TIGHT PERIODIC WAVELET FRAMES AND APPROXIMATION ORDERS

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Abstract. A systematic study on tight periodic wavelet frames and their approximation orders is conducted. We identify a necessary and sufficient condition, in terms of refinement masks, for applying the unitary extension principle for periodic functions to construct tight wavelet frames. Then a theory on the approximation orders of truncated tight frame series is established, which facilitates the construction of tight periodic wavelet frames with desirable approximation orders. Finally, a notion of vanishing moments for periodic wavelets, which is missing in the current literature, is introduced and related to frame approximation orders and sparse representations of locally smooth functions. As illustrations, the results are applied to two classes of examples: one is band-limited and the other is time-localized.

1. Introduction

The setup of tight wavelet frames provides great flexibility in approximating and representing periodic functions. Fundamental issues involved include the construction of tight periodic wavelet frames, the approximation powers of such wavelet frames, and whether the wavelet frames lead to sparse representations of locally smooth periodic functions. This paper discusses all these aspects. It gives a systematic study on tight periodic wavelet frames and approximation orders of the truncated frame series. It introduces a notion of vanishing moments; and for tight periodic wavelet frames, it relates vanishing moments to approximation orders as well as to sparse representations of locally smooth functions.

As motivation of our study, we note that wavelet frames on the real line have recently found many applications in image restoration. Surveys of wavelet frames and their applications were provided in [19, 37]. The interested reader could consult, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 17] for details of tight frame based algorithms developed for various image restoration problems. When periodic boundary conditions are used in the implementation

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of these algorithms, we are essentially applying tight periodic wavelet frames generated from tight wavelet frames on the real line. This illustrates a certain need, as well as being a more general and natural approach, to study periodic wavelets directly, instead of periodizing wavelets on the real line.

Some of the earliest papers on periodic wavelets in this direction include [14, 33, 34]. The study was extended to multidimensional multiwavelets in [20, 21], the former for orthonormal and Riesz multiwavelets and the latter for biorthogonal multiwavelets. Leveraging on the tools developed in [20], general theories for wavelet frames of periodic functions of one or higher dimensions were established in [12, 22, 23]. Different from the typical situation on the real line, periodic wavelets and wavelet frames are nonstationary in nature and the subspaces generated by them at various levels are of finite dimensions. This opens opportunities for either more general results or simpler analytical arguments. As such, the topic has its own characteristics and it is not possible to deduce desired results from their counterparts, if any, of the real line case.

In recent years, wavelets on the real line or a higher-dimensional Euclidean space were also investigated under the nonstationary setting due to the additional flexibilities available. In [30], the nonstationary approach made possible further desirable properties such as spectral approximation orders of compactly supported infinitely differentiable wavelets. In [36], an extensive study on nonstationary tight frames of this flavor on a multidimensional Euclidean space was given. In addition, nonstationary tight wavelet frames on unbounded intervals were discussed in [13].

This paper could be viewed as further or parallel developments of earlier investigations on several different fronts, while establishing results tailored to the periodic setting. Based on the unitary extension principle for periodic functions derived in [23] and Fourier-based techniques in [24, 25, 27, 28, 30, 35], it gives a necessary and sufficient condition, in terms of refinement masks, which is easy to verify for generating tight periodic wavelet frames. The paper also establishes a theory for approximation orders of truncated tight periodic frame series, constructs tight periodic wavelet frames with desirable approximation orders, and relates the concept of approximation orders to vanishing moments. In this sense, it is a continuation of [16] which contains a complete study from approximation orders to vanishing moments for the usual non-periodic case. The general approach developed here for tight periodic wavelet frames with spectral approximation orders is applicable to both band-limited constructions as well as time-localized examples. For the latter, the paper parallels the construction of nonstationary tight wavelet frames on the real line in [30].

This paper is organized as follows. In Section 2, we first recall from [23] the unitary extension principle for periodic functions motivated by the unitary extension principle of [35], which provides a flexible and convenient way of constructing tight wavelet frames. Then a construction procedure with periodic sequences of refinement masks as the starting point is formulated in Theorem 2.2. A necessary and sufficient condition in terms of these refinement masks for applying the unitary extension principle is identified in Corollary 2.1. A general theorem (Theorem 3.2) is presented in Section 3 to give a necessary and sufficient condition for the frame approximation order of a tight wavelet frame constructed via the setup of Theorem 2.2. As far as we know, no notion of vanishing moments for periodic functions is available in the literature and no obvious candidate is in sight. In Section 4, drawing inspiration from the real line case, we formulate a notion of vanishing moments for a sequence of periodic functions, and establish a theorem (Theorem 4.1) to characterize the interplay between approximation orders and vanishing moments for tight periodic wavelet frames. We further propose a concept of global vanishing moments to link up with sparse representations of locally smooth functions. The paper concludes in Section 5 with two classes of examples, one with good frequency-localization and the other with good time-localization. Both classes of tight wavelet frames enjoy the desirable properties of spectral frame approximation order and (global) vanishing moments of arbitrarily high order.

2. Tight wavelet frames

We begin with the unitary extension principle and formulate a general procedure for constructing tight wavelet frames. The emphasis is on having refinement masks as the starting point. The conditions for this can be easily verified and also provide insight to the refinement masks that enable the construction process.

Let $L^2[0,2\pi]$ be the space of all 2π-periodic square-integrable complex-valued functions over the real line $\mathbb{R}$, with inner product $\langle\cdot,\cdot\rangle$ given by $\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$, where $f,g\in L^2[0,2\pi]$, and norm $\|\cdot\|_2 := \langle\cdot,\cdot\rangle^{1/2}$. For a function $f\in L^2[0,2\pi]$, we express its Fourier series as $\sum_{n\in\mathbb{Z}}\widehat{f}(n)e^{in\cdot}$, where $\widehat{f}(n) := \langle f, e^{in\cdot}\rangle$, $n\in\mathbb{Z}$, are its Fourier coefficients.

For $k\ge 0$ and $\ell\in\mathbb{Z}$, we define the $\frac{2\pi\ell}{2^k}$-*shift operator* $T_k^\ell : L^2[0,2\pi]\mapsto L^2[0,2\pi]$ by

$$T_k^\ell f := f(\cdot - \tfrac{2\pi\ell}{2^k}), \quad f\in L^2[0,2\pi].$$

For $f\in L^2[0,2\pi]$, since f is periodic, it suffices to consider the shifts $T_k^\ell f$, $\ell\in\mathcal{R}_k$, where $\mathcal{R}_k$ is given by

$$\mathcal{R}_k := \{-2^{k-1}+1, -2^{k-1}+2, \dots, 2^{k-1}\}.$$

Let $\mathcal{S}(2^k)$ be the space of all 2^k-periodic complex sequences a_k, that is, $a_k(\ell+2^kp)=a_k(\ell)$ for all $\ell,p\in\mathbb{Z}$. We denote the discrete Fourier transform of $a_k\in\mathcal{S}(2^k)$ by $\widehat{a_k}(j) := \sum_{\ell\in\mathcal{R}_k} a_k(\ell)e^{-\frac{2\pi i\ell j}{2^k}}$, where $j\in\mathcal{R}_k$. The sequence $\widehat{a_k}$ also lies in $\mathcal{S}(2^k)$.

Now, consider positive integers ρ_k, $k\ge 0$, and functions ϕ_0, ψ_k^m, $k\ge 0$, $m=1,2,\dots,\rho_k$, in $L^2[0,2\pi]$. We say that the collection $\{\phi_0\}\cup\{T_k^\ell\psi_k^m : k\ge 0,\ m=1,2,\dots,\rho_k,\ \ell\in\mathcal{R}_k\}$ forms a *normalized tight wavelet frame*, or simply *tight wavelet frame*, for the space $L^2[0,2\pi]$

if

$$\|f\|_2^2 = |\langle f, \phi_0\rangle|^2 + \sum_{k=0}^{\infty}\sum_{m=1}^{\rho_k}\sum_{\ell\in\mathcal{R}_k} |\langle f, T_k^\ell \psi_k^m\rangle|^2, \quad f \in L^2[0, 2\pi]. \tag{2.1}$$

In this case, the functions ψ_k^m are called *periodic wavelets*, or simply *wavelets*.

Our construction of wavelets is based on a sequence of *refinable functions* $\{\phi_k\}_{k\ge 0}$ in $L^2[0, 2\pi]$, which satisfies for every $k \ge 0$, the *periodic refinement equation*

$$\phi_k = \sqrt{2} \sum_{\ell\in\mathcal{R}_{k+1}} a_{k+1}(\ell) T_{k+1}^\ell \phi_{k+1} \tag{2.2}$$

for some $a_{k+1} \in \mathcal{S}(2^{k+1})$. For each $k \ge 0$, the wavelets $\psi_k^m \in L^2[0, 2\pi]$, $m = 1, 2, \ldots, \rho_k$, with ρ_k being some positive integer, are given by the *periodic wavelet equation*

$$\psi_k^m = \sqrt{2} \sum_{\ell\in\mathcal{R}_{k+1}} b_{k+1}^m(\ell) T_{k+1}^\ell \phi_{k+1}, \tag{2.3}$$

where $b_{k+1}^m \in \mathcal{S}(2^{k+1})$, $m = 1, 2, \ldots, \rho_k$. (To simplify subsequent formulas, we introduce here the normalizing constant $\sqrt{2}$ in (2.2) and (2.3). In [20, 21, 22, 23], this constant was absorbed into the coefficients of the periodic refinement and wavelet equations.) It is well known (see for instance [20, 22, 23]) that (2.2) and (2.3) can be formulated equivalently in the Fourier domain as

$$\widehat{\phi_k}(n) = \sqrt{2}\, \widehat{a_{k+1}}(n) \widehat{\phi_{k+1}}(n), \quad n \in \mathbb{Z}, \tag{2.4}$$

and

$$\widehat{\psi_k^m}(n) = \sqrt{2}\, \widehat{b_{k+1}^m}(n) \widehat{\phi_{k+1}}(n), \quad n \in \mathbb{Z}, \tag{2.5}$$

where $m = 1, 2, \ldots, \rho_k$. The periodic sequences $\widehat{a_{k+1}}$ and $\widehat{b_{k+1}^m}$, $m = 1, 2, \ldots, \rho_k$, in $\mathcal{S}(2^{k+1})$ are known as *refinement mask* and *wavelet masks* respectively.

The unitary extension principle, first introduced in [35], provides a useful approach for constructing tight wavelet frames for the space $L^2(\mathbb{R}^d)$ of square-integrable functions on $\mathbb{R}^d$. It entails choosing appropriate wavelet masks with respect to a given refinement mask. A periodic analog of the unitary extension principle was obtained in [23] for the space $L^2([0, 2\pi]^d)$, with a complete characterization of the choice of wavelet masks. This result ([23, Theorem 2.2]) specialized to the context on hand is as follows.

Theorem 2.1. *Suppose that $\phi_k \in L^2[0, 2\pi]$, $k \ge 0$, satisfy* (2.4) *for some $\widehat{a_{k+1}} \in \mathcal{S}(2^{k+1})$, and*

$$\lim_{k\to\infty} 2^k |\widehat{\phi_k}(n)|^2 = 1, \quad n \in \mathbb{Z}. \tag{2.6}$$

For each $k \ge 0$ with some positive integer ρ_k, let $\psi_k^m \in L^2[0, 2\pi]$, $m = 1, 2, \ldots, \rho_k$, be as defined in (2.5)*, where $\widehat{b_{k+1}^m} \in \mathcal{S}(2^{k+1})$, $m = 1, 2, \ldots, \rho_k$. Define the $\rho_k \times 2$ matrices*

$$M_k(j) := \begin{pmatrix} \widehat{b_{k+1}^1}(j) & \widehat{b_{k+1}^1}(j+2^k) \\ \vdots & \vdots \\ \widehat{b_{k+1}^{\rho_k}}(j) & \widehat{b_{k+1}^{\rho_k}}(j+2^k) \end{pmatrix}, \quad j \in \mathcal{R}_k. \tag{2.7}$$

If for every $k \geq 0$,

$$M_k(j)^* M_k(j) = I_2 - \begin{pmatrix} \overline{\widehat{a_{k+1}}(j)} \\ \overline{\widehat{a_{k+1}}(j+2^k)} \end{pmatrix} \begin{pmatrix} \widehat{a_{k+1}}(j) & \widehat{a_{k+1}}(j+2^k) \end{pmatrix}, \quad j \in \mathcal{R}_k, \tag{2.8}$$

where $M_k(j)^*$ denotes the conjugate transpose of $M_k(j)$, then the collection $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ forms a tight wavelet frame for $L^2[0, 2\pi]$.

As noted in [23], the refinement equation (2.4) for $k \geq 0$ and the condition (2.6) imply that the sequence of subspaces $\{V_k\}_{k\geq 0}$, defined by $V_k := \langle\{T_k^\ell \phi_k : \ell \in \mathcal{R}_k\}\rangle$, forms a *multiresolution analysis* of $L^2[0, 2\pi]$, that is, $V_k \subseteq V_{k+1}$ for every $k \geq 0$ and $\overline{\bigcup_{k\geq 0} V_k} = L^2[0, 2\pi]$.

For each $k \geq 0$, observe that the determinant of the 2×2 matrix on the right-hand side of (2.8) is $1 - |\widehat{a_{k+1}}(j)|^2 - |\widehat{a_{k+1}}(j + 2^k)|^2$. Therefore, by (2.8),

$$|\widehat{a_{k+1}}(j)|^2 + |\widehat{a_{k+1}}(j + 2^k)|^2 \leq 1, \quad j \in \mathcal{R}_k. \tag{2.9}$$

That is, if (2.8) holds for some $M_k(j)$ in (2.7), then (2.9) must hold. Conversely, if (2.9) holds, then it is straightforward to see that the 2×2 matrix on the right-hand side of (2.8) is positive semi-definite. Therefore, we can factorize this matrix and obtain a 2×2 matrix $M_k(j)$ satisfying (2.8). In other words, with $\rho_k = 2$, we can construct $\widehat{b_{k+1}^m}$, $m = 1, 2$, for $M_k(j)$ in (2.7) to satisfy (2.8). These observations may also be made via [23, Theorem 4.1].

