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TRANSFER OF UNITARY REPRESENTATIONS*

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Introduction. This paper has as its primary purpose a clear exposition of the idea in [21] which we describe as transfer between real forms of a semi-simple Lie group over $\mathbb{C}$. The basic point is that there are representations of one real form that can be more easily understood in the context of another real from. An interesting example is a minimal representation (that is annihilated by the Joseph ideal) of a split group over $\mathbb{R}$. In the case when the complexification admits a Hermitian symmetric real form, minimal representations of that real form are part of the "analytic continuation of the holomorphic discrete series" (cf. [5]). For example, in the case of $SO(4,4)$ or split E_7 one can "transfer" the holomorphic minimal representations of $SO(6,2)$ and Hermitian symmetric E_7 respectively. If there is no Hermitian symmetric real form then since there is always a quaternionic real form one can do a similar transfer. We feel that this exposition is necssary since the original discussion had many misprints which could be confusing and also lacked, in some cases, proper reference to earlier related work. The secondary purpose is to give some new examples of its applicability. These examples give more evidence of a possible deep connection between the notion of transfer and Howe's theory of dual pairs. We will now give a description of the paper.

Let $G_{\mathbb{C}}$ be a connected, simply connected semi-simple Lie group over $\mathbb{C}$. Let G and G' be two real forms of $G_{\mathbb{C}}$ with respective maximal compact subgroups K and K'. Let θ and θ' be corresponding Cartan involutions looked upon as automorphisms of $G_{\mathbb{C}}$. We assume that $\theta\theta' = \theta'\theta$. Set $M = K \cap K'$. Let $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}_{\mathbb{C}}$ denote the Lie algebras of $G, G', G_{\mathbb{C}}$ respectively. We shall use similar notation throughout the article thus the Lie groups will be denoted by capital letters and the corresponding Lie algebra by the corresponding lower case fraktur letter.

Let (π, V) be an irreducible admissible $(\mathfrak{g}', K')$-module. Then we may look upon it as a $(\mathfrak{g}_{\mathbb{C}}, M)$-module. We can apply the Zuckerman functors $(\Gamma_M^K)^i$ to V and get $(\mathfrak{g}, K)$-modules

$$V^i = (\Gamma_M^K)^i(V).$$

See [16] for a comprehensive treatment of the Zuckerman functors. Since V has an infinitesimal character, the modules V^i will also have the same infinitesimal character. Therefore any finite dimensional K-invariant subspace of V^i will generate an admissible $(\mathfrak{g}, K)$-submodule of V^i. Observing that if $\mathrm{Ann}(V)$ is the annihilator of V in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, then $\mathrm{Ann}(V^i) \supset \mathrm{Ann}(V)$. Thus V^i has Gelfand-Kirillov dimension less than or equal to that of V. This simple method of obtaining admissible modules is what we mean by the most basic method of transfer. In the special case that V is unitarizable $(\mathfrak{g}', K')$-module with some special properties as a $\mathfrak{k}_{\mathbb{C}}$-module one can make these transferred submodules much more explicit.

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We will now assume that V is unitarizable as a $(\mathfrak{g}', K')$-module and as a $(\mathfrak{k}_{\mathbb{C}}, M)$-module it splits into a direct sum of irreducible submodules :

$$V = \bigoplus_j V_j,$$

with V_j a finite direct sum of irreducible $(\mathfrak{k}_{\mathbb{C}}, M)$-modules L_j and each L_j is unitarizable as a $(\mathfrak{k}_1, M)$-module. Here $\mathfrak{k}_1 = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}'$ is a real form of $\mathfrak{k}_{\mathbb{C}}$ with $\theta'|_{\mathfrak{k}_{\mathbb{C}}}$ as the Cartan involution and it contains $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{k}'$ as the corresponding maximal compact subalgebra. Then there will be a decomposition of V^i into $(\mathfrak{g}, K)$-submodules corresponding to irreducible M-submodules W of the exterior power $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ (Proposition 1.5), which (by results of Kumaresan, Parthasarathy and Vogan-Zuckerman) when non-zero pick out certain θ'-stable parabolic subalgebras of $\mathfrak{k}_{\mathbb{C}}$. Furthermore if there exists an involutive unitary operator on V which realizes the action of $\theta\theta'$, then each piece of V^i has a $(\mathfrak{g}, K)$-invariant, non-degenerate Hermitian form with a positivity test (Proposition 2.5), which is effective in many interesting cases (Corollary 2.7 and Theorem 4.1). In this paper, the only example that will be studied in detail will involve the case when V and V_j are highest weight modules. However, in earlier work of the first named author with B. Gross [8, 9] these ideas have been used to transfer quaternionic unitary representations for E_n for $n = 6, 7, 8$ from the quaternionic real form to the split real form (in particular, giving a new construction of the minimal representations of the split groups) and also give the Blattner type branching formulas for the restriction of small discrete series for arbitrary simple Lie groups over $\mathbb{R}$ to certain (non-compact) symmetric subgroups.

The idea of comparing unitary representations for two real forms of a semi-simple Lie algebra was initiated by Enright [3]. In it he considers $G' = G_0 \times G_0$, where G_0 is a simply connected, connected Lie group with Lie algebra $\mathfrak{g}_0$ and it is of Hermitian symmetric type, and G is the simply connected, connected complex Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. The representations π of $G_0 \times G_0$ are in the analytic continuation of the holomorphic discrete series, and the representations of G obtained are the components of degenerate principal series. This was extended in much greater generality in parts of [6] (Theorem 7.2 and Theorem 7.3), with G' of Hermitian symmetric type and the representations π of G' in the part of analytic continuation of the holomorphic discrete series where the modules remain irreducible. In both of the mentioned cases, the derived functor modules vanish except at the middle degree, and in the middle degree it carries a unitarizable representation of G (or zero). When π is a general unitarizable highest weight module, the derived functor modules may be non-zero in degrees below the middle dimension. The situation was considered in [7] and partial results were obtained when both G' and G are of Hermitian symmetric type. The work of the first named author [21] extended the earlier work mentioned above and developed the comparison techniques in a more general framework. As mentioned above, the main aim of the current article is to make [21] more accessible. In the last section, we give an example of a connection of the construction here with Howe's theory of dual pairs. This may be considered as providing a global description of the representations obtained or as describing certain theta lifts as derived functor modules.

In this transfer of unitary representations between real forms, several Lie algebras play a role. It is instructive to arrange them in a diagram consisting of four diamonds (Figure 1). Here σ and σ' are respectively the complex conjugations of $\mathfrak{g}_{\mathbb{C}}$ with respect

$$\begin{tikzcd}[column sep=small]
& & \mathfrak{g}_{\mathbb{C}} \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\sigma'"] & & \\
& \mathfrak{g} \arrow[dl, dash, "\theta"'] \arrow[dr, dash, "\sigma'"] & & \mathfrak{g}' \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\theta'"] & \\
\mathfrak{k}_{\mathbb{C}} \arrow[dr, dash, "\sigma'"] & & \mathfrak{g} \cap \mathfrak{g}' \arrow[dl, dash, "\theta"'] \arrow[dr, dash, "\theta'"] & & \mathfrak{k}'_{\mathbb{C}} \arrow[dl, dash, "\sigma"'] \\
& \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}' \arrow[dr, dash, "\theta'"] & & \mathfrak{g} \cap \mathfrak{k}'_{\mathbb{C}} \arrow[dl, dash, "\theta"'] & \\
& & \mathfrak{k} \cap \mathfrak{k}' & &
\end{tikzcd}$$

FIG. 1. *A diamond of Lie algebras*

to $\mathfrak{g}$ and $\mathfrak{g}'$, and they are assumed to satisfy the compatibility conditions

$$\sigma\sigma' = \sigma'\sigma, \quad \theta\sigma' = \sigma'\theta, \quad \theta'\sigma = \sigma\theta'.$$

1. A decomposition of Zuckerman functors. We will begin this section in more generality than the introduction.

Let G be a real reductive Lie group, and K be a fixed maximal compact subgroup. Let $M \subset K$ be a compact subgroup.

