COMPONENTWISE POLYNOMIAL SOLUTIONS AND DISTRIBUTION SOLUTIONS OF REFINEMENT EQUATIONS

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Abstract. In this paper we present an example of a refinement equation such that up to a multiplicative constant it has a unique compactly supported distribution solution while it can simultaneously have a compactly supported componentwise constant function solution that is not locally integrable. This leads to the conclusion that in general the componentwise polynomial solution cannot be globally identified with the unique compactly supported distribution solution of the same refinement equation. We further show that any compactly supported componentwise polynomial solution to a given refinement equation with the dilation factor 2 must coincide, after a proper normalization, with the unique compactly supported distribution solution to the same refinement equation. This is a direct consequence of a general result stating that any compactly supported componentwise polynomial refinable function with the dilation factor 2, without assuming that the refinable function is locally integrable in advance, must be a finite linear combination of the integer shifts of some B-spline.

In this paper, we start with an example showing that a compactly supported componentwise polynomial solution of a refinement equation may not coincide globally with its compactly supported distribution solution in general. However, this is not the case when the dilation factor is 2. In fact, we show that any compactly supported componentwise polynomial solution of a refinement equation with the dilation factor 2 can be globally identified with its compactly supported distribution solution as a consequence of a general result. As in [2], a componentwise polynomial is defined as follows:

Definition 1. A compactly supported function ϕ defined on $\mathbb{R}$ is *a componentwise polynomial* if there exists an open set G such that the Lebesgue measure of $\mathbb{R}\backslash G$ is zero and the restriction of ϕ on any connected open component of G coincides with some polynomial.

It is clear that a compactly supported spline is a componentwise polynomial, since the open set G in Definition 1 is a union of finitely many connected open intervals. A componentwise polynomial has an analytic expression up to a set of measure zero, since it is a polynomial on each connected component of G. The concept of componentwise polynomials was first introduced and studied in [1, 21] under the term of local polynomials.

In [2], a few examples of compactly supported componentwise polynomial refinable functions are given. In particular, examples of componentwise constant refinable functions, which satisfy either orthogonality or interpolation property and which are continuous and symmetric, are given in [2]. Additional examples of componentwise linear refinable functions that are differentiable and symmetric are also given in [2]. Next, we present another example of a componentwise constant that is a compactly supported measurable function solution of a refinement equation but it cannot be regarded globally as a compactly supported distribution solution of the same refinement equation.

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For a positive integer $M \ge 2$ and a finitely supported sequence $\{h(k)\}_{k\in\mathbb{Z}}$, we say that ϕ is a compactly supported *distribution solution* of the refinement equation

$$\phi = M \sum_{k\in\mathbb{Z}} h(k)\phi(M\cdot -k) \tag{1}$$

if the compactly supported distribution ϕ satisfies the refinement equation (1) in the distribution sense. It is well known that if the mask $H(\xi) := \sum_{k\in\mathbb{Z}} h(k)e^{-ik\xi}$ satisfies $H(0) = 1$, then the refinement equation (1) has a unique compactly supported distribution solution with the normalization condition $\hat{\phi}(0) = 1$ (see e.g. [4]). In fact, the compactly supported distribution solution ϕ can be obtained via its Fourier transform $\hat{\phi}$ which is defined by the infinite product:

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} H(M^{-j}\xi), \qquad \xi \in \mathbb{R}. \tag{2}$$

Here, the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$, $\xi \in \mathbb{R}$ and can be naturally extended to tempered distributions.

We say that a (Lebesgue) measurable function $\phi : \mathbb{R} \mapsto \mathbb{C}$ satisfies the refinement equation (1) in the sense of almost everywhere, if

$$\phi(x) = M \sum_{k\in\mathbb{Z}} h(k)\phi(Mx - k), \qquad a.e.\ x \in \mathbb{R}. \tag{3}$$

Such a measurable function ϕ is called a *measurable function solution* of the refinement equation in (1). It is not clear so far whether a compactly supported measurable function solution to the refinement equation (3) is unique up to a multiplicative constant.