The condition (2.8) can also be equivalently rewritten as

$$\begin{cases} |\widehat{a_{k+1}}(j)|^2 + \displaystyle\sum_{m=1}^{\rho_k} |\widehat{b_{k+1}^m}(j)|^2 = 1, \\ \widehat{a_{k+1}}(j)\overline{\widehat{a_{k+1}}(j + 2^k)} + \displaystyle\sum_{m=1}^{\rho_k} \widehat{b_{k+1}^m}(j)\overline{\widehat{b_{k+1}^m}(j + 2^k)} = 0, \end{cases} \quad j \in \mathcal{R}_{k+1}. \tag{2.10}$$

Due to the ease of satisfying (2.10), for our construction of tight wavelet frames, we focus on identifying refinable functions ϕ_k, $k \geq 0$, with desirable properties. The starting point here is refinement masks $\widehat{a_k}$, $k \geq 1$, where it turns out that the condition (2.9) for Theorem 2.1 has important ramifications. We shall identify conditions that should be imposed on the refinement masks so that Theorem 2.1 and its corresponding construction procedure are applicable. For this, we first establish the existence of a symbolic solution to (2.4) for the given refinement masks $\widehat{a_k}$, $k \geq 1$.

Lemma 2.1. *Suppose that $\widehat{a_k} \in \mathcal{S}(2^k)$, $k \geq 1$, satisfy*

$$\sum_{k=1}^{\infty} \big|1 - |\widehat{a_k}(n)|\big| < \infty, \quad n \in \mathbb{Z}. \tag{2.11}$$

Then there exists $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\ge 0}$*, with* $\widehat{\phi_k}(n)\in\mathbb{C}$*, such that the refinement equation* (2.4) *holds for every* $k\ge 0$*, and*

$$|\widehat{\phi_k}(n)| = 2^{-k/2}\prod_{r=k+1}^{\infty}|\widehat{a_r}(n)| := 2^{-k/2}\lim_{N\to\infty}\prod_{r=k+1}^{N}|\widehat{a_r}(n)|,\quad n\in\mathbb{Z},\ k\ge 0. \tag{2.12}$$

Proof. It is straightforward to see that (2.11) implies that each of the infinite products in (2.12) converges absolutely. To avoid confusion, for each $k\ge 0$ and $n\in\mathbb{Z}$, we denote by (the magnitude) $m_k(n)$ the value of the infinite product in (2.12). That is,

$$m_k(n) := 2^{-k/2}\lim_{N\to\infty}\prod_{r=k+1}^{N}|\widehat{a_r}(n)|,\quad n\in\mathbb{Z},\ k\ge 0. \tag{2.13}$$

So, all $m_k(n)$ are well-defined nonnegative numbers. Now we recursively construct the phase factors $p_k(n)$ of $\widehat{\phi_k}(n)$. Let $p_0(n), n\in\mathbb{Z}$, be arbitrary complex numbers such that $|p_0(n)|=1$, and recursively define

$$p_{k+1}(n) := p_k(n)\frac{\overline{\widehat{a_{k+1}}(n)}}{|\widehat{a_{k+1}}(n)|},\quad n\in\mathbb{Z},\ k\ge 0, \tag{2.14}$$

where in case $\widehat{a_{k+1}}(n)=0$ we simply pick $p_{k+1}(n)$ as an arbitrary complex number such that $|p_{k+1}(n)|=1$. Define

$$\widehat{\phi_k}(n) := m_k(n)p_k(n),\quad n\in\mathbb{Z},\ k\ge 0. \tag{2.15}$$

Noting that $m_k(n) = \sqrt{2}|\widehat{a_{k+1}}(n)|m_{k+1}(n)$, we can easily verify that (2.4) and (2.12) are satisfied. ■

The above proof is constructive. It gives a procedure of constructing $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\ge 0}$ by (2.15) via (2.13) and (2.14). We further note that when

$$\sum_{k=1}^{\infty}|1-\widehat{a_k}(n)|<\infty,\quad n\in\mathbb{Z}, \tag{2.16}$$

the following infinite products are well defined:

$$\widehat{\phi_k}(n) := 2^{-k/2}\prod_{r=k+1}^{\infty}\widehat{a_r}(n),\quad n\in\mathbb{Z},\ k\ge 0. \tag{2.17}$$

Consequently, if (2.17) holds, then both (2.4) and (2.12) are satisfied. As (2.17) is a more convenient way of defining $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\ge 0}$ when compared to (2.15) via (2.13) and (2.14), it is not surprising that (2.16) is a stronger condition than (2.11). This is obvious because $\big|1-|\widehat{a_k}(n)|\big|\le|1-\widehat{a_k}(n)|$ for all $k\ge 1$ and $n\in\mathbb{Z}$. Under the stronger assumption of (2.16), $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\ge 0}$ can be defined by (2.17) and still agrees with the construction by (2.15) as long as we take $p_k(n)=\frac{\prod_{r=k+1}^{\infty}\widehat{a_r}(n)}{\prod_{r=k+1}^{\infty}|\widehat{a_r}(n)|}$ (for $\prod_{r=k+1}^{\infty}|\widehat{a_r}(n)|=0$, we pick $p_k(n)$ as an arbitrary complex number with $|p_k(n)|=1$). Of course, there are many other choices of $\{p_k(n)\}_{n\in\mathbb{Z},k\ge 0}$ that satisfy (2.14).

The result below identifies conditions on the refinement masks $\widehat{a_k}$, $k \geq 1$, for which the corresponding refinable functions are in $L^2[0,2\pi]$ and satisfy the hypothesis of Theorem 2.1.

Proposition 2.1. *Suppose that* $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\geq 0}$*, with* $\widehat{\phi_k}(n) \in \mathbb{C}$*, satisfies* (2.4) *and* (2.12)*, where* $\widehat{a_k} \in \mathcal{S}(2^k)$*,* $k \geq 1$*. Then the following hold.*

(a) *Assume* (2.9) *for every* $k \geq 0$*. Then for each* $k \geq 0$*, the inequality*

$$2^k \sum_{p\in\mathbb{Z}} |\widehat{\phi_k}(j+2^k p)|^2 \leq 1, \quad j \in \mathcal{R}_k, \tag{2.18}$$

holds. Consequently, $2^k|\widehat{\phi_k}(n)|^2 \leq 1$ *for every* $n \in \mathbb{Z}$*, and* $\sum_{n\in\mathbb{Z}} |\widehat{\phi_k}(n)|^2 \leq 1$*. In particular, with* $\widehat{\phi_k}(n)$*,* $n \in \mathbb{Z}$*, as Fourier coefficients, the function* ϕ_k *lies in* $L^2[0,2\pi]$*.*

(b) *Assume that* (2.11) *holds,* $\sup_{k\geq 0} 2^k|\widehat{\phi_k}(n)|^2 < \infty$ *for every* $n \in \mathbb{Z}$*, and*

$$\widehat{a_k}(0) = 1, \quad k \geq 1. \tag{2.19}$$

Then (2.6) *must hold.*

Proof. The proof of item (a) is the same as in [20, 30]. We briefly describe the argument here for completeness. By (2.12), for any positive integer M, we have

$$2^k \sum_{p=-M}^{M} |\widehat{\phi_k}(j+2^k p)|^2 = \lim_{N\to\infty} \sum_{p=-M}^{M} \prod_{r=k+1}^{N} |\widehat{a_r}(j+2^k p)|^2.$$

For large N such that $2^{N-k} \geq 2M+1$, it can be easily deduced by induction and (2.9) that

$$\sum_{p=-M}^{M} \prod_{r=k+1}^{N} |\widehat{a_r}(j+2^k p)|^2 \leq \sum_{p=-M}^{2^{N-k}-1-M} \prod_{r=k+1}^{N} |\widehat{a_r}(j+2^k p)|^2 \leq 1.$$

Consequently, $2^k \sum_{p=-M}^{M} |\widehat{\phi_k}(j+2^k p)|^2 \leq 1$. Letting M go to infinity, we have (2.18).

Next, we prove item (b) by a similar argument as in [26, Section 5]. First, it is not difficult to see that (2.11) is equivalent to

$$\sum_{k=1}^{\infty} \left|1 - |\widehat{a_k}(n)|^2\right| < \infty, \quad n \in \mathbb{Z}, \tag{2.20}$$

since (2.11) implies $\sup_{k\geq 1} |\widehat{a_k}(n)| < \infty$.

Since the infinite products in (2.12) converge, for $k \geq 0$ and $n \in \mathbb{Z}$, by (2.12), we have

$$1 - 2^k|\widehat{\phi_k}(n)|^2 = \prod_{r=k+1}^{\infty} |\widehat{a_r}(0)|^2 - \prod_{r=k+1}^{\infty} |\widehat{a_r}(n)|^2,$$

where (2.19) is also used. Applying the identity

$$\prod_{r=k+1}^{L} |\widehat{a_r}(0)|^2 - \prod_{r=k+1}^{L} |\widehat{a_r}(n)|^2 = \left(\prod_{r=k+1}^{L-1} |\widehat{a_r}(0)|^2\right)\left(|\widehat{a_L}(0)|^2 - |\widehat{a_L}(n)|^2\right) + \left(\prod_{r=k+1}^{L-1} |\widehat{a_r}(0)|^2 - \prod_{r=k+1}^{L-1} |\widehat{a_r}(n)|^2\right)|\widehat{a_L}(n)|^2$$

repeatedly over L, we see that

$$\prod_{r=k+1}^{\infty} |\widehat{a_r}(0)|^2 - \prod_{r=k+1}^{\infty} |\widehat{a_r}(n)|^2 = \sum_{\tau=k+1}^{\infty}\left(\prod_{r=k+1}^{\tau-1} |\widehat{a_r}(0)|^2\right)\left(|\widehat{a_\tau}(0)|^2 - |\widehat{a_\tau}(n)|^2\right)\left(\prod_{r=\tau+1}^{\infty} |\widehat{a_r}(n)|^2\right).$$

Setting $C_n := \sup_{k\ge 0} 2^k|\widehat{\phi_k}(n)|^2 < \infty$, $\prod_{r=\tau+1}^{\infty} |\widehat{a_r}(n)|^2 = 2^\tau|\widehat{\phi_\tau}(n)|^2 \le C_n$. Then we apply (2.19) to deduce that

$$\left|1 - 2^k|\widehat{\phi_k}(n)|^2\right| \le \sum_{\tau=k+1}^{\infty}\left|1 - |\widehat{a_\tau}(n)|^2\right|\left(\prod_{r=\tau+1}^{\infty} |\widehat{a_r}(n)|^2\right) \le C_n \sum_{\tau=k+1}^{\infty}\left|1 - |\widehat{a_\tau}(n)|^2\right|. \tag{2.21}$$

Now by (2.20), we have $\lim_{k\to\infty}\sum_{\tau=k+1}^{\infty} |1 - |\widehat{a_\tau}(n)|^2| = 0$. Therefore, (2.6) holds. ∎

The above proposition says that whenever the refinement masks satisfy (2.9), (2.11) and (2.19), their corresponding refinable functions satisfy the conditions for Theorem 2.1. This enables the standard construction process via the unitary extension principle to be applicable. Thus combining Theorem 2.1, Lemma 2.1, and Proposition 2.1, we obtain the following result for constructing tight wavelet frames for $L^2[0, 2\pi]$, under assumptions entirely in terms of refinement masks.

Theorem 2.2. *Suppose that $\widehat{a_{k+1}} \in \mathcal{S}(2^{k+1})$, $k \ge 0$, satisfy (2.9), (2.11) and (2.19), and $\widehat{b^m_{k+1}} \in \mathcal{S}(2^{k+1})$, $k \ge 0$ and $m = 1, 2, \dots, \rho_k$, satisfy (2.10). Define refinable functions ϕ_k, $k \ge 0$, by (2.15) via (2.13) and (2.14), and wavelets ψ^m_k, $k \ge 0$ and $m = 1, 2, \dots, \rho_k$, by (2.5). Then the collection $\{\phi_0\} \cup \{T^\ell_k\psi^m_k : k \ge 0,\ m = 1, 2, \dots, \rho_k,\ \ell \in \mathcal{R}_k\}$ forms a tight wavelet frame for $L^2[0, 2\pi]$.*

By Lemma 2.1, it is not difficult to see that any $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z},k\ge 0}$ satisfying (2.2) and (2.12) must be generated by (2.15) via (2.13) and (2.14). We shall see later in Proposition 2.2 that both (2.11) and (2.12) are necessary conditions for constructing tight wavelet frames.

In the setup of Theorem 2.2, the refinement masks are assumed to satisfy the conditions (2.9), (2.11) and (2.19). These conditions imply (2.6), which is a basic assumption in the unitary extension principle (Theorem 2.1). We have seen earlier that (2.9) is necessary and sufficient for (2.10) (formulated equivalently as (2.8) in Theorem 2.1) to hold for some $\widehat{b^m_{k+1}} \in \mathcal{S}(2^{k+1}), m = 1, 2, \dots, \rho_k$. We shall now show that for a tight wavelet frame constructed from refinement and wavelet masks satisfying (2.10), the conditions (2.6) and (2.11) are both necessary. This provides a certain converse to Theorem 2.2.

