SMALL SUPPORT SPLINE RIESZ WAVELETS IN LOW DIMENSIONS

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Abstract. In [B. Han and Z. Shen, *SIAM J. Math. Anal.*, 38 (2006), 530–556], a family of univariate short support Riesz wavelets was constructed from uniform B-splines. A bivariate spline Riesz wavelet basis from the Loop scheme was derived in [B. Han and Z. Shen, *J. Fourier Anal. Appl.*, 11 (2005), 615–637]. Motivated by these two papers, we develop in this article a general theory and a construction method to derive small support Riesz wavelets in low dimensions from refinable functions. In particular, we obtain small support spline Riesz wavelets from bivariate and trivariate box splines. Small support Riesz wavelets are desirable for developing efficient algorithms in various applications. For example, the short support Riesz wavelets from [B. Han and Z. Shen, *SIAM J. Math. Anal.*, 38 (2006), 530–556] were used in a surface fitting algorithm of [M.J. Johnson, Z. Shen and Y.H. Xu, *J. Approx. Theory*, 159 (2009), 197–223], and the Riesz wavelet basis from the Loop scheme was used in a very efficient geometric mesh compression algorithm in [A. Khodakovsky, P. Schröder and W. Sweldens, *Proceedings of SIGGRAPH*, 2000].

1. Introduction and Main Results

In this paper we shall develop a general theory and a construction method to derive Riesz wavelets in low dimensions from refinable functions. As an application, we shall construct in this paper small support spline Riesz wavelets from bivariate and trivariate box splines. As we shall see later, the construction is very simple; the corresponding wavelets and their masks have a very simple form. Similar to [39, 29], the construction is essentially from [40, 41], where orthogonal wavelets with exponential decay and pre-wavelets with compact support were constructed. However, the mathematical analysis to establish the Riesz property of the constructed system is very much technically involved so that the previous available results in the literature are not sufficient. We start this adventure by introducing the simple construction method. To do so, let us introduce the definition of a Riesz wavelet first.

1.1. Definition of Riesz wavelets in Sobolev spaces.

For a real number τ, by $H^\tau(\mathbb{R}^d)$ we denote the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{H^\tau(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2(1+\|\xi\|^2)^\tau \, d\xi < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Here the Fourier transform $\hat{f}$ for $f \in L_1(\mathbb{R}^d)$ used in this paper is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}\,dx$, $\xi \in \mathbb{R}^d$ where $x \cdot \xi$ is the inner product of the two vectors x and ξ in $\mathbb{R}^d$. The Fourier transform can be naturally extended to square

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integrable functions and tempered distributions. Note that $H^\tau(\mathbb{R}^d)$ is a Hilbert space under the inner product:

$$\langle f, g\rangle_{H^\tau(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1+\|\xi\|^2)^\tau\, d\xi, \qquad f,\, g \in H^\tau(\mathbb{R}^d). \tag{1.1}$$

Moreover, for each $g \in H^{-\tau}(\mathbb{R}^d)$, $\langle f, g\rangle := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$, for all $f \in H^\tau(\mathbb{R}^d)$ defines a linear functional on $H^\tau(\mathbb{R}^d)$. The spaces $H^\tau(\mathbb{R}^d)$ and $H^{-\tau}(\mathbb{R}^d)$ are known to form a pair of dual spaces. The Hilbert space $L^2(\mathbb{R}^d)$ is a special case of $H^\tau(\mathbb{R}^d)$ with $\tau = 0$.

Denote $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. For given $\phi, \psi^1, \dots, \psi^s \in H^\tau(\mathbb{R}^d)$, the properly normalized wavelet system in $H^\tau(\mathbb{R}^d)$ is defined as:

$$\begin{aligned} \mathrm{WS}^\tau(\phi;\psi^1,\dots,\psi^s) &:= \{\phi_{0,k} : k\in\mathbb{Z}^d\} \cup \left\{\psi^{\ell,\tau}_{j,k} : j\in\mathbb{N}_0,\ k\in\mathbb{Z}^d, \ell=1,\dots,s\right\}\\ \text{with } \phi_{0,k} &:= \phi(\cdot - k) \ \text{ and } \ \psi^{\ell,\tau}_{j,k} := 2^{j(d/2-\tau)}\psi^\ell(2^j\cdot - k). \end{aligned} \tag{1.2}$$

We say that $\mathrm{WS}^\tau(\phi;\psi^1,\dots,\psi^s)$ is *a Riesz basis* in the Sobolev space $H^\tau(\mathbb{R}^d)$ if

(1) the linear combinations of elements in $\mathrm{WS}^\tau(\phi;\psi^1,\dots,\psi^s)$ is dense in $H^\tau(\mathbb{R}^d)$;

(2) $\mathrm{WS}^\tau(\phi;\psi^1,\dots,\psi^s)$ is a *Riesz sequence* in $H^\tau(\mathbb{R}^d)$, that is, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1\left[\sum_{k\in\mathbb{Z}^d}|v_k|^2 + \sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}^d}|w^\ell_{j,k}|^2\right] &\leqslant \left\|\sum_{k\in\mathbb{Z}^d} v_k\phi_{0,k} + \sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}^d} w^\ell_{j,k}\psi^{\ell,\tau}_{j,k}\right\|^2_{H^\tau(\mathbb{R}^d)}\\ &\leqslant C_2\left[\sum_{k\in\mathbb{Z}^d}|v_k|^2 + \sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}^d}|w^\ell_{j,k}|^2\right] \end{aligned} \tag{1.3}$$

holds for all finitely supported sequences $\{v_k\}_{k\in\mathbb{Z}^d}$ and $\{w^\ell_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}^d,\ell=1,\dots,s}$.

