. When $r = n$, we get the full ‘‘Capelli identity’’:

$$\det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdots & E_{1n} \\ \cdots & \cdots & \cdots \\ E_{n1} & \cdots & E_{nn} \end{pmatrix} S = \det \begin{pmatrix} X_{nn} & \cdots & X_{n1} \\ \cdots & \cdots & \cdots \\ X_{1n} & \cdots & X_{11} \end{pmatrix} S.$$

Proof. We shall sketch the proof since it is very similar to the proof of Theorem A1.2.

Denote the matrix on the left hand side of the identity by R_r . We need to interchange the r th column of R_r with all the $(n-r)$ columns on the right (using commutation relations). After the interchange, this r th column will appear in the last column, and then we can use the given operator equation to replace the last column to get various X_{ir} ’s. Then we have to do exactly the same thing for the $(r-1)$ th column, and so on. We deal with the r th column first.

We note that the entry of the r th column of R_r is given by $E_{kr} + (n-r)\epsilon\delta_{kr}$ for $1 \leq k \leq n$. We examine the commutation relations of $E_{kr} + (n-r)\epsilon\delta_{kr}$ with entries of the j th column of R_r , for $r+1 \leq j \leq n$. We have

$$[E_{kr} + (n-r)\epsilon\delta_{kr}, X_{lj}] = [E_{kr}, X_{lj}] = \begin{cases} 0, & l \neq r, \\ \epsilon X_{kj}, & l = r. \end{cases}$$

Thus for each column on the right, there is exactly one entry with non-zero commutation relations with $E_{kr} + (n-r)\epsilon\delta_{kr}$. All of them are in the r th row of R_r .

Using an argument similar to the one for the proof of Theorem A1.2, we get

$$\det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdot & E_{1r} & X_{1,r+1} & \cdot & X_{1n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ E_{r1} & \cdot & E_{rr} + (n-r)\epsilon & X_{r,r+1} & \cdot & X_{rn} \\ E_{r+1,1} & \cdot & E_{r+1,i} & X_{r+1,r+1} & \cdot & X_{r+1,n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ E_{n1} & \cdot & E_{nr} & X_{n,r+1} & \cdot & X_{nn} \end{pmatrix} = \det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdot & E_{1,r-1} & X_{1,r+1} & \cdot & X_{1n} & E_{1r} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots & \cdots \\ E_{r-1,1} & \cdot & E_{r-1,r-1} + (n-r+1)\epsilon & X_{r-1,r+1} & \cdot & X_{r-1,n} & E_{r-1,r} \\ E_{r+1,1} & \cdot & E_{r+1,r-1} & X_{r+1,r+1} & \cdot & X_{r+1,n} & E_{r+1,r} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots & \cdots \\ E_{n1} & \cdot & E_{n,r-1} & X_{n,r+1} & \cdot & X_{nn} & E_{nr} \\ E_{r1} & \cdot & E_{r,r-1} & X_{r,r+1} & \cdot & X_{rn} & E_{rr} \end{pmatrix}.$$

Thus using the given operator equations, we obtain

$$\det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdot & E_{1r} & X_{1,r+1} & \cdot & X_{1n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ E_{r1} & \cdot & E_{rr} + (n-r)\epsilon & X_{r,r+1} & \cdot & X_{rn} \\ E_{r+1,1} & \cdot & E_{r+1,r} & X_{r+1,r+1} & \cdot & X_{r+1,n} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots \\ E_{n1} & \cdot & E_{nr} & X_{n,r+1} & \cdot & X_{nn} \end{pmatrix} S = \det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdot & E_{1,r-1} & X_{1,r+1} & \cdot & X_{1n} & X_{1r} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots & \cdots \\ E_{r-1,1} & \cdot & E_{r-1,r-1} + (n-r+1)\epsilon & X_{r-1,r+1} & \cdot & X_{r-1,n} & X_{r-1,r} \\ E_{r+1,1} & \cdot & E_{r+1,r-1} & X_{r+1,r+1} & \cdot & X_{r+1,n} & X_{r+1,r} \\ \cdots & \cdot & \cdots & \cdots & \cdot & \cdots & \cdots \\ E_{n1} & \cdot & E_{n,r-1} & X_{n,r+1} & \cdot & X_{nn} & X_{nr} \\ E_{r1} & \cdot & E_{r,r-1} & X_{r,r+1} & \cdot & X_{rn} & X_{rr} \end{pmatrix} S.$$

Now we deal with the $(r-1)$ th column of the new matrix in the above identity, and try to interchange with the r th, $(r+1)$ th columns of this new matrix, and so on. Continuing this way as what we have done for the r th column of R_r , we arrive at the identity in the theorem. \square

A3 Variations: Row determinant and symmetric square representation

We shall deduce similar identities of Capelli type for row determinant, when the relevant commutation relations are

$$[E_{ij}, X_{kl}] = \epsilon \delta_{il} X_{kj},$$

as opposed to

$$[E_{ij}, X_{kl}] = \epsilon \delta_{jk} X_{il},$$

for column determinant.