Proposition 2.2. *Let* $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ *be a tight wavelet frame for* $L^2[0, 2\pi]$ *generated via* (2.4) *and* (2.5), *where* $\widehat{a_{k+1}} \in \mathcal{S}(2^{k+1})$ *and* $\widehat{b_{k+1}^m} \in \mathcal{S}(2^{k+1})$, $k \geq 0$, $m = 1, 2, \ldots, \rho_k$, *are refinement and wavelet masks satisfying* (2.10). *Then* (2.6), (2.11), *and* (2.12) *must hold.*

Proof. Since $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ forms a tight wavelet frame for $L^2[0, 2\pi]$ with (2.4), (2.5) and (2.10) being satisfied, it is not difficult to show that (2.6) must hold. The main ideas for this can be found in [23, 24, 35]. Essentially, (2.10) implies that for $k \geq 1$,

$$\sum_{\ell \in \mathcal{R}_k} |\langle f, T_k^\ell \phi_k \rangle|^2 = |\langle f, \phi_0 \rangle|^2 + \sum_{r=0}^{k-1} \sum_{m=1}^{\rho_r} \sum_{\ell \in \mathcal{R}_r} |\langle f, T_r^\ell \psi_r^m \rangle|^2, \quad f \in L^2[0, 2\pi].$$

For a fixed $n \in \mathbb{Z}$, choose $f := e^{in\cdot}$. Then $\sum_{\ell \in \mathcal{R}_k} |\langle f, T_k^\ell \phi_k \rangle|^2 = 2^k |\widehat{\phi_k}(n)|^2$ and it follows from (2.1) that

$$\lim_{k \to \infty} 2^k |\widehat{\phi_k}(n)|^2 = \lim_{k \to \infty} \left(|\langle f, \phi_0 \rangle|^2 + \sum_{r=0}^{k-1} \sum_{m=1}^{\rho_r} \sum_{\ell \in \mathcal{R}_r} |\langle f, T_r^\ell \psi_r^m \rangle|^2 \right) = \|f\|_2^2 = 1.$$

We now deduce from (2.4) and (2.6) that (2.11) and (2.12) must hold as well. Fix $n \in \mathbb{Z}$. Applying (2.4) repeatedly gives

$$2^{k/2} |\widehat{\phi_k}(n)| = 2^{(k+1)/2} |\widehat{\phi_{k+1}}(n)|\, |\widehat{a_{k+1}}(n)| = 2^{N/2} |\widehat{\phi_N}(n)| \prod_{r=k+1}^{N} |\widehat{a_r}(n)|, \quad N > k \geq 0. \tag{2.22}$$

By (2.6), we have $\lim_{N \to \infty} 2^{N/2} |\widehat{\phi_N}(n)| = 1$. Consequently, it follows from (2.22) that the limit in (2.12) exists and (2.12) holds. In addition, (2.6) shows that there exists a positive integer K_n such that $2^{k/2} |\widehat{\phi_k}(n)| > 0$ for all $k \geq K_n$. Therefore, we have $\lim_{N \to \infty} \prod_{r=k+1}^{N} |\widehat{a_r}(n)| = 2^{k/2} |\widehat{\phi_k}(n)| > 0$ for all $k \geq K_n$. In particular, $|\widehat{a_r}(n)| > 0$ for all $r > K_n$. Since (2.10) implies (2.9), we conclude that

$$0 < |\widehat{a_r}(n)| \leq 1, \quad r > K_n. \tag{2.23}$$

Consequently, it follows from $\lim_{N \to \infty} \prod_{r=k+1}^{N} |\widehat{a_r}(n)| > 0$ that

$$\sum_{r=k+1}^{\infty} -\ln |\widehat{a_r}(n)| < \infty, \quad k \geq K_n. \tag{2.24}$$

Using the simple fact that $1 - x \leq -\ln x$ for all $0 < x \leq 1$, we now deduce from (2.23) and (2.24) that

$$\sum_{r=k+1}^{\infty} \big|1 - |\widehat{a_r}(n)|\big| = \sum_{r=k+1}^{\infty} 1 - |\widehat{a_r}(n)| \leq \sum_{r=k+1}^{\infty} -\ln |\widehat{a_r}(n)| < \infty, \quad k \geq K_n.$$

Hence, (2.11) must be satisfied. ■

In light of the results in Theorem 2.2 and Proposition 2.2, we are ready to write down a necessary and sufficient condition for refinement masks that can be used to construct tight wavelet frames with the unitary extension principle.

Corollary 2.1. *Suppose that* $\widehat{a_{k+1}} \in \mathcal{S}(2^{k+1})$, $k \geq 0$, *satisfy* (2.19). *Then the following two conditions are equivalent.*

(i) *There exist refinable functions* ϕ_k *and wavelet masks* $\widehat{b_{k+1}^m} \in \mathcal{S}(2^{k+1})$, $k \geq 0$, $m = 1, 2, \ldots, \rho_k$, *which satisfy* (2.4) *and* (2.10), *and generate via* (2.5) *a tight wavelet frame* $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ *for* $L^2[0, 2\pi]$.

(ii) *Both* (2.9) *and* (2.11) *hold.*

Among all the conditions imposed on the refinement masks in Theorem 2.2, (2.11) seems to be rather demanding, even though it is a necessary condition for tight wavelet frames. Fortunately, this is not the case and the different scenarios depicted in the following proposition demonstrate that both (2.11) and (2.16) can be easily satisfied.

Proposition 2.3. *For* $k \geq 1$, *let* $\widehat{a_k} \in \mathcal{S}(2^k)$. *Then the following hold.*

(a) *Assume that* (2.19) *holds,* $\sup_{k \geq 1, n \in \mathbb{Z}} |\widehat{a_k}(n)| < \infty$, *and*

$$\sum_{k=1}^{\infty} 2^{-k} \deg(\widehat{a_k}) < \infty, \tag{2.25}$$

where for $k \geq 1$,

$$\deg(\widehat{a_k}) := \min\left\{L \in \{0, 1, \ldots, 2^{k-1} - 1\} : a_k(\ell) = 0 \text{ for } \ell \in \mathcal{R}_k \backslash \{-L, \ldots, L\}\right\} \tag{2.26}$$

with a_k *the inverse discrete Fourier transform of* $\widehat{a_k}$, *and the convention of setting* $\deg(\widehat{a_k}) := 2^{k-1}$ *if no* L *satisfies the criterion in* (2.26). *Then* (2.11) *and* (2.16) *hold.*

(b) *Assume that for every* $n \in \mathbb{Z}$, *there exist positive constants* C_n, K_n *and* α_n *for which*

$$\big|1 - |\widehat{a_k}(n)|\big| \leq C_n 2^{-\alpha_n k}, \quad k \geq K_n. \tag{2.27}$$

Then (2.11) *holds.*

(c) *Assume that for every* $n \in \mathbb{Z}$, *there exist positive constants* C_n, K_n *and* α_n *for which*

$$|1 - \widehat{a_k}(n)| \leq C_n 2^{-\alpha_n k}, \quad k \geq K_n. \tag{2.28}$$

Then (2.16) *holds.*

Proof. Suppose that the conditions in item (a) hold. We shall prove both (2.11) and (2.16). Since (2.16) implies (2.11), it suffices to establish (2.16).

By (2.25), we have $\lim_{k\to\infty} 2^{-k}\deg(\widehat{a_k}) = 0$. Therefore, there exists a positive integer K such that for every $k \geq K$, $\deg(\widehat{a_k}) < 2^{k-1}$. Consequently, $\{-\deg(\widehat{a_k}), \ldots, \deg(\widehat{a_k})\} \subseteq \mathcal{R}_k$ and we may express $\widehat{a_k} \in \mathcal{S}(2^k)$ as

$$\widehat{a_k}(j) = \sum_{\ell=-\deg(\widehat{a_k})}^{\deg(\widehat{a_k})} a_k(\ell) e^{-\frac{2\pi i \ell j}{2^k}}, \quad j \in \mathcal{R}_k.$$

Define the trigonometric polynomial

$$A_k(\xi) := \sum_{\ell=-\deg(\widehat{a_k})}^{\deg(\widehat{a_k})} a_k(\ell) e^{-i\ell\xi}, \quad \xi \in \mathbb{R}.$$

Therefore,

$$\widehat{a_k}(n) = A_k(\tfrac{2\pi n}{2^k}), \quad n \in \mathbb{Z}. \tag{2.29}$$

For any $\xi_0 \in \mathbb{R}$ and $n \in \mathbb{Z}$, we deduce that

$$\left|A_k(\xi_0) - A_k(\tfrac{2\pi n}{2^k})\right| = \left|\int_{\frac{2\pi n}{2^k}}^{\xi_0} A_k'(\xi)\,d\xi\right| \leq (\max_{0\leq\xi\leq 2\pi} |A_k'(\xi)|)|\xi_0 - \tfrac{2\pi n}{2^k}|. \tag{2.30}$$

In addition, Bernstein's inequality for trigonometric polynomials yields

$$\max_{0\leq\xi\leq 2\pi} |A_k'(\xi)| \leq \deg(\widehat{a_k}) \max_{0\leq\xi\leq 2\pi} |A_k(\xi)|.$$

Combining this with (2.29) and (2.30), we obtain

$$|A_k(\xi_0) - \widehat{a_k}(n)| \leq \deg(\widehat{a_k}) \left|\xi_0 - \tfrac{2\pi n}{2^k}\right| \max_{0\leq\xi\leq 2\pi} |A_k(\xi)|. \tag{2.31}$$

We shall now show that $\max_{0\leq\xi\leq 2\pi} |A_k(\xi)|$ is uniformly bounded over all $k \geq K$. For a fixed $k \geq K$, take $\xi_0 \in [0, 2\pi]$ to be a point for which $|A_k(\xi_0)| = \max_{0\leq\xi\leq 2\pi} |A_k(\xi)|$ and choose $n \in \mathbb{Z}$ such that $|\xi_0 - \frac{2\pi n}{2^k}| \leq \frac{2\pi}{2^k}$. Then it follows from (2.31) that

$$\left(1 - 2\pi 2^{-k}\deg(\widehat{a_k})\right) \max_{0\leq\xi\leq 2\pi} |A_k(\xi)| \leq |\widehat{a_k}(n)| \leq C := \sup_{\tau\geq 1, \nu\in\mathbb{Z}} |\widehat{a_\tau}(\nu)| < \infty.$$

Since $\lim_{k\to\infty} 2^{-k}\deg(\widehat{a_k}) = 0$ from (2.25), this implies that there exists a positive constant C' for which

$$\max_{0\leq\xi\leq 2\pi} |A_k(\xi)| \leq C', \quad k \geq K. \tag{2.32}$$

Finally, for each $k \geq K$, take $\xi_0 = 0$ and n an arbitrary integer in (2.31). Then using (2.19), (2.29), (2.31), and (2.32), we obtain

$$|1 - \widehat{a_k}(n)| \leq 2\pi C'|n|\, 2^{-k}\deg(\widehat{a_k}).$$

Therefore, we conclude that

$$\sum_{k=K}^{\infty} |1 - \widehat{a_k}(n)| \leq 2\pi C'|n| \sum_{k=K}^{\infty} 2^{-k}\deg(\widehat{a_k}) \leq 2\pi C'|n| \sum_{k=1}^{\infty} 2^{-k}\deg(\widehat{a_k}) < \infty$$

for every $n \in \mathbb{Z}$. This implies (2.16) and completes the proof of item (a).

For item (b), it is trivial to see that if (2.27) holds, then (2.11) holds since for each $n \in \mathbb{Z}$,

$$\sum_{k=K_n}^{\infty} \big|1 - |\widehat{a_k}(n)|\big| \le C_n \sum_{k=K_n}^{\infty} 2^{-\alpha_n k} = \frac{C_n 2^{\alpha_n(1-K_n)}}{2^{\alpha_n} - 1} < \infty.$$

Similarly, (2.28) trivially implies (2.16), giving item (c). ∎

Finally, we note that the two conditions (2.9) and (2.19) assumed in Theorem 2.2 imply that $\widehat{a_k}(2^{k-1}) = 0$. By the periodicity of $\widehat{a_k}$,

$$\widehat{a_k}(2^{k-1} + 2^k q) = 0, \quad q \in \mathbb{Z}. \tag{2.33}$$

Now for $k \ge 0$, if $\{\widehat{\phi_k}(n)\}_{n\in\mathbb{Z}}$ is defined by (2.15) via (2.13) and (2.14), then

$$|\widehat{\phi_k}(2^k p)| = 2^{-k/2}\delta_{p0}, \quad p \in \mathbb{Z}, \tag{2.34}$$

where δ_{p0} takes the value 1 if $p = 0$, and 0 otherwise. (In the next section, we will see that (2.34) leads to a basic assumption associated with approximation orders of tight wavelet frames.) To verify (2.34), we observe from (2.12) and (2.19) that $|\widehat{\phi_k}(0)| = 2^{-k/2}$. For $p \in \mathbb{Z}\backslash\{0\}$, if p is odd, by writing $p = 2q+1$ where $q \in \mathbb{Z}$, (2.33) implies that $\widehat{a_{k+1}}(2^k p) = \widehat{a_{k+1}}(2^k + 2^{k+1} q) = 0$ and so $\widehat{\phi_k}(2^k p) = 0$. On the other hand, if p is even, dividing by 2 repeatedly, we obtain $p = 2^\lambda(2q+1)$ for some positive integer λ and some $q \in \mathbb{Z}$. Similar to the case when p is odd, $\widehat{a_{k+\lambda+1}}(2^k p) = 0$ and therefore $\widehat{\phi_k}(2^k p) = 0$.

3. Frame approximation orders

For our study of approximation orders of truncated tight frame series, we begin with some basic notions. For $\nu \in \mathbb{R}$, let $H^\nu[0, 2\pi]$ be the Sobolev space of all 2π-periodic tempered distributions f such that $\|f\|^2_{H^\nu[0,2\pi]} := \sum_{n\in\mathbb{Z}} (1+n^2)^\nu |\widehat{f}(n)|^2 < \infty$. For $\nu \ge 0$, the Sobolev seminorm $|f|_{H^\nu[0,2\pi]}$ is defined by $|f|^2_{H^\nu[0,2\pi]} := \sum_{n\in\mathbb{Z}} |n|^{2\nu} |\widehat{f}(n)|^2$, where $f \in H^\nu[0, 2\pi]$. It is not difficult to see that for $\nu \ge 0$, $2^{\min\{1-\nu,0\}} \|f\|^2_{H^\nu[0,2\pi]} \le \|f\|_2^2 + |f|^2_{H^\nu[0,2\pi]} \le 2^{\max\{1-\nu,0\}} \|f\|^2_{H^\nu[0,2\pi]}$.

For $k \ge 1$, the *frame approximation operator* Q_k associated with the truncation of a tight wavelet frame $\{\phi_0\} \cup \{T_r^\ell \psi_r^m : r \ge 0,\ m = 1, 2, \ldots, \rho_r,\ \ell \in \mathcal{R}_r\}$ at level k is defined to be

$$Q_k(f) := \langle f, \phi_0\rangle \phi_0 + \sum_{r=0}^{k-1} \sum_{m=1}^{\rho_r} \sum_{\ell\in\mathcal{R}_r} \langle f, T_r^\ell \psi_r^m\rangle T_r^\ell \psi_r^m, \quad f \in L^2[0, 2\pi]. \tag{3.1}$$

While tight periodic wavelet frames were investigated in [22, 23], the approximation order of a truncated tight wavelet frame series of the form (3.1), or a truncated orthonormal wavelet series, has yet to be addressed. On the other hand, its counterpart for tight wavelet frames on the real line was already well studied in [16, 30] (see also [27] for the case of multiwavelet frames). Following [16, 27, 30] for the space $L^2(\mathbb{R})$, for $\nu \ge 0$, we say that a tight wavelet frame $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \ge 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ for $L^2[0, 2\pi]$ provides *frame approximation order* ν if there exist a positive constant C and a positive integer K such that for all $k \geq K$,

$$\|f - Q_k(f)\|_2 \leq C 2^{-k\nu} |f|_{H^\nu[0,2\pi]}, \quad f \in H^\nu[0, 2\pi]. \tag{3.2}$$

In addition, we say that a tight wavelet frame provides the *spectral frame approximation order* if it provides frame approximation order ν for every positive number ν.