For an open set G and a measurable function f on $\mathbb{R}$, we say that f is locally integrable on G if f is integrable on every compact set that is contained inside G. Let g be a distribution and f be a measurable function on $\mathbb{R}$. For an open set G of $\mathbb{R}$, we say that the distribution g can be identified with the measurable function f on G (or equivalently, the measurable function f can be identified with the distribution g on G) if f is locally integrable on G and $g(h) = \int_{\mathbb{R}} f(x)h(x)dx$ for every $C^\infty(\mathbb{R})$ function h whose support is a compact set inside G. If a distribution g can be identified with a measurable function f on $\mathbb{R}$, then we simply say that the distribution g can be globally identified with the measurable function f.

It has been extensively investigated in the literature that under what conditions, the unique compactly supported distribution solution of (1) will be a function in certain function spaces (e.g., $L_1(\mathbb{R})$, $L_2(\mathbb{R})$, $C(\mathbb{R})$, or Sobolev spaces). It is essentially done by investigating the convergence of a cascade algorithm, which is closely related to the infinite product in (2), in various function spaces. The detail can be found in, for example, [4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16]. However, little attention has been given in the literature to the inverse direction, i.e., for a given compactly supported measurable function solution of the refinement equation (1), whether it can be always globally identified with the unique compactly supported distribution solution of the same refinement equation. Next, we demonstrate by an example that a refinement equation can have a unique compactly supported distribution solution (of course, up to a multiplicative constant), while at the same time it has a compactly supported measurable function solution that cannot be globally identified with its compactly supported distribution solution on $\mathbb{R}$. The construction is done by deriving a compactly supported componentwise constant refinable function that is not in $L_1(\mathbb{R})$, hence, not locally integrable. Since a measurable function can be globally identified with a distribution on $\mathbb{R}$ only if it is locally integrable, this function solution cannot be regarded globally as the compactly supported distribution solution of the same refinement equation.

Example 2. Let $c \in \mathbb{R}$ be an arbitrary fixed constant. Consider the refinement equation

$$\phi(x) = (1-c)\phi(3x+1) + \phi(3x) + \phi(3x-1) + c\phi(3x-2), \qquad a.e.\ x \in \mathbb{R}. \tag{4}$$

Then it is clear that up to a multiplicative constant this equation has a unique compactly supported distribution solution. On the other hand, a compactly supported componentwise constant function ϕ that satisfies the above refinement equation in the sense of almost everywhere can be constructed as follows: First, set $\mathcal{O} := \cup_{k=1}^{\infty} \cup_{\epsilon_j \in \{0,1\}, 1 \le j \le k-1, \epsilon_k = 0} A_{(\epsilon_1,\dots,\epsilon_k)}$, where $A_{(\epsilon_1,\dots,\epsilon_k)}$ are open intervals defined by

$$A_{(\epsilon_1,\dots,\epsilon_k)} := \left(\sum_{j=1}^{k} 3^{-j}\epsilon_j + 2^{-1}3^{-k} - 2^{-1} \ , \ \sum_{j=1}^{k} 3^{-j}\epsilon_j + 3^{-k} - 2^{-1} \right)$$

for $k \ge 1, \epsilon_j \in \{0,1\}$ and $1 \le j \le k$. Then

$$G := \mathcal{O} \cup (0, 1/2) \cup (\mathcal{O} + 1) \cup (-\infty, -1/2) \cup (1, \infty)$$

is an open set and $\mathbb{R}\backslash G$ has measure zero.

