SOME ASPECTS OF UNITARY REPRESENTATIONS OF CLASSICAL GROUPS

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Introduction

A fundamental problem in representation theory is to determine the unitary dual $\hat{G}$ of a given (real) reductive Lie group G, namely the collection of equivalent classes of all irreducible unitary representations of G. For a compact Lie group, irreducible unitary representations were parameterized long time ago through the work of Cartan and Weyl (Cartan's Theorem of Highest Weight and Weyl's Character Formula). Unfortunately if the reductive Lie group is not compact, its unitary representations are either 1-dimensional or can only be found in infinite-dimensional spaces. Thus a major task is to invent ways of constructing (new) representations and determine when the representations produced are unitary. In this article, we will first review three important and well-known constructions: parabolic induction, cohomological induction and dual pair correspondence. We will also discuss a comparison technique called transfer of unitary representations between real forms.

As mentioned, the most (and the only) obvious unitary representations of any reductive Lie group G are its unitary characters. Starting from this collection, other unitary representations may then be constructed through application of general procedures, and so on. Indeed, the spherical unitary principal series are obtained by normalized parabolic induction from unitary characters of a minimal parabolic subgroup and are naturally associated with certain hyperbolic coadjoint orbits; Unitary representations associated to elliptic coadjoint orbits are obtained by cohomological induction from unitary characters of the centralizers of elliptic elements (the $A_{\mathfrak{q}}(\lambda)$'s), among them there

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are the discrete series representations; It is therefore reasonable to expect that unitary representations obtained by dual pair correspondence of unitary characters should be special as well. The second purpose of this article is to examine this distinguished class of unitary representations (for classical groups) and discuss their connections to the various other general constructions and to the philosophy of unipotent representations, as expounded by Vogan [16].

1. Constructing representations from those of subgroups

Two main methods of constructing representations of a reductive Lie group are parabolic induction and cohomological induction. Both start from representations of subgroups and yield unitary representations under appropriate conditions.

We introduce some notations. For any Lie group G, denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{g}$. The trivial representation of G will be denoted by $\mathbb{1}_G$ or just $\mathbb{1}$ if no confusion shall arise in the context. Similar notations will be used throughout the article.

1.1. **Parabolic induction.**

Let us give the induction construction in its simpliest form.

Let G be a group and $B \subseteq G$ be a subgroup. Given a representation (σ, W) of B, one considers the space

$$\mathrm{Ind}_B^G(\sigma) = \left\{ f : G \mapsto W \mid f(xb) = \sigma(b)^{-1} f(x), \quad x \in G, b \in B \right\}.$$

Then G acts on $\mathrm{Ind}_B^G(\sigma)$ by left translation. This is called the representation of G induced by σ.

Remark 1.1.1. (1) *G acts on the space $\mathbb{C}(G)$ of all functions on G by left translation (the regular representation). The left action of G preseves the space $\mathrm{Ind}_B^G(\sigma)$ because left and right multiplications on G commute with each other.*

(2) *Let $X = G/B$ and*

$$\mathcal{V}_W : G \times_B W \to X$$

be the homogeneous vector bundle over X associated to the representation σ of B on W. Then the function space $\mathrm{Ind}_B^G(\sigma)$ may be identified with the space of sections of $\mathcal{V}_W$.

(3) *If G is a nilpotent Lie group, then induction from (appropriate) subgroups of G will yield all irreducible unitary representations of G.*

Definition 1.1.2. (1) *A linear reductive Lie group is a closed subgroup of $GL(n,\mathbb{R})$ (for some n) such that $g \in G$ if and only if ${}^t g \in G$.*
(2) *A reductive Lie group is a finite covering of a linear reductive Lie group.*

Remark 1.1.3. *A reductive Lie group G comes with it a Cartan involution θ. Its fixed point group $K = G^\theta$ is a maximal compact subgroup. For a linear reductive Lie group G in $GL(n,\mathbb{R})$, we have $\theta(g) = {}^t g^{-1}$ ($g \in G$) and $K = G \cap O(n)$.*

Let G be a reductive Lie group with a fixed maximal compact subgroup K. A manageable class of representations of G are the so-called admissible representations, those whose restriction to K contains any $\delta \in \hat{K}$ only finite number of times. All irreducible unitary representations are admissible. We will only be concerned with admissible representations.

The most relevant subgroups of G to induce representations are the parabolic subgroups. These are closed subgroups P of G such that G/P are compact manifolds (called generalized flag varieties). A parabolic subgroup P has a Langlands decomposition $P = MAN$, where A is a vector group, MA is the Levi component of P, and N is nilpotent. We take σ to be an admissible representation of M, and $\nu \in \mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \oplus i\mathfrak{a}^*$. Denote

$$I(P;\sigma,\nu) = (smooth) - Ind_P^G(\sigma \otimes e^{\nu+\rho} \otimes \mathbb{1}_N).$$

Here and as usual, $\rho \in \mathfrak{a}^*$ denotes the half sum of all positive restricted roots (counted with multiplicity), and $e^{\nu+\rho}$ denotes the quasi-character of A with differential $\nu + \rho$. We use smooth induction, namely we take $Ind_P^G(\mathrm{W})$ to consist of C^∞ functions from G with values in the space of smooth vectors W^∞ of W, which satisfy the required transformation property under P.

Facts: (see [9])
(1) $I(P;\sigma,\nu)$ is admissible.
(2) If σ is unitary and $\nu \in i\mathfrak{a}^*$, then $I(\sigma,\nu)$ is unitary.

If P is a minimal parabolic subgroup $P_{min} = M_{min}A_{min}N_{min}$, then M_{min} is compact ($M_{min} = Z_K(A_{min})$, the centralizer of A_{min} in K). We may thus take σ to be finite dimensional. The representations $I(P_{min};\sigma,\nu)$ are called the (non-unitary) principal series.