Consider a sequence of refinable functions $\{\phi_k\}_{k\geq 0}$ in $L^2[0, 2\pi]$ satisfying (2.4) and (2.6). Suppose that $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ is a tight wavelet frame for $L^2[0, 2\pi]$ constructed via Theorem 2.1. In other words, for every $k \geq 0$, the $\rho_k \times 2$ matrices $M_k(j)$, $j \in \mathcal{R}_k$, in (2.7) satisfy (2.8). By applying the characterization on decomposition and reconstruction in [23, Theorem 2.1] (for the real line case, see [15, Theorem 2.2], [27, Section 3], as well as [16, 24, 27, 29] for more details), this is equivalent to

$$\sum_{\ell\in\mathcal{R}_{k+1}} \langle f, T_{k+1}^\ell \phi_{k+1}\rangle T_{k+1}^\ell \phi_{k+1} = \sum_{\ell\in\mathcal{R}_k} \langle f, T_k^\ell \phi_k\rangle T_k^\ell \phi_k + \sum_{m=1}^{\rho_k} \sum_{\ell\in\mathcal{R}_k} \langle f, T_k^\ell \psi_k^m\rangle T_k^\ell \psi_k^m, \quad f \in L^2[0, 2\pi]. \tag{3.3}$$

Following the ideas in [16], for $k \geq 0$, the *quasi-interpolation operator* P_k associated with the refinable function ϕ_k at level k is defined to be

$$P_k(f) := \sum_{\ell\in\mathcal{R}_k} \langle f, T_k^\ell \phi_k\rangle T_k^\ell \phi_k, \quad f \in L^2[0, 2\pi]. \tag{3.4}$$

Applying (3.3) repeatedly, we see that for $k \geq 1$,

$$\sum_{\ell\in\mathcal{R}_k} \langle f, T_k^\ell \phi_k\rangle T_k^\ell \phi_k = \langle f, \phi_0\rangle \phi_0 + \sum_{r=0}^{k-1} \sum_{m=1}^{\rho_r} \sum_{\ell\in\mathcal{R}_r} \langle f, T_r^\ell \psi_r^m\rangle T_r^\ell \psi_r^m, \quad f \in L^2[0, 2\pi], \tag{3.5}$$

that is,

$$P_k(f) = Q_k(f), \quad f \in L^2[0, 2\pi],$$

where Q_k is the frame approximation operator defined in (3.1). Hence the study of frame approximation orders provided by the tight wavelet frame $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \geq 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ can be conducted in terms of the quasi-interpolation operators P_k, $k \geq 1$. This will be the approach that we take here. The following lemma gives some basic properties of these operators.

Lemma 3.1. *For $k \geq 0$ and $\phi_k \in L^2[0, 2\pi]$, let P_k be as defined in* (3.4)*. Then for $f \in L^2[0, 2\pi]$, the Fourier coefficients of $P_k(f)$ are given by*

$$\widehat{P_k(f)}(n) = 2^k \widehat{\phi_k}(n) \sum_{p\in\mathbb{Z}} \widehat{f}(n + 2^k p) \overline{\widehat{\phi_k}(n + 2^k p)}, \quad n \in \mathbb{Z}. \tag{3.6}$$

Consequently, $P_k(1) = 1$ if and only if

$$\widehat{\phi_k}(0) \overline{\widehat{\phi_k}(2^k p)} = 2^{-k} \delta_{p0}, \quad p \in \mathbb{Z}. \tag{3.7}$$

Proof. By writing the Fourier series of ϕ_k as $\sum_{j\in\mathcal{R}_k}\sum_{p\in\mathbb{Z}}\widehat{\phi_k}(j+2^kp)e^{i(j+2^kp)\cdot}$, standard arguments in periodic wavelets (see for instance [12, Lemma 1.1]), which involve summing roots of unity, show that for $f\in L^2[0,2\pi]$,

$$\sum_{\ell\in\mathcal{R}_k}\langle f,T_k^\ell\phi_k\rangle T_k^\ell\phi_k = 2^k\sum_{j\in\mathcal{R}_k}\left(\sum_{p\in\mathbb{Z}}\widehat{f}(j+2^kp)\overline{\widehat{\phi_k}(j+2^kp)}\right)\sum_{q\in\mathbb{Z}}\widehat{\phi_k}(j+2^kq)e^{i(j+2^kq)\cdot}$$
$$= 2^k\sum_{n\in\mathbb{Z}}\widehat{\phi_k}(n)\left(\sum_{p\in\mathbb{Z}}\widehat{f}(n+2^kp)\overline{\widehat{\phi_k}(n+2^kp)}\right)e^{in\cdot}.$$

Then the Fourier coefficients of $P_k(f)$ in (3.4) are given by (3.6).

If f is the constant function 1, then its Fourier coefficients are given by $\widehat{f}(n)=\delta_{n0}$, $n\in\mathbb{Z}$. In this case, by (3.6),

$$P_k(1) = 2^k\sum_{p\in\mathbb{Z}}\widehat{\phi_k}(-2^kp)\overline{\widehat{\phi_k}(0)}e^{-i2^kp\cdot},$$

and therefore $P_k(1)=1$ if and only if (3.7) holds. ∎

As noted at the end of Section 2, the conditions imposed in Theorem 2.2 imply that for each $k\ge 0$, (2.34) holds which leads to (3.7). The condition (3.7), which amounts to the operator P_k reproducing constant functions (that is, it has approximation order 1), will be a basic assumption that we make on refinable functions. The following theorem is about general approximation order of P_k. The corresponding result for $L^2(\mathbb{R})$ was given in [31, 32]. However, the result for $L^2[0,2\pi]$ cannot be derived by simply applying [32, Theorem 2.1]. Nevertheless, some ideas from the proof of [32, Theorem 2.1] motivate our arguments here.

Theorem 3.1. *For $k\ge 1$ and $\phi_k\in L^2[0,2\pi]$, let P_k be as defined in* (3.4) *and suppose that* (3.7) *holds. Then for every $\nu\ge 0$,*

$$\|P_k(f)-f\|_2^2 \le \max\{2C_{k,1}+4C_{k,2},\, 2C_{k,3}+4C_{k,4}+2^{1+2\nu}\}\,2^{-2k\nu}|f|^2_{H^\nu[0,2\pi]},\quad f\in H^\nu[0,2\pi], \tag{3.8}$$

where

$$C_{k,1} := \max_{j\in\mathcal{R}_k\setminus\{0\}} |2^{-k}j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)^2, \tag{3.9}$$

$$C_{k,2} := \max_{j\in\mathcal{R}_k\setminus\{0\}} 2^{2k}|2^{-k}j|^{-2\nu}|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{Z}\setminus\{0\}}|\widehat{\phi_k}(j+2^kq)|^2, \tag{3.10}$$

$$C_{k,3} := \max_{j\in\mathcal{R}_k} 2^{2k}|\widehat{\phi_k}(j)|^2\sum_{p\in\mathbb{Z}\setminus\{0\}}|2^{-k}j+p|^{-2\nu}|\widehat{\phi_k}(j+2^kp)|^2, \tag{3.11}$$

$$C_{k,4} := \max_{j\in\mathcal{R}_k} 2^{2k}\sum_{q\in\mathbb{Z}\setminus\{0\}}|\widehat{\phi_k}(j+2^kq)|^2\sum_{p\in\mathbb{Z}\setminus\{0\}}|2^{-k}j+p|^{-2\nu}|\widehat{\phi_k}(j+2^kp)|^2. \tag{3.12}$$

Proof. For $\nu \geq 0$ and $f \in H^\nu[0,2\pi]$, it follows from (3.6) of Lemma 3.1 that for $n \in \mathbb{Z}$,

$$\widehat{P_k(f)}(n) - \widehat{f}(n) = 2^k \widehat{\phi_k}(n) \sum_{p\in\mathbb{Z}} \widehat{f}(n+2^k p)\overline{\widehat{\phi_k}(n+2^k p)} - \widehat{f}(n). \tag{3.13}$$

Consider

$$\|P_k(f) - f\|_2^2 = \sum_{j\in\mathcal{R}_k} |\widehat{P_k(f)}(j) - \widehat{f}(j)|^2 + \sum_{j\in\mathcal{R}_k}\sum_{q\in\mathbb{Z}\setminus\{0\}} |\widehat{P_k(f)}(j+2^k q) - \widehat{f}(j+2^k q)|^2. \tag{3.14}$$

For the first sum on the right-hand side of (3.14), observe that by (3.13), for $j \in \mathcal{R}_k$,

$$\widehat{P_k(f)}(j) - \widehat{f}(j) = \left(2^k|\widehat{\phi_k}(j)|^2 - 1\right)\widehat{f}(j) + 2^k\widehat{\phi_k}(j) \sum_{p\in\mathbb{Z}\setminus\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)},$$

and so

$$\begin{aligned}\sum_{j\in\mathcal{R}_k} |\widehat{P_k(f)}(j) - \widehat{f}(j)|^2 \leq 2\sum_{j\in\mathcal{R}_k}\left(1 - 2^k|\widehat{\phi_k}(j)|^2\right)^2 |\widehat{f}(j)|^2 + \\ 2\sum_{j\in\mathcal{R}_k} 2^{2k}|\widehat{\phi_k}(j)|^2 \left|\sum_{p\in\mathbb{Z}\setminus\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)}\right|^2.\end{aligned} \tag{3.15}$$

Now, using (3.7),

$$\begin{aligned} 2\sum_{j\in\mathcal{R}_k}\left(1 - 2^k|\widehat{\phi_k}(j)|^2\right)^2 |\widehat{f}(j)|^2 &= 2\sum_{j\in\mathcal{R}_k\setminus\{0\}} |j|^{-2\nu}\left(1 - 2^k|\widehat{\phi_k}(j)|^2\right)^2 |j|^{2\nu}|\widehat{f}(j)|^2 \\ &\leq 2C_{k,1}\, 2^{-2k\nu}\sum_{j\in\mathcal{R}_k} |j|^{2\nu}|\widehat{f}(j)|^2, \end{aligned} \tag{3.16}$$

where $C_{k,1}$ is as defined in (3.9). Next, employing the Cauchy-Schwarz inequality,

$$\begin{aligned} &2\sum_{j\in\mathcal{R}_k} 2^{2k}|\widehat{\phi_k}(j)|^2 \left|\sum_{p\in\mathbb{Z}\setminus\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)}\right|^2 \\ &\leq 2\sum_{j\in\mathcal{R}_k} 2^{2k}|\widehat{\phi_k}(j)|^2 \left(\sum_{p\in\mathbb{Z}\setminus\{0\}} |j+2^k p|^{2\nu}|\widehat{f}(j+2^k p)|^2\right) \times \\ &\qquad \left(2^{-2k\nu}\sum_{p\in\mathbb{Z}\setminus\{0\}} |2^{-k}j + p|^{-2\nu}|\widehat{\phi_k}(j+2^k p)|^2\right) \\ &\leq 2C_{k,3}\, 2^{-2k\nu}\sum_{j\in\mathcal{R}_k}\sum_{p\in\mathbb{Z}\setminus\{0\}} |j+2^k p|^{2\nu}|\widehat{f}(j+2^k p)|^2, \end{aligned} \tag{3.17}$$

where $C_{k,3}$ is given by (3.11).

For the multiple sums on the right-hand side of (3.14), again using (3.13), we have for $j \in \mathcal{R}_k$ and $q \in \mathbb{Z}\backslash\{0\}$,

$$\widehat{P_k(f)}(j+2^k q) - \widehat{f}(j+2^k q) = 2^k \widehat{\phi_k}(j+2^k q) \sum_{p\in\mathbb{Z}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)} - \widehat{f}(j+2^k q),$$

and thus

$$\begin{aligned}
&\sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{P_k(f)}(j+2^k q) - \widehat{f}(j+2^k q)|^2 \\
&\le 2\sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^k q)|^2 + 2\sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2 \left|\sum_{p\in\mathbb{Z}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)}\right|^2 \\
&\le 2\sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^k q)|^2 + 4\sum_{j\in\mathcal{R}_k} |\widehat{f}(j)|^2|\widehat{\phi_k}(j)|^2 \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2 + \\
&\quad 4\sum_{j\in\mathcal{R}_k} \left|\sum_{p\in\mathbb{Z}\backslash\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)}\right|^2 \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2. \qquad (3.18)
\end{aligned}$$

Note that $|2^{-k}j+q| \ge 2^{-1}$ for all $j \in \mathcal{R}_k$ and $q \in \mathbb{Z}\backslash\{0\}$. Therefore, by $\nu \ge 0$, it is straightforward to see that

$$2\sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^k q)|^2 \le 2^{1+2\nu}2^{-2k\nu} \sum_{j\in\mathcal{R}_k} \sum_{q\in\mathbb{Z}\backslash\{0\}} |j+2^k q|^{2\nu}|\widehat{f}(j+2^k q)|^2. \qquad (3.19)$$

Using (3.7),

$$\begin{aligned}
&4\sum_{j\in\mathcal{R}_k} |\widehat{f}(j)|^2|\widehat{\phi_k}(j)|^2 \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2 \\
&= 4\sum_{j\in\mathcal{R}_k\backslash\{0\}} |j|^{2\nu}|\widehat{f}(j)|^2\,|j|^{-2\nu}|\widehat{\phi_k}(j)|^2 \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2 \\
&\le 4C_{k,2}\,2^{-2k\nu} \sum_{j\in\mathcal{R}_k\backslash\{0\}} |j|^{2\nu}|\widehat{f}(j)|^2, \qquad (3.20)
\end{aligned}$$

where $C_{k,2}$ is given by (3.10). By similar arguments as those leading to (3.17), with $C_{k,4}$ as in (3.12),

$$\begin{aligned}
&4\sum_{j\in\mathcal{R}_k} \left|\sum_{p\in\mathbb{Z}\backslash\{0\}} \widehat{f}(j+2^k p)\overline{\widehat{\phi_k}(j+2^k p)}\right|^2 \sum_{q\in\mathbb{Z}\backslash\{0\}} 2^{2k}|\widehat{\phi_k}(j+2^k q)|^2 \\
&\le 4C_{k,4}\,2^{-2k\nu} \sum_{j\in\mathcal{R}_k} \sum_{p\in\mathbb{Z}\backslash\{0\}} |j+2^k p|^{2\nu}|\widehat{f}(j+2^k p)|^2. \qquad (3.21)
\end{aligned}$$

To complete the proof, we substitute (3.16) and (3.17) into (3.15), and (3.19)–(3.21) into (3.18). Then combining the resulting inequalities gives

$$\|P_k(f)-f\|_2^2 \le (2C_{k,1}+4C_{k,2})2^{-2k\nu}\sum_{j\in\mathcal{R}_k} j^{2\nu}|\widehat{f}(j)|^2 + (2C_{k,3}+4C_{k,4}+2^{1+2\nu})2^{-2k\nu}\sum_{j\in\mathcal{R}_k}\sum_{p\in\mathbb{Z}\setminus\{0\}}|j+2^kp|^{2\nu}|\widehat{f}(j+2^kp)|^2,$$

which implies (3.8). ∎

Theorem 3.1 shares some similarity to the real line case in [32, Theorem 2.1] for $L^2(\mathbb{R})$. In Theorem 3.1 for the periodic case, we used ϕ_k for both the primal and dual parts in the quasi-interpolation operator P_k. The set $\mathcal{R}_k$ plays the role of the interval $[-\pi,\pi]$ in [32]. In the frequency domain, a periodic function in $L^2[0,2\pi]$ becomes a sequence (of Fourier coefficients) on $\mathbb{Z}$ and this makes the proof in the periodic case simpler. As mentioned earlier, while our proof for Theorem 3.1 adapts some ideas from the proof of [32, Theorem 2.1], the result cannot be directly deduced from the corresponding result for the $L^2(\mathbb{R})$ case.