Wavelets are normally obtained from *a refinable function* $\phi : \mathbb{R}^d \to \mathbb{C}$, which satisfies the refinement equation

$$\phi = 2^d\sum_{k\in\mathbb{Z}^d} a(k)\phi(2\cdot - k), \tag{1.4}$$

where $a : \mathbb{Z}^d \to \mathbb{C}$ is a sequence on $\mathbb{Z}^d$, called the *mask* for ϕ. Define the Fourier series associated with a by $\hat{a}(\xi) := \sum_{k\in\mathbb{Z}^d} a(k)e^{-ik\cdot\xi}$, $\xi\in\mathbb{R}^d$. Then the refinement equation in the time domain in (1.4) can be rewritten in the frequency domain as follows:

$$\hat{\phi}(\xi) = \hat{a}(\xi/2)\hat{\phi}(\xi/2), \qquad a.e.\ \xi\in\mathbb{R}^d. \tag{1.5}$$

For a compactly supported tempered distribution ϕ on $\mathbb{R}^d$, we say that ϕ has *stable integer shifts* (or simply, ϕ is stable) if $(\hat{\phi}(\xi + 2\pi k))_{k\in\mathbb{Z}^d}$ is not the zero sequence for all $\xi\in\mathbb{R}^d$. In general, a non-redundant dyadic wavelet system such as a Riesz wavelet basis is often generated from a refinable function ϕ with stable integer shifts. Moreover, the system has $2^d - 1$ wavelet generators $\psi^1, \dots, \psi^{2^d-1}$ which are derived from the refinable function ϕ in the following way:

$$\widehat{\psi^\ell}(\xi) := \widehat{b^\ell}(\xi/2)\hat{\phi}(\xi/2), \qquad \ell = 1,\dots,2^d-1, \qquad \xi\in\mathbb{R}^d \tag{1.6}$$

such that

$$\det \mathcal{P}_{[\hat{a},\widehat{b^1},\dots,\widehat{b^{2^d-1}}]}(\xi) \neq 0, \qquad a.e.\ \xi\in\mathbb{R}^d \tag{1.7}$$

with

$$\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi) := \begin{bmatrix} \hat{a}(\xi+2\pi\omega_0) & \cdots & \hat{a}(\xi+2\pi\omega_{2^d-1}) \\ \widehat{b^1}(\xi+2\pi\omega_0) & \cdots & \widehat{b^1}(\xi+2\pi\omega_{2^d-1}) \\ \vdots & \ddots & \vdots \\ \widehat{b^{2^d-1}}(\xi+2\pi\omega_0) & \cdots & \widehat{b^{2^d-1}}(\xi+2\pi\omega_{2^d-1}) \end{bmatrix}, \tag{1.8}$$

where $\Omega_{2I_d} := \{\omega_0, \ldots, \omega_{2^d-1}\} = \{\omega/2 \;:\; \omega \in [0,1]^d \cap \mathbb{Z}^d\}$ with $\omega_0 := 0$ and I_d denotes the $d \times d$ identity matrix. In fact, in order to have a Riesz wavelet basis $\mathrm{WS}^\tau(\phi; \psi^1, \ldots, \psi^{2^d-1})$ in $H^\tau(\mathbb{R}^d)$, it is necessary, but not sufficient, that the refinable function ϕ is stable and the condition in (1.7) is satisfied. This is a well known fact, e.g., see [11, 16, 22, 23, 26, 28, 29, 30, 43].

1.2. **Small support Riesz wavelets and known constructions in low dimensions.** Generally, the low-pass mask $\hat{a}$ is given and one has to find the high-pass wavelet masks $\widehat{b^1}, \ldots, \widehat{b^{2^d-1}}$ such that (1.7) is satisfied. For dimension $d = 1$, a well-known method to construct $\widehat{b^1}$ ([14]) is

$$\widehat{b^1}(\xi) := e^{-i\xi}\overline{\hat{a}(\xi+\pi)}, \qquad \xi \in \mathbb{R}. \tag{1.9}$$

Consequently, $\det \mathcal{P}_{[\hat{a},\widehat{b^1}]}(\xi) = -e^{-i\xi}(|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2)$. In the case that $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is an orthonormal system in $L_2(\mathbb{R})$, it is well-known that one must have $|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 = 1$ and $\mathrm{WS}^0(\phi;\psi)$, with the high-pass wavelet mask in (1.9), is an orthonormal wavelet basis for $L_2(\mathbb{R})$. When ϕ is a uniform B-spline, then such a construction leads to a Riesz wavelet basis as shown in [28]. More generally, when ϕ is a pseudo-spline, such a construction also leads to a Riesz wavelet basis as shown in [13]. However, as it was pointed out by [22], there exists an example that such a construction does not lead to a Riesz wavelet basis even if the underlying refinable function has stable integer shifts and can be arbitrarily smooth.

As pointed out in [28], the wavelet constructed from a uniform B-spline by the wavelet mask in (1.9) has the smallest support among all the wavelets that have the same order of vanishing moments. In fact, the pre-wavelet constructed in [10] from a uniform B-spline of the same order has the support that is about twice the support of that in [28]. The supports of the wavelets in [28] are also much smaller than those of spline Riesz wavelets constructed via the biorthogonal wavelet construction of [12].

Suppose that $\hat{a}$ is a mask with real coefficients and symmetry such that

$$\overline{\hat{a}(\xi)} = e^{-ic_a\cdot\xi}\hat{a}(\xi) \qquad \text{for some } \; c_a \in \mathbb{Z}^d. \tag{1.10}$$

For dimensions $d = 2$ and $d = 3$, a simple and interesting construction has been proposed in [40, 41] to derive $\widehat{b^1}, \ldots, \widehat{b^{2^d-1}}$ from the symmetric mask $\hat{a}$ satisfying (1.10). This construction is crucial for the construction here and for those constructions of the small support wavelet from the Loop scheme given in [29, 39] as well. More precisely, for $d = 2$, the high-pass wavelet masks $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ are defined to be

$$\begin{aligned} &\widehat{b^1}(\xi_1,\xi_2) := e^{-i(\xi_1+\xi_2)}\overline{\hat{a}(\xi_1+\pi,\xi_2)}, \qquad \widehat{b^2}(\xi_1,\xi_2) := e^{-i\xi_2}\overline{\hat{a}(\xi_1,\xi_2+\pi)}, \\ &\widehat{b^3}(\xi_1,\xi_2) := e^{-i\xi_1}\overline{\hat{a}(\xi_1+\pi,\xi_2+\pi)}. \end{aligned} \tag{1.11}$$