Theorem A3.1. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying the commutation relations

$$[E_{ij}, X_{kl}] = \epsilon \delta_{il} X_{kj}.$$

For each $0 \leq r \leq n$, define a matrix Q_r of order $n \times n$ as follows:

$$Q_r = \begin{pmatrix} X_{11} & \cdots & X_{1r} & X_{1n} & \cdots & X_{1,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r1} & \cdots & X_{rr} & X_{rn} & \cdots & X_{r,r+1} \\ E_{n1} & \cdots & E_{nr} & E_{nn} + (n-r-1)\epsilon & \cdots & E_{n,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r+1,1} & \cdots & E_{r+1,r} & E_{r+1,n} & \cdots & E_{r+1,r+1} \end{pmatrix}.$$

Suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } r+1 \leq i \leq n, 1 \leq j \leq n.$$

Then we have

$$\text{rdet}(X)S = \text{rdet}(Q_r)S.$$

Proof. Let $F_{ij} = E_{ji}$, $Y_{ij} = X_{ji}$ so that $F = E^t$, $Y = X^t$. Then we have

$$[F_{ij}, Y_{kl}] = [E_{ji}, X_{lk}] = \epsilon \delta_{jk} X_{li} = \epsilon \delta_{jk} Y_{il}.$$

Further we have $Y_{ij}S = F_{ij}S$ for $1 \leq i \leq n$, $r+1 \leq j \leq n$. Thus $F = (F_{ij})_{n \times n}$, $Y = (Y_{ij})_{n \times n}$ satisfy the hypothesis of Theorem A1.2.

The current theorem now follows from Theorem A1.2 and the fact that $\text{rdet}(A) = \det(A^t)$ for any $n \times n$ matrix A . \square

In a similar way, we obtain (from Theorem A2.1)

Theorem A3.2. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V satisfying the commutation relations

$$[E_{ij}, X_{kl}] = \epsilon \delta_{il} X_{kj}.$$

Fix $0 \leq r \leq n$, and suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq r, 1 \leq j \leq n.$$

Then we have

$$\begin{aligned} & rdet \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdots & E_{1r} & E_{1,r+1} & \cdots & E_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r1} & \cdots & E_{rr} + (n-r)\epsilon & E_{r,r+1} & \cdots & E_{rn} \\ X_{r+1,1} & \cdots & X_{r+1,r} & X_{r+1,r+1} & \cdots & X_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n1} & \cdots & X_{nr} & X_{n,r+1} & \cdots & X_{nn} \end{pmatrix} S \\ &= rdet \begin{pmatrix} X_{r+1,r+1} & \cdots & X_{r+1,n} & X_{r+1,r} & \cdots & X_{r+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n,r+1} & \cdots & X_{nn} & X_{nr} & \cdots & X_{n1} \\ X_{r,r+1} & \cdots & X_{rn} & X_{rr} & \cdots & X_{r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,r+1} & \cdots & X_{1n} & X_{1r} & \cdots & X_{11} \end{pmatrix} S. \end{aligned}$$

Next we shall be concerned with the ‘‘symmetric square representation’’. Here we are given two matrices of linear operators acting on the same vector space such that

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{jk}X_{il} + \delta_{jl}X_{ki}), \end{aligned}$$

or (corresponding to the row determinant)

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{ik}X_{jl} + \delta_{il}X_{kj}). \end{aligned}$$

where ϵ is a complex number. We also have some element S in the vector space satisfying equations of the form

$$E_{ij}S = X_{ij}S.$$

We will state the following theorems without any proofs, since they are identical to proofs of Theorems A1.2, A2.1, A3.1, A3.2. The reason is essentially as follows: Because of the way the entries of relevant matrices are positioned, the non-zero commutation relations that we use in interchanging the two columns (in the first case) turn out to ignore the term $\epsilon\delta_{jl}X_{ki}$, namely as far as these interchanges are concerned, the commutation relations behave just like $[E_{ij}, X_{kl}] = \epsilon\delta_{jk}X_{il}$.