Suppose that we are given a $(\mathfrak{g}_{\mathbb{C}}, M)$-module (π, V). By applying the Zuckerman functors $(\Gamma_M^K)^i$ to V, we produce $(\mathfrak{g}_{\mathbb{C}}, K)$-modules

$$\Gamma^i(V) = (\Gamma_M^K)^i(V). \tag{1.1}$$

We recall (a form of) this construction [4, 2, 22].

Let $\mathcal{H}(K)$ denote the algebra of matrix coefficients of finite-dimensional unitary representations of K, when viewed as an algebra under convolution it is called the Hecke algebra for K. However, we will only be using the operation of multiplication as functions on K. The space $\mathcal{H}(K)$ has two K-module structures: left multiplication l_K and right multiplication r_K. We view the tensor product $V \otimes \mathcal{H}(K)$ as a $(\mathfrak{k}, M)$-module under the action $\pi \otimes l_K$. Let $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ be the relative Lie algebra cohomology of $\mathfrak{k}, M$ with coefficients in $V \otimes \mathcal{H}(K)$. We define actions of $\mathfrak{g}_{\mathbb{C}}$ and K on $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ as follows.

Identify $V \otimes \mathcal{H}(K) = \mathcal{H}(K, V)$ as a space of V-valued functions on K, by $v \otimes f \mapsto F$ where $F(k) = f(k)v$. This space can be described as the space of all functions F from K to V such that

$$\dim \operatorname{span}\{F(k) | k \in K\} < \infty$$

and if $\lambda \in V^*$ then the function $g(k) = \lambda(F(k))$ is K-finite. Then $\mathfrak{g}$ acts on $\mathcal{H}(K, V)$ by

$$(\mu(X)F)(k) = \pi(Ad(k)X)F(k), \tag{1.2}$$

where $k \in K$, $X \in \mathfrak{g}_{\mathbb{C}}$ and $F \in \mathcal{H}(K, V)$. It is easy to check that $\mu(X)$ defines a $(\mathfrak{k}, M)$-homomorphism of $\mathcal{H}(K, V)$ to itself. Thus $\mu(X)$ induces an endomorphism, $\bar{\mu}(X)$, on the relative cohomology spaces and with respect to this action $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ is a $\mathfrak{g}_{\mathbb{C}}$-module.

We note that K acts on $V \otimes \mathcal{H}(K)$ by $I \otimes r_K$ commuting with $\pi \otimes l_K$, and so induces a K-module structure on $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$. With respect to the actions $(\bar{\mu}, I \otimes r_K)$, each $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ is a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. The critical point here is that all of the homological algebra is done in the context of $(\mathfrak{k}, M)$ cohomology and so the action of K is easily computed. We will spend a significant part of this paper studying the implications of this observation. In the earlier work of the first named author with Enright in [4] it was shown that the action of $\mathfrak{g}$ was a consequence of naturality. One "unwinds" the action described in [4, 22] then the action described above is obvious.

We also note that the cochain complex defining $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ is given as

$$C^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K)) = \mathrm{Hom}_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), V \otimes \mathcal{H}(K)).$$

The actions $(\mu, I \otimes r_K)$ of $\mathfrak{g}_{\mathbb{C}}$ and K are actually defined on $C^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$, but these actions do not make $C^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. The problem is that the derived action of $X \in \mathfrak{k}$ does not agree with the action of X as an element of $\mathfrak{g}$ (in Equation (1.2)). But the induced actions $(\bar{\mu}, I \otimes r_K)$ on $H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ are compatible, and so we have a $(\mathfrak{g}_{\mathbb{C}}, K)$-module. The Zuckerman modules are then

$$\Gamma^i(V) = H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K)). \tag{1.3}$$

We now come to the main assumptions of this article.

(A1): There is a real form $\mathfrak{k}_1$ of $\mathfrak{k}_{\mathbb{C}}$ and a real reductive group K_1 such that $K_1 \supset M$ and (K_1, M) is a symmetric pair of noncompact type.

(A2): As a $(\mathfrak{k}_{\mathbb{C}}, M)$-module, V splits into a direct sum

$$V = \bigoplus_j V_j, \qquad \text{with} \;\; V_j \simeq m_j L_j \tag{1.4}$$

a finite direct sum of irreducible $(\mathfrak{k}_{\mathbb{C}}, M)$-modules L_j and each L_j is unitarizable as a $(\mathfrak{k}_1, M)$-module.

An interesting case where (A1) and (A2) are satisfied is the subject of [6]. In that paper the authors consider a family of generalized Verma modules $N(\lambda + t\xi)$ associated to a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ such that the nilradical $\mathfrak{u}$ satisfies $[\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{u}] = 0$ and $N(\lambda + t\xi)$ is irreducible for $t \leq 0$. See §4 of [6] for details. We will be giving several examples of this phenomenon in this article. Also many examples were analyzed in [21].

In the setting of the introduction, the restriction $\theta'|_{\mathfrak{k}_{\mathbb{C}}}$ is an involution of $\mathfrak{k}_{\mathbb{C}}$. The corresponding real form of $\mathfrak{k}_{\mathbb{C}}$ is $\mathfrak{k}_1 = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}'$, with the complexified Cartan decomposition

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{o}_k.$$

Here $\mathfrak{o}_k = \mathfrak{o} \cap \mathfrak{k}_{\mathbb{C}}$, $\mathfrak{o} = (\mathfrak{p}')_{\mathbb{C}}$, and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ is the Cartan decomposition of $\mathfrak{g}'$. Thus the assumption (A1) is satisfied. If V is unitarizable as a $(\mathfrak{g}', K')$-module, and it is M-admissible, then the assumption (A2) is satisfied. More generally (A2) will be satisfied if $K' \supset B$, for a compact subgroup $B \subset K$ and V is B-admissible. This condition holds true for unitary highest weight representations of a Hermitian symmetric real form when B contains the center of K'. We shall have more to say on this case in § 4. It has also been studied for quaternionic representations of a quaternionic real form of $G_{\mathbb{C}}$ when B is the "quaternionic $SU(2)$" in [8]. In that reference it was shown how one can transfer the minimal representation of quaternionic E_8 to split E_8. The same procedure will work for E_6 and E_7. In addition, in [9] it is shown that "small discrete series" are admissible for small subgroup, B, of a maximal compact subgroup. In that paper transfer was used in order to derive a Blattner type multiplicty formula for the restriction of a small discrete series to a symmetric subgroup containing the group B.

For the rest of this paper, we shall take both $(A1)$ and $(A2)$ as standing hypotheses.

For each $\gamma \in \hat{K}$, fix $F_\gamma \in \gamma$. Recall that Peter-Weyl theorem implies that as a (K, K)-bimodule (under l_K, r_K)

$$\mathcal{H}(K) \simeq \bigoplus_{\gamma \in \hat{K}} F_\gamma^* \otimes F_\gamma.$$

Note that the coboundary operator d on $C^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K))$ commutes with $I \otimes r_K$. Thus we have the decomposition (as K-modules)

$$H^i(\mathfrak{k}, M; \mathcal{H}(K) \otimes V) \simeq \bigoplus_{\gamma \in \hat{K}} H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K)(\gamma)).$$

Here $\mathcal{H}(K)(\gamma)$ denotes the γ-isotypic component of $\mathcal{H}(K)$ with respect to the right action r_K, and is isomorphic to $F_\gamma^* \otimes F_\gamma$ under l_K, r_K.