Then it can be proven easily (see e.g. [40, 41]) that

$$\overline{\mathcal{P}_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi)}^T \mathcal{P}_{[\hat{a},\widehat{b^1},\widehat{b^2},\widehat{b^3}]}(\xi) = \left(\sum_{\omega\in\Omega_{2I_2}} |\hat{a}(\xi+2\pi\omega)|^2\right) I_4.$$

For $d=3$, the high-pass wavelet masks $\widehat{b^1},\ldots,\widehat{b^7}$ are defined to be

$$\begin{aligned}
&\widehat{b^1}(\xi_1,\xi_2,\xi_3):=e^{-i(\xi_1+\xi_2)}\overline{\hat a(\xi_1+\pi,\xi_2,\xi_3)},\ \ \widehat{b^2}(\xi_1,\xi_2,\xi_3):=e^{-i(\xi_2+\xi_3)}\overline{\hat a(\xi_1,\xi_2+\pi,\xi_3)},\\
&\widehat{b^3}(\xi_1,\xi_2,\xi_3):=e^{-i(\xi_1+\xi_3)}\overline{\hat a(\xi_1,\xi_2,\xi_3+\pi)},\ \ \widehat{b^4}(\xi_1,\xi_2,\xi_3):=e^{-i\xi_1}\overline{\hat a(\xi_1+\pi,\xi_2+\pi,\xi_3)},\\
&\widehat{b^5}(\xi_1,\xi_2,\xi_3):=e^{-i\xi_2}\overline{\hat a(\xi_1,\xi_2+\pi,\xi_3+\pi)},\ \ \widehat{b^6}(\xi_1,\xi_2,\xi_3):=e^{-i\xi_3}\overline{\hat a(\xi_1+\pi,\xi_2,\xi_3+\pi)},\\
&\widehat{b^7}(\xi_1,\xi_2,\xi_3):=e^{-i(\xi_1+\xi_2+\xi_3)}\overline{\hat a(\xi_1+\pi,\xi_2+\pi,\xi_3+\pi)}.
\end{aligned} \tag{1.12}$$

Then we have $\overline{\mathcal{P}_{[\hat a,\widehat{b^1},\ldots,\widehat{b^7}]}(\xi)}^T\mathcal{P}_{[\hat a,\widehat{b^1},\ldots,\widehat{b^7}]}(\xi)=\Big(\sum_{\omega\in\Omega_{2I_3}}|\hat a(\xi+2\pi\omega)|^2\Big)I_8$.

These constructions of wavelet masks in the bivariate and trivariate cases are very simple. However, due to a well-known result in geometry, it was shown in [40] that there is no such simple explicit formula for $d>3$.

1.3. **Main subjects in this article.** With the choice of the high-pass filters in (1.11) or (1.12), if the integer shifts of a compactly supported refinable function ϕ with $\hat\phi(0)=1$ form an orthonormal system in $L_2(\mathbb{R}^d)$, then $\mathrm{WS}^0(\phi;\psi^1,\ldots,\psi^{2^d-1})$ is an orthonormal basis of $L_2(\mathbb{R}^d)$ (see [40, 41]). When a special bivariate box spline generated by Loop scheme is used, it was proven in [29] that such a construction leads to a Riesz basis in $L_2(\mathbb{R}^2)$. This motivates our adventure here to see whether this is true for more general bivariate and trivariate box splines.

This simple explicit form of wavelet masks has the same number of nonzero terms as that of the corresponding refinement masks. Hence, the supports of the wavelets defined in (1.6) have the same size as the support of the corresponding compactly supported refinable function. Note that the supports of pre-wavelets constructed in [41] are about twice the support of the corresponding refinable function. The supports of Riesz wavelets in this paper are also much more smaller than those of the spline Riesz wavelets constructed via the biorthogonal wavelet construction of [16, 32]. Furthermore, it is well known that stable uniform B-splines and more general stable box splines are among those refinable functions that have small supports for a given regularity order. The above constructions lead to small support wavelets for a given order of regularity or vanishing moments when stable refinable splines are used.

An important family of refinable functions consists of the box splines. For a given $d\times n$ (direction) matrix Ξ of full rank with integer entries and $n\geqslant d$, the Fourier transform of its associated box spline M_Ξ is given by

$$\widehat{M_\Xi}(\xi):=\prod_{k\in\Xi}\frac{1-e^{-ik\cdot\xi}}{ik\cdot\xi},\qquad \xi\in\mathbb{R}^d, \tag{1.13}$$

where $k\in\Xi$ means that $k\in\mathbb{Z}^d$ is a column vector of Ξ and k goes through all the columns of Ξ once and only once. The box spline M_Ξ is refinable and its refinement mask is given by

$$\widehat{a^\Xi}(\xi):=\prod_{k\in\Xi}\frac{1+e^{-ik\cdot\xi}}{2}. \tag{1.14}$$

Box splines are symmetric (to some point in $\mathbb{Z}^d/2$) and belong to $C^{\nu_\infty(\Xi)-1}$, where $\nu_\infty(\Xi)+1$ is the minimum number of columns that can be discarded from Ξ to obtain a matrix of rank $<d$. Furthermore, the integer shifts of a box spline are stable, whenever the matrix Ξ is a unimodular matrix, that is, every basis of columns from Ξ has determinant ±1.

In this paper, we are particularly interested in box splines with stable integer shifts and symmetry. In dimension one, if Ξ consists of a $1\times r$ row vector with all its components being 1, then the box spline M_Ξ is the well-known B-spline of order r, which has stable integer shifts and the Fourier series of its mask being $2^{-r}(1+e^{-i\xi})^r$. With the choice of a wavelet mask $\widehat{b^1}$ in (1.9), it has been proved in [28, Theorem 2.2] that $\mathrm{WS}^0(\phi;\psi^1)$ forms a Riesz wavelet basis in $L_2(\mathbb{R})$ for any $r\in\mathbb{N}$. As an example of applications of the analysis developed here, we show that the general analysis here can also be used to provide another simple proof of [28, Theorem 2.2]. This shows that although the development of the analysis is technical, it is powerful.