The importance of the principal series representations may be seen from the following result (due to Casselman).

Theorem 1.1.4. *(see [9]) Every irreducible admissiable representation of G is infinitesimally equivalent to a subrepresentation of some $I(P_{min};\sigma,\nu)$, where $\sigma \in \hat{M}$, and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$.*

Note that the above theorem is not a classification (or parametrization) theorem, as it is extremely complicated businesss to determine the composition series of principal series representations.

For $\nu \in i\mathfrak{a}^*$, the representation $I(P_{min};\sigma,\nu)$ is admissible and unitary and so is a finite direct sum of irreducible unitary (sub)representations. But in fact for $\sigma = \mathbb{1}$, we have the following

Theorem 1.1.5. *([10]) The representation*

$$J(\nu) = I(P_{min};\mathbb{1},\nu), \qquad \nu \in i\mathfrak{a}^*, \tag{1.1.6}$$

is irreducible.

The representations $J(\nu)$ $(\nu \in i\mathfrak{a}^*)$ are called spherical unitary principal series.

Remark 1.1.7. *An attractive feature of the class of spherical unitary principal series is its easy description. It is obtained from unitary characters $\mathbb{1} \otimes e^{\nu} \otimes \mathbb{1}$ of P_{min} by normalized parabolic induction. As there are no reducibilities involved, no complications arise whatsoever. Other representations to be discussed in this article have a similar attractive feature.*

1.2. **Cohomological induction.** As remarked previously, representations obtained by parabolic induction can be realized geometrically in the space of sections of homogeneous vector bundles over a generalized flag variety G/P. Cohomologically induced representations, on the other hand, have the corresponding geometric objects cohomology spaces associated to homogeneous holomorphic vector bundles over a non-compact complex variety G/L (more precisely algebraic analog of such objects). Harish-Chandra, Schmid, Zuckerman and Vogan are some of the main contributors of the theory.

As always we have a reductive Lie group G with a fixed maximal compact subgroup K. Let θ be the corresponding Cartan involution. The first ingredient in the cohomological induction construction is a θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ (the complexification of $\mathfrak{g}$). This means the following: write the Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, then $\theta(\mathfrak{q}) = \mathfrak{q}$ and $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{l}_{\mathbb{C}}$. Let

$$L = \{g \in G \mid Ad(g)\mathfrak{q} \subseteq \mathfrak{q}\}.$$

A special property of the homogeneous space G/L is that it has a G-invariant complex structure so that $\mathfrak{u}$, $\bar{\mathfrak{u}}$ are the anti-holomorphic and holomorphic part of the tangent space at eL, respectively.

The idea is that an admissible representation of L will yield a homogeneous holomorphic vector bundle over G/L. We can then look for representations of G in various cohomology spaces such as sheaf and L^2 cohomologies associated to the holomorphic bundle. However formidable analytic difficulties arose in trying to make sense of this. Zuckerman overcame all the difficulties by working purely algebraically, and in the process introduced his famous functors.

We shall start with an admissible $(\mathfrak{l}, M)$-module V, where $M = L \cap K$ is a maximal compact subgroup of L. From the work of Harish-Chandra, this is the same as giving an (infinitesimal equivalent class of) admissible representation of L.

It is customary to first normalize: $V^\sharp = V \otimes \wedge^{top}\mathfrak{u}$. This is still a $(\mathfrak{l}, M)$-module, but we shall view it as a $(\mathfrak{q}, M)$-module with $\mathfrak{u}$ acting trivially. From that, we form the so-called produced $(\mathfrak{g}_\mathbb{C}, M)$-module

$$X = pro_{(\mathfrak{q},M)}^{(\mathfrak{g}_\mathbb{C},M)}(V^\sharp) = Hom_\mathfrak{q}(\mathcal{U}(\mathfrak{g}_\mathbb{C}), V^\sharp)_{M-finite}.$$

Roughly speaking, X may be interpretted as the space of M-finite formal power series sections of the homogeneous holomorphic vector bundle over G/L associated to $V^\sharp$.

Denote $\Gamma = \Gamma_M^K$ the Zuckerman functor from M to K. This functor yields a $(\mathfrak{g}_\mathbb{C}, K)$-module $\Gamma(X)$ from a $(\mathfrak{g}_\mathbb{C}, M)$-module X; when K is connected and simply connected, $\Gamma(X)$ just picks out the subspace of $\mathfrak{k}$-finite vectors of X, which has an obvious $(\mathfrak{g}_\mathbb{C}, K)$-action. In general, $\Gamma(X)$ may be described as the maximal $(\mathfrak{g}_\mathbb{C}, M)$-submodule of X on which the $\mathfrak{k}$-action can be exponentiated to K.

A crucial observation here is that Γ is a left-exact functor, and so it has right derived functors Γ^i, for $i \in \mathbb{Z}_{\geq 0}$. These are the so-called Zuckerman functors. See [14] for a complehensive treatment of these functors.

Remark 1.2.1. *For a vector bundle, taking the (global) sections is a left-exact functor. Its right derived functors yield the cohomologies of the vector bundle.*

Denote

$$R_\mathfrak{q}^i(V) = \Gamma^i(X) = \Gamma^i(pro_{(\mathfrak{q},M)}^{(\mathfrak{g}_\mathbb{C},M)}(V^\sharp)).$$

They will be refered to as derived functor modules or with some abuse of terminology cohomologically induced representations.

The following theorem is due to Zuckerman and Vogan (see [15]), with contribution in first part of (2) by Enright-Wallach [2]. We refer the reader to [14] for the terminoloy.