We now use the decomposition of V in (1.4) to obtain

$$H^i(\mathfrak{k}, M; V \otimes \mathcal{H}(K)) \simeq \bigoplus_j \bigoplus_{\gamma \in \hat{K}} H^i(\mathfrak{k}, M; V_j \otimes \mathcal{H}(K)(\gamma)).$$

For simplicity, we assume that G has compact center and that M and K are connected. The center $Z(K)$ of K is the same as the center of K_1 and it acts on a representation of type $\gamma \in \hat{K}$ and L_j by unitary characters (the central characters). For each j, let $\hat{K}_j$ denote the set of those $\gamma \in \hat{K}$ which has the same central character and the same $\mathfrak{k}_{\mathbb{C}}$-infinitesimal character as that of L_j. Set

$$\Omega(V) = \bigoplus_j \bigoplus_{\gamma \in \hat{K}_j} V_j \otimes \mathcal{H}(K)(\gamma),$$
$$X(V) = \bigoplus_j \bigoplus_{\gamma \notin \hat{K}_j} V_j \otimes \mathcal{H}(K)(\gamma).$$

Obviously, $V \otimes \mathcal{H}(K) = \Omega(V) \oplus X(V)$. Note that this decomposition depends on the decomposition of V in (1.4).

Wigner's Lemma [1] implies that

$$H^i(\mathfrak{k}, M; X(V)) = 0.$$

Thus $H^i(\mathfrak{k}, M; \mathcal{H}(K) \otimes V) = H^i(\mathfrak{k}, M; \Omega(V))$. As (K_1, M) is a symmetric pair of non-compact type, the coboundaries for the $(\mathfrak{k}_1, M)$-cohomologies of unitary representations are all zero. See [1]. We therefore have

$$H^i(\mathfrak{k}, M; \mathcal{H}(K) \otimes V) = H^i(\mathfrak{k}, M; \Omega(V)) = \mathrm{Hom}_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V)).$$

With a bit more work we may summarize by the following

Proposition 1.5. *Suppose that $M \subset K$ is a compact subgroup satisfying (A1) and (π, V) is a $(\mathfrak{g}_{\mathbb{C}}, M)$-module satifying (A2).*

1. *We have*

$$H^i(\mathfrak{k}, M; \Omega(V)) = Hom_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V)),$$

with the $(\mathfrak{g}_{\mathbb{C}}, K)$-action $(\nu, I \otimes r_K)$, where

$$\nu(X)\omega = P_\Omega(\bar{\mu}(X)\omega), \text{ for } X \in \mathfrak{g}_{\mathbb{C}}, \omega \in Hom_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V)),$$

and P_Ω is the projection of $V \otimes \mathcal{H}(K)$ onto $\Omega(V)$ corresponding to the decomposition $V \otimes \mathcal{H}(K) = \Omega(V) \oplus X(V)$. Thus we have

$$(\Gamma_M^K)^i(V) = H^i(\mathfrak{k}, M; \Omega(V)) = Hom_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V)). \tag{1.6}$$

2. *Let W be a M-submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$. Then*

$$Hom_M(W, \Omega(V))$$

is a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of $Hom_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V))$.

If W is a M submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ then we will use the notation $\Gamma_W(V)$ for the $(\mathfrak{g}_{\mathbb{C}}, K)$-module $\mathrm{Hom}_M(W, \Omega(V))$. If $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}) = \sum_\delta W_\delta^i$ is a decomposition of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ into irreducible M-modules, then we have the decomposition

$$(\Gamma_M^K)^i(V) = H^i(\mathfrak{k}, M; \Omega(V)) = \sum_\delta \Gamma_{W_\delta^i}(V),$$

as $(\mathfrak{g}, K)$-modules. As K-modules, we have

$$\Gamma_W(V) \simeq \bigoplus_j \bigoplus_{\gamma \in \hat{K}_j} \mathrm{Hom}_M(W, V_j \otimes F_\gamma^*)\ F_\gamma. \tag{1.7}$$

2. Unitarizability. In this section, we assume beyond (A1) and (A2) that

(A3): there is a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form on the $(\mathfrak{g}_{\mathbb{C}}, M)$-module V, denoted by $\langle\ ,\ \rangle$.

Remark 2.1. *If V is a generalized Verma module, one constructs a $(\mathfrak{g}, M)$-invariant form in the standard way (the Shapavalov form). In some other cases, such a $(\mathfrak{g}, M)$-invariant form can often be "transfered" from a $(\mathfrak{g}', M)$-invariant form, where $\mathfrak{g}'$ is another real form of $\mathfrak{g}_{\mathbb{C}}$. See the proofs of Proposition 2.5 and Theorem 4.1.*

Since the form is $(\mathfrak{g}, M)$-invariant, it is $(\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}, M)$-invariant and so $\langle V_i, V_j \rangle = 0$ if $i \neq j$ (Schur's Lemma). On $\mathcal{H}(K)$ we put the L^2-inner product which we also denote by $\langle\ ,\ \rangle$. On $V \otimes \mathcal{H}(K)$, we put the tensor product Hermitian form for which we also use the notation $\langle\ ,\ \rangle$. Note that $\langle \Omega(V), X(V) \rangle = 0$ and so this form is non-degenerate on $\Omega(V)$. On $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$, we put the Hermitian inner product corresponding to $-B|_{\mathfrak{k} \times \mathfrak{k}}$, where B is a fixed $\mathrm{Ad}(G)$-invariant non-degenerate bilinear form on $\mathfrak{g}$ such that $-B(\theta X, X) > 0$ for X a non-zero element of $\mathfrak{g}$.

Identity

$$\mathrm{Hom}_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), \Omega(V)) = ((\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes \Omega(V))^M$$

as a subspace of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes \Omega(V)$. We put on the latter the tensor product Hermitian form, which is clearly non-degenerate. It will be denoted (as usual) by $\langle\ ,\ \rangle$. Since M is compact, and all the relevant forms are M-invariant, we see by an easy averaging argument that the restriction of $\langle\ ,\ \rangle$ to $(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes \Omega(V))^M$ is non-degenerate.

A straight forward calculation yields

Lemma 2.2. *Let W be an M-submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$, then the Hermitian form $\langle\ ,\ \rangle$ on $\Gamma_W(V)$ is $(\mathfrak{g}, K)$-invariant and non-degenerate.*

We now carry out an analysis of when the form $\langle\ ,\ \rangle$ on V induces a positive-definite form on $\Gamma_W(V)$. We recall some facts about unitary representations with $(\mathfrak{k}_1, M)$-cohomology.

Let L be an irreducible $(\mathfrak{k}_1, M)$-module which is unitarizable. Let F be an irreducible finite-dimensional $(\mathfrak{k}_1, M)$-module such that L and F have the same central character and the same infinitesimal character. Let W be an irreducible M-submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ and assume that

$$\operatorname{Hom}_M(\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}), L \otimes F^*) \neq 0.$$

We fix a maximal torus T in M with the Lie algebra $\mathfrak{t}$ and let

$$\mathfrak{h} = \{X \in \mathfrak{k}_{\mathbb{C}} |\ [X, \mathfrak{t}] = 0\}$$

be the corresponding fundamental Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. Denote θ' the Cartan involution of $\mathfrak{k}_{\mathbb{C}}$ corresponding to M. Then the result of Vogan-Zuckerman [18] implies that there exists a θ'-stable parabolic subalgebra, $\mathfrak{q} \supset \mathfrak{h}$, of $\mathfrak{k}_{\mathbb{C}}$, with the following properties: let $\mathfrak{u}$ be the nilradical of $\mathfrak{q}$ and let

$$\mathfrak{u}_n = \{X \in \mathfrak{u} |\ \theta'(X) = -X\}$$

be the non-compact part of $\mathfrak{u}$. Set $\rho_{\mathfrak{q},n}(h) = \frac{1}{2}\operatorname{tr}(ad(h)|_{\mathfrak{u}_n}$ for $h \in \mathfrak{t}$. Then W has highest weight $2\rho_{\mathfrak{q},n}$ with respect to any system of positive roots of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ compactible with $\mathfrak{q} \cap \mathfrak{m}_{\mathbb{C}}$. Let $\gamma_{M,\mathfrak{q}}(0)$ denote the corresponding equivalence class of irreducible M-modules.