Next, we construct a componentwise constant polynomial ϕ on G as follows. Let $\phi(x) = 1$ on $(0, 1/2)$ and $\phi(x) = 0$, $x \in (-\infty, -1/2) \cup (1, \infty)$. We further require that ϕ should satisfy the normalization condition:

$$\phi(x) + \phi(x+1) = 1 \quad a.e.\ x \in (-1/2, 1/2). \tag{5}$$

Then, the values of ϕ on $\mathcal{O} + 1$ can be defined by (5) from the values of ϕ on $\mathcal{O}$. Hence, we only need to define the values of ϕ on $\mathcal{O}$ or $A_{(\epsilon_1,\dots,\epsilon_k)}$. This is done iteratively. To be a solution of the refinement equation (4), it is clear that $\phi(x) = 1 - c$, $x \in A_{(0)}$. Since ϕ is constant on the interval $A_{(0)}$, we simply write it as $\phi(A_{(0)}) = 1 - c$. Similarly, $\phi(A_{(1)}) = 1$. For other intervals in $\mathcal{O}$, the values of ϕ can be defined iteratively by

$$\phi(A_{(0,\epsilon_1,\dots,\epsilon_k)}) = (1-c)\phi(A_{(\epsilon_1,\dots,\epsilon_k)}), \qquad \phi(A_{(1,\epsilon_1,\dots,\epsilon_k)}) = (1-c) + c\phi(A_{(\epsilon_1,\dots,\epsilon_k)}). \tag{6}$$

This leads to a function ϕ supported on $[-1/2, 1]$ whose restriction on any open connected interval contained in G coincides with some constant. Furthermore, ϕ satisfies the refinement equation (4) in the sense of almost everywhere on $\mathbb{R}$ by the above construction.

Next, we choose c so that the resulting ϕ is not in $L_1(\mathbb{R})$. For simplicity, we write $A_k := A_{(\epsilon_1,\dots,\epsilon_k)}$ where $\epsilon_{k-1} = 1$ and $\epsilon_j = 0$ for all $1 \le j \le k$, $j \ne k-1$. Then,

$$A_k = (3^{-k+1} + 2^{-1}3^{-k} - 2^{-1}, 3^{-k+1} + 3^{-k} - 2^{-1}), \qquad k \ge 2$$

and $\phi(A_k) = (1-c)^{k-1}(1+c)$ by the iterative formula (6). This leads to

$$\int_{\mathbb{R}} |\phi(x)|dx \ge \int_{\cup_{k\ge 2} A_k} |\phi(x)|dx = \frac{|1+c|}{6} \sum_{k=1}^{\infty} \frac{|1-c|^k}{3^k}.$$

If $|1-c| \ge 3$, then the corresponding componentwise constant ϕ that satisfies the refinement equation (4) will not be in $L_1(\mathbb{R})$. Therefore, it is not locally integrable. This implies that this compactly supported measurable function solution ϕ of (4) cannot be globally identified with the compactly supported distribution solution of (4). Moreover, a detailed calculation shows that $\phi \in L_p(\mathbb{R})$ for $0 < p < \infty$ if and only if $|c|^p + |1-c|^p < 3$. Similarly, $\phi \in C(\mathbb{R})$ if and only if $\max(|c|, |1-c|) < 1$ (e.g., see [7, 8, 9]). By a similar argument as in [1, 21], we remark that this compactly supported measurable function solution ϕ of (4) could be identified with the compactly supported distribution solution of (4) on the open set G. The smoothness of the compactly supported distribution solution of (4) is also discussed in [3].

We note that all the examples of refinable componentwise polynomials in [2] and the above example are measurable function solutions of some refinement equations with a dilation factor $M \ge 3$. Our next result asserts that any componentwise polynomial refinable function with the dilation factor 2 must be a finite linear combination of the integer shifts of some B-spline. Therefore, it is impossible to construct a compactly supported componentwise polynomial refinable function with the dilation factor 2 that does not coincide globally with the corresponding compactly supported distribution solution. This result looks similar to Theorem 9 of [18], but they are different. We cannot use the result of [18], since the refinable function there is assumed to be integrable with various other conditions, e.g., the linear independence of the refinable function. This implies that [18] starts with a compactly supported integrable refinable function that coincides globally with the corresponding compactly supported distribution solution, while we use the next result to conclude that any compactly supported componentwise polynomial refinable function with the dilation factor 2 must coincide globally with its compactly supported distribution solution. Note that a compactly supported componentwise polynomial may not be integrable, as demonstrated by Example 2. We further acknowledge that some techniques here were already used in [18].