Theorem 1.2.2. *Assume that V has an infinitesimal character λ and $\lambda+\rho(\mathfrak{q})$ is in the good range with respect to $\mathfrak{q}$. Denote $s = \dim_{\mathbb{C}}(K/M)$ (the middle dimension). Then*

(1)
$$R^i_{\mathfrak{q}}(V) = 0, \qquad \textit{for} \;\; i \neq s;$$

(2) *If V carries a non-degenerate $(\mathfrak{l}, M)$-invariant Hermitian form, then $R^s_{\mathfrak{q}}(V)$ carries a non-degenerate $(\mathfrak{g}, K)$-invariant Hermitian form. Furthermore if the Hermitian form on V is positive definite, then the corresponding Hermitian form on $R^s_{\mathfrak{q}}(V)$ is positive definite.*

(3) *If V is irreducible, then $R^s_{\mathfrak{q}}(V)$ is irreducible.*

We now describe the simplest class of unitary representations obtained by the derived functor constructions. Let $T \subset K$ be a maximal torus. Its centralizer $H = Z_G(T)$ is a (fundamental) Cartan subgroup of G, which gives rise to a root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi\subset\mathfrak{h}^*_{\mathbb{C}}} \mathfrak{g}_{\mathbb{C},\alpha}$.

Let $\lambda \in i\mathfrak{t}^*$, and define a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{q}(\lambda) = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi,(\lambda,\alpha)\geq 0} \mathfrak{g}_{\mathbb{C},\alpha}.$$

The corresponding group L is $L(\lambda) = \{g \in G | Ad^*(g)(\lambda) = \lambda\}$.

Let $\mathbb{C}_\lambda$ be the unitary character of $L(\lambda)$ with differential λ. We shall denote

$$A_{\mathfrak{q}}(\lambda) = R^s_{\mathfrak{q}(\lambda)}(\mathbb{C}_\lambda), \qquad \lambda \in i\mathfrak{t}^*. \tag{1.2.3}$$

Remark 1.2.4. (1) *When λ is regular, the representations $A_{\mathfrak{q}}(\lambda)$ are in the fundamental series and are tempered, namely their K-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for $\epsilon > 0$. If in addition rank $(G) =$ rank (K), then $L(\lambda) = T$ is a compact Cartan subgroup. The corresponding $A_{\mathfrak{q}}(\lambda)$ are in the discrete series, namely their K-finite matrix coefficients are in $L^2(G)$. All discrete series representations of G (if they exist) are obtained this way (see* [21]*).*

(2) *According to Vogan-Zuckerman* [18]*, the collection of representations $\{A_{\mathfrak{q}}(\lambda)\}$ is precisely the set of irreducible unitary representations of G having non-zero continuous-cohomology (with twisted coefficients). By Matsushima's formula, they are thus responsible for ordinary cohomologies in locally symmetric spaces.*

2. Constructing representations from those of dual partners

Another powerful way of constructing representations is through dual pair correspondence, as expounded by Howe [4]. This is a correspondence of representations between two subgroups of a real symplectic group.

Let $(W, <, >)$ be a real symplectic vector space, and $Sp(W) = Sp(W, <, >)$ the symplectic group. It has a truely distinguished unitary representation, with many names in the literature. It is called an oscillator representation, or metaplectic representation, harmonic representation, Weil (or Segal-Shale-Weil) representation. The representation depends on a non-trivial unitary character ψ of $\mathbb{R}$, and will be denoted by ω_ψ. Its existance may be understood as follows.

Given the symplectic space W, there is an associated Heisenberg group $H(W)$. The symplectic group $Sp(W)$ operates on $H(W)$ fixing its one-dimensional center $\mathbb{R}$. On the other hand, the Stone-von Neumann Theorem asserts that there is a unique irreducible unitary representation π_ψ of $H(W)$ with ψ as the central character. For each $g \in Sp(W)$, the representation $g^{-1} \cdot \pi_\psi$ clearly has the same central character ψ and so there exists a unitary operator $\omega_\psi(g)$ such that

$$\pi_\psi(g \cdot h) = \omega_\psi(g)\pi_\psi(h)\omega_\psi(g)^{-1}, \qquad g \in Sp(W),\, h \in H(W).$$

This gives rise to the oscillator representation ω_ψ, as a projective representation of $Sp(W)$. It turns out that ω_ψ can be made into a (true) representation of the unique non-trivial double cover $Mp(W)$ of the symplectic group $Sp(W)$ (called the metaplectic group).

In quantum mechnics terms, the Stone-von Neumann Theorem says in a sense that the only "reasonable" operators satisfying the Heisenberg commutation relations $[X_i, Y_j] = \delta_{ij} I$ $(1 \le i, j \le N)$ are the momentum operators $p_i = \frac{\partial}{\partial x_i}$ and the position operators $q_i = x_i$ (multiplication by coordinates). Since the symplectic group is the symmetry group of the canonical commutation relations (CCR), the uniqueness of CCR implies that for each element of the symplectic group, there must be some unitary operator linking the original CCR to the transformed one (by g).

Definition 2.1. *A pair of subgroups (G, G') of the symplectic group $Sp(W)$ is called a reductive dual pair if (i) G' is the centralizer of G in $Sp(W)$ and vice versa; (ii) Both G and G' act (absolutely) reductively on W.*

Here is the main construction of reductive dual pairs (called type 1): Let D be one of the division algebras over $\mathbb{R}$, namely $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$, with either the standard or the trival involution $\natural$. Let V be a right D vector space equipped with a $\natural$-hermitian form $(\ ,\ )$, and V' be a left D vector space equipped with a $\natural$-skew-hermitian form $(\ ,\ )'$. Consider the tensor product $W = V \otimes_D V'$, which we equip with the following symplectic form $<,> = \mathrm{Re}\ ((\ ,\ ) \otimes (\ ,\ )'^{\natural})$. Note that although the tensor product of forms over $\mathbb{H}$ does not make sense, when you take the real part, you do get a well-defined $\mathbb{R}$-bilinear (and this case symplectic) form. Let G (respectively G') be the isometry group of $(\ ,\ )$ (respectively $(\ ,\ )'$). Then (G, G') form a reductive dual pair in $Sp(W)$.