We fix $\mathfrak{b}$, a θ'-stable Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ such that $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{q}$. If Λ is a dominant integral element of $\mathfrak{h}^*$ with respect to $\mathfrak{b}$ that is also T-integral, we define $\gamma_{M,\mathfrak{q}}(\Lambda)$ to be the equivalence class of irreducible M-modules with highest weight

$$\Lambda|_{\mathfrak{t}} + 2\rho_{\mathfrak{q},n}.$$

The result of Vogan-Zuckerman [18] now implies that if Λ is the highest weight of F with respect to $\mathfrak{b}$, then there exists such a $\mathfrak{q}$ with dim $F^{\mathfrak{u}} = 1$ and such that L is isomorphic to $A_{\mathfrak{q}}(\Lambda)$. Furthermore

$$\operatorname{Hom}_M(W, L \otimes F^*) = \operatorname{Hom}_M(W, L(\gamma_{M,\mathfrak{q}}(\Lambda)) \otimes F^*(\gamma_{M,\nu}{}^*)), \tag{2.3}$$

where $\nu = \Lambda|_{\mathfrak{t}}$. Here the notation $L(\gamma)$ and $F^*(\gamma)$ indicate isotypic component with respect to the M-type γ. Also, $\gamma_{M,\nu}$ is the equivalence class of irreducible M-module with highest weight ν with respect to $\mathfrak{b} \cap \mathfrak{m}_{\mathbb{C}}$. Finally the equality means that if we expand L and F^* into isotypic components with respect to M, then only the indicated term contributes. Note that $\gamma_{M,\mathfrak{q}}(\Lambda)$ is the unique minimal M-type of $A_{\mathfrak{q}}(\Lambda)$ in the sense of Vogan.

Applying (2.3) to each of the terms in the decomposition

$$\Gamma_W(V) \simeq \bigoplus_j \bigoplus_{\gamma\in\hat{K}_j} \mathrm{Hom}_M(W, V_j \otimes F_\gamma^*)\ F_\gamma,$$

we have proved the following result.

Proposition 2.4. *Let $\langle\ ,\ \rangle$ be a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form on V and let W be an irreducible M-submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$.*

1. *Suppose that $W \notin \gamma_{M,\mathfrak{q}}(0)$ for all $\mathfrak{q}$ (in $\mathfrak{k}_{\mathbb{C}}$). Then $\Gamma_W(V) = 0$.*
2. *Suppose $W \in \gamma_{M,\mathfrak{q}}(0)$ for some $\mathfrak{q}$ and the restriction of the form $\langle\ ,\ \rangle$ to the minimal M-type spaces*

$$V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j)) \times V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j))$$

is positive definite for each j such that $L_j \simeq A_{\mathfrak{q}}(\Lambda_j)$. Then $\Gamma_W(V)$ is unitary.

We conclude this section with a general unitarizability result.

We will use the notation of § 1. Recall that $M \subset K$ is a compact subgroup satisfying (A1) and (π, V) is a $(\mathfrak{g}_{\mathbb{C}}, M)$-module satisfying (A2). Let θ' be the Cartan involution of $\mathfrak{k}_{\mathbb{C}}$ corresponding to M. Assume that θ' extends to an involutive automorphism of $\mathfrak{g}$. Then $\theta\theta' = \theta'\theta$. Let

$$\mathfrak{p} = \{X \in \mathfrak{g} |\ \theta(X) = -X\}$$

and set $\mathfrak{g}_u = \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}}$, which is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. Let τ denote complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}_u$. Then $\sigma' = \tau\theta' = \theta'\tau$ is an involutive antilinear automorphism and it defines a corresponding real form $\mathfrak{g}'$ of $\mathfrak{g}_{\mathbb{C}}$. Thus σ' is the complex conjugation in $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{g}'$. Let G' be a connected real reductive group with Lie algebra $\mathfrak{g}'$ and maximal compact subgroup K' corresponding to the Cartan involution θ' with $\{k \in K' |\ \theta(k) = k\} \simeq M$. We will therefore think of M as a subgroup of K'. We have by construction

$$\sigma'|_{\mathfrak{g}} = \theta\theta'.$$

The following result gives a general criterion on "transfer of unitary structures".

Proposition 2.5. *Suppose that (π, V) is an irreducible unitarizable $(\mathfrak{g}', K')$-module with invariant inner product $(\ ,\ )$. Assume that there exists an involutive unitary operator T on V such that*

$$\pi(\theta\theta'(X)) = T\pi(X)T^{-1}, \quad \textit{and} \quad T\pi(m) = \pi(m)T, \tag{2.6}$$

where $X \in \mathfrak{g}_{\mathbb{C}}$, $m \in M$. Let $W \in \gamma_{M,\mathfrak{q}}(0)$ for some $\mathfrak{q}$, and assume further that for each j such that $L_j \simeq A_{\mathfrak{q}}(\Lambda_j)$, the corresponding minimal M-type spaces $V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j))$ all lie in V_+ or all lie in V_- where

$$V_\pm = \{v \in V |\ Tv = \pm v\}.$$

Then $\Gamma_W(V)$ is unitary.

Proof. Define a new form $\langle\ ,\ \rangle$ by $\langle v, w\rangle = (Tv, w)$, for $v, w \in V$. We claim that it is a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form.

We have for $X \in \mathfrak{g}$,

$$\begin{aligned} \langle Xv, w\rangle &= (T(Xv), w) = (\theta\theta'(X)Tv, w) \\ &= (\sigma'(X)Tv, w) = -(Tv, Xw) = -\langle v, Xw\rangle \end{aligned}$$

This proves the $\mathfrak{g}$-invariance. The rest of the claim is obvious.

By replacing T by $-T$ if necessary, we may assume that the minimal M-type spaces $V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j)$ all lie in the $+1$ eigenspace V_+ of T. Obviously the two forms $\langle\ ,\ \rangle$ and $(\ ,\ )$ coincide on V_+, so the positivity of $\langle\ ,\ \rangle$ on V_+ is the same as the positivity of $(\ ,\ )$ on V_+.

The result now follows from Proposition (2.4). □

We specialize to the following situation: suppose that $\theta\theta' \in \mathrm{Ad}(G)$. As $\theta\theta'$ preserves K, there will be some $m_0 \in K$ such that $\theta\theta' = \mathrm{Ad}(m_0)$. This implies that $\theta'(m_0) = m_0$, and so $m_0 \in M$. Note that $\mathrm{Ad}(m_0^2) = I$ and $\mathrm{Ad}(m_0)|_M = I$. Thus $m_0^2 \in Z(G) = Z(G')$ and $m_0 \in Z(M)$.

Let χ be the central character of V and set

$$T = \frac{1}{\chi(m_0^2)^{\frac{1}{2}}}\pi(m_0),$$

where $\chi(m_0^2)^{\frac{1}{2}}$ is a fixed choice of square root of $\chi(m_0^2)$. Then T satisfies the hypothesis of Proposition (2.5). We therefore have the following

Corollary 2.7. *Suppose that (π, V) is an irreducible unitarizable $(\mathfrak{g}', K')$-module with invariant inner product $(\ ,\ )$. Assume that $\theta\theta' = Ad(m_0)$ for $m_0 \in M$. Let $W \in \gamma_{M,\mathfrak{q}}(0)$ for some $\mathfrak{q}$, and assume further that for each j such that $L_j \simeq A_{\mathfrak{q}}(\Lambda_j)$, the action of m_0 on the corresponding minimal M-type spaces $V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j))$ is of the same multiple (out of the two possible choices $\pm\chi(m_0^2)^{\frac{1}{2}}$). Then $\Gamma_W(V)$ is unitary.*

3. The case when the L_j are unitary highest weight modules. In this section, we examine the case when all the L_j's in the decomposition of $V|_{(\mathfrak{k}_{\mathbb{C}},M)}$ in (1.4) are unitary highest weight modules. To conclude unitarity of $\Gamma_W(V)$ (Proposition 2.4), we need to know the minimal M-types of such modules. We shall adhere to the usual practice of denoting a real reductive group by G. The relevant group we have in mind is actually K_1 (and for G' in § 4).