Theorem 3. *Let $H(\xi) := \sum_{k\in\mathbb{Z}} h(k)e^{-ik\xi}$ be a 2π-periodic trigonometric polynomial with $H(0) = 1$. Suppose that ϕ is a compactly supported nontrivial componentwise polynomial function satisfying the refinement equation (3) in the sense of almost everywhere with the dilation factor $M = 2$. Then ϕ must be a finite linear combination of the integer shifts of some B-spline.*

Proof. Without loss of generality, we assume that $H(\xi) = \sum_{k=0}^{N} h(k)e^{-ik\xi}$, that is, its coefficient sequence $\{h(k)\}_{k\in\mathbb{Z}}$ is supported inside $[0, N]$ with $h(0)h(N) \neq 0$. Consequently, the refinement equation (3) becomes

$$\phi(x) = 2\sum_{k=0}^{N} h(k)\phi(2x-k), \qquad a.e.\ x \in \mathbb{R}. \tag{7}$$

For a measurable function f on $\mathbb{R}$, the essential support of f is defined to be ess-supp$(f) := \mathbb{R}\backslash G_f$, where G_f is the union of all open intervals (a, b) such that $f(x) = 0$ for almost every $x \in (a, b)$. Note that (7) implies ess-supp$\phi \subseteq \frac{1}{2}[0, N] + \frac{1}{2}$ess-supp ϕ. Since ϕ is compactly supported, we now conclude that ϕ is essentially supported inside $[0, N]$.

Let $\tilde{H}(z) := \sum_{k\in\mathbb{Z}} h(k)z^k$. Then $\tilde{H}$ is a polynomial such that $H(\xi) = \tilde{H}(e^{-i\xi})$ and $\tilde{H}(1) = 1$. We first consider the case that $\tilde{H}$ has no symmetric zeros in $\mathbb{C}\backslash\{0\}$, that is, $\tilde{H}(z)$ and $\tilde{H}(-z)$ do not vanish simultaneously for any $z \in \mathbb{C}\backslash\{0\}$. Let B_0 and B_1 be the $N \times N$ matrices defined by

$$B_0 := \big(2h(2j-k)\big)_{0\le j,k\le N-1}, \qquad B_1 := \big(2h(2j-k+1)\big)_{0\le j,k\le N-1}.$$

Denote

$$\Phi(x) := [\phi(x), \phi(x+1), \dots, \phi(x+N-1)]^T, \qquad x \in (0,1).$$

It follows from the refinement equation (7) that

$$\Phi\left(\frac{x}{2}\right) = B_0\Phi(x) \quad \text{and} \quad \Phi\left(\frac{x+1}{2}\right) = B_1\Phi(x), \qquad a.e.\ x \in (0,1). \tag{8}$$

Since $\tilde{H}(z)$ and $\tilde{H}(-z)$ do not vanish simultaneously, it follows from [17, Lemma 1] or [21, Lemma 2.7.1] that the matrices B_0 and B_1 must be invertible. Therefore, the relation in (8) can be rewritten as

$$\Phi(x) = [B_0]^{-1}\Phi\left(\frac{x}{2}\right), \qquad a.e.\ x \in (0,1) \tag{9}$$

and

$$\Phi(x) = [B_1]^{-1}\Phi\left(\frac{x+1}{2}\right), \qquad a.e.\ x \in (0,1). \tag{10}$$

Suppose that Φ is a vector of polynomials on a nonempty open interval (a, b) with $(a, b) \subseteq (0, 1)$. Clearly, either $a < 1/2$, or $b > 1/2$, or both hold true. Now we can take one of the following two steps:

Step 1: If $a < 1/2$, then we apply (9) and we see that Φ must be a vector of polynomials on $(\tilde{a}, \tilde{b}) := (2a, 2b) \cap (0, 1)$. It is evident that $(\tilde{a}, \tilde{b}) = (2a, 2b)$ if $b < 1/2$ and $(\tilde{a}, \tilde{b}) = (2a, 1)$ if $b \ge 1/2$.