For the standard involution of D, the reductive dual pairs are as follows:

$$(G, G') = \begin{cases} (O(p,q), Sp(2n,\mathbb{R})), & D = R, \\ (U(p,q), U(m,n)), & D = \mathbb{C}, \\ (Sp(p,q), O^*(2n)), & D = \mathbb{H}. \end{cases}$$

The real significance of the above setting derives from Howe's remarkable discovery that when you restrict an oscillator representation to a reductive dual pair (G, G'), its spectrum will produce a one-to-one correspondence between (certain) representations of G and G'. We shall make the statement precise.

We fix a realization of $\omega = \omega_\psi$ on a Hilbert space $\mathcal{H}$ and we let ω^∞ the smooth representation on $\mathcal{H}^\infty$ (the space of smooth vectors of $\mathcal{H}$). For any subgroup E of $Sp(W)$, let $\widetilde{E}$ be the inverse image of E under the projection map $Mp(W) \to Sp(W)$.

Denote by $\mathrm{Irr}(\widetilde{G})$ the set of infinitesimal equivalence classes of irreducible admissible representations of $\widetilde{G}$, and $R(\widetilde{G}, \omega)$ the subset of those in $\mathrm{Irr}(\widetilde{G})$ which are realizable as quotients by $\widetilde{G}$-invariant closed subspaces of $\mathcal{H}^\infty$. Likewise for any subgroup E of $Sp(W)$.

Definition 2.2. *Two irreducible admissible representations of π of $\widetilde{G}$ and π' of $\widetilde{G'}$ are said to be in duality correspondence if there exists a non-trivial $\widetilde{G} \times \widetilde{G'}$-intertwining map*

$$\mathcal{H}^\infty \to \pi \otimes \pi'$$

with closed kernel, namely if $\pi \otimes \pi' \in R(\widetilde{G} \cdot \widetilde{G'}, \omega)$.

Theorem 2.3. *(Howe Duality Theorem* [6]*) The duality correspondence $\pi \leftrightarrow \pi'$ defined above is a bijection between $R(\widetilde{G}, \omega)$ and $R(\widetilde{G'}, \omega)$. Moreover, an element $\pi \otimes \pi'$ of $R(\widetilde{G} \cdot \widetilde{G'}, \omega)$ occurs as a quotient of ω^∞ with closed kernel in a unique way.*

Remark 2.4. *A remarkable fact is that exactly the same duality result holds for p-adic groups (conjectured by Howe and proven by Waldspurger for $p \neq 2$). Howe in fact made a global duality conjecture (part I and part II, and part I follows from the local duality conjecture together with some of his results on correspondence between Hecke algebras). See* [4]*. It is this global version where the theory of θ-series fits in. For this reason, the dual pair correspondence of representations for real and p-adic groups are also called local theta correspondence (besides the names Howe duality correspondence and Howe quotient correspondence).*

We shall write the correspondence as $\pi' = \theta(\pi)$ or $\pi = \theta(\pi')$. They are said to be theta lift of each other. We may sometimes write $\pi' = \theta_{G \to G'}(\pi)$.

The representation $\pi' = \theta(\pi)$ is described more concretely as follows: given $\pi \in R(\widetilde{G}, \omega)$, denote Q the set of $\widetilde{G}$-intertwining maps from ω^∞ to π' with closed kernel. Let $N_\pi = \bigcap_{\varphi \in Q} \ker\, \varphi$, which is the minimal $\widetilde{G}$-invariant closed subspace of $\mathcal{H}^\infty$ such that $\mathcal{H}^\infty / N_\pi$ is a multiple of π. It is easy to see that we can write

$$\mathcal{H}^\infty / N_\pi \simeq \pi \otimes \omega(\pi), \tag{2.5}$$

where $\omega(\pi)$ is a representation of $\widetilde{G'}$. The Howe Duality Theorem asserts that $\omega(\pi)$ has a unique irreducible quotient π' so that $\pi \otimes \pi' \in R(\widetilde{G} \cdot \widetilde{G'}, \omega)$. Of course $\pi' = \theta(\pi)$, the theta lift of π. The representation $\omega(\pi)$ is called Howe's maximal quotient of π, which is a quasi-simple admissible representation of $\widetilde{G'}$ of finite length.

Remark 2.6. *One may view Howe's maximal quotient $\omega(\pi')$ as something like an induced representation, and $\theta(\pi)$ will then be like the Langlands quotient of the induced representation (if it has a unique irreducible quotient).*

We now address the important question of unitarity. We will be mainly concerned with type 1 dual pairs. The main results are due to Li [12].

Definition 2.7. *A type 1 reductive dual pair (G, G') is in stable range (respectively strictly stable range) with G the smaller member, if there exists a totally isotropic subspace V_1' of V' such that $\dim_D(V) \leq \dim_D(V_1')$ (respectively $\dim_D(V) < \dim_D(V_1')$. The condition will be denoted by $2G \leq G'$ (respectively $2G < G'$).*

Let $\mathbb{Z}_2 = \{\pm 1\}$ be the kernel of the projection map $Mp(W) \to Sp(W)$. Denote by $\widehat{\widetilde{G}}_{genuine}$ the set of equivalent classes of irreducible unitary representations of $\widetilde{G}$, whose restriction to $\mathbb{Z}_2$ is a multiple of the unique non-trivial character of $\mathbb{Z}_2$. Likewise for G'.