For the moment, let G be a connected real reductive group with compact center, and K is a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Assume that $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian symmetric pair. Thus we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

with $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ an $\mathrm{Ad}(K)$-invariant decomposition, and $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0$. We note that $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+$ is a (maximal) parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$ and $\mathfrak{b}_k$ be a Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ such that $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{b}_k$. Let Φ_k^+ be the corresponding system of positive roots of $\Phi(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let

$$\Phi_n^+ = \{\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})|\ (\mathfrak{g}_{\mathbb{C}})_\alpha \subset \mathfrak{p}^+\}.$$

Then $\Phi^+ = \Phi_k^+ \cup \Phi_n^+$ is a positive system of $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Set

$$\rho_k = \frac{1}{2} \sum_{\alpha \in \Phi_k^+} \alpha, \qquad \rho_n = \frac{1}{2} \sum_{\alpha \in \Phi_n^+} \alpha,$$

and $\rho = \rho_k + \rho_n$.

If F is an irreducible unitary K-module, set

$$N(F) = \mathcal{U}(\mathfrak{g}_\mathbb{C}) \otimes_{\mathcal{U}(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+)} F \tag{3.1}$$

to be the corresponding generalized Verma module. Here F is regarded as a $\mathcal{U}(\mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+)$-module with $\mathfrak{p}^+ F = 0$. Let $L(F)$ denote the unique irreducible quotient of $N(F)$. As usual, $N(F)$ and $L(F)$ are $(\mathfrak{g}, K)$-modules with $\mathfrak{g}$ acting by left multiplication and K acting by

$$k(x \otimes f) = \mathrm{Ad}(k)x \otimes kf, \qquad \text{for } x \in \mathcal{U}(\mathfrak{g}_\mathbb{C}), f \in F.$$

We will now introduce the invariant Hermitian form (Shapavalov form) on $N(F)$. Let

$$p: \mathcal{U}(\mathfrak{g}_\mathbb{C}) \to \mathcal{U}(\mathfrak{k}_\mathbb{C})$$

be the projection given by the decomposition:

$$\mathcal{U}(\mathfrak{g}_\mathbb{C}) = \mathcal{U}(\mathfrak{k}_\mathbb{C}) \oplus (\mathfrak{p}^- \mathcal{U}(\mathfrak{g}_\mathbb{C}) + \mathcal{U}(\mathfrak{g}_\mathbb{C})\mathfrak{p}^+).$$

Let σ be the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$. For $X \in \mathfrak{g}_\mathbb{C}$ define $X^* = -\sigma(X)$ and extend this action to a conjugate linear antiautomorphism of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$: i.e.,

$$(xy)^* = y^* x^*, \ 1^* = 1, \qquad \text{with } \ x, y \in \mathcal{U}(\mathfrak{g}_\mathbb{C}).$$

If $x, y \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$, $f, g \in F$ then we set

$$(x \otimes f, y \otimes g) = (p(y^* x) f, g).$$

Then $(\ ,\ )$ defines a $(\mathfrak{g}, K)$-invariant Hermitian form on $N(F)$. As is well-known the radical of this form is $M(F)$, the unique maximal proper submodule of $N(F)$. Therefore $(\ ,\ )$ induces a $(\mathfrak{g}, K)$-invariant non-degenerate Hermitian form on

$$L(F) = N(F)/M(F). \tag{3.2}$$

Such a form is unique (up to a scalar multiple).

Let λ be the highest weight of F with respect to $\mathfrak{b}_k$. If $\gamma \in \hat{K}$ then let λ_γ denote the highest weight of any representative of γ with respect to $\mathfrak{b}_k$. We fix an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}_\mathbb{C}$ with $-B(\theta X, X) > 0$ for X a non-zero element of $\mathfrak{g}$, as before. On $i\mathfrak{t}^*$, we will use the inner product $(\ ,\ )$ that is dual to $B|_{i\mathfrak{t} \times i\mathfrak{t}}$. The following result is (essentially) due to Parthasarathy [14]

Proposition 3.3. *Assume that $L(F)$ is unitarizable.*

1. *F is the unique minimal K-type in the sense of Vogan. Namely if $\gamma \in \hat{K}$ is such that $L(F)(\gamma) \neq 0$ then*

$$||\lambda_\gamma + 2\rho_k|| \geq ||\lambda + 2\rho_k||,$$

with equality if and only if $\lambda_\gamma = \lambda$.

2. *Suppose in addition that $\lambda+\rho$ is integral and regular. Let $s \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (the Weyl group) be such that $s(\lambda+\rho)$ is dominant. Then there exits a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ depending only on s such that $L(F) \simeq A_{\mathfrak{q}}(\xi)$ for an appropriate ξ.*

Remark 3.4. *We may decompose $L(F)$ into even and odd parts, as follows: let T be the following (well-defined) involutive endomorphism of $N = N(F)$:*

$$T(x \otimes f) = \theta(x) \otimes f, \qquad x \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}}), f \in F.$$

Since θ commutes with $*$ and p, we have $(Tv, Tw) = (v, w)$ for $v, w \in N$. Thus T preserves the maximal proper submodule of N and "pushes down" to an involutive endomorphism of $L = L(F)$, still denoted by T. Assume that L is unitarizable. We have the (orthogonal) eigenspace decomposition of T:

$$L = L_+ \oplus L_-,$$

where $L_\pm = \{v \in L | Tv = \pm v\}$ are the even and odd parts. Note also that $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and so we have the orthogonal direct sum decomposition

$$L = L^{\mathfrak{p}^+} \oplus \mathfrak{p}^- L.$$

Clearly the minimal K-type of $L(F)$ is $F \simeq L^{\mathfrak{p}^+} \subset L_+$ and $L_- \subset \mathfrak{p}^- L$. Considerations of evenness and oddness (with respect to some involutive operator T) were used in a number of ways in [3] and [6] to arrive at unitarity results. C.f. Proposition 2.5 and Theorem 4.1.

We now return to the situation of the previous two sections: $M \subset K$ is a compact subgroup satisfying (A1) and (π, V) is a $(\mathfrak{g}_{\mathbb{C}}, M)$-module satisfying (A2) and possessing a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form $\langle\ ,\ \rangle$ (as in (A3)). In addition, we assume that $(\mathfrak{k}_1, M)$ satisfies the hypotheses above for $(\mathfrak{g}, K)$, namely the pair $(\mathfrak{k}_1, M)$ is of Hermitian symmetric type. We write

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{o}_k^+ \oplus \mathfrak{o}_k^-$$

for the decomposition used above for $\mathfrak{g}_{\mathbb{C}}$.

In view of Proposition 3.3, Proposition 2.4 now implies the following

Proposition 3.5. *Assume that in the decomposition (1.4), each $L_j \simeq L(F_j)$, a unitarizable $(\mathfrak{k}_1, M)$-module with highest weight. Assume further that the $(\mathfrak{g}, M)$-invariant Hermitian form $\langle\ ,\ \rangle$ is positive definite on the following highest weight space*

$$V^{\mathfrak{o}_k^+} = \{v \in V | \mathfrak{o}_k^+ v = 0\}.$$

Then $\Gamma_W(V)$ is either zero or unitarizable for each irreducible M-submodule W of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$.