Step 2: If $b > 1/2$, then we apply (10) and we see that Φ must be a vector of polynomials on $(\tilde{a}, \tilde{b}) := (2a-1, 2b-1) \cap (0, 1)$. It is evident that $(\tilde{a}, \tilde{b}) = (2a-1, 2b-1)$ if $a > 1/2$ and $(\tilde{a}, \tilde{b}) = (0, 2b-1)$ if $a \le 1/2$.

In other words, after applying either Step 1 or Step 2, we see that either $\tilde{a} = 0$, or $\tilde{b} = 1$, or the length of the new interval $(\tilde{a}, \tilde{b})$ doubles that of the original interval (a, b).

Since ϕ is a compactly supported componentwise polynomial, for any nonempty open interval $(c, d) \subseteq (0, 1)$, it is not difficult to see that there must exist a nonempty subinterval $(a, b) \subseteq (c, d)$ such that $\phi_{|_{(a,b)+k}}$ is a polynomial (the polynomial on every piece may vary with the integer k) for every integer $k \in \mathbb{Z}$. That is, Φ is a vector of polynomials on a nonempty open interval (a, b) and $(a, b) \subseteq (0, 1)$.

Now applying either Step 1 or Step 2 no more than $\log_2 \frac{1}{b-a}$ times, we must end up with either $\tilde{a} = 0$ or $\tilde{b} = 1$. So, without loss of generality, we can assume that Φ is a vector of polynomials on interval (a, b) with either $a = 0$ or $b = 1$. Let n be the smallest integer such that $2^n(b - a) \geq 1$. If $a = 0$, then we apply Step 1 n times and if $b = 1$, we apply Step 2 n times. For both cases, since the length of the new interval doubles after every application of either Step 1 or Step 2, it is easy to verify that we must end up with the final new interval $(\tilde{a}, \tilde{b}) = (0, 1)$. Consequently, we proved that Φ is a vector of polynomials on $(0, 1)$. In other words, ϕ is a piecewise polynomial with integer knots. Since ϕ is a piecewise polynomial refinable function, by [13] (also cf. [6, 19, 22]), ϕ must be a finite linear combination of the integer shifts of some B-spline.

When $\tilde{H}$ has symmetric zeros in $\mathbb{C}\backslash\{0\}$, one can find a new mask $P(\xi) = \sum_{k=0}^{N} q(k)e^{-ik\xi}$ with $P(0) = 1$, such that $\tilde{P}(z) := \sum_{k=0}^{N} q(k)z^k$ has no symmetric zeros in $\mathbb{C}\backslash\{0\}$. Furthermore, there is a compactly supported componentwise polynomial refinable function ψ, such that ϕ is a finite linear combination of the integer shifts of ψ.

Indeed, suppose that $\tilde{H}$ has symmetric zeros in $\mathbb{C}\backslash\{0\}$, that is, suppose that $\tilde{H}(z_0) = \tilde{H}(-z_0) = 0$ for some $z_0 \in \mathbb{C}\backslash\{0\}$. Since $\tilde{H}(1) = 1$, we have $z_0 \neq 1, -1$. Therefore, we can write

$$\tilde{H}(z) = \frac{z^2 - z_0^2}{1 - z_0^2} Q(z) \tag{11}$$

for a Laurent polynomial $Q(z) := \sum_{k \in \mathbb{Z}} q(k)z^k$ with $Q(1) = 1$ and $q(k) = 0$ for all $k < 0$ and $k \geq N$. Now we deduce from (11) that

$$h(j) = \frac{1}{1 - z_0^2}[q(j-2) - z_0^2 q(j)], \qquad j \in \mathbb{Z}. \tag{12}$$