Based on Theorem 3.1, we now present a necessary and sufficient condition on the frame approximation order for smooth functions provided by the tight wavelet frames of Theorem 2.2.

Theorem 3.2. *Suppose that $\widehat{a_{k+1}}$ and $\widehat{b^m_{k+1}}$, $k\ge 0$, $m=1,2,\dots,\rho_k$, are refinement and wavelet masks satisfying the hypothesis of Theorem 2.2. Let ϕ_k and ψ_k^m, $k\ge 0$, $m=1,2,\dots,\rho_k$, be refinable functions and wavelets generated as in Theorem 2.2. For $\nu\ge 0$, the tight wavelet frame $\{\phi_0\}\cup\{T_k^\ell\psi_k^m : k\ge 0,\ m=1,2,\dots,\rho_k,\ \ell\in\mathcal{R}_k\}$ provides frame approximation order ν as in (3.2), if and only if, there exist a positive constant C, independent of k, and a positive integer K such that*

$$\max\{C_{k,1},C_{k,2}\}\le C,\quad k\ge K, \tag{3.22}$$

where $C_{k,1}$ and $C_{k,2}$ are defined in (3.9) and (3.10) respectively. Moreover, (3.22) is satisfied when there exist positive constants $\epsilon,\tilde{C},K$ with $0<\epsilon\le 1/2$ such that

$$2^{2k\nu}\max_{j\in(\mathcal{R}_k\cap(-2^k\epsilon,2^k\epsilon])\setminus\{0\}}|j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)\le\tilde{C},\quad k\ge K. \tag{3.23}$$

Proof. We start with the sufficiency. First, as a consequence of the hypothesis of Theorem 2.2, it follows from Lemma 2.1 and Proposition 2.1 that for every $k\ge K$, (2.18) holds, which leads to $2^k|\widehat{\phi_k}(j)|^2\le 1$ for all $j\in\mathcal{R}_k$. By $\nu\ge 0$, we have $|2^{-k}j+p|^{-2\nu}\le 2^{2\nu}$ for all $j\in\mathcal{R}_k$ and $p\in\mathbb{Z}\setminus\{0\}$ as $|2^{-k}j+p|\ge 2^{-1}$. Consequently, by (2.18), for every $j\in\mathcal{R}_k$,

$$2^{2k}|\widehat{\phi_k}(j)|^2\sum_{p\in\mathbb{Z}\setminus\{0\}}|2^{-k}j+p|^{-2\nu}|\widehat{\phi_k}(j+2^kp)|^2\le 2^{2\nu}\left(2^k|\widehat{\phi_k}(j)|^2\right)\left(2^k\sum_{p\in\mathbb{Z}}|\widehat{\phi_k}(j+2^kp)|^2\right)\le 2^{2\nu}$$

and

$$2^{2k}\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2\sum_{p\in\mathbb{Z}\backslash\{0\}}|2^{-k}j+p|^{-2\nu}|\widehat{\phi_k}(j+2^kp)|^2\le 2^{2\nu}\left(2^k\sum_{p\in\mathbb{Z}}|\widehat{\phi_k}(j+2^kp)|^2\right)^2\le 2^{2\nu}.$$

Therefore, we have $C_{k,3}\le 2^{2\nu}$ and $C_{k,4}\le 2^{2\nu}$, where $C_{k,3}$ and $C_{k,4}$ are defined in (3.11) and (3.12) respectively. Now by our assumption in (3.22), it follows from Theorem 3.1 that

$$\|P_k(f)-f\|_2^2\le\max\{6C, 6\times 2^{2\nu}+2^{1+2\nu}\}2^{-2k\nu}|f|^2_{H^\nu[0,2\pi]},\quad f\in H^\nu[0,2\pi],\ k\ge K.$$

Since $Q_k=P_k$ for all $k\ge 1$, we conclude that the tight wavelet frame $\{\phi_0\}\cup\{T_k^\ell\psi_k^m: k\ge 0,\ m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{R}_k\}$ provides frame approximation order ν.

For the necessity, suppose that there are positive constants C and K such that

$$\|Q_k(f)-f\|_2^2=\|P_k(f)-f\|_2^2\le C2^{-2k\nu}|f|^2_{H^\nu[0,2\pi]},\quad f\in H^\nu[0,2\pi],\ k\ge K. \tag{3.24}$$

For $k\ge K$ and a fixed $j\in\mathcal{R}_k$, consider $f\in L^2[0,2\pi]$ such that $\widehat{f}(n)=\delta_{nj}$ for $n\in\mathbb{Z}$. Then by (3.13), for any $j'\in\mathcal{R}_k$ and $q\in\mathbb{Z}$, we have

$$\begin{aligned}\widehat{P_k(f)}(j'+2^kq)&=2^k\widehat{\phi_k}(j'+2^kq)\sum_{p\in\mathbb{Z}}\widehat{f}(j'+2^kq+2^kp)\overline{\widehat{\phi_k}(j'+2^kq+2^kp)}\\&=2^k\widehat{\phi_k}(j'+2^kq)\overline{\widehat{\phi_k}(j')}\widehat{f}(j').\end{aligned}$$

Now by (3.14), we obtain

$$\begin{aligned}\|P_k(f)-f\|_2^2=&\sum_{j'\in\mathcal{R}_k}\left(1-2^k|\widehat{\phi_k}(j')|^2\right)^2|\widehat{f}(j')|^2+\\&\sum_{j'\in\mathcal{R}_k}|\widehat{f}(j')|^2 2^{2k}|\widehat{\phi_k}(j')|^2\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j'+2^kq)|^2\\=&\left(1-2^k|\widehat{\phi_k}(j)|^2\right)^2+2^{2k}|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2.\end{aligned}$$

Moreover, $|f|^2_{H^\nu[0,2\pi]}=|j|^{2\nu}$. Hence, we deduce from (3.24) and the above that

$$\left(1-2^k|\widehat{\phi_k}(j)|^2\right)^2+2^{2k}|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2\le C2^{-2k\nu}|j|^{2\nu},\quad j\in\mathcal{R}_k,\ k\ge K. \tag{3.25}$$

Consequently, by the definitions of $C_{k,1}$ and $C_{k,2}$ in (3.9) and (3.10), (3.22) holds.

Now we prove that if (3.23) holds, then we must have (3.22). As noted in the beginning of the proof, (2.18) is satisfied, and so for every $j\in\mathcal{R}_k$, $2^k|\widehat{\phi_k}(j)|^2\le 1$ and

$$2^k\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2\le 1-2^k|\widehat{\phi_k}(j)|^2\le 1.$$

For $k \geq K$, denote $\mathcal{R}_{k,\epsilon} := \mathcal{R}_k \cap (-2^k\epsilon, 2^k\epsilon]$. It is evident that for $j \in \mathcal{R}_k \backslash \mathcal{R}_{k,\epsilon}$, we have $|j|^{-2\nu} \leq (2^k\epsilon)^{-2\nu}$ and therefore, by $0 \leq 1 - 2^k|\widehat{\phi_k}(j)|^2 \leq 1$,

$$\max_{j\in\mathcal{R}_k\backslash\mathcal{R}_{k,\epsilon}} |2^{-k}j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)^2 \leq 2^{2k\nu}\max_{j\in\mathcal{R}_k\backslash\mathcal{R}_{k,\epsilon}} |j|^{-2\nu} \leq 2^{2k\nu}2^{-2k\nu}\epsilon^{-2\nu} = \epsilon^{-2\nu}.$$

Noting that $|2^{-k}j|^{-2\nu} \geq 1$ for all $j \in \mathcal{R}_k\backslash\{0\}$ and observing that $\mathcal{R}_{k,\epsilon} \subseteq \mathcal{R}_k$, we have

$$\max_{j\in\mathcal{R}_{k,\epsilon}\backslash\{0\}} |2^{-k}j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)^2 \leq \left(2^{2k\nu}\max_{j\in\mathcal{R}_{k,\epsilon}\backslash\{0\}} |j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)\right)^2 \leq \tilde{C}^2,$$

where we used (3.23) in the last inequality. Hence, $C_{k,1} \leq \max\{\epsilon^{-2\nu}, \tilde{C}^2\} < \infty$. In addition, by (2.18) and $2^k|\widehat{\phi_k}(j)|^2 \leq 1$ for all $j \in \mathcal{R}_k\backslash\mathcal{R}_{k,\epsilon}$, we have

$$\max_{j\in\mathcal{R}_k\backslash\mathcal{R}_{k,\epsilon}} 2^{2k}|2^{-k}j|^{-2\nu}|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2 \leq 2^{2k\nu}\max_{j\in\mathcal{R}_k\backslash\mathcal{R}_{k,\epsilon}}|j|^{-2\nu} \leq \epsilon^{-2\nu}.$$

By $2^k\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2 \leq 1 - 2^k|\widehat{\phi_k}(j)|^2$ for all $j \in \mathcal{R}_{k,\epsilon}\backslash\{0\}$, we deduce that

$$\begin{aligned}\max_{j\in\mathcal{R}_{k,\epsilon}\backslash\{0\}} 2^{2k}|2^{-k}j|^{-2\nu}|\widehat{\phi_k}(j)|^2\sum_{q\in\mathbb{Z}\backslash\{0\}}|\widehat{\phi_k}(j+2^kq)|^2 &\leq 2^{2k\nu}\max_{j\in\mathcal{R}_{k,\epsilon}\backslash\{0\}}|j|^{-2\nu}\left(1-2^k|\widehat{\phi_k}(j)|^2\right)\\ &\leq \tilde{C},\end{aligned}$$

where we again used (3.23) in the last inequality. Thus $C_{k,2} \leq \max\{\epsilon^{-2\nu}, \tilde{C}\}$. In conclusion, (3.22) holds with $C := \max\{\tilde{C}, \tilde{C}^2, \epsilon^{-2\nu}\}$. ■

Under the hypothesis of Theorem 2.2, in particular (2.9), (2.19) and (2.20), it follows from the proof of Proposition 2.1 that (2.21) is satisfied with $C_n = 1$ for all $n \in \mathbb{Z}$. That is, for every $k \geq 0$, we have

$$\left|1-2^k|\widehat{\phi_k}(n)|^2\right| \leq \sum_{\tau=k+1}^{\infty}\left|1-|\widehat{a_\tau}(n)|^2\right|, \quad n \in \mathbb{Z}.$$

Thus whenever there exist positive constants ϵ, C', K with $\epsilon \leq 1/2$ for which

$$2^{2k\nu}\max_{j\in(\mathcal{R}_k\cap(-2^k\epsilon,2^k\epsilon])\backslash\{0\}}|j|^{-2\nu}\left(\sum_{\tau=k+1}^{\infty}\left|1-|\widehat{a_\tau}(j)|^2\right|\right) \leq C', \quad k \geq K, \tag{3.26}$$

(3.23) holds. This gives the following result on frame approximation order in terms of refinement masks.

Corollary 3.1. *Let $\widehat{a_{k+1}}$ and $\widehat{b_{k+1}^m}$, $k \geq 0$, $m = 1, 2, \dots, \rho_k$, be refinement and wavelet masks satisfying the hypothesis of Theorem 2.2, and ϕ_k and ψ_k^m, $k \geq 0$, $m = 1, 2, \dots, \rho_k$, their corresponding refinable functions and wavelets. For $\nu \geq 0$, assume that there exist positive constants ϵ, C', K with $\epsilon \leq 1/2$ for which (3.26) holds. Then the tight wavelet frame $\{\phi_0\} \cup \{T_k^\ell\psi_k^m : k \geq 0,\ m = 1, 2, \dots, \rho_k,\ \ell \in \mathcal{R}_k\}$ provides frame approximation order ν.*

The frame approximation order describes the approximation power to smooth functions for the truncated frame series. It is also related to vanishing moments and sparse representations of locally smooth functions, which will be addressed in the next section.

4. Vanishing moments

We first recall the notion of vanishing moments for stationary wavelets on the real line to motivate an analogous concept for periodic wavelets. For a compactly supported function $\psi \in L^2(\mathbb{R})$, we say that ψ has ν vanishing moments (for a nonnegative integer ν) if $\int_{\mathbb{R}} \psi(x)x^\kappa dx = 0$ for all $\kappa = 0, 1, \ldots, \nu - 1$. In the frequency domain, this is equivalent to saying that $\widehat{\psi}(0) = \widehat{\psi}'(0) = \cdots = \widehat{\psi}^{(\nu-1)}(0) = 0$; that is, all the derivatives of $\widehat{\psi}$, the Fourier transform of ψ, vanish at 0 for orders up to $\nu - 1$. For a smooth function f that belongs to C^ν near a fixed point x_0 in $\mathbb{R}$, using the Taylor expansion of f at x_0, one can easily deduce that

$$|\langle f, 2^{k/2}\psi(2^k \cdot -\ell_k)\rangle|^2 = O(2^{-(2\nu+1)k})$$

as $\lim_{k\to\infty} 2^{-k}\ell_k = x_0$. Hence, vanishing moments give sparsity of the coefficients of a tight wavelet frame for locally smooth functions, or more precisely, for piecewise smooth functions. This sparsity leads to the power of the sparse approximation of piecewise smooth functions through a nonlinear approximation scheme. For a given wavelet frame, while its approximation order describes the approximation power to globally smooth functions, its vanishing moments reflect the approximation power to piecewise smooth functions. Furthermore, these two concepts are closely related, as shown in [16].