Theorem 2.8. *([12]) Suppose that (G, G') is in the stable range with G the smaller member, then $\widehat{\widetilde{G}}_{genuine} \subset R(\widetilde{G}, \omega)$ and Howe duality correspondence gives rise to an injection*

$$\theta : \quad \widehat{\widetilde{G}}_{genuine} \hookrightarrow \widehat{\widetilde{G'}}_{genuine}.$$

Remark 2.9. (1) *With respect to the so-called Fell topology on the unitary dual, the above injection is actually a topological embedding.*

(2) *According to [13], the collection $\{\theta_{G \to G'}(\pi)\} \subset \widehat{\widetilde{G'}}_{genuine}$, where $2G < G'$ and $\pi \in \widehat{\widetilde{G}}_{genuine}$, may be characterized as irreducible unitary representations of low rank (in the sense of Howe).*

Proposition 2.10. *([25]) Suppose that (G, G') is in the stable range with G the small member. Then for any genuine unitary character χ of $\widetilde{G}$, its Howe quotient $\omega(\chi)$ is already irreducible. Thus $\omega(\chi) = \theta(\chi)$ is irreducible and unitary. (We exclude the case of $(G, G') = (Sp(2n, \mathbb{R}), O(2n, 2n))$ and $\chi = \mathbb{1}_{Sp(2n,\mathbb{R})}$.)*

One may wish to compare the above result with that of Kostant (in §1.1) on the irreducibility of spherical unitary principal series. In §4 we will examine this special class of unitary representations (of $\widetilde{G'}$), namely theta lifts of (genuine) unitary characters of $\widetilde{G}$ in the stable range.

3. Comparing representations of different real forms of a (same) complex group

In this section, we will discuss a comparison technique for unitary representations of two real forms of a semi-simple complex Lie group, which is called transfer. The idea was initiated by Enright in a special case [1] and was developped by Enright-Parthasarathy-Wallach-Wolf [3] and then systematically by Wallach [20]. We shall follow the exposition in Wallach-Zhu [22]

Let $G_{\mathbb{C}}$ be a connected, simply connected semi-simple Lie group over $\mathbb{C}$. Let G and G' be two real forms of $G_{\mathbb{C}}$ with respective maximal compact subgroups K and K'. Let θ and θ' be corresponding Cartan involutions looked upon as automorphisms of $G_{\mathbb{C}}$. We assume that

$\theta\theta' = \theta'\theta$. Set $M = K \cap K'$. Let $\mathfrak{g}, \mathfrak{g}', \mathfrak{g}_{\mathbb{C}}$ denote the Lie algebras of $G, G', G_{\mathbb{C}}$ respectively.

Let (π, V) be an irreducible unitarizable $(\mathfrak{g}', K')$-module. Then we may look upon it as a $(\mathfrak{g}_{\mathbb{C}}, M)$-module. We can apply the Zuckerman functors $(\Gamma_M^K)^i$ to V and get $(\mathfrak{g}, K)$-modules

$$V^i = (\Gamma_M^K)^i(V).$$

Since V has an infinitesimal character, the modules V^i will also have the same infinitesimal character. Therefore any finite dimensional K-invariant subspace of V^i will generate an admissible $(\mathfrak{g}, K)$-submodule of V^i. This simple method of obtaining admissible modules is what we mean by the most basic method of transfer.

In order to make meaningful statements on the structure of transfered submodules in particular on unitarity, we will assume that V satisfies the following (strong) condition.

A: As a $(\mathfrak{k}_{\mathbb{C}}, M)$-module, V splits into a direct sum

$$V = \bigoplus_j V_j, \qquad \text{with } \ V_j \simeq m_j L_j$$

a finite direct sum of irreducible $(\mathfrak{k}_{\mathbb{C}}, M)$-modules L_j and each L_j is unitarizable as a $(\mathfrak{k}_1, M)$-module. Here $\mathfrak{k}_1 = \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{g}'$ is a real form of $\mathfrak{k}_{\mathbb{C}}$ with $\theta'|_{\mathfrak{k}_{\mathbb{C}}}$ as the Cartan involution and it contains $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{k}'$ as the corresponding maximal compact subalgebra.

Let us recall some facts about unitary representations with $(\mathfrak{k}_1, M)$-cohomology.

Let L be an irreducible $(\mathfrak{k}_1, M)$-module which is unitarizable. Let F be an irreducible finite-dimensional $(\mathfrak{k}_1, M)$-module such that L and F have the same central character and the same infinitesimal character. Let W be an irreducible M-submodule of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ and assume that

$$\mathrm{Hom}_M(W, L \otimes F^*) \neq 0.$$

We fix a maximal torus T in M with the Lie algebra $\mathfrak{t}$ and let $\mathfrak{h} = \{X \in \mathfrak{k}_{\mathbb{C}} | \ [X, \mathfrak{t}] = 0\}$ be the corresponding fundamental Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. Denote θ' the Cartan involution of $\mathfrak{k}_{\mathbb{C}}$ corresponding to M. Then the result of Vogan-Zuckerman [18] implies that there exists a θ'-stable parabolic subalgebra, $\mathfrak{q} \supset \mathfrak{h}$, of $\mathfrak{k}_{\mathbb{C}}$, with the following properties: let $\mathfrak{u}$ be the nilradical of $\mathfrak{q}$ and let $\mathfrak{u}_n = \{X \in \mathfrak{u} | \ \theta'(X) = -X\}$ be the non-compact part of $\mathfrak{u}$. Set $\rho_{\mathfrak{q},n}(h) = \frac{1}{2}\mathrm{tr}(ad(h)|_{\mathfrak{u}_n}$ for $h \in \mathfrak{t}$.

Then W has highest weight $2\rho_{\mathfrak{q},n}$ with respect to any system of positive roots of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ compatible with $\mathfrak{q} \cap \mathfrak{m}_{\mathbb{C}}$. Let $\gamma_{M,\mathfrak{q}}(0)$ denote the corresponding equivalence class of irreducible M-modules.