Define $F(x) := \sum_{k \in \mathbb{Z}} z_0^{2k-2}\phi(x + k), x \in \mathbb{R}$. Since ϕ is compactly supported, the infinite sum in the definition of $F(x)$ is in fact finite and therefore, $F(x)$ is well defined on any bounded set. Furthermore, we claim that

$$F(x) = \sum_{k \in \mathbb{Z}} z_0^{2k-2}\phi(x + k) = 0, \qquad a.e.\ x \in \mathbb{R}. \tag{13}$$

Indeed, since ϕ satisfies the refinement equation in (7), by (12), we have

$$\begin{aligned}
F(x) &= \sum_{k \in \mathbb{Z}} z_0^{2k-2}\phi(x + k) \\
&= 2\sum_{k \in \mathbb{Z}} z_0^{2k-2} \sum_{j \in \mathbb{Z}} h(j)\phi(2(x + k) - j) \\
&= \frac{2}{1 - z_0^2} \sum_{k \in \mathbb{Z}} z_0^{2k-2} \sum_{j \in \mathbb{Z}} [q(j-2) - z_0^2 q(j)]\phi(2x + 2k - j) \\
&= \frac{2}{1 - z_0^2} \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} [z_0^{2k-2} q(j + 2k - 2) - z_0^{2k} q(j + 2k)]\phi(2x - j) \\
&= \frac{2}{1 - z_0^2} \sum_{j \in \mathbb{Z}} \phi(2x - j) \sum_{k \in \mathbb{Z}} [z_0^{2k-2} q(j + 2k - 2) - z_0^{2k} q(j + 2k)] \\
&= 0.
\end{aligned}$$

Define a function ψ by

$$\psi(x) := \sum_{k=1}^{\infty} z_0^{2k-2}\phi(x+k), \qquad x \in \mathbb{Z}. \tag{14}$$

Since ϕ is essentially supported inside $[0, N]$, the function ψ is well defined. Next, we show that ψ is also compactly supported. By the definition of ψ in (14), the essential support of ψ is contained inside $\cup_{k=1}^{\infty}[\text{ess-supp}(\phi) - k] \subseteq (-\infty, N-1]$. On the other hand, by (13), we see that

$$\psi(x) = -\sum_{k=-\infty}^{0} z_0^{2k-2}\phi(x+k), \qquad a.e.\ x \in \mathbb{R}, \tag{15}$$

which implies that the essential support of ψ is contained inside $\cup_{k=-\infty}^{0}[\text{ess-supp}(\phi) - k] \subseteq [0, \infty)$. Therefore, the essential support of ψ must be contained inside $(-\infty, N-1] \cap [0, \infty) = [0, N-1]$. Hence, ψ is compactly supported. Note that on $[0, N-1]$, the sum in (14) is a finite sum, hence, ψ is a compactly supported componentwise polynomial. Moreover, by the definition of ψ in (14), it is easy to see that

$$\psi(x) = \phi(x+1) + z_0^2\psi(x+1), \qquad a.e.\ x \in \mathbb{R}.$$

This leads to

$$\phi(x) = \psi(x-1) - z_0^2\psi(x), \qquad a.e.\ x \in \mathbb{R}. \tag{16}$$

That is, ϕ is a finite linear combination of the integer shifts of ψ. Hence, if ψ is a finite linear combination of the integer shifts of a B-spline, then so is ϕ. The relation in (16) also implies that ψ is nontrivial if ϕ is nontrivial.