Theorem A3.3. *Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V such that*

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{jk}X_{il} + \delta_{jl}X_{ki}). \end{aligned}$$

For each $0 \leq r \leq n$, define a matrix P_r of order $n \times n$ as follows:

$$P_r = \begin{pmatrix} X_{11} & \cdots & X_{1r} & E_{1n} & \cdots & E_{1,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r1} & \cdots & X_{rr} & E_{rn} & \cdots & E_{r,r+1} \\ X_{n1} & \cdots & X_{nr} & E_{nn} + (n-r-1)\epsilon & \cdots & E_{n,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r+1,1} & \cdots & X_{r+1,r} & E_{r+1,n} & \cdots & E_{r+1,r+1} \end{pmatrix}.$$

Suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq n, r+1 \leq j \leq n.$$

Then we have

$$\det(X)S = \det(P_r)S.$$

Theorem A3.4. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V such that

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{jk}X_{il} + \delta_{jl}X_{ki}). \end{aligned}$$

Fix $0 \leq r \leq n$, and suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq n, 1 \leq j \leq r.$$

Then we have

$$\begin{aligned} & \det \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdots & E_{1r} & X_{1,r+1} & \cdots & X_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r1} & \cdots & E_{rr} + (n-r)\epsilon & X_{r,r+1} & \cdots & X_{rn} \\ E_{r+1,1} & \cdots & E_{r+1,r} & X_{r+1,r+1} & \cdots & X_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{n1} & \cdots & E_{nr} & X_{n,r+1} & \cdots & X_{nn} \end{pmatrix} S \\ &= \det \begin{pmatrix} X_{r+1,r+1} & \cdots & X_{r+1,n} & X_{r+1,r} & \cdots & X_{r+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n,r+1} & \cdots & X_{nn} & X_{nr} & \cdots & X_{n1} \\ X_{r,r+1} & \cdots & X_{rn} & X_{rr} & \cdots & X_{r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,r+1} & \cdots & X_{1n} & X_{1r} & \cdots & X_{11} \end{pmatrix} S. \end{aligned}$$

Theorem A3.5. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V such that

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{ik}X_{jl} + \delta_{il}X_{kj}). \end{aligned}$$

For each $0 \leq r \leq n$, define a matrix Q_r of order $n \times n$ as follows:

$$Q_r = \begin{pmatrix} X_{11} & \cdots & X_{1r} & X_{1n} & \cdots & X_{1,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{r1} & \cdots & X_{rr} & X_{rn} & \cdots & X_{r,r+1} \\ E_{n1} & \cdots & E_{nr} & E_{nn} + (n-r-1)\epsilon & \cdots & E_{n,r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r+1,1} & \cdots & E_{r+1,r} & E_{r+1,n} & \cdots & E_{r+1,r+1} \end{pmatrix}.$$

Suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } r+1 \leq i \leq n, 1 \leq j \leq n.$$

Then we have

$$\text{rdet}(X)S = \text{rdet}(Q_r)S.$$

Theorem A3.6. Let $E = (E_{ij})_{n \times n}$, $X = (X_{ij})_{n \times n}$ be matrices of linear operators on a vector space V such that

$$\begin{aligned} X_{ij} &= X_{ji}, \\ [E_{ij}, X_{kl}] &= \epsilon(\delta_{ik}X_{jl} + \delta_{il}X_{kj}). \end{aligned}$$

Fix $0 \leq r \leq n$, and suppose that $S \in V$ satisfies

$$X_{ij}S = E_{ij}S, \quad \text{for } 1 \leq i \leq r, 1 \leq j \leq n.$$

Then we have

$$\begin{aligned} & \text{rdet} \begin{pmatrix} E_{11} + (n-1)\epsilon & \cdots & E_{1r} & E_{1,r+1} & \cdots & E_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{r1} & \cdots & E_{rr} + (n-r)\epsilon & E_{r,r+1} & \cdots & E_{rn} \\ X_{r+1,1} & \cdots & X_{r+1,r} & X_{r+1,r+1} & \cdots & X_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n1} & \cdots & X_{nr} & X_{n,r+1} & \cdots & X_{nn} \end{pmatrix} S \\ &= \text{rdet} \begin{pmatrix} X_{r+1,r+1} & \cdots & X_{r+1,n} & X_{r+1,r} & \cdots & X_{r+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{n,r+1} & \cdots & X_{nn} & X_{nr} & \cdots & X_{n1} \\ X_{r,r+1} & \cdots & X_{rn} & X_{rr} & \cdots & X_{r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{1,r+1} & \cdots & X_{1n} & X_{1r} & \cdots & X_{11} \end{pmatrix} S. \end{aligned}$$

A4 Relationship with the Capelli identity

We recall the Capelli identity (see [29], [15]).