4. The case when V is a unitarizable highest weight module. In this section we will be dealing with an important special case in which we will be able to give an easily calculated criterion for unitarizability. Let $\mathfrak{g}$ be semi-simple with Cantan involution θ. Let $\mathfrak{g}'$ be another real form of $\mathfrak{g}_{\mathbb{C}}$ with Cartan involution θ'.

Assume that $\theta\theta' = \theta'\theta$. In addition we assume that the pair $(\mathfrak{g}', \mathfrak{k}')$ is Hermitian symmetric with

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{o}^+ \oplus \mathfrak{o}^-,$$

$\mathfrak{o} = (\mathfrak{p}')_{\mathbb{C}} = \mathfrak{o}^+ \oplus \mathfrak{o}^-$, $\mathfrak{o}^\pm$ abelian and $[\mathfrak{k}'_{\mathbb{C}}, \mathfrak{o}^\pm] \subset \mathfrak{o}^\pm$. We further assume that there is a connected semi-simple Lie group G' with Lie algebra $\mathfrak{g}'$ having maximal compact subgroup K' with algebra $\mathfrak{k}'$. We assume also that θ defines an automorphism of G' and that

$$M = \{k \in K' | \; \theta(k) = k\}.$$

Clearly we have

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{o}_k^+ \oplus \mathfrak{o}_k^-,$$

where $\mathfrak{o}_k^\pm = \{X \in \mathfrak{o}^\pm | \theta(X) = X\}$.

THEOREM 4.1. *Let $V = L(F)$ be a unitarizable $(\mathfrak{g}', K')$-module of highest weight. We assume that*

$$\theta(\mathfrak{o}^+) = \mathfrak{o}^+, \tag{4.2}$$

$$\theta(k')f = k'f, \qquad k' \in K', f \in F. \tag{4.3}$$

Then $\Gamma_W(V)$ is either zero or unitarizable for each irreducible M-submodule W of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$.

REMARK 4.4. *The condition (4.2) ensures that θ fixes the center of K' (as opposed to sending an element of the center to its inverse). Geometrically this means that K_1/M is a complex submanifold of G'/K'.*

Proof. Note that (A1) is automatic in our setting. We have already observed that the condition (4.2) implies that θ acts as the identity on the center $Z(K')$ of K' and so $M \supset Z(K')$. Since V is admissible as a $Z(K')$-module, this implies that the hypothesis (A2) is also satisfied. Furthermore all the L_j's in the decomposition of $V|_{(\mathfrak{k}_{\mathbb{C}}, M)}$ in (1.4) are unitary highest weight modules for the pair $(\mathfrak{k}_1, M)$. See [12].

If $x \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ and $f \in F$, then set

$$T(x \otimes f) = \theta\theta'(x) \otimes f.$$

Since $\theta\theta'(\mathfrak{k}'_{\mathbb{C}}) = \mathfrak{k}'_{\mathbb{C}}$ and $\theta\theta'(\mathfrak{o}^+) = \mathfrak{o}^+$, (4.3) implies that T "pushes down" to an involutive endomorphism of $N = N(F)$. Note that $\theta\theta'$ commutes with the operators $*$ and p (defined in § 3), the condition (4.3) also implies that $(Tv, Tw) = (v, w)$ for $v, w \in N$. Here $(\; , \;)$ is the canonical $(\mathfrak{g}', K')$-invariant Hermitian form on N defined previously. Thus T preserves the maximal proper submodule of N and "pushes down" to an involutive endomorphism of $L = L(F)$, still denoted by T.

Note that T satisfies the required properties in (2.6) of Proposition (2.5). As in its proof, we define a new form $\langle \; , \; \rangle$ by $\langle v, w\rangle = (Tv, w)$, for $v, w \in V$ and we see that it is a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form.

We have the eigenspace decomposition of $\theta|_{\mathfrak{o}^\pm}$:

$$\mathfrak{o}^\pm = \mathfrak{o}_k^\pm \oplus \mathfrak{o}_n^\pm,$$

where $\mathfrak{o}_n^{\pm} = \{X \in \mathfrak{o}^{\pm} | \theta(X) = -X\}$. Note that $(\mathfrak{o}_k^{+})^* = \mathfrak{o}_k^{-}$ and so

$$L = L^{\mathfrak{o}_k^{+}} \oplus \mathfrak{o}_k^{-} L,$$

an orthogonal direct sum decomposition with respect to both $(\ ,\ )$ and $\langle\ ,\ \rangle$.

We also have the orthogonal eigenspace decomposition of T:

$$N = N_+ \oplus N_-, \qquad L = L_+ \oplus L_-,$$

where $N_\pm = \{v \in N | Tv = \pm v\}$ and $L_\pm = \{v \in L | Tv = \pm v\}$. Since $N = \mathcal{U}(\mathfrak{o}^-) \otimes F = \mathcal{U}(\mathfrak{o}_k^-)\mathcal{U}(\mathfrak{o}_n^-) \otimes F$ and since

$$\theta\theta'|_{\mathfrak{o}_k^-} = -I, \qquad \text{and} \qquad \theta\theta'|_{\mathfrak{o}_n^-} = I,$$

we have

$$\mathcal{U}(\mathfrak{o}_n^-) \otimes F \subset N_+, \quad \text{and so} \quad N_- \subset \mathfrak{o}_k^- N.$$

As L is a $\mathfrak{g}_{\mathbb{C}}$ (and T-module) quotient of N, we have

$$L_- \subset \mathfrak{o}_k^- L.$$

Taking the orthogonal complement, we have $L^{\mathfrak{o}_k^+} \subset L_+$. This implies that the forms $(\ ,\ )$ and $\langle\ ,\ \rangle$ are identical on $L^{\mathfrak{o}_k^+}$.

In view of Proposition (3.5), the result follows. □

5. A family of examples. In this section we will analyse a specific family of examples where the two real forms are Hermitian symmetric, indeed isomorphic to the real symplectic group. Special cases of the material of this section were studied in [21]. We consider $(\mathfrak{g}, K) \simeq (Sp(2n,\mathbb{R}), U(n))$ and $(\mathfrak{g}', K') \simeq (Sp(2n,\mathbb{R}), U(n))$. For both groups we need to go to the two-fold covering of $Sp(2n,\mathbb{R})$, denoted by $\widetilde{Sp}(2n,\mathbb{R})$. We label the roots Φ of $\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}$ (in the usual way) as $\epsilon_i \pm \epsilon_j$, $i \neq j$ and $\pm 2\epsilon_i$, $1 \leq i, j \leq n$. Choose the system of positive roots Φ^+ so that the simple roots are

$$\alpha_1 = 2\epsilon_n, \alpha_2 = \epsilon_1 - \epsilon_2, ..., \alpha_n = \epsilon_{n-1} - \epsilon_n.$$

Let r, s be non-negative integers with $r + s = n$. Set

$$\begin{cases} \eta_i = \epsilon_i, & 1 \leq i \leq r, \\ \eta_{r+i} = -\epsilon_{n-i+1}, & 1 \leq i \leq s. \end{cases}$$

Let

$$H = \frac{1}{2}\sum_{i=1}^{n} \eta_i, \quad \text{and} \quad H' = \frac{1}{2}\sum_{i=1}^{r} \eta_i - \frac{1}{2}\sum_{i=r+1}^{n} \eta_i.$$

We take

$$\theta = e^{\pi i \operatorname{ad} H}, \quad \theta' = e^{\pi i \operatorname{ad} H'}$$

to define the Cartan involutions of the corresponding pairs.