We fix $\mathfrak{b}_k$, a θ'-stable Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ such that $\mathfrak{h} \subset \mathfrak{b}_k \subset \mathfrak{q}$. If Λ is a dominant integral element of $\mathfrak{h}^*$ with respect to $\mathfrak{b}_k$ that is also T-integral, we define $\gamma_{M,\mathfrak{q}}(\Lambda)$ to be the equivalence class of irreducible M-modules with highest weight

$$\Lambda|_{\mathfrak{t}} + 2\rho_{\mathfrak{q},n}.$$

The result of Vogan-Zuckerman [18] now implies that if $F = F_\Lambda$ is of highest weight Λ with respect to $\mathfrak{b}_k$, then there exists such a $\mathfrak{q}$ with $\dim F^{\mathfrak{u}} = 1$ and such that L is isomorphic to $A_{\mathfrak{q}}(\Lambda)$ (see §1.2 for its definition). Note that $\gamma_{M,\mathfrak{q}}(\Lambda)$ is the unique minimal M-type of $A_{\mathfrak{q}}(\Lambda)$ in the sense of Vogan.

The following summaries some basic results on the transfer of unitary representations (see [20, 22]). As noted, we assume that V satisfies the condition (A).

Theorem 3.1. *For each irreducible M-submodule W of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$, there is a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule $\Gamma_W(V)$ of $(\Gamma_M^K)^i(V)$ with the following properties:*

(1) *If $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}) = \sum_\delta W_\delta^i$ is a decomposition of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$ into irreducible M-modules, then we have the decomposition*

$$(\Gamma_M^K)^i(V) = \sum_\delta \Gamma_{W_\delta^i}(V),$$

as $(\mathfrak{g}, K)$-modules.

(2) *If $W \notin \gamma_{M,\mathfrak{q}}(0)$ for all $\mathfrak{q}$ (in $\mathfrak{k}_{\mathbb{C}}$). Then $\Gamma_W(V) = 0$.*

(3) *If $W \in \gamma_{M,\mathfrak{q}}(0)$ for some $\mathfrak{q}$, then*

$$\Gamma_W(V) \simeq \bigoplus_{\substack{j,\Lambda \\ L_j \simeq A_{\mathfrak{q}}(\Lambda)}} Hom_M(W, V_j \otimes F_\Lambda^*)\ F_\Lambda,$$

as a K-module.

(4) *If $\langle\ ,\ \rangle$ is a $(\mathfrak{g}, M)$-invariant non-degenerate Hermitian form on V, then there is a natural $(\mathfrak{g}, K)$-invariant non-degenerate Hermitian form on $\Gamma_W(V)$, which is induced by $\langle\ ,\ \rangle$ on V and by the L^2-inner product of*

$$\mathcal{H}(K) \simeq \bigoplus_\Lambda F_\Lambda^* \otimes F_\Lambda.$$

Here $\mathcal{H}(K)$ denotes the algebra of matrix coefficients of finite-dimensional unitary representations of K.

(5) *If the restriction of the form* $\langle\ ,\ \rangle$ *to the minimal* M*-type spaces*

$$V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j)) \times V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j))$$

is positive definite for each j *such that* $L_j \simeq A_{\mathfrak{q}}(\Lambda_j)$. *Then* $\Gamma_W(V)$ *is unitary.*

The conditions in (4) and (5) of Theorem 3.1 are satisfied in the following situation.

Proposition 3.2. *([22]) Suppose that* (π, V) *is an irreducible unitarizable* $(\mathfrak{g}', K')$*-module with invariant inner product* $(\ ,\ )$. *Assume that there exists an involutive unitary operator* T *on* V *such that*

$$\pi(\theta\theta'(X)) = T\pi(X)T^{-1}, \quad \text{and} \quad T\pi(m) = \pi(m)T,$$

where $X \in \mathfrak{g}_{\mathbb{C}}$, $m \in M$. *Let* $W \in \gamma_{M,\mathfrak{q}}(0)$ *for some* $\mathfrak{q}$, *and assume further that for each* j *such that* $L_j \simeq A_{\mathfrak{q}}(\Lambda_j)$, *the corresponding minimal* M*-type spaces* $V_j(\gamma_{M,\mathfrak{q}}(\Lambda_j))$ *all lie in* V_+ *or all lie in* V_- *where*

$$V_\pm = \{v \in V |\ Tv = \pm v\}.$$

Then $\Gamma_W(V)$ *is unitary.*

The operator T in the above proposition may be called a "transfer operator". In the case of unitarizable highest wieght modules, such a transfer operator is provided for by $\theta\theta'$ as an automorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Theorem 3.3. *(see [22]) Assume that the pair* $(\mathfrak{g}', \mathfrak{k}')$ *is Hermitian symmetric so that*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{o}^+ \oplus \mathfrak{o}^-,$$

$\mathfrak{o}^\pm$ *abelian and* $[\mathfrak{k}'_{\mathbb{C}}, \mathfrak{o}^\pm] \subset \mathfrak{o}^\pm$. *Let* $V = L(F)$ *be a unitarizable* $(\mathfrak{g}', K')$*-module of highest weight, where* $F = V^{\mathfrak{o}^+}$ *is an irreducible unitary* K'*-module. We further assume that*

$$\theta(\mathfrak{o}^+) = \mathfrak{o}^+,$$

$$\theta(k')f = k'f, \qquad k' \in K', f \in F.$$

Then $\Gamma_W(V)$ *is either zero or unitarizable for each irreducible* M*-submodule* W *of* $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$.