The above standard notion of vanishing moments for stationary tight wavelet frames on the real line is very simple and can be easily checked by finding the order of zeros of the function $\widehat{\psi}$ at the origin. As shown in [16, 27], for stationary tight wavelet frames on the real line, there is a close relation between the order of vanishing moments and the frame approximation order. However, the situation is quite different for nonstationary settings and a straightforward generalization is not feasible. In fact, the nonstationary tight wavelet frame on the real line constructed in [30, Theorem 1.3] is derived from the up-functions using the masks for B-splines and all its generating wavelet functions ψ_k^1, ψ_k^2, ψ_k^3 have the property $\widehat{\psi_k^1}(0) = \widehat{\psi_k^2}(0) = \widehat{\psi_k^3}(0) = 0$ for all k. In other words, all these nonstationary wavelet generators have at least one "vanishing moment" in the classical sense. However, such a nonstationary tight wavelet frame, as demonstrated in [30, Theorem 1.3], does not have any frame approximation order. This implies that the relation between the classical definition of vanishing moments, as defined by the property of polynomial cancellation, and frame approximation order no longer applies to nonstationary tight wavelet frames on the real line.

There are even more difficulties in extending the notion of vanishing moments for stationary wavelets on the real line to functions in $L^2[0, 2\pi]$. In addition to the nonstationary setting of periodic wavelets, in the frequency domain, $\widehat{\psi}'(0)$ does not make sense since the Fourier series of $\psi \in L^2[0,2\pi]$ is a discrete sequence $\{\widehat{\psi}(n)\}_{n\in\mathbb{Z}}$. On the other hand, recall that in the time domain, the notion of vanishing moments for stationary wavelets is simply to annihilate algebraic polynomials up to certain order so that $\int_{\mathbb{R}} \psi(x)x^\kappa dx = 0$ for all $\kappa = 0, 1, \dots, \nu-1$. Therefore, a very appealing first approach is to require that the periodic wavelet functions annihilate some natural basis functions. Since polynomials x^κ are not naturally included in $L^2[0,2\pi]$, it is quite tempting and also seems reasonable to replace the role of polynomials by trigonometric polynomials. More precisely, the corresponding notion for $\psi \in L^2[0,2\pi]$ could look like: $\int_0^{2\pi} \psi(x)e^{-i\kappa x}dx = 0$ for all $\kappa = 0, 1, \dots, \nu-1$. Unfortunately, such a notion cannot be applicable for several reasons. First of all, one of the main features of wavelets and wavelet frames is the scaling property. For algebraic polynomials, after a scaling, the space Π_κ of all polynomials of degree no more than κ remains the same. More precisely, if $p \in \Pi_\kappa$, then $p(2\cdot)$, $p(2^{-1}\cdot)$ also belong to Π_κ. However, denoting $\mathrm{TP}_\kappa := \{1, e^{\pm i\xi}, \dots, e^{\pm i\kappa\xi}\}$, it is straightforward to see that the linear space TP_κ is no longer invariant under scaling. Secondly, even if the wavelet generators of a tight periodic wavelet frame can indeed make all the elements in TP_κ vanished, one cannot simply expect that such a notion of vanishing moments will lead to frame approximation order. In other words, similar to the nonstationary real line case, there could be problems in preserving the desired connection between vanishing moments and frame approximation orders.

To the best of our knowledge, no notion of vanishing moments for functions in $L^2[0,2\pi]$ exists in the literature. In this section, we shall introduce a form of vanishing moments for periodic functions in $L^2[0,2\pi]$ with the goal of having a close relation to frame approximation orders and sparse representations of piecewise smooth functions.

To motivate our formulation, consider a compactly supported function $\psi \in L^2(\mathbb{R})$. Note that there is a positive integer K such that all the supports of $2^{k/2}\psi(2^k\cdot)$ are contained inside $[-\pi,\pi]$ for $k \ge K$. Therefore, we can regard them as restrictions of 2π-periodic functions to $[-\pi,\pi]$ and denote $\psi_k^{per} \in L^2[0,2\pi]$ such that $\psi_k^{per}(x) := 2^{k/2}\psi(2^k x)$ for $x \in (-\pi,\pi]$. It is a simple calculation to show that $\widehat{\psi_k^{per}}(n) = \frac{2^{-k/2}}{2\pi}\widehat{\psi}(2^{-k}n)$ for $n \in \mathbb{Z}$. We now show that ψ has ν vanishing moments if and only if

$$\widehat{\psi_k^{per}}(0) = 0 \quad \text{and} \quad \sup_{j\in\mathcal{R}_k\backslash\{0\}} |j|^{-2\nu}2^k|\widehat{\psi_k^{per}}(j)|^2 \le C2^{-2k\nu}, \quad k \ge K, \tag{4.1}$$

where C is a positive constant. Indeed, since ψ has ν vanishing moments, there exists a positive constant C_0 such that $|\widehat{\psi}(\xi)| \le C_0|\xi|^\nu$ for all $\xi \in [-1/2,1/2]$. (Here we have used the fact that ψ is a compactly supported tempered distribution and therefore $\widehat{\psi}$ is an analytic function.) Consequently, by $\widehat{\psi_k^{per}}(n) = \frac{2^{-k/2}}{2\pi}\widehat{\psi}(2^{-k}n)$ for $n \in \mathbb{Z}$, we conclude that $\widehat{\psi_k^{per}}(0) = 0$, and for $j \in \mathcal{R}_k\backslash\{0\}$,

$$|j|^{-2\nu}2^k|\widehat{\psi_k^{per}}(j)|^2 = |j|^{-2\nu}\frac{1}{(2\pi)^2}|\widehat{\psi}(2^{-k}j)|^2 \le |j|^{-2\nu}\frac{1}{(2\pi)^2}C_0^2|2^{-k}j|^{2\nu} = \frac{C_0^2}{(2\pi)^2}2^{-2k\nu}.$$

Thus, (4.1) holds with $C := \frac{C_0^2}{(2\pi)^2}$.

Conversely, if (4.1) holds, then we have $\widehat{\psi}(0) = 0$ and for $k \ge K$,

$$C 2^{-2k\nu} \ge |j|^{-2\nu} 2^k |\widehat{\psi_k^{per}}(j)|^2 = |j|^{-2\nu} \frac{1}{(2\pi)^2} |\widehat{\psi}(2^{-k}j)|^2, \quad j \in \mathcal{R}_k \backslash \{0\}.$$

In other words,

$$|\widehat{\psi}(2^{-k}j)|^2 \le (2\pi)^2 C |2^{-k}j|^{2\nu}, \quad j \in \mathcal{R}_k, \ k \ge K.$$

Since $\{2^{-k}\mathcal{R}_k : k \ge K\}$ is dense in $[-1/2, 1/2]$ and $\widehat{\psi}$ is continuous, we must have $|\widehat{\psi}(\xi)| \le 2\pi\sqrt{C}|\xi|^\nu$ for all $\xi \in [-1/2, 1/2]$. That is, ψ, as a function on the real line, must have ν vanishing moments.

Motivated by (4.1), we now introduce a notion of vanishing moments for 2π-periodic functions and study its relation to frame approximation order. Let $\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k \ge 0}$ be a sequence of functions in $L^2[0, 2\pi]$. We say that $\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k \ge 0}$ has ν *vanishing moments* if there exist positive constants C and K, independent of k and j, such that

$$\widehat{\psi_k^1}(0) = \cdots = \widehat{\psi_k^{\rho_k}}(0) = 0 \quad \text{and} \quad \max_{j \in \mathcal{R}_k \backslash \{0\}} \sum_{m=1}^{\rho_k} |j|^{-2\nu} 2^k |\widehat{\psi_k^m}(j)|^2 \le C 2^{-2k\nu}, \quad k \ge K. \quad (4.2)$$

While the number of vanishing moments ν for the real line case is always a nonnegative integer, our formulation here for the periodic case allows ν to be any nonnegative real number. We should emphasize that here we are only using the term "vanishing moments" to describe the property (4.2), instead of providing a definition that is of similar form to the classical stationary case over the real line. As to be seen in due course, the definition (4.2) can be related to frame approximation order and sparse representations of piecewise smooth functions in $L^2[0, 2\pi]$, which are two desirable consequences of traditional vanishing moments.

The definition (4.2) has an equivalent compact form of

$$\sum_{m=1}^{\rho_k} 2^k |\widehat{\psi_k^m}(j)|^2 \le C |2^{-k}j|^{2\nu}, \quad j \in \mathcal{R}_k, \ k \ge K. \quad (4.3)$$

From (4.3), as $|2^{-k}j| \le 1/2$ for all $j \in \mathcal{R}_k$, whenever the vanishing moment condition (4.2) holds for some positive number ν, it holds for every positive number μ no greater than ν.

In Theorem 2.2, we have a general procedure via refinement and wavelet masks to construct tight wavelet frames for $L^2[0, 2\pi]$. Theorem 3.2 characterizes the frame approximation order provided by such a tight wavelet frame. We shall now discuss the intrinsic relationship between frame approximation order and vanishing moments.

Theorem 4.1. *Let ϕ_k and ψ_k^m, $k \ge 0$, $m = 1, 2, \dots, \rho_k$, be refinable functions and wavelets generated as in Theorem 2.2 such that the tight wavelet frame $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \ge 0, \ m = 1, 2, \dots, \rho_k, \ \ell \in \mathcal{R}_k\}$ provides frame approximation order $\nu > 0$. Then the sequence $\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k \ge 0}$ has at least $\nu/2$ vanishing moments. Conversely, if $\{\psi_k^m : m =$*

$1,2,\ldots,\rho_k\}_{k\ge 0}$ *obtained from Theorem 2.2 has* ν *vanishing moments, then the tight wavelet frame* $\{\phi_0\}\cup\{T_k^\ell\psi_k^m : k\ge 0,\ m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{R}_k\}$ *provides frame approximation order at least* ν*.*

Proof. It follows from (2.1) and (3.5) that for $k\ge 0$,

$$\sum_{\ell\in\mathcal{R}_k}|\langle f,T_k^\ell\phi_k\rangle|^2+\sum_{r=k}^{\infty}\sum_{m=1}^{\rho_r}\sum_{\ell\in\mathcal{R}_r}|\langle f,T_r^\ell\psi_r^m\rangle|^2=\|f\|_2^2,\quad f\in L^2[0,2\pi]. \tag{4.4}$$

Note that for $r\ge k$ and $m=1,2,\ldots,\rho_r$,

$$\sum_{\ell\in\mathcal{R}_r}|\langle f,T_r^\ell\psi_r^m\rangle|^2=\sum_{n\in\mathbb{Z}}\sum_{p\in\mathbb{Z}}2^r\overline{\widehat{f}(n)}\widehat{f}(n+2^rp)\widehat{\psi_r^m}(n)\overline{\widehat{\psi_r^m}(n+2^rp)}. \tag{4.5}$$

For a fixed $j\in\mathbb{Z}$, consider $f\in L^2[0,2\pi]$ such that $\widehat{f}(n)=\delta_{nj}$ for $n\in\mathbb{Z}$. Then the identity in (4.5) implies that $\sum_{\ell\in\mathcal{R}_r}|\langle f,T_r^\ell\psi_r^m\rangle|^2=2^r|\widehat{\psi_r^m}(j)|^2$. Consequently, by (4.4),

$$\sum_{r=k}^{\infty}\sum_{m=1}^{\rho_r}2^r|\widehat{\psi_r^m}(j)|^2=1-2^k|\widehat{\phi_k}(j)|^2,\quad j\in\mathbb{Z}. \tag{4.6}$$

Since the tight wavelet frame $\{\phi_0\}\cup\{T_k^\ell\psi_k^m : k\ge 0,\ m=1,2,\ldots,\rho_k,\ \ell\in\mathcal{R}_k\}$ provides frame approximation order ν, as shown in the proof of Theorem 3.2, (3.25) must hold for some positive constants C and K. Therefore, it follows from (4.6) that

$$\begin{aligned}\sum_{m=1}^{\rho_k}2^k|j|^{-\nu}|\widehat{\psi_k^m}(j)|^2&\le\sum_{r=k}^{\infty}\sum_{m=1}^{\rho_r}2^r|j|^{-\nu}|\widehat{\psi_r^m}(j)|^2\\&=|j|^{-\nu}(1-2^k|\widehat{\phi_k}(j)|^2)\le\sqrt{C}2^{-k\nu},\quad j\in\mathcal{R}_k\backslash\{0\},\ k\ge K.\end{aligned}$$

In addition, (3.25) gives $1-2^k|\widehat{\phi_k}(0)|^2=0$ for $k\ge K$. Thus by (4.6), we must have $2^k|\widehat{\psi_k^m}(0)|=0$ for all $k\ge K$ and $m=1,\ldots,\rho_k$. Therefore, $\{\psi_k^m : m=1,2,\ldots,\rho_k\}_{k\ge 0}$ has at least $\nu/2$ vanishing moments.

Conversely, if $\{\psi_k^m : m=1,2,\ldots,\rho_k\}_{k\ge 0}$ has ν vanishing moments, then it is easy to see from (4.2) and (4.6) that (3.23) holds with $\epsilon:=1/2$ and $\tilde{C}:=C2^{2\nu}/(2^{2\nu}-1)$, where we have used the fact that $\mathcal{R}_r\subseteq\mathcal{R}_{r+1}$. By Theorem 3.2, the tight wavelet frame here provides frame approximation order at least ν. ■

In the proof of the converse direction in Theorem 4.1, we note that the result is also valid if we assume the following slightly weaker version of (4.2):

$$\sum_{m=1}^{\rho_k}2^k|\widehat{\psi_k^m}(0)|^2\le C2^{-2k\nu}\quad\text{and}\quad\max_{j\in\mathcal{R}_k\backslash\{0\}}\sum_{m=1}^{\rho_k}|j|^{-2\nu}2^k|\widehat{\psi_k^m}(j)|^2\le C2^{-2k\nu},\quad k\ge K, \tag{4.7}$$

where C and K are positive constants, independent of k and j. It is obvious that the vanishing moment condition (4.2) implies (4.7).

The condition (4.7) is related to sparse representations of piecewise smooth functions. In this connection, we obtain results on general sequences of functions before returning to tight wavelet frames at the end of the section. We begin with the following auxiliary result which will be used in due course.