Let $P(M_{m,n})$ be the algebra of polynomial functions on the space of $m \times n$ matrices. We have the polarization operators

$$E_{ij} = \sum_{t=1}^m x_{ti} \frac{\partial}{\partial x_{tj}}, \quad 1 \leq i, j \leq n,$$

where $x = (x_{ij})_{m \times n}$ is a typical element of $M_{m,n}$. Then the Capelli identity says

$$\det(E_{ij} + (n-i)\delta_{ij}) = \sum \det(x_{i_a j_b}) \det\left(\frac{\partial}{\partial x_{i_a j_b}}\right),$$

where the sum is over all pairs of n -tuples $1 \leq i_1 < i_2 < \cdots < i_n \leq m$, $1 \leq j_b = b \leq n$, and $\det(x_{i_a j_b})$ is the determinant of $n \times n$ minor of x formed by the rows indicated by i_a and columns indicated by j_b . The equality is an equality of operators on $P(M_{m,n})$.

Remark A4.1. *The above is the way which appears in the literature. But actually what Weyl proved in his book is*

$$\det(E_{n+1-i, n+1-j} + (n-i)\delta_{ij}) = \sum \det(x_{i_a j_b}) \det\left(\frac{\partial}{\partial x_{i_a j_b}}\right).$$

To see how our identities imply that of Capelli's, we shall employ the technique of doubling the variables [12].

Introduce $y = (y_{ij})_{m \times n}$, another typical element of $M_{m,n}$, and consider $P(M_{m,n} \oplus M_{m,n})$, the algebra of polynomial functions on $M_{m,n} \oplus M_{m,n}$. We identify $P(M_{m,n} \oplus M_{m,n})$ as the algebra of polynomial functions in variables x_{ij} and y_{ij} , where $1 \leq i \leq m$, $1 \leq j \leq n$.

There is a (polarization) map from $P(M_{m,n})$ to $P(M_{m,n} \oplus M_{m,n})$:

$$f \in P(M_{m,n}) \mapsto h_f \in P(M_{m,n} \oplus M_{m,n}) : h_f(x_{ij}, y_{ij}) = f\left(\frac{x_{ij} + y_{ij}}{2}\right).$$

Clearly $h_f(x_{ij}, x_{ij}) = f(x_{ij})$, for $f \in P(M_{m,n})$.

Define $U \subseteq P(M_{m,n} \oplus M_{m,n})$ by

$$U = \{h \in P(M_{m,n} \oplus M_{m,n}) | h = h_f \text{ for some } f \in P(M_{m,n})\}.$$

Thus U is a copy of $P(M_{m,n})$ in $P(M_{m,n}) \oplus P(M_{m,n})$.

Define

$$D_{ij} = \sum_{t=1}^m x_{ti} \frac{\partial}{\partial y_{tj}}, \quad 1 \leq i, j \leq n.$$

We observe that

- (1) $D_{ij}h = E_{ij}h, \quad h \in U,$
- (2) $[E_{ij}, D_{kl}] = \delta_{jk}D_{il}.$

Therefore Theorem A1.2 (for $r = 0$) and Theorem A2.1 (for $r = n$) imply that

$$\begin{aligned} \det(E_{n+1-i, n+1-j} + (n-i)\delta_{ij})h &= \det(D_{ij})h, \\ \det(E_{i,j} + (n-i)\delta_{ij})h &= \det(D_{n+1-i, n+1-j})h, \end{aligned}$$

where $h \in U$.

Denote $x^t = (x_{ji})_{n \times m}$, $\partial y = (\frac{\partial}{\partial y_{ij}})_{m \times n}$ and $D = (D_{ij})_{n \times n}$. Clearly

$$D = x^t \partial y.$$

Note that the entries of D , x^t and ∂y commute among themselves. Thus we have

$$\det(D_{ij}) = \det(D_{n+1-i, n+1-j}) = \sum \det(x_{i_a j_b}) \det\left(\frac{\partial}{\partial y_{i_a j_b}}\right),$$

where the sum is over all pairs of n -tuples, as before. So we obtain

$$\begin{aligned} \det(E_{n+1-i, n+1-j} + (n-i)\delta_{ij})h &= \det(E_{i,j} + (n-i)\delta_{ij})h \\ &= \sum \det(x_{i_a j_b}) \det\left(\frac{\partial}{\partial y_{i_a j_b}}\right)h, \quad h \in U. \end{aligned}$$

Finally we apply the above identities to $h = h_f$, where $f \in P(M_{m,n})$, and then we specialize at $y_{ij} = x_{ij}$, and we get the Capelli identity valid for any $f \in P(M_{m,n})$, both the version in the literature and the version in Weyl's book.

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