4. More on the representations $\theta_{G\to G'}(\chi)$, $2G \leq G'$

We now examine the special class of unitary representations (of $\widetilde{G'}$), which are theta lifts of (genuine) unitary characters χ of $\widetilde{G}$ in the stable range. We will examine their connections with representations obtained by parabolic induction (degenerate principal series) and transfer of unitary representations (through Zuckerman functors).

In order to explain the ideas and results efficiently, we will examine the case

$$(G, G') = (O(p,q), Sp(2n,\mathbb{R})) \subset Sp(2n(p+q),\mathbb{R}).$$

We will be interested in the representations of $\widetilde{Sp}(2n,\mathbb{R})$. To respect the convention (of calling the group G) within the representation theory community, we shall change into a different notation $(H,G) = (O(p,q), Sp(2n,\mathbb{R}))$. The stable range condition $p+q \leq n$ will only be imposed later.

In this case, the oscillator representation can be realized on $\mathcal{H} = L^2(M_{p+q,n}(\mathbb{R}))$ through the Schrodinger model, with the group $H = O(p,q)$ acting in the obvious way. The space of smooth vectors $\mathcal{H}^\infty$ is then $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$, the space of Schwartz class functions on $M_{p+q,n}(\mathbb{R})$. Thus the Howe quotient $\omega(\chi) = \omega^{p,q}(\chi)$ is realized as the maximal quotient of $\mathcal{S}(M_{p+q,n}(\mathbb{R}))$ on which $O(p,q)$ acts by the character χ, namely

$$\omega^{p,q}(\chi) = \mathcal{S}(M_{p+q,n}(\mathbb{R}))/\overline{<((\omega-\chi)(O(p,q))\mathcal{S}(M_{p+q,n}(\mathbb{R}))>}, \quad (4.1)$$

where $\overline{<((\omega-\chi)(O(p,q))\mathcal{S}(M_{p+q,n}(\mathbb{R}))>}$ denotes the closure of the span of $((\omega-\chi)(O(p,q))\mathcal{S}(M_{p+q,n}(\mathbb{R}))$ in the usual Frechet topology of the Schwartz space. It is thus dual to

$$\mathcal{S}^*(M_{p+q,n}(\mathbb{R}))^{O(p,q);\chi} = \{\Phi \in \mathcal{S}^*(M_{p+q,n}(\mathbb{R})) \,|\, h\cdot\Phi = \chi(h)\Phi,\, h \in O(p,q)\},$$

the space of χ-eigendistributions for $H = O(p,q)$ on $M_{p+q,n}(\mathbb{R})$. For $\chi = 1\!\!1$, the Howe quotient $\omega^{p,q}(1\!\!1)$ is generally called the space of H-coinvariants, and it is dual to the space of H-invariant distributions on $M_{p+q,n}(\mathbb{R})$. We refer the interested reader to [23, 7] for results on eigendistributions.

As mentioned previously, $\omega^{p,q}(\chi)$ is a quasi-simple admissible representation of $\widetilde{Sp}(2n,\mathbb{R})$, and it has a unique irreducible quotient $\theta^{p,q}(\chi)$.

Facts: ([11])

(1) $\omega^{p,q}(\chi)$ is $\widetilde{K}$-free (with an explicitly given set of $\widetilde{K}$-types).
(2) There is a natural embedding

$$\omega^{p,q}(1\!\!1) \hookrightarrow Ind_{\widetilde{M}N}^{\widetilde{G}}(\chi_0^{p-q} \otimes |\ |^{\frac{p+q}{2}} \otimes 1\!\!1),$$

where $M \simeq GL(n,\mathbb{R})$, and χ_0 is the character of $\widetilde{M}$ of order 4 given by

$$\chi_0(a,\epsilon) = \epsilon \cdot \begin{cases} 1, & \text{if} \quad \det(a) > 0, \\ i, & \text{if} \quad \det(a) < 0. \end{cases}$$

Remark 4.2. *Note that for a fixed m and $\alpha = 0,1,2,3$, all Howe quotients $\omega^{p,q}(\mathbb{1})$ with $p+q=m$ and $p-q \equiv \alpha \pmod 4$ are embedded into the same degenerate principal series $Ind_{\widetilde{M}N}^{\widetilde{G}}(\chi_0^\alpha \otimes |\ |^{\frac{m}{2}} \otimes \mathbb{1})$. The composition series of $\omega^{p,q}(\mathbb{1})$ and their relationship with the structure of $Ind_{\widetilde{M}N}^{\widetilde{G}}(\chi_0^\alpha \otimes |\ |^s \otimes 1)$ are made explicit in* [11].

Now we impose the stable range condition: $p+q \leq n$. Thus

$$\omega^{p,q}(\chi) = \theta^{p,q}(\chi).$$

For $\chi = \mathbb{1}$, apart from their description as coinvariants, the collection of representations $\theta^{p,q}(\mathbb{1})$ for $p+q<n$ coincide with some unipotent representations constructed by Sahi previously [19].

The following result (almost) characterizes the representations $\theta^{p,q}(\chi)$ $(p+q<n)$ in terms of $\mathcal{V}(Ann(\pi))$, the associated variety of the annihilator of π. We refer the reader to [16], where one can find many important ideas of Vogan on associated varieties and unipotent representations.

Theorem 4.3. *([8]) Let π be an irreducible unitary representation of a finite cover of $Sp(2n,\mathbb{R})$ such that $\mathcal{V}(Ann(\pi))$ is equal to the closure of the nilpotent orbit $[\underbrace{2,...,2}_{m},\underbrace{1,...,1}_{2n-2m}]$, where $m<n$. Then there is (i) a pair of integers (p,q) with $p+q=m$ and (ii) an irreducible finite dimensional unitary representation σ of $O(p,q)$ such that $\pi \simeq \theta^{p,q}(\sigma)$.*

Needless to say that the only irreducible finite dimensional unitary representations of $O(p,q)$ (for $pq \neq 0$) are unitary characters. Note that the information carried in $\mathcal{V}(Ann(\pi))$ is in general quite weak, but in the current case of small nilpotent orbits it is in fact very strong.