Next, we prove that ψ is refinable with the finitely supported mask $\{p(k)\}_{k\in\mathbb{Z}}$, where

$$\tilde{P}(z) := \frac{z - z_0^2}{1 - z_0^2} Q(z) = \sum_{k\in\mathbb{Z}} p(k)z^k.$$

Clearly, $\tilde{P}(1) = 1$ by $\tilde{H}(1) = 1$ and (11). Furthermore, we can easily deduce from the above relation that

$$p(j) = \frac{1}{1 - z_0^2}[q(j-1) - z_0^2 q(j)], \qquad j \in \mathbb{Z}. \tag{17}$$

It remains to prove that

$$\psi(x) = 2\sum_{j\in\mathbb{Z}} p(j)\psi(2x - j), \qquad a.e.\ x \in \mathbb{R}. \tag{18}$$

By the definition of ψ in (14), we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} p(j)\psi(2x-j) &= \sum_{j\in\mathbb{Z}} p(j) \sum_{k=1}^{\infty} z_0^{2k-2}\phi(2x+k-j) \\
&= \sum_{k=1}^{\infty}\sum_{j\in\mathbb{Z}} p(j) z_0^{2k-2}\phi(2x+k-j) \\
&= \sum_{k=1}^{\infty}\sum_{j\in\mathbb{Z}} z_0^{2k-2} p(j+k)\phi(2x-j) \\
&= \sum_{j\in\mathbb{Z}} \phi(2x-j) \sum_{k=1}^{\infty} z_0^{2k-2} p(j+k).
\end{aligned}$$

By (17), we have

$$\sum_{k=1}^{\infty} z_0^{2k-2} p(j+k) = \frac{1}{1-z_0^2} \sum_{k=1}^{\infty} z_0^{2k-2}[q(j+k-1) - z_0^2 q(j+k)]$$
$$= \frac{1}{1-z_0^2} \sum_{k=1}^{\infty} [z_0^{2k-2} q(j+k-1) - z_0^{2k} q(j+k)]$$
$$= \frac{1}{1-z_0^2} q(j).$$

Therefore, we have

$$\sum_{j\in\mathbb{Z}} p(j)\psi(2x-j) = \frac{1}{1-z_0^2} \sum_{j\in\mathbb{Z}} q(j)\phi(2x-j), \qquad a.e.\ x \in \mathbb{R}. \tag{19}$$

On the other hand, by the definition of ψ in (14), we have

$$2^{-1}\psi(x) = 2^{-1} \sum_{k=1}^{\infty} z_0^{2k-2} \phi(x+k)$$
$$= \sum_{k=1}^{\infty} z_0^{2k-2} \sum_{j\in\mathbb{Z}} h(j)\phi(2x+2k-j)$$
$$= \sum_{k=1}^{\infty} \sum_{j\in\mathbb{Z}} z_0^{2k-2} h(j+2k)\phi(2x-j)$$
$$= \sum_{j\in\mathbb{Z}} \phi(2x-j) \sum_{k=1}^{\infty} z_0^{2k-2} h(j+2k).$$

By (12), we deduce that

$$\sum_{k=1}^{\infty} z_0^{2k-2} h(j+2k) = \frac{1}{1-z_0^2} \sum_{k=1}^{\infty} z_0^{2k-2}[q(j+2k-2) - z_0^2 q(j+2k)]$$
$$= \frac{1}{1-z_0^2} \sum_{k=1}^{\infty} [z_0^{2k-2} q(j+2k-2) - z_0^{2k} q(j+2k)] = \frac{1}{1-z_0^2} q(j).$$

Therefore, we conclude that

$$2^{-1}\psi(x) = \frac{1}{1-z_0^2} \sum_{j\in\mathbb{Z}} q(j)\phi(2x-j), \qquad a.e.\ x \in \mathbb{R}. \tag{20}$$

Combining the identities in (19) and (20), we see that (18) is verified.