The positive roots of K are given by

$$\eta_i - \eta_j, \quad 1 \leq i < j \leq n.$$

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{p}'_{\mathbb{C}}$ be the (complexified) Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$. Let $\Phi^+_{k'}$ and $\Phi^+_{n'}$ be respectively the positive roots in $\mathfrak{k}'_{\mathbb{C}}$ and $\mathfrak{p}'_{\mathbb{C}}$. Write $\mathfrak{o} = \mathfrak{p}'_{\mathbb{C}}$ and choose $\mathfrak{o}^+$ so that

$$\Phi^+_{n'} = \{\alpha \in \Phi | (\mathfrak{g}_{\mathbb{C}})_\alpha \in \mathfrak{o}^+\}.$$

Then we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{o} \oplus \mathfrak{o}^-.$$

Write $\mathfrak{o}^\pm_k = \mathfrak{o}^\pm \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{o}^\pm_n = \mathfrak{o}^\pm \cap \mathfrak{p}$, as before. Then we have

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{o}_k \oplus \mathfrak{o}^-_k.$$

We set

$$\mathfrak{l}_{\mathbb{C}} = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{o}^+_n \oplus \mathfrak{o}^-_n$$

and $\mathfrak{l} = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{g}$. Then $\mathfrak{l} = \mathfrak{g} \cap \mathfrak{g}'$ and $(\mathfrak{l}, M)$ is a symmetric pair of Hermitian symmetric type. We have

$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \simeq \mathfrak{sp}(2r, \mathbb{R}) \oplus \mathfrak{sp}(2s, \mathbb{R}).$$

The simple roots for $\mathfrak{l}_1$ are

$$\eta_1 - \eta_2, ..., \eta_{r-1} - \eta_r, 2\eta_r$$

and those for $\mathfrak{l}_2$ are

$$-2\eta_{r+1}, \eta_{r+1} - \eta_{r+2}, ..., \eta_{n-1} - \eta_n.$$

We have

$$M = K \cap K' \simeq U(r) \times U(s).$$

We assume that we have chosen $\mathfrak{p}^+$ such that

$$\mathfrak{o}^+_n \cap \mathfrak{p}^+ \subset (\mathfrak{l}_1)_{\mathbb{C}}.$$

Then Harish-Chandra's strongly orthogonal roots for $\mathfrak{l}_1$ are $2\eta_r, ..., 2\eta_1$ and those for $\mathfrak{l}_2$ are $-2\eta_{r+1}, ..., -2\eta_n$.

Let $\mathfrak{b}_m$ be Borel subalgebra of $\mathfrak{m}_{\mathbb{C}}$ fixed by our choice of positive roots and let $\mathfrak{n}^+_m$ be the nilradical of $\mathfrak{b}_m$. We shall need to use some (non-zero) highest weight vectors

$$v^+_i \in S(\mathfrak{o}^-_n \cap \mathfrak{p}^+)(-2\eta_r - \cdots - 2\eta_{r-i-1})^{\mathfrak{n}^+_m}$$

and

$$v^-_i \in S(\mathfrak{o}^-_n \cap \mathfrak{p}^-)(2\eta_{r+1} + \cdots + 2\eta_{q+i})^{\mathfrak{n}^+_m}.$$

Here the notation S indicates the symmetric algebra and $S(\mathfrak{o}^-_n \cap \mathfrak{p}^+)(\lambda)$ denotes the M-isotypic component of $S(\mathfrak{o}^-_n \cap \mathfrak{p}^+)$ with highest weight λ, and similar notations apply likewise. Then

$$S(\mathfrak{o}^-_n)^{\mathfrak{n}^+_m} = \mathbb{C}[v^+_1, ..., v^+_r, v^-_1, ..., v^-_s].$$

This follows from a well-known result of Schmid [15].

Note that $\alpha_1 = 2\epsilon_n$ is the unique simple root in $\Phi^+_{n'}$. Let $\Lambda_1 \in \mathfrak{h}^*$ be defined by

$$\frac{2(\Lambda_1, \alpha_i)}{(\alpha_i, \alpha_i)} = \delta_{i1}.$$

In terms of $\eta_1, ..., \eta_n$, we have

$$\Lambda_1 = \eta_1 + \cdots + \eta_r - \eta_{r+1} - \cdots - \eta_n.$$

We shall consider unitary highest weight module $L(-\frac{m}{2}\Lambda_1)$ of $G = \widetilde{Sp}(2n, \mathbb{R})$ of highest weight $-\frac{m}{2}\Lambda_1$, m a natural number. For $m \geq n$, it was studied in [6], and so we will concentrate on the cases $m < n$.

From the analysis in [19], we have (as a K'-module)

$$L(-\frac{m}{2}\Lambda_1) \simeq \mathbb{C}_{-\frac{m}{2}\Lambda_1} \otimes S(\mathfrak{o}^-)/ \sum_{k \geq m+1} S(\mathfrak{o}^-)V_k,$$

where $\mathbb{C}_{-\frac{m}{2}\Lambda_1}$ is the 1-dimensional K'-module of weight $-\frac{m}{2}\Lambda_1$,

$$V_k = S(\mathfrak{o}^-)(-\gamma_1 - \cdots - \gamma_k),$$

and $\gamma_1 < \cdots < \gamma_n$ is the Harish-Chandra's system of strongly orthogonal roots for $\Phi^+_{n'}$. From this, we may conclude that

$$L(-\frac{m}{2}\Lambda_1)^{\mathfrak{o}^+_k} \simeq \mathbb{C}_{-\frac{m}{2}\Lambda_1} \otimes S(\mathfrak{o}^-_n)/ \sum_{k \geq m+1} S(\mathfrak{o}^-_n)q(V_k),$$

where $q : S(\mathfrak{o}^-) \mapsto S(\mathfrak{o}^-_n)$ is the surjective algebra homomorphism with kernel $\ker q = \mathfrak{o}^-_k S(\mathfrak{o}^-)$.

The M-highest weight vectors in $\sum_{k \geq m+1} S(\mathfrak{o}^-_n)q(V_k)$ are computed to be of the form

$$\sum_{i+j \geq m+1} \mathbb{C}[v^+_1, ..., v^+_r, v^-_1, ..., v^-_s]v^+_i v^-_j.$$

This implies that the M-highest weight vectors in $L(-\frac{m}{2}\Lambda_1)^{\mathfrak{o}^+_k}$ are of the form

$$\mathbb{C}_{-\frac{m}{2}\Lambda_1} \otimes \mathbb{C}[v^+_1, ..., v^+_r, v^-_1, ..., v^-_s]/ \sum_{i+j \geq m+1} \mathbb{C}[v^+_1, ..., v^+_r, v^-_1, ..., v^-_s]v^+_i v^-_j.$$

Let μ be such a highest weight, then

$$\mu = -\frac{m}{2}\Lambda_1 - 2k_1\eta_r - \cdots - 2k_r\eta_1 + 2l_1\eta_{r+1} + \cdots + 2l_s\eta_n,$$

where

$$k_1 \geq \cdots \geq k_r \geq 0, \qquad l_1 \geq \cdots \geq l_s \geq 0,$$

and

$$k_i l_j = 0, \qquad \text{if } i + j \geq m + 1.$$

Let $1 \le p \le s$ be the largest j such that $l_j > 0$. Clearly $p \le m$.
Set $q = m - p$. Then $k_i = 0$ for any $i \ge m + 1 - p = q + 1$.
Denote

$$\hat{M}_{p,q} = \{\mu | k_1 \ge \cdots \ge k_q \ge 0,\ l_1 \ge \cdots \ge l_p > 0\}.$$

Then we get a disjoint union decomposition:

$$M\text{-types of } \ L(-\frac{m}{2}\Lambda_1)^{\mathfrak{o}_k^+} = \coprod_{p+q=m} \hat{M}_{p,q}.$$

Take $\mu \in \hat{M}_{p,q}$:

$$\mu = -\frac{m}{2}\Lambda_1 - 2k_1\eta_r - \cdots - 2k_q\eta_{r-q+1} + 2l_1\eta_{r+1} + \cdots + 2l_p\eta_{r+p}.$$