Lemma 4.1. *For ψ_k^m, $k \ge 0$, $m = 1, 2, \dots, \rho_k$, be functions in $L^2[0, 2\pi]$. For $\mu \ge 0$, $k \ge 0$ and $f \in H^\mu[0, 2\pi]$, the following inequality holds:*

$$\sum_{m=1}^{\rho_k} |\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle|^2 \le 4 \left(\sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 \right) |\widehat{f}(0)|^2 + \max\{4D_{k,1}, 2D_{k,2}\} |f|^2_{H^\mu[0,2\pi]}, \tag{4.8}$$

where $\ell_{k,m} \in \mathbb{Z}$, $m = 1, 2, \dots, \rho_k$, and

$$D_{k,1} := \sum_{j \in \mathcal{R}_k \backslash \{0\}} \sum_{m=1}^{\rho_k} |j|^{-2\mu} |\widehat{\psi_k^m}(j)|^2, \tag{4.9}$$

$$D_{k,2} := \max_{j \in \mathcal{R}_k} 2^{-(2\mu-1)k} \sum_{p \in \mathbb{Z} \backslash \{0\}} \sum_{m=1}^{\rho_k} |2^{-k} j + p|^{-2\mu} |\widehat{\psi_k^m}(j + 2^k p)|^2. \tag{4.10}$$

Proof. Note that for $f \in H^\mu[0, 2\pi]$, $m = 1, 2, \dots, \rho_k$ and $\ell_{k,m} \in \mathbb{Z}$,

$$\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \overline{\widehat{\psi_k^m}(n)} e^{i 2\pi 2^{-k} \ell_{k,m} n}. \tag{4.11}$$

Therefore, we have

$$\begin{aligned}
\sum_{m=1}^{\rho_k} |\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle|^2 &\le \sum_{m=1}^{\rho_k} \left(\sum_{j \in \mathcal{R}_k} \sum_{p \in \mathbb{Z}} |\widehat{f}(j + 2^k p) \widehat{\psi_k^m}(j + 2^k p)| \right)^2 \\
&= \sum_{m=1}^{\rho_k} \left(\sum_{j \in \mathcal{R}_k} |\widehat{f}(j) \widehat{\psi_k^m}(j)| + \sum_{j \in \mathcal{R}_k} \sum_{p \in \mathbb{Z} \backslash \{0\}} |\widehat{f}(j + 2^k p) \widehat{\psi_k^m}(j + 2^k p)| \right)^2 \\
&\le 2I_1 + 2I_2,
\end{aligned}$$

where $I_1 := \sum_{m=1}^{\rho_k} \left(\sum_{j \in \mathcal{R}_k} |\widehat{f}(j) \widehat{\psi_k^m}(j)| \right)^2$ and

$$I_2 := \sum_{m=1}^{\rho_k} \left(\sum_{j \in \mathcal{R}_k} \sum_{p \in \mathbb{Z} \backslash \{0\}} |\widehat{f}(j + 2^k p) \widehat{\psi_k^m}(j + 2^k p)| \right)^2.$$

It is easy to see that

$$\begin{aligned}
I_1 &\le 2|\widehat{f}(0)|^2 \sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 + 2\sum_{m=1}^{\rho_k}\left(\sum_{j\in\mathcal{R}_k\backslash\{0\}} |\widehat{f}(j)\widehat{\psi_k^m}(j)|\right)^2 \\
&\le 2|\widehat{f}(0)|^2 \sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 + 2\sum_{m=1}^{\rho_k}\left(\sum_{j\in\mathcal{R}_k\backslash\{0\}} |\widehat{f}(j)|^2|j|^{2\mu}\right)\left(\sum_{j\in\mathcal{R}_k\backslash\{0\}} |j|^{-2\mu}|\widehat{\psi_k^m}(j)|^2\right) \\
&\le 2|\widehat{f}(0)|^2 \sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 + 2\left(\sum_{j\in\mathcal{R}_k\backslash\{0\}} |\widehat{f}(j)|^2|j|^{2\mu}\right)\left(\sum_{j\in\mathcal{R}_k\backslash\{0\}}\sum_{m=1}^{\rho_k} |j|^{-2\mu}|\widehat{\psi_k^m}(j)|^2\right).
\end{aligned}$$

We also have

$$\begin{aligned}
I_2 &\le 2^k \sum_{m=1}^{\rho_k}\sum_{j\in\mathcal{R}_k}\left(\sum_{p\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^kp)\widehat{\psi_k^m}(j+2^kp)|\right)^2 \\
&\le 2^k \sum_{j\in\mathcal{R}_k}\sum_{m=1}^{\rho_k}\left(\sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^kq)|^2|j+2^kq|^{2\mu}\right)\left(\sum_{p\in\mathbb{Z}\backslash\{0\}} |j+2^kp|^{-2\mu}|\widehat{\psi_k^m}(j+2^kp)|^2\right) \\
&\le \sum_{j\in\mathcal{R}_k}\left(\sum_{q\in\mathbb{Z}\backslash\{0\}} |\widehat{f}(j+2^kq)|^2|j+2^kq|^{2\mu}\right)\left(2^k\sum_{p\in\mathbb{Z}\backslash\{0\}}\sum_{m=1}^{\rho_k} |j+2^kp|^{-2\mu}|\widehat{\psi_k^m}(j+2^kp)|^2\right).
\end{aligned}$$

Consequently, (4.8) holds. ■

We shall employ Lemma 4.1 to demonstrate the relationship between vanishing moments and sparsity of expansion coefficients. To this end, we need to derive appropriate bounds of $D_{k,1}$ and $D_{k,2}$ defined by (4.9) and (4.10). The quantity $D_{k,1}$ depends on the values of $\widehat{\psi_k^m}$ on the set $\mathcal{R}_k\backslash\{0\}$ and it will be taken care of by the assumption (4.7). On the other hand, $D_{k,2}$ is based on the values of $\widehat{\psi_k^m}$ outside the set $\mathcal{R}_k$. In order to handle this, we extend the assumption (4.7) globally beyond $\mathcal{R}_k$. To be more precise, for $\nu \ge 0$, we say that a sequence of functions $\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k\ge 0}$ in $L^2[0, 2\pi]$ has *global vanishing moments of order* ν if there exist positive constants C and K, independent of k and j, such that

$$\sum_{m=1}^{\rho_k} 2^k|\widehat{\psi_k^m}(0)|^2 \le C2^{-2k\nu} \quad \text{and} \quad \sum_{m=1}^{\rho_k} |n|^{-2\nu}2^k|\widehat{\psi_k^m}(n)|^2 \le C2^{-2k\nu}, \quad n\in\mathbb{Z}\backslash\{0\},\ k\ge K. \tag{4.12}$$

Essentially, (4.12) combines (4.7) with some decay properties of $\widehat{\psi_k^m}$ outside the set $\mathcal{R}_k$.

Proposition 4.1. *Suppose that* $\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k\ge 0} \subseteq L^2[0, 2\pi]$ *has global vanishing moments of order* ν *as in* (4.12) *for some* $\nu > 0$*, where* C *and* K *are positive constants. Then for any* $\mu > \nu + 1/2$*, there exists a positive constant* $C_{\mu,\nu,\psi}$*, depending only on* μ, ν *and*

$\{\psi_k^m : m = 1, 2, \dots, \rho_k\}_{k\ge 0}$, *such that*

$$\sum_{r=k}^{\infty}\sum_{m=1}^{\rho_r} |\langle f, T_r^{\ell_{r,m}}\psi_r^m\rangle|^2 \le C_{\mu,\nu,\psi} 2^{-(2\nu+1)k}\left(|\widehat{f}(0)|^2 + |f|^2_{H^\mu[0,2\pi]}\right), \quad f \in H^\mu[0, 2\pi],\ k \ge K, \tag{4.13}$$

where $\ell_{k,m} \in \mathbb{Z}$ *for* $k \ge K$ *and* $m = 1, 2, \dots, \rho_k$.

Proof. By (4.8) in Lemma 4.1, for $f \in H^\mu[0, 2\pi]$, $k \ge K$, $m = 1, 2, \dots, \rho_k$ and $\ell_{k,m} \in \mathbb{Z}$, we have

$$\sum_{m=1}^{\rho_k} |\langle f, T_k^{\ell_{k,m}}\psi_k^m\rangle|^2 \le 4\left(\sum_{m=1}^{\rho_k}|\widehat{\psi_k^m}(0)|^2\right)|\widehat{f}(0)|^2 + \max\{4D_{k,1}, 2D_{k,2}\}|f|^2_{H^\mu[0,2\pi]}.$$

It follows from (4.12) that

$$\sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 \le C2^{-(2\nu+1)k}$$

and

$$\begin{aligned} D_{k,1} &= \sum_{j\in\mathcal{R}_k\setminus\{0\}} 2^{-k}|j|^{-2(\mu-\nu)} \sum_{m=1}^{\rho_k} |j|^{-2\nu}2^k|\widehat{\psi_k^m}(j)|^2 \le \sum_{j\in\mathcal{R}_k\setminus\{0\}} 2^{-k}|j|^{-2(\mu-\nu)}C2^{-2k\nu} \\ &\le 2^{-(2\nu+1)k}C\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|^{-2(\mu-\nu)} = C'2^{-(2\nu+1)k}, \end{aligned}$$

where $C' := C\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|^{-2(\mu-\nu)} < \infty$ by $\mu - \nu > 1/2$. In addition, for any $j \in \mathcal{R}_k$,

$$\begin{aligned} &2^{-(2\mu-1)k}\sum_{p\in\mathbb{Z}\setminus\{0\}}\sum_{m=1}^{\rho_k}|2^{-k}j+p|^{-2\mu}|\widehat{\psi_k^m}(j+2^kp)|^2 \\ &= \sum_{p\in\mathbb{Z}\setminus\{0\}}|j+2^kp|^{-2(\mu-\nu)}\sum_{m=1}^{\rho_k}|j+2^kp|^{-2\nu}2^k|\widehat{\psi_k^m}(j+2^kp)|^2 \\ &\le C2^{-2k\mu}\sum_{p\in\mathbb{Z}\setminus\{0\}}|2^{-k}j+p|^{-2(\mu-\nu)} \le C''2^{-2k\mu}, \end{aligned}$$

where

$$C'' := C\sup_{x\in[-1/2,1/2]}\sum_{p\in\mathbb{Z}\setminus\{0\}}|x+p|^{-2(\mu-\nu)} < \infty.$$

Thus we have $D_{k,2} \le C''2^{-2k\mu}$. Therefore,

$$\max\{4D_{k,1}, 2D_{k,2}\} \le 2^{-(2\nu+1)k}\max\{4C', 2^{1-2(\mu-\nu-1/2)k}C''\} \le 2^{-(2\nu+1)k}\max\{4C', 2C''\},$$

where we have again used the fact that $\mu > \nu + 1/2$. Consequently,

$$\sum_{m=1}^{\rho_k}|\langle f, T_k^{\ell_{k,m}}\psi_k^m\rangle|^2 \le \tilde{C}2^{-(2\nu+1)k}\left(|\widehat{f}(0)|^2 + |f|^2_{H^\mu[0,2\pi]}\right) \tag{4.14}$$

with $\tilde{C} := \max\{4C, 4C', 2C''\}$. Now summing (4.14), it is easy to see that (4.13) holds with $C_{\mu,\nu,\psi} := \tilde{C}/(1-2^{-(2\nu+1)}) < \infty$. ■

The definition of global vanishing moments in (4.12) combines the consequence of vanishing moments in (4.7) with some regularity of $\psi_k^1, \ldots, \psi_k^{\rho_k}$ reflected by the decay of their Fourier coefficients. This leads to (4.14) which quantifies sparse representations for locally smooth functions. Another possible candidate for the definition of global vanishing moments is

$$\sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 + \sum_{m=1}^{\rho_k} \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{-2\nu} |\widehat{\psi_k^m}(n)|^2 \le C 2^{-(2\nu+1)k}, \quad k \ge K, \tag{4.15}$$

where C and K are positive constants. It is clear that (4.15) implies (4.12). Comparing with (4.12), higher regularity of $\psi_k^1, \ldots, \psi_k^{\rho_k}$, given by faster decay of their Fourier coefficients, is assumed in (4.15).

In Proposition 4.1, the global vanishing moments of order ν in (4.12) leads to (4.14) for functions in $H^\mu[0,2\pi]$ where $\mu > \nu + 1/2$. The following result shows that the stronger assumption of (4.15) gives (4.14) with $\mu = \nu$. The converse is also explored, again with $\mu = \nu$.

Proposition 4.2. *Let $\{\psi_k^m : m = 1, 2, \ldots, \rho_k\}_{k\ge0} \subseteq L^2[0,2\pi]$ satisfy (4.15) for some positive constants C and K. Then (4.14) holds with $\mu = \nu$ and $\tilde{C} = C$ for all $f \in H^\mu[0,2\pi]$, $k \ge K$, $m = 1, 2, \ldots, \rho_k$ and $\ell_{k,m} \in \mathbb{Z}$. Conversely, if (4.14) holds with $\mu = \nu$ and $\tilde{C} < \infty$, then (4.12) holds with $C = \tilde{C}$.*

Proof. By (4.11), for $f \in H^\nu[0,2\pi]$, $k \ge K$, $m = 1, 2, \ldots, \rho_k$ and $\ell_{k,m} \in \mathbb{Z}$, we have

$$\begin{aligned}\sum_{m=1}^{\rho_k} |\langle f, T^{\ell_{k,m}} \psi_k^m \rangle|^2 &\le \sum_{m=1}^{\rho_k} \left(|\widehat{f}(0)||\widehat{\psi_k^m}(0)| + \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{f}(n)||\widehat{\psi_k^m}(n)| \right)^2 \\ &\le \sum_{m=1}^{\rho_k} \left(|\widehat{f}(0)|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} |\widehat{f}(n)|^2 |n|^{2\nu} \right) \times \\ &\qquad \left(|\widehat{\psi_k^m}(0)|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{-2\nu} |\widehat{\psi_k^m}(n)|^2 \right) \\ &\le C 2^{-(2\nu+1)k} \left(|\widehat{f}(0)|^2 + |f|^2_{H^\nu[0,2\pi]} \right),\end{aligned}$$

where (4.15) is used in the last inequality. Hence, (4.14) holds with $\mu = \nu$ and $\tilde{C} = C$.