Remark 4.4. *For σ as in Theorem 4.3, the associated variety of $\theta^{p,q}(\sigma)$ is the closure of a certain nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}_{p,q}$, whose $G_{\mathbb{C}}$ extension is of the type $[\underbrace{2,...,2}_{p+q},\ \underbrace{1,...,1}_{2n-2(p+q)}]$. As a $K_{\mathbb{C}}$-module, the coordinate ring $\mathbb{C}[\overline{\mathbb{O}_{p,q}}] \simeq \sum_\lambda \tau_\lambda^n$, where τ_λ^n denotes a copy of the irreducible representation of $K_{\mathbb{C}} \simeq GL(n,\mathbb{C})$ with the highest weight λ, and the summation is over $\lambda = (2\alpha_1,...,2\alpha_p,0,...,0,-2\beta_q,...,-2\beta_1)$, where $\alpha_1 \geq \cdots \geq \alpha_p \geq 0$, $\beta_1 \geq \cdots \geq \beta_q \geq 0$ are integers. The $\widetilde{K}$-structure of the representation $\pi = \theta^{p,q}(\mathbb{1})$ satisfies*

$$\pi|_{\widetilde{K}} \simeq (\det)^{\frac{p-q}{2}} \otimes \mathbb{C}[\overline{\mathbb{O}_{p,q}}].$$

The representation π may be justifiably called the unipotent representation associated to the nilpotent $K_{\mathbb{C}}$-orbit $\mathbb{O}_{p,q}$.

If $pq \neq 0$, there exist four characters of $O(p,q)$. They are denoted by $1^{\xi,\eta}$ (with $\xi, \eta \in \{0,1\}$), which are characterized by the following conditions

$$1^{\xi,\eta}|_{O(p)} = (\det)^{\xi}, \qquad 1^{\xi,\eta}|_{O(q)} = (\det)^{\eta}.$$

Clearly $\mathbb{1} = 1^{0,0}$ and $\det = 1^{1,1}$.

We now explain how various $\theta^{p,q}(\chi)$ are related to each other by the transfer construction. Denote $m = p+q$. Then $\theta^{m,0}(\mathbb{1})$ is a unitary lowest weight module of lowest weight $\frac{m}{2}\Lambda_1$, and $\theta^{0,m}(\mathbb{1})$ is a unitary highest weight module of highest weight $-\frac{m}{2}\Lambda_1$. Here $\Lambda_1 = (1,...,1)$ in usual coordinates. We denote the latter by $L(-\frac{m}{2}\Lambda_1)$, to which we shall apply the transfer.

We are back in the setting of §3. We have $(\mathfrak{g}, K) \simeq (\mathfrak{sp}(2n,\mathbb{R}), U(n))$ and $(\mathfrak{g}', K') \simeq (\mathfrak{sp}(2n,\mathbb{R}), U(n))$. For both groups we will need to go to the metaplectic cover $\widetilde{Sp}(2n,\mathbb{R})$.

Let r, s be non-negative integers with $r+s = n$. We can embed G and G' into $G_{\mathbb{C}} = Sp(2n,\mathbb{C})$ so that $M = K \cap K' \simeq U(r) \times U(s)$.

Let $p \leq s$ and $q \leq r$ with $p+q = m$. Then there is a certain irreducible M-submodule $W_{p,q}$ of $\wedge^i(\mathfrak{k}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})$, where $i = rs - (r-q)(s-p)$ (below the middle dimension rs). The corresponding θ'-stable parabolic $\mathfrak{q}$ of $\mathfrak{k}_{\mathbb{C}}$ is determined by $\lambda_{p,q} = (\underbrace{0,...,0}_{r-q}, \underbrace{1,...,1}_{q}, \underbrace{-1,...,-1}_{p}, \underbrace{0,...,0}_{s-p})$. We refer the reader to [22] for details. Write $\Gamma_{p,q} = \Gamma_{W_{p,q}}$ in the notation of §3. Then we have the following

Conjecture 4.5. *([22]) Let $p \leq s$, $q \leq r$ and $p+q = m \leq n$. Then we have*

$$\Gamma_{p,q}(L(-\frac{m}{2}\Lambda_1)) \simeq \theta^{p,q}(1^{\xi,\eta}),$$

where $\xi \equiv r-q$, $\eta \equiv s-p$ (mod 2).

We shall end this article by stating a simple yet very useful result on representations with scalar K-types.

For a vector space W, denote by $S(W)$ the symmetric algebra over W, and $S_r(W)$ the subspace of $S(W)$ consisting of homogeneous elements of degree r, where $r \in \mathbb{Z}_{\geq 0}$. Write $S^r(W) = \sum_{0 \leq k \leq r} S_k(W)$.

Theorem 4.6. *([24]) Let G be a connected noncompact semisimple Lie group with finite center, and K be a maximal compact subgroup. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ be the complexified Cartan decomposition and let $\psi : S(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ be the symmetrization map. Suppose that*

$$\psi(S^r(\mathfrak{g}_{\mathbb{C}})^G) + \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}} \supseteq \psi(S^r(\mathfrak{p}_{\mathbb{C}})^K), \qquad r \in \mathbb{Z}_{\geq 0}$$

(true at least for all classical G), and $\rho \in \hat{K}$ is one-dimensional. Then for an irreducible Harish-Chandra module V with a non-zero ρ-isotypic component, the infinitesimal character of V determines V up to infinitesimal equivalence.

From the above theorem, it is easily checked that the stated conjecture is true in the special case when $\xi = \eta = 0$. The representations concerned will then contain a (same) one-dimensional K-type, and so are determined by their infinitesimal characters.

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