We have $\rho_k = \sum_{j=1}^n \frac{n-(2j-1)}{2}\eta_j$. In terms of the η coordinates, we have

$$\mu + \rho_k = (\lambda_1, ..., \lambda_n) + (\underbrace{0, ..., 0}_{r-q}, \underbrace{-2k_q, ..., -2k_1}_{q}, \underbrace{2l_1, ..., 2l_p}_{p}, \underbrace{0, ..., 0}_{s-p}),$$

where

$$\lambda_j = \begin{cases} \frac{n-(2j-1)}{2} - \frac{m}{2}, & 1 \le j \le r, \\ \frac{n-(2j-1)}{2} + \frac{m}{2}, & r+1 \le j \le n. \end{cases}$$

If $q < r$, then we require that

$$2l_p + \frac{n - (2(r+p) - 1)}{2} + \frac{m}{2} > \frac{n-1}{2} - \frac{m}{2},$$

and if $p < s$, then we require that

$$-\frac{n-1}{2} + \frac{m}{2} > -2k_q + \frac{n - (2(r-q)+1)}{2} - \frac{m}{2}.$$

The stated conditions are respectively

$$2l_p > r - q - 1, \text{ if } q < r,$$

and

$$2k_q > s - p - 1, \text{ if } p < s.$$

The point here is that $\mu + \rho_k$ is regular if both of the above conditions are satisfied.

The regular weights μ in $\hat{M}_{p,q}$ are all in the same K-Weyl chamber. Let $s = s_{p,q}$ be the following permutation so that $s(\mu + \rho_k)$ is dominant regular:

$$\begin{aligned}
&\eta_{r+1} \mapsto \eta_1, ..., \eta_{r+p} \mapsto \eta_p, \qquad (p \text{ of them}),\\
&\eta_1 \mapsto \eta_{p+1}, ..., \eta_{r-q} \mapsto \eta_{p+r-q}, \qquad (r-q \text{ of them}),\\
&\eta_{p+r+1} \mapsto \eta_{p+r-q+1}, ..., \eta_n \mapsto \eta_{n-q}, \qquad (s-p \text{ of them}),\\
&\eta_{r-q+1} \mapsto \eta_{n-q+1}, ..., \eta_r \mapsto \eta_n, \qquad (q \text{ of them}).
\end{aligned}$$

The length of s is equal to

$$(r-q)p+qp+q(s-p)=rs-(r-q)(s-p).$$

Note that the "middle" dimension is $\frac{1}{2}\dim(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})=rs$.

One checks that the corresponding θ'-stable parabolic $\mathfrak{q}$ of $\mathfrak{k}_{\mathbb{C}}$ is determined by the wight

$$\lambda_{p,q}=-(\eta_{r-q+1}+\cdots+\eta_r)+(\eta_{r+1}+\cdots+\eta_{r+p}).$$

In turn $\mathfrak{q}$ corresponds to an irreducible M-submodule $W_{p,q}$ of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$, where $i=rs-(r-q)(s-p)$.

Write

$$\Gamma_{p,q}=\Gamma_{W_{p,q}}.$$

Using (1.7) and (2.3) or a derived functor analog of Borel-Weil-Bott theorem (see [4], Proposition 6.3 and [7], Corollary 3.2), we see that the highest weights of K-types of $\Gamma_{p,q}(L(-\frac{m}{2}\Lambda_1))$ are those ν such that

$$\begin{aligned}\nu+\rho_k&=s(\mu+\rho_k)\\&=(g_1,...,g_n)+(\underbrace{2l_1,...,2l_p}_{p},0,...,0,\underbrace{-2k_q,...,-2k_1}_{q}),\end{aligned}$$

where

$$g_i=\begin{cases}\frac{n-2(r+i)-1}{2}+\frac{m}{2}, & 1\le i\le p,\\ \frac{n-2(i-p)-1}{2}-\frac{m}{2}, & p+1\le i\le p+r-q,\\ \frac{n-2(i+q)-1}{2}+\frac{m}{2}, & p+r-q+1\le i\le n-q,\\ \frac{n-2(i-s)-1}{2}-\frac{m}{2}, & n-q+1\le i\le n,\end{cases}$$

and

$$k_1\ge\cdots\ge k_q\ge[\frac{s-p}{2}],\qquad l_1\ge\cdots\ge l_p\ge[\frac{r-q}{2}].$$

We therefore have

$$\nu=(h_1,...,h_n)+(\underbrace{2l_1,...,2l_p}_{p},0,...,0,\underbrace{-2k_q,...,-2k_1}_{q}),$$

where

$$h_i=\begin{cases}-r+\frac{m}{2}=\frac{p-q}{2}-(r-q), & 1\le i\le p,\\ p-\frac{m}{2}=\frac{p-q}{2}, & p+1\le i\le p+r-q,\\ -q+\frac{m}{2}=\frac{p-q}{2}, & p+r-q+1\le i\le n-q,\\ s-\frac{m}{2}=\frac{p-q}{2}+(s-p), & n-q+1\le i\le n.\end{cases}$$

Let $\xi,\eta\in\{0,1\}$ such that

$$\xi\equiv r-q \pmod 2,\qquad \eta\equiv s-p \pmod 2.$$

Then $r-q=2[\frac{r-q}{2}]+\xi$, $s-p=2[\frac{s-p}{2}]+\eta$.

We therefore conclude that the K-types of $\Gamma_{p,q}(L(-\frac{m}{2}\Lambda_1))$ are of the form

$$\nu=\frac{p-q}{2}1_n+(\underbrace{2l_1'+\xi,...,2l_p'+\xi}_{p},0,...,0,\underbrace{-2k_q'-\eta,...,-2k_1'-\eta}_{q}),$$

where

$$l_1'\geq\cdots\geq l_p'\geq 0,\qquad k_1'\geq\cdots\geq k_q'\geq 0.$$

At this point we have constructed a family of unitarizable $(\mathfrak{g},K)$ modules $\Gamma_{p,q}(L(-\frac{m}{2}\Lambda_1)$ for $p\leq s, q\leq r$ and $p+q=m\leq n$. We will now give a conjectural interpretation of these unitary representations.

We consider the reductive dual pair $(O(p,q),Sp(2n,\mathbb{R}))$ in the sense of Howe [10]. For $\xi,\eta\in\{0,1\}$, we define a character $1^{\xi,\eta}$ of $O(p,q)$ by the conditions

$$1^{\xi,\eta}|_{O(p)}=(\det)^{\xi},\qquad 1^{\xi,\eta}|_{O(q)}=(\det)^{\eta}.$$

Then the general theory of Howe [11] assigns a representation $\theta^{p,q}(1^{\xi,\eta})$ of $\widetilde{Sp}(2n,\mathbb{R})$ to the character $1^{\xi,\eta}$ (and to any irreducible admissible representation of $O(p,q)$). The representations $\theta^{p,q}(1^{\xi,\eta})$ are called theta lifts of $1^{\xi,\eta}$ and are all unitary when $p+q\leq n$ (the stable range condition, see [13]).

Conjecture 5.1. *Let $p\leq s$, $q\leq r$ and $p+q=m\leq n$. Then we have*

$$\Gamma_{p,q}(L(-\frac{m}{2}\Lambda_1))\simeq\theta^{p,q}(1^{\xi,\eta}),$$

where $\xi\equiv r-q$, $\eta\equiv s-p$ (mod 2).

Note that $L(-\frac{m}{2}\Lambda_1)\simeq\theta^{0,m}(1^{0,0})$. It is easily checked that the conjecture is true in the special case when $\xi=\eta=0$. The representations concerned will then contain a (same) 1-dimensional K-type, and so are determined by their infinitesimal characters. We refer the reader to [23] for this as well as discussions on $\theta^{p,q}(1^{\xi,\eta})$ and other related representations arising from the formalism of Howe correspondence.

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