In dimension two, as a well-known example of bivariate box splines with stable integer shifts and symmetry, Ξ consists of the columns $(1,0)^T,(0,1)^T,(1,1)^T$ with equal multiplicity r. That is, the centered box spline denoted by $\phi^{r,2d}=M_\Xi$ belongs to C^{2r-2} and its Fourier transform is

$$\widehat{\phi^{r,2d}}(\xi_1,\xi_2)=\frac{\sin^r(\xi_1/2)}{(\xi_1/2)^r}\frac{\sin^r(\xi_2/2)}{(\xi_2/2)^r}\frac{\sin^r(\xi_1/2+\xi_2/2)}{(\xi_1/2+\xi_2/2)^r} \tag{1.15}$$

with its mask given by

$$\widehat{a^{r,2d}}(\xi_1,\xi_2):=\cos^r\left(\frac{\xi_1}{2}\right)\cos^r\left(\frac{\xi_2}{2}\right)\cos^r\left(\frac{\xi_1+\xi_2}{2}\right). \tag{1.16}$$

There are more choices of box splines defined on $\mathbb{R}^3$ with stable integer shifts and symmetry. Interested readers can find more details on box splines with stable integer shifts and symmetry in [1, Page 90]. In this paper, we consider one class of the box splines with stable integer shifts and symmetry with the direction matrix whose columns come from the columns of

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{bmatrix} \tag{1.17}$$

with equal multiplicity r. Let Ξ be the above matrix in (1.17). For $r\in\mathbb{N}$, we are interested in refinable functions ϕ given by

$$\widehat{\phi^{r,3d}}(\xi)=(\widehat{M_\Xi}(\xi))^r \tag{1.18}$$

with its mask

$$\widehat{a^{r,3d}}(\xi):=\Big(\prod_{k\in\Xi}\frac{1+e^{-ik\cdot\xi}}{2}\Big)^r. \tag{1.19}$$

With ϕ being one of the above box splines with stable integer shifts and with the choice of the wavelet masks $\widehat{b^1},\dots,\widehat{b^{2^d-1}}$ in (1.11) and (1.12) for dimensions $d=2,3$, it is quite natural and interesting to ask whether $\mathrm{WS}^\tau(\phi;\psi^1,\dots,\psi^{2^d-1})$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^d)$ for some $\tau\in\mathbb{R}$ for all multiplicities r, since such Riesz wavelets have high smoothness with respect to their very small support.

Small support spline wavelets are widely used in many applications, because they are simple and have small supports. For example, in the bivariate case, if we pick $r=2$, the corresponding box spline is the basis function of the Loop scheme around the regular vertices in computer graphics. The Loop scheme is an algorithm to generate a smooth subdivision surface from an initial triangular mesh (see [37]). The Loop scheme around the regular vertices is the subdivision scheme derived from the mask in (1.15) with $r=2$. It is well-known that the basis function of the Loop scheme is C^2 around regular vertices and is only C^1 around irregular vertices ([39, 37]). One of the main ideas of [39] is to build up a wavelet system with small support by using the methods of [40, 41] without using an explicit dual refinable function. By this, one still has a fast wavelet reconstruction algorithm, while the wavelet decomposition is obtained by solving a system of linear equations numerically. Since in applications the reconstruction is normally "online" that needs to be fast and decomposition is "off-line" whose speed is not as urgent and critical as the "online" counterpart, this choice is reasonable and feasible. The wavelets ψ^1, ψ^2 and ψ^3 used in [39] are defined by (1.6) with $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ being defined in (1.11). It was proven in [29] that its corresponding wavelet system forms a Riesz basis for $L_2(\mathbb{R}^2)$. The analysis developed in this paper shows that the wavelet system generated by ψ^1, ψ^2 and ψ^3 defined in (1.6) with $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ being defined by (1.11) from any arbitrary box spline given in (1.15) forms a Riesz basis for $L_2(\mathbb{R}^2)$. These wavelets can be useful when the fast reconstruction is needed, while the decomposition can be done "off line". Another such an example is that small support spline wavelets in [28] are used to derive fast algorithms for smooth surface fitting from scattered noisy data in [36].

Another class of small support wavelet systems that are widely used in applications are spline tight wavelet frame systems constructed by the unitary extension principle of [42]. The spline wavelet tight frame systems are redundant and self dual systems with small support. They are used to derive efficient and fast algorithms for various image restoration problems including deblurring and blind deblurring, denoise, inpainting, and image decomposition. The interested reader should consult [2, 3, 4, 5, 6, 7, 8, 9] for details. Small support spline tight frame wavelets are also used to reconstruct the scene (a visible piecewise smooth surface) from a set of scattered noisy and possibly sparse range data in [33], that is a challenging problem in robotic navigation and computer graphics. The tight frame theory, the unitary extension principle, and the corresponding decomposition and reconstruction algorithms can be found in e.g. [15, 42]. For a short survey, please see [44].

1.4. **The fundamental quantity $\nu_p(\hat{a}, 2I_d)$ in wavelet analysis.** To state our main results of this paper, let us introduce an important quantity. For two finitely supported sequences u, v on $\mathbb{Z}^d$, their convolution $u * v$ is defined to be

$$[u * v](n) := \sum_{k\in\mathbb{Z}^d} u(n-k)v(k), \qquad n \in \mathbb{Z}^d.$$

That is, $\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi)$. For $1 \leqslant p \leqslant \infty$, we define $\|u\|^p_{\ell_p(\mathbb{Z}^d)} := \sum_{k\in\mathbb{Z}^d} |u(k)|^p$ with the usual modification for $p = \infty$. Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\beta = (\beta_1, \dots, \beta_d)^T \in \mathbb{N}_0^d$, $|\beta| := \beta_1 + \cdots + \beta_d$ and $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$ is the standard partial differential operator, where ∂_j denotes the partial derivative with respect to the jth variable. For a 2π-periodic trigonometric polynomial $\hat{a}$ in d-variables, we say that $\hat{a}$ has κ *sum rules* if

$$\partial^\beta \hat{a}(2\pi\omega) = 0, \qquad \beta \in \mathbb{N}_0^d \quad \text{with} \quad |\beta| < \kappa, \omega \in \Omega_{2I_d}\backslash\{0\}. \tag{1.20}$$

Let $1 \leqslant p \leqslant \infty$. Now we recall the definition of an important quantity $\nu_p(\hat{a}, 2I_d)$ from [19]. For a 2π-periodic trigonometric polynomial $\hat{a}$ such that $\hat{a}$ has κ but not $\kappa+1$ sum rules, for an integer $\tilde{\kappa} \geqslant \kappa$, $\nu_p(\hat{a}, 2I_d)$ is defined to be ([19, Page 61])

$$\nu_p(\hat{a}, 2I_d) := d(1/p-1) - \log_2 \max_{\beta\in\mathbb{N}_0^d,\, |\beta|<\tilde{\kappa}} \limsup_{n\to\infty} \|a_n * \nabla^\beta \delta\|^{1/n}_{\ell_p(\mathbb{Z}^d)}, \tag{1.21}$$

where the two finitely supported sequences $\nabla^\beta\delta$ and a_n are defined via the frequency domain by

$$\widehat{\nabla^\beta\delta}(\xi) := (1-e^{-i\xi_1})^{\beta_1} \cdots (1-e^{-i\xi_d})^{\beta_d} \quad \text{and} \quad \widehat{a_n}(\xi) := \hat{a}(2^{n-1}\xi)\cdots\hat{a}(2\xi)\hat{a}(\xi) \tag{1.22}$$

for $\beta = (\beta_1, \dots, \beta_d)^T \in \mathbb{N}_0^d$ and $\xi = (\xi_1, \dots, \xi_d)^T \in \mathbb{R}^d$. The quantity $\nu_p(\hat{a}, 2I_d)$ is independent of the choice of $\tilde{\kappa} \geqslant \kappa$ and it is known ([19, (4.7)]) that

$$\nu_q(\hat{a}, 2I_d) + d(1/p-1/q) \leqslant \nu_p(\hat{a}, 2I_d) \leqslant \nu_q(\hat{a}, 2I_d), \qquad 1 \leqslant q \leqslant p \leqslant \infty. \tag{1.23}$$

The definition of the quantity $\nu_p(\hat{a}, 2I_d)$ can be generalized to $\nu_p(\hat{a}, M)$ for a matrix mask $a : \mathbb{Z}^d \to \mathbb{C}^{r\times r}$ and a general $d \times d$ dilation matrix M, see [19, Page 61] for detail. The quantity $\nu_p(\hat{a}, M)$ plays a fundamental role in many aspects of wavelet analysis. To give readers some rough ideas about the importance of the quantity $\nu_p(\hat{a}, M)$, in the following we list some applications of the quantity $\nu_p(\hat{a}, M)$ in wavelet analysis without going into too many technical details and definitions.

(1) A vector subdivision scheme (or equivalently a vector cascade algorithm) associated with a matrix mask $\hat{a}$ and an expansive dilation matrix M converges in the space $L_p(\mathbb{R}^d)$ if and and only $\nu_p(\hat{a}, M) > 0$, see [19, Theorem 4.3] and [17, 21, 25] as well as references therein;
(2) Let κ be a positive integer and M be an expansive isotropic dilation matrix M (that is, M is similar to a diagonal matrix with all its diagonal entries having the same modulus > 1). A vector subdivision scheme with a matrix mask $\hat{a}$ and the dilation matrix M converges in the Sobolev space $W_p^\kappa(\mathbb{R}^d)$ (that is, $f \in W_p^\kappa(\mathbb{R}^d)$ means that all its mth mixed derivatives of f belong to $L_p(\mathbb{R}^d)$) if and only if $\nu_p(\hat{a}, M) > \kappa$, see [19, Theorem 4.3], [21, Theorem 3.1] and references therein;
(3) Let ϕ be a nontrivial compactly supported refinable vector of distributions satisfying $\hat{\phi}(M^T\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$. Then $\nu_p(\hat{a}, M) \leqslant \nu_p(\phi)$. Moreover, $\nu_p(\hat{a}, M) = \nu_p(\phi)$ if M is isotropic and the integer shifts of $\phi \in L_p(\mathbb{R}^d)$ are stable, where $\nu_p(\phi) = \sup\{\tau \in \mathbb{R} \,:\, \phi \in W_p^\tau(\mathbb{R}^d)\}$ measures the L_p smoothness of ϕ, see [19, Page 69] and [21, Theorem 4.1];
(4) $\nu_\infty(\hat{a}, M)$ is used in [19, Corollary 5.2] to provide a complete characterization of a refinable function vector ϕ with a matrix mask $\hat{a}$ and an isotropic dilation matrix M such that ϕ is a Hermite interpolant. See [18, 19] for more details on refinable Hermite interpolants;
(5) $\nu_2(\hat{a}, M)$ plays a central role in the investigation of the stability of a wavelet system in various Sobolev spaces, see [23, 26, 27, 28, 29, 30] as well as section 4 of this paper.

Moreover, $\nu_2(\hat{a}, M)$ can be numerically computed, e.g., see [20, 19, 23]. For more details on the quantity $\nu_p(\hat{a}, M)$ and its importance in wavelet analysis, see [17, 19, 20, 21, 23, 26, 27, 28, 29, 30] and numerous references therein.

Since it is unavoidable for us to deal with masks $\hat{a}$ which are not 2π-trigonometric polynomials, in the following we shall extend the definition of $\nu_2(\hat{a}, 2I_d)$ to the general setting. For two 2π-periodic trigonometric polynomials $\hat{a}$ and $\mathring{\hat{a}}$ such that $|\hat{a}(\xi)| \leqslant |\mathring{\hat{a}}(\xi)|$ for all $\xi \in \mathbb{R}^d$, by the definition of $\nu_2(\hat{a}, 2I_d)$, we always have (e.g., see [23]) $\nu_2(\mathring{\hat{a}}, 2I_d) \leqslant \nu_2(\hat{a}, 2I_d)$. However, $\nu_p(\mathring{\hat{a}}, 2I_d) \leqslant \nu_p(\hat{a}, 2I_d)$ may no longer hold for $p \neq 2$.

For a general 2π-periodic measurable function $\hat{a}$ that is not a 2π-periodic trigonometric polynomial, we now define a similar quantity $\mu_2(\hat{a}, 2I_d)$. For a 2π-periodic function $\hat{a}$ such that $\lim_{j\to\infty} \hat{a}(2^{-j}\xi) = \hat{a}(0)$ for almost every $\xi \in \mathbb{R}^d$, we define

$$\mu_2(\hat{a}, 2I_d) := \sup\{\nu_2(\mathring{\hat{a}}, 2I_d) \,:\, \mathring{\hat{a}} \in \mathcal{U}_{\hat{a}}\}, \tag{1.24}$$

where $\mathcal{U}_{\hat{a}}$ denotes the set of all 2π-periodic trigonometric polynomial $\mathring{\hat{a}}$ such that $\mathring{\hat{a}}(0) = \hat{a}(0)$ and $|\hat{a}(\xi)| \leqslant |\mathring{\hat{a}}(\xi)|$ for almost every $\xi \in \mathbb{R}^d$. If such a 2π-periodic trigonometric polynomial $\mathring{\hat{a}}$ does not exist, we simply define $\mu_2(\hat{a}, 2I_d) := -\infty$. If $\hat{a}$ is a 2π-periodic trigonometric polynomial, now it is evident that $\mu_2(\hat{a}, 2I_d) = \nu_2(\hat{a}, 2I_d)$. Hence, $\mu_2(\hat{a}, 2I_d)$ is a generalization of $\nu_2(\hat{a}, 2I_d)$.

1.5. **Main results of the article.** For every positive integer r, let $\hat{a} := \widehat{a^{r,2d}}$ in (1.16) be the mask for the three directional bivariate box spline $\phi := \phi^{r,2d}$ in (1.15). Let $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ be defined in (1.11) and ψ^1, ψ^2, ψ^3 as in (1.6) with $d = 2$. Denote

$$\widehat{\tilde{a}^{r,2d}}(\xi) := \frac{\widehat{a^{r,2d}}(\xi)}{\mathbf{q}^{r,2d}(2\xi)}, \qquad \xi \in \mathbb{R}^2 \tag{1.25}$$

with

$$\mathbf{q}^{r,2d}(2\xi_1, 2\xi_2) := \sum_{\gamma_1=0}^{1}\sum_{\gamma_2=0}^{1} |\widehat{a^{r,2d}}(\xi_1 + \pi\gamma_1, \xi_2 + \pi\gamma_2)|^2. \tag{1.26}$$

With all these, we present the following two main results of this paper. The first one is for the bivariate three directional box splines in (1.15).

Theorem 1.1. *The system $WS^\tau(\phi; \psi^1, \psi^2, \psi^3)$, as defined in* (1.2) *with $d = 2$ and $s = 3$, is a Riesz wavelet basis in $H^\tau(\mathbb{R}^2)$ for all $\tau \in O_2 := (-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), 2r - 1/2)$. Furthermore, O_2 is not empty.*

With some further efforts, we are able to prove that the wavelets ψ^1, ψ^2, ψ^3 in (1.6) from bivariate three directional box splines in (1.15) form a Riesz basis in $L_2(\mathbb{R}^2)$, i.e. we are able to prove that $\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) > 0$. This generalizes the main result of [29] where a special bivariate three directional box spline (i.e. $r = 2$ in (1.16)) from the Loop scheme is used to derive a Riesz wavelet basis in $L_2(\mathbb{R}^2)$ with small support; such a basis is used in a very efficient geometric mesh compression algorithm as proposed in [39].

Theorem 1.1 can be further generalized to trivariate box splines. However, as we shall see, the analysis is even more difficult and technical.

For every positive integer r, let $\hat{a} := \widehat{a^{r,3d}}$ in (1.19) be the mask for the trivariate box spline $\phi := \phi^{r,3d}$ in (1.18). Let $\widehat{b^1}, \ldots, \widehat{b^7}$ be defined in (1.12) and $\psi^1, \ldots, \psi^7$ as in (1.6) with $d = 3$. Denote

$$\widehat{\tilde{a}^{r,3d}}(\xi) := \frac{\widehat{a^{r,3d}}(\xi)}{\mathbf{q}^{r,3d}(2\xi)}, \qquad \xi \in \mathbb{R}^3 \tag{1.27}$$

with

$$\mathbf{q}^{r,3d}(2\xi_1, 2\xi_2, 2\xi_3) := \sum_{\gamma_1=0}^{1}\sum_{\gamma_2=0}^{1}\sum_{\gamma_3=0}^{1} |\widehat{a^{r,3d}}(\xi_1 + \pi\gamma_1, \xi_2 + \pi\gamma_2, \xi_3 + \pi\gamma_3)|^2. \tag{1.28}$$

Then we have:

Theorem 1.2. *The system $WS^\tau(\phi; \psi^1, \ldots, \psi^7)$, as defined in* (1.2) *with $d = 3$ and $s = 7$, is a Riesz wavelet basis in $H^\tau(\mathbb{R}^3)$ for all $\tau \in O_3 := (-\mu_2(\widehat{\tilde{a}^{r,3d}}, 2I_3), 2r - 1/2)$. Furthermore, O_3 is not empty.*

An outline of the paper is as follows. After presenting several auxiliary results that we shall prove in section 4, we prove Theorems 1.1 and 1.2 in section 2. In section 3, we shall discuss Riesz wavelets in $L_2(\mathbb{R}^d)$ for $d = 1$ and $d = 2$ by providing some examples of Riesz wavelets in L_2. In particular, we are able to show that the Riesz wavelets in Sobolev spaces in Theorem 1.1 are in fact also Riesz wavelets in $L_2(\mathbb{R}^2)$ for all $r \in \mathbb{N}$. In section 4, we shall develop some general results on Riesz wavelets in Sobolev spaces. Finally, in section 5 we prove several inequalities on bivariate and trivariate trigonometric polynomials which play a key role in our study of Riesz wavelets in Sobolev spaces from box splines in low dimensions.

2. Riesz Wavelets in Sobolev Spaces from Box Splines

We shall prove Theorems 1.1 and 1.2 in this section. For this, we first present some auxiliary results which are special cases of more general results in section 4.

For Riesz wavelets in Sobolev spaces, we have the following result, which is a special case of Theorem 4.1.

Corollary 2.1. *Let $\hat{a}, \widehat{b^1}, \ldots, \widehat{b^{2^d-1}}$ be 2π-periodic trigonometric polynomials in d-variables such that $\det \mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$, where $\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}$ is defined in (1.8). Define $\hat{\tilde{a}}(\xi)$ to be the $(1,1)$-entry of the matrix $\overline{\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi)}^{-1}$. Suppose that $\hat{a}(0) = \hat{\tilde{a}}(0) = 1$. Define ϕ and $\psi^1, \ldots, \psi^{2^d-1}$ by*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \hat{a}(2^{-j}\xi) \quad \text{and} \quad \widehat{\psi^\ell}(2\xi) := \widehat{b^\ell}(\xi)\hat{\phi}(\xi),\ \xi \in \mathbb{R}^d,\ \ell = 1, \ldots, 2^d - 1. \tag{2.1}$$

Then for any $\tau \in (-\mu_2(\hat{\tilde{a}}, 2I_d), \nu_2(\hat{a}, 2I_d))$, $WS^\tau(\phi; \psi^1, \ldots, \psi^{2^d-1})$ is a Riesz wavelet basis for $H^\tau(\mathbb{R}^d)$.

In order to find Riesz wavelets in Sobolev spaces, a major task is to estimate the quantity $\mu_2(\hat{\tilde{a}}, 2I_d)$ and $\mu_2(\hat{a}, 2I_d)$. To prove Theorems 1.1 and 1.2, we need the following result, which is a direct consequence of Theorem 4.5.

Proposition 2.2. *Let $\hat{a}$ be a 2π-periodic trigonometric polynomial such that*

$$\sum_{\omega \in \Omega_{2I_d}} |\hat{a}(\xi/2 + 2\pi\omega)| > 0 \qquad \forall\, \xi \in \mathbb{R}^d. \tag{2.2}$$

For every positive integer r, define a mask $\widehat{\tilde{a}^r}$ by

$$\widehat{\tilde{a}^r}(\xi) := \frac{(\hat{a}(\xi))^r}{\mathbf{q}^r(2\xi)} \quad \text{with} \quad \mathbf{q}^r(\xi) := \sum_{\omega \in \Omega_{2I_d}} |\hat{a}(\xi/2 + 2\pi\omega)|^{2r}. \tag{2.3}$$

Then

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant (r-1)[\nu_1(\hat{a}, 2I_d) - d] + \nu_2(\hat{a}, 2I_d) + r \log_2 \min\{\mathbf{q}^1(\xi) : \xi \in \mathbb{R}^d\}. \tag{2.4}$$

With these two auxiliary results, we are now ready to prove Theorem 1.1 as follows:

Proof of Theorem 1.1. To show that $\mathrm{WS}^\tau(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^d)$ for some $\tau \in \mathbb{R}$, by Corollary 2.1, we have to prove that the interval $(-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), \nu_2(\hat{a}^{r,2d}, 2I_2))$ is nonempty which is equivalent to

$$\nu_2(\widehat{a^{r,2d}}, 2I_2) + \mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) > 0. \tag{2.5}$$

It is well known that $\nu_2(\widehat{a^{r,2d}}, 2I_2) = 2r - 1/2$ and $\nu_1(\widehat{a^{r,2d}}, 2I_2) = 2r$ from box spline theory (see e.g.[1]). We now use Proposition 2.2 to estimate $\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2)$. By [29, Lemma 1] for which a simple proof will be given in Lemma 3.3 of this paper, we have $7/16 \leqslant \mathbf{q}^{1,2d}(\xi)$, for all $\xi \in \mathbb{R}^2$. Noting that $\nu_1(\hat{a}, 2I_2) = 2$ and $\nu_2(\hat{a}, 2I_2) = 3/2$ where $\hat{a} = \widehat{a^{1,2d}}$, and applying (2.4) of Proposition 2.2, we have

$$\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) \geqslant 0 + \nu_2(\hat{a}, 2I_2) + r \log_2(7/16) \geqslant 3/2 + r \log_2(7/16).$$

Consequently, by $\nu_2(\widehat{a^{r,2d}}, 2I_2) = 2r - 1/2$, we have

$$\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) + \nu_2(\widehat{a^{r,2d}}, 2I_2) \geqslant 1 + r\log_2(7/4) > 1 \tag{2.6}$$

for all positive integers r. That is, the open interval $O_2 = (-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), 2r - 1/2)$ is nonempty. Now by Corollary 2.1, $\mathrm{WS}^\tau(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^2)$ for all $\tau \in \mathbb{R}$ such that $\tau \in O_2$.

We note that the interval $(-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), 2r - 1/2)$ can be very large when r is large as shown in (2.6). This implies that the bivariate box spline wavelet basis constructed here can be a Riesz basis in a wide range of Sobolev spaces. In particular, we will show in the next section that they are Riesz bases in $L_2(\mathbb{R}^2)$ as well.

Next, we prove Theorem 1.2 for the trivariate case. For this, we need the following lemma whose proof will be delayed until the last section of this paper.

Lemma 2.3. *Let* $\mathbf{q}^{2,3d}$ *be defined in* (1.28). *Then*

$$\frac{1}{32} \leqslant \mathbf{q}^{2,3d}(\xi) \leqslant 1 \qquad \forall\, \xi \in \mathbb{R}^3. \tag{2.7}$$

With this, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.3, we have $1/32 \leqslant \mathbf{q}^{2,3d}(\xi) \leqslant 1$ for all $\xi \in \mathbb{R}^3$. Therefore, we have $\mathbf{q}^{1,3d}(\xi) \geqslant \sqrt{\mathbf{q}^{2,3d}(\xi)} \geqslant 2^{-5/2}$. For every $n \in \mathbb{N}$, applying Proposition 2.2 with $\hat{a} = \widehat{a^{n,3d}}$ and $r = 1$, we have $\nu_2(\widehat{a^{n,3d}}, 2I_3) = 2n - 1/2$ and

$$\mu_2(\widehat{\tilde{a}^{n,3d}}, 2I_3) \geqslant 0 + \nu_2(\widehat{a^{n,3d}}, 2I_3) + \log_2 \min_{\xi\in\mathbb{R}^3} \mathbf{q}^{n,3d}(\xi) = 2n - 1/2 + \log_2 \min_{\xi\in\mathbb{R}^3} \mathbf{q}^{n,3d}(\xi). \tag{2.8}$$

If $n = 1$, since $\mathbf{q}^{1,3d}(\xi) \geqslant 2^{-5/2}$, by (2.8), we have $\mu_2(\widehat{\tilde{a}^{1,3d}}, 2I_3) \geqslant 2 - 1/2 - 5/2 = -1$. Hence we obtain $\mu_2(\widehat{\tilde{a}^{1,3d}}, 2I_3) + \nu_2(\widehat{a^{1,3d}}, 2I_3) \geqslant (-1) + (2 - 1/2) = 1/2 > 0$. If $n \geqslant 2$, then by Cauchy-Schwarz inequality, we have $\mathbf{q}^{n,3d}(\xi) \geqslant \left[\mathbf{q}^{2,3d}(\xi)\right]^{n/2} 2^{3(1-n/2)} \geqslant 2^{-5n/2}2^{3-3n/2} = 2^{3-4n}$. Hence by (2.8) we obtain

$$\mu_2(\widehat{\tilde{a}^{n,3d}}, 2I_3) \geqslant 2n - 1/2 + (3 - 4n) = 5/2 - 2n.$$

Therefore we have $\mu_2(\widehat{\tilde{a}^{n,3d}}, 2I_3) + \nu_2(\widehat{a^{n,3d}}, 2I_3) \geqslant (5/2 - 2n) + (2n - 1/2) = 2 > 0$. Consequently, for all $r \in \mathbb{N}$, we have $\mu_2(\widehat{\tilde{a}^{r,3d}}, 2I_3) + \nu_2(\widehat{a^{r,3d}}, 2I_3) > 0$. That is, the open interval $O_3 = (-\mu_2(\widehat{\tilde{a}^{r,3d}}, 2I_2), 2r - 1/2)$ is not empty. Now by Corollary 2.1, $\mathrm{WS}^\tau(\phi; \psi^1, \ldots, \psi^7)$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^3)$ for all $\tau \in O_3$.

3. Riesz Wavelets in $L_2(\mathbb{R}^2)$

In this section we discuss Riesz wavelets in $L_2(\mathbb{R}^2)$. In particular, we will show that the Riesz wavelets in Sobolev spaces in Theorem 1.1 are in fact also Riesz wavelets in $L_2(\mathbb{R}^2)$ for all $r \in \mathbb{N}$.

We start with the following simple observation. Let $\widehat{\tilde{a}^r}$ and $\mathbf{q}^r$ be defined in (2.3) for a given mask $\hat{a}$. If

$$\mathbf{q}^1(\xi) = \sum_{\omega\in\Omega_{2I_d}} |\hat{a}(\xi/2 + 2\pi\omega)|^2 \geqslant \rho > 0 \qquad \forall\, \xi \in \mathbb{R}^d, \tag{3.1}$$

then by (2.4) of Proposition 2.2, we have

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant (r-1)[\nu_1(\hat{a}, 2I_d) - d] + \nu_2(\hat{a}, 2I_d) + r\log_2 \rho. \tag{3.2}$$

Furthermore, when the mask satisfies

$$\sum_{\omega\in\Omega_{2I_d}} |\hat{a}(\xi/2+2\pi\omega)| \geqslant 1, \tag{3.3}$$

we can easily deduce that (3.1) holds with $\rho = 2^{-d}$ and therefore, (3.2) becomes

$$\mu_2(\widehat{\hat{a}^r}, 2I_d) \geqslant (r-1)[\nu_1(\hat{a}, 2I_d) - 2d] + \nu_2(\hat{a}, 2I_d) - d.$$

Note that we always have $\nu_2(\hat{a}, 2I_d) \geqslant \nu_1(\hat{a}, 2I_d) - d/2$. So, if (3.3) holds and $\nu_1(\hat{a}, 2I_d) \geqslant 2d$, then for any positive integer r, we must have

$$\begin{aligned}\mu_2(\widehat{\hat{a}^r}, 2I_d) &\geqslant (r-1)[\nu_1(\hat{a}, 2I_d) - 2d] + \nu_2(\hat{a}, 2I_d) - d \geqslant \nu_2(\hat{a}, 2I_d) - d\\ &\geqslant \nu_1(\hat{a}, 2I_d) - 3d/2 \geqslant d/2 > 0.\end{aligned}$$

We remark that there are many masks satisfying (3.3). For example, an interpolatory mask $\hat{a}$ satisfies $\sum_{\omega\in\Omega_{2I_d}} \hat{a}(\xi/2+2\pi\omega) = 1$ and consequently, (3.3) holds.

With the above observations, we are able to reproduce [28, Theorem 2.2] by a simple argument as shown in the next example when the univariate B-splines are used. This example is given here to demonstrate the power of the technical analysis given in the next section.

Example 3.1. *Let $\hat{a}(\xi) = (1+e^{-i\xi})/2$. For any positive integer r, define $\widehat{a^r}(\xi) := (\hat{a}(\xi))^r$ and $\widehat{\hat{a}^r}, \mathbf{q}^r$ as in (2.3). Define*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a^r}(2^{-j}\xi) \quad \textit{and} \quad \hat{\psi}(2\xi) := e^{-i\xi}\overline{\widehat{a^r}(\xi+\pi)}\hat{\phi}(\xi). \tag{3.4}$$

Then ϕ is the B-spline of order r and $WS^0(\phi;\psi)$ is a Riesz wavelet basis in $L_2(\mathbb{R})$ for all positive integers r.

Proof. Since $|\hat{a}(\xi)|^2 + |\hat{a}(\xi+\pi)|^2 = 1$, (3.1) is obviously true with $\rho = 1$. Note that $\nu_1(\hat{a}, 2) = 1$ and $\nu_2(\hat{a}, 2) = 1/2$. Consequently, by (3.2), we have

$$\mu_2(\widehat{\hat{a}^r}, 2) \geqslant (r-1)[\nu_1(\hat{a}, 2) - 1] + \nu_2(\hat{a}, 2) = 1/2 > 0.$$

Since $\nu_2(\widehat{a^r}, 2) = 2r-1$, by Corollary 2.1, $\mathrm{WS}^0(\phi;\psi)$ is a Riesz wavelet