Conversely, suppose that (4.14) holds with $\mu = \nu$ and $\tilde{C} < \infty$. For a fixed integer j, consider $f \in L^2[0,2\pi]$ such that $\widehat{f}(n) = \delta_{nj}$ for $n \in \mathbb{Z}$. Then $\widehat{f}(0) = \delta_{0j}$ and $|f|^2_{H^\nu[0,2\pi]} = |j|^{2\nu}$. For $k \ge K$, $m = 1, 2, \ldots, \rho_k$ and $\ell_{k,m} \in \mathbb{Z}$, it follows from (4.11) that $|\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle| = |\widehat{\psi_k^m}(j)|$. Taking $j = 0$, (4.14) gives $\sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(0)|^2 \le \tilde{C} 2^{-(2\nu+1)k}$. On the other hand, when

$j \neq 0$, (4.14) implies the second inequality in (4.12) with $C = \tilde{C}$. ■

It is also easy to see that (4.15) is satisfied if for some $\tilde{\mu} < \nu - 1/2$,

$$\sum_{m=1}^{\rho_k} 2^k |\widehat{\psi_k^m}(0)|^2 \le C 2^{-2k\nu} \quad \text{and} \quad \sum_{m=1}^{\rho_k} |n|^{-2\tilde{\mu}} 2^k |\widehat{\psi_k^m}(n)|^2 \le C 2^{-2k\nu}, \quad n \in \mathbb{Z}\backslash\{0\},\ k \ge K, \tag{4.16}$$

where C and K are positive constants. In comparison to (4.12), faster decay of the values of $\widehat{\psi_k^m}(n)$ is assumed in (4.16), which enables it to imply (4.15).

Finally, we illustrate how vanishing moments are related to sparsity in representations of locally smooth functions, or piecewise smooth functions. Let f be a function that is smooth in a neighborhood of x_0 in $\mathbb{R}$, say $[x_0 - 2\varepsilon, x_0 + 2\varepsilon]$ for some $\varepsilon > 0$. Suppose that $\{\psi_k^m : m = 1, 2, \ldots, \rho_k\}_{k \ge 0}$ has locality; that is, there are a positive integer K and some integers $\ell_{k,m}$, $m = 1, 2, \ldots, \rho_k$, $k \ge K$, for which the supports of $T_k^{\ell_{k,m}} \psi_k^m$ are contained inside $[x_0 - \varepsilon, x_0 + \varepsilon]$ for all $k \ge K$ and $m = 1, 2, \ldots, \rho_k$. Let h be a compactly supported C^∞-function such that $h(x) = 1$ for $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ and $h(x) = 0$ for $x \notin [x_0 - 2\varepsilon, x_0 + 2\varepsilon]$. Then fh is smooth everywhere and $\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle = \langle fh, T_k^{\ell_{k,m}} \psi_k^m \rangle$. By (4.14), we see that

$$\sum_{m=1}^{\rho_k} |\langle f, T_k^{\ell_{k,m}} \psi_k^m \rangle|^2 = \sum_{m=1}^{\rho_k} |\langle fh, T_k^{\ell_{k,m}} \psi_k^m \rangle|^2 \le C 2^{-(2\nu+1)k}, \quad k \ge K,$$

where C is some positive number. This recovers the notion of vanishing moments for compactly supported functions on the real line.

Returning to tight periodic wavelet frames, suppose that ϕ_k and ψ_k^m, $k \ge 0$, $m = 1, 2, \ldots, \rho_k$, are refinable functions and wavelets constructed as in Theorem 2.2, for which the tight wavelet frame $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \ge 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ provides frame approximation order $2\nu > 0$. By Theorem 4.1, the sequence $\{\psi_k^m : m = 1, 2, \ldots, \rho_k\}_{k \ge 0}$ has at least ν vanishing moments, which gives (4.7), where C and K are positive constants. Furthermore, in view of (2.18) and (4.6) supported by the setup of Theorem 2.2, for every $k \ge K$,

$$2^k \sum_{m=1}^{\rho_k} |\widehat{\psi_k^m}(n)|^2 \le 1 - 2^k |\widehat{\phi_k}(n)|^2 \le 1, \quad n \in \mathbb{Z}. \tag{4.17}$$

For $n \in \mathbb{Z}\backslash\mathcal{R}_k$, it can be written in the form $n = j + 2^k p$ where $j \in \mathcal{R}_k$ and $p \in \mathbb{Z}\backslash\{0\}$. Since $|2^{-k} j + p| \ge 2^{-1}$, it follows from (4.17) that

$$\sum_{m=1}^{\rho_k} |j + 2^k p|^{-2\nu} 2^k |\widehat{\psi_k^m}(j + 2^k p)|^2 \le 2^{2\nu} 2^{-2k\nu} \sum_{m=1}^{\rho_k} 2^k |\widehat{\psi_k^m}(j + 2^k p)|^2 \le 2^{2\nu} 2^{-2k\nu}.$$

Combining the above, we see that $\{\psi_k^m : m = 1, 2, \ldots, \rho_k\}_{k \ge 0}$ has global vanishing moments of order ν defined as in (4.12). Therefore by Proposition 4.1, (4.14) holds. If $\{\psi_k^m : m = 1, 2, \ldots, \rho_k\}_{k \ge 0}$ has locality, this enables sparse representations for locally smooth functions.

5. Examples

To illustrate the theory developed, we study two classes of examples that fit into the tight frame setup provided by Theorem 2.2 and also satisfy the results for frame approximation order in Theorem 3.2 and vanishing moments in Theorem 4.1. One class is well localized in the frequency domain and in fact band-limited. The other class has good localization in the time domain, which is analogous to compactly supported functions for the real line case. Given their respective strengths, the two classes complement each other in terms of potential applications.

Example 5.1. In [23, Example 4.1], using the unitary extension principle for $L^2[0,2\pi]$ (stated as Theorem 2.1 here), a parametric family of tight trigonometric polynomial wavelet frames was constructed. The underlying refinable functions ϕ_k, $k \geq 0$, for the construction are as follows. Let $\{J_k\}_{k\geq 0}$ and $\{L_k\}_{k\geq 0}$ be two strictly increasing sequences of nonnegative integers satisfying $J_k \leq L_k \leq 2^{k-1}$. The refinable functions

$$\phi_k = \sum_{n=-L_k}^{L_k} \widehat{\phi}_k(n) e^{in\cdot}, \quad k \geq 0,$$

are trigonometric polynomials, which satisfy for $k \geq 0$, $\widehat{\phi_k}(n) = \frac{1}{\sqrt{2^k}}$ for $n \in \{-J_k, \ldots, J_k\}$, and $\widehat{\phi_k}(n) = \widehat{\phi_k}(-n) > 0$ with

$$0 < \frac{\widehat{\phi_k}(n)}{\widehat{\phi_{k+1}}(n)} < \sqrt{2} \tag{5.1}$$

for $n \in \{-L_k, \ldots, L_k\} \backslash \{-J_k, \ldots, J_k\}$. The parametric family of tight wavelet frames obtained in [23] is made possible through the use of the Householder transformation and the values $\frac{\widehat{\phi_k}(n)}{\widehat{\phi_{k+1}}(n)}$ in (5.1).

First let us put the trigonometric polynomials ϕ_k, $k \geq 0$, into the context of Theorem 2.2. For $k \geq 1$, define $\widehat{a_k} \in \mathcal{S}(2^k)$ by setting

$$\widehat{a_k}(j) := \begin{cases} 1, & \text{if } j \in \{-J_{k-1}, \ldots, J_{k-1}\}, \\ \sigma_k(j), & \text{if } j \in \{-L_{k-1}, \ldots, L_{k-1}\} \backslash \{-J_{k-1}, \ldots, J_{k-1}\}, \\ 0, & \text{if } j \in \mathcal{R}_k \backslash \{-L_{k-1}, \ldots, L_{k-1}\}, \end{cases} \tag{5.2}$$

where $0 < \sigma_k(j) = \sigma_k(-j) < 1$ for $j \in \{-L_{k-1}, \ldots, L_{k-1}\} \backslash \{-J_{k-1}, \ldots, J_{k-1}\}$. Then all $\widehat{a_k}$ satisfy (2.9), and the infinite products in (2.17) are precisely the Fourier coefficients of the trigonometric polynomials ϕ_k. The convergence of the infinite products in (2.17) is also guaranteed by (2.16) which, by Proposition 2.3, is a consequence of (2.28). For (2.28), take a fixed integer n. Since $\{J_k\}_{k\geq 0}$ is a strictly increasing sequence, there exists a positive integer K_n such that $n \in \{-J_{K_n-1}, \ldots, J_{K_n-1}\}$. Thus for $k \geq K_n$, as $n \in \{-J_{k-1}, \ldots, J_{k-1}\}$, it follows from (5.2) that $\widehat{a_k}(n) = 1$ and so (2.28) holds for any positive constants C_n and α_n.

By Theorem 2.2, for any functions ψ_k^m, $k \ge 0$, $m = 1, 2, \ldots, \rho_k$, in $L^2[0, 2\pi]$ as defined in (2.5), where $\widehat{b_{k+1}^m} \in \mathcal{S}(2^{k+1})$, $k \ge 0$, $m = 1, 2, \ldots, \rho_k$, satisfy (2.10), the collection of trigonometric polynomials $\{\phi_0\} \cup \{T_k^\ell \psi_k^m : k \ge 0,\ m = 1, 2, \ldots, \rho_k,\ \ell \in \mathcal{R}_k\}$ forms a tight wavelet frame for $L^2[0, 2\pi]$. Under the extra condition that

$$\liminf_{k\to\infty} 2^{-k} J_k > 0, \tag{5.3}$$

we now show that this tight wavelet frame for $L^2[0, 2\pi]$ has the spectral frame approximation order. The condition (5.3) can be easily satisfied; for instance, by $J_k = 2^{k-1} - 1$ or $J_k = 2^{k-2}$ for sufficiently large k. By (5.3), there exist positive constants ϵ and K with $\epsilon < 1/2$ such that $2^{-k} J_k \ge \epsilon$ for every $k \ge K$; that is, $J_k \ge \epsilon 2^k$. Since $1 - 2^k |\widehat{\phi_k}(j)|^2 = 0$ for all $j \in \{-J_k, \ldots, J_k\}$, it follows that $1 - 2^k |\widehat{\phi_k}(j)|^2 = 0$ for all $j \in \mathcal{R}_k \cap (-2^k \epsilon, 2^k \epsilon]$. Hence for any $\nu > 0$, (3.23) is satisfied and by Theorem 3.2, the tight wavelet frame has frame approximation order ν. (Alternatively, we may use (5.2) to verify the condition (3.26) and employ Corollary 3.1.) In other words, it possesses the spectral frame approximation order. Furthermore, Theorem 4.1 and the discussion at the end of Section 4 imply that such a tight wavelet frame has (global) vanishing moments of arbitrarily high order.

Our second example is based on time-localized refinable functions. More precisely, it is constructed from the masks for pseudo-splines of type II with order (s, l) given in [18]. The masks were also used in [30] to obtain compactly supported symmetric nonstationary tight frames for $L^2(\mathbb{R})$, which have infinite order of smoothness and the spectral frame approximation order.

Example 5.2. For positive integers s, l, we denote

$$P_{s,l}(x) := \sum_{\kappa=0}^{l-1} \binom{s+\kappa-1}{\kappa} x^\kappa = \sum_{\kappa=0}^{l-1} \frac{(s+\kappa-1)!}{\kappa!(s-1)!} x^\kappa, \quad x \in \mathbb{R}.$$

The masks for pseudo-splines of type II with order (s, l) in [18] are given by

$$H_{s,l}(\xi) := \cos^{2s}(\xi/2) P_{s,l}(\sin^2(\xi/2)), \quad \xi \in \mathbb{R},\ s \ge 1,\ l = 1, \ldots, s.$$

In [30], nonstationary real-valued compactly supported tight wavelet frames for $L^2(\mathbb{R})$ were constructed from sequences of refinable pseudo-spline masks A_k, $k \ge 1$, of the form

$$A_k(\xi) := H_{s_k, l_k}(\xi), \quad \xi \in \mathbb{R},\ k \ge 1,$$

where

$$1 \le l_k \le s_k, \qquad \lim_{k\to\infty} s_k = \infty, \qquad \sum_{k=1}^{\infty} 2^{-k} s_k < \infty. \tag{5.4}$$

Then $A_k(0) = 1$ and

$$|A_k(\xi)|^2 + |A_k(\xi+\pi)|^2 \le 1, \quad \xi \in \mathbb{R}, \tag{5.5}$$

for every $k \ge 1$. For $k \ge 1$, we define $\widehat{a_k} \in \mathcal{S}(2^k)$ by setting

$$\widehat{a_k}(j) := A_k\left(\tfrac{2\pi j}{2^k}\right), \quad j \in \mathcal{R}_k. \tag{5.6}$$

Then (2.9) follows from (5.5). Note that $\deg(\widehat{a_k}) = s_k$ and therefore, by (5.4), (2.25) holds. Moreover, it is obvious that $\widehat{a_k}(0) = 1$ and $|\widehat{a_k}(n)| \le 1$ for all $k \ge 1$ and $n \in \mathbb{Z}$. By Proposition 2.3, we see that (2.16) is satisfied and therefore, (2.17) holds. In contrast to Example 5.1, here we apply item (a) instead of item (c) of Proposition 2.3. With the aid of (2.9), Proposition 2.1 ensures that (2.17) leads to a sequence of refinable functions $\{\phi_k\}_{k\ge0}$ in $L^2[0, 2\pi]$.

It was shown in [30, Lemma 3.3] that for any $\nu > 0$, there exist positive constants C and K such that

$$0 \le 1 - |A_k(\xi)|^2 \le C|\xi|^{2\nu}, \quad \xi \in [-\pi, \pi], \ k \ge K.$$

When $k \ge K$, it then follows from (5.6) that

$$\sum_{\tau=k+1}^{\infty} \left|1 - |\widehat{a_\tau}(j)|^2\right| \le \sum_{\tau=k+1}^{\infty} C|2\pi 2^{-\tau} j|^{2\nu} = 2^{-2k\nu}|j|^{2\nu} C(2\pi)^{2\nu}/(2^{2\nu} - 1), \quad j \in \mathcal{R}_k\backslash\{0\}.$$

That is, we have

$$2^{2k\nu} \max_{j\in\mathcal{R}_k\backslash\{0\}} |j|^{-2\nu} \left(\sum_{\tau=k+1}^{\infty} \left|1 - |\widehat{a_\tau}(j)|^2\right|\right) \le C', \quad k \ge K,$$

where $C' := C(2\pi)^{2\nu}/(2^{2\nu} - 1) < \infty$. Hence, (3.26) is satisfied with $\epsilon := 1/2$. Now by Corollary 3.1, any tight wavelet frame generated as in Theorem 2.2 provides frame approximation order ν. Since ν is an arbitrary positive number, the tight wavelet frame provides the spectral frame approximation order. Similar to the arguments in Example 5.1, we also conclude that such a wavelet frame has (global) vanishing moments of arbitrarily high order.

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