Note that the degree of the polynomial $\tilde{P}$ is one degree lower than that of $\tilde{H}$. Note that $\tilde{P}$ either does not vanish at z_0 and $-z_0$ simultaneously, or the order of symmetric zero at z_0 and $-z_0$ is reduced by one. If $\tilde{P}$ still has symmetric zeros, we can continue this procedure until that the resulting mask $\tilde{P}$ has no symmetric zeros in $\mathbb{C}\backslash\{0\}$. Therefore, the conclusion follows from the case that $\tilde{H}$ has no symmetric zeros in $\mathbb{C}\backslash\{0\}$. □

Let ϕ be a compactly supported measurable function solution to the refinement equation (7) in Theorem 3. In fact, the above proof shows that if there is a nonempty open interval $(a, b) \subseteq [0, 1]$ such that ϕ coincides with some polynomial on $(a+k, b+k)$ for every $k \in \mathbb{Z}$, then ϕ must be a finite linear combination of the integer shifts of some B-spline. Using a similar argument as in [20], the same conclusion could still hold if the nonempty open interval (a, b) is replaced by a measurable subset of $[0, 1]$ with a positive measure.

References

[1] N. Bi, L. Debnath and Q. Sun, Asymptotic behavior of M-band scaling functions of Daubechies type, *Z. Anal. Anwendungen*, **17** (1998), 813–830.

[2] N. Bi, B. Han and Z. Shen, Examples of refinable componentwise polynomials, *Appl. Comput. Harmon. Anal.*, **22** (2007), 368–373.

[3] X. Dai, D. Huang and Q. Sun, Some properties of five-coefficient refinement equation. *Arch. Math.* **66** (1996), 299–309.

[4] I. Daubechies, Ten lectures on wavelets. CBMS-NSF Regional Conference Series in Applied Mathematics, **61**, SIAM, Philadelphia, (1992).

[5] I. Daubechies and J. C. Lagarias, Two-scale difference equations. I. Existence and global regularity of solutions, *SIAM J. Math. Anal.*, **22** (1991), 1388–1410.

[6] X. Gao, S. L. Lee, and Q. Sun, Eigenvalues of scaling operators and a characterization of B-splines. *Proc. Amer. Math. Soc.* **134** (2006), 1051–1057

[7] B. Han, Symmeric orthonormal scaling functions and wavelets with dilation factor 4, *Adv. Comput. Math.*, **3** (1998), 221–247.

[8] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, *J. Approx. Theory*, **124** (2003), 44–88.

[9] B. Han, Solutions in Sobolev spaces of vector refinement equations with a general dilation matrix, *Adv. Comput. Math.*, **24** (2006), 375–403.

[10] B. Han, Refinable functions and cascade algorithms in weighted spaces with Hölder continuous masks, *SIAM J. Math. Anal.* **40** (2008), 70–102.

[11] B. Han and R. Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, *SIAM J. Math. Anal.*, **29** (1998), 1177–1999

[12] Q. T. Jiang and Z. Shen, On existence and week stability of matrix refinable functions, *Constr. Approx.*, **15** (1999), 337–353.

[13] W. Lawton, S. L. Lee and Z. Shen, Convergence of multidimensional cascade algorithm, *Numer. Math.*, **78** (1998), 427–438.

[14] W. Lawton, S. L. Lee and Z. Shen, Complete characterization of refinable spline, *Adv. Comput. Math.*, **3** (1995), 137–145.

[15] A. Ron and Z. Shen, The Sobolev regularity of refinable functions, *J. Approx. Theory*, **106** (2000), 185–225.

[16] Q. Sun, Convergence and boundedness of cascade algorithm in Besov spaces and Triebel-Lizorkin spaces. I, *Adv. Math. (China)*, **29** (2000), 507–526.

[17] Q. Sun, Two-scale difference equation: local and global linear independence, (1991), manuscript.

[18] D. Huang and Q. Sun, Affine similarity of refinable functions, *Approx. Theory and its Appl.* **15(3)** (1999), 81-91.

[19] Q. Sun, Refinable functions with compact support. *J. Approx. Theory* **86** (1996), 240–252.

[20] Q. Sun, Local reconstruction for sampling in shift-invariant space, *Adv. Comput. Math.*, to appear.

[21] Q. Sun, N. Bi and D. Huang, *An introduction to multiband wavelets.* Zhejiang University Press, 2001.

[22] Q. Sun and Z. Zhang, A characterization of compactly supported both m and n refinable distributions. *J. Approx. Theory* **99** (1999), 198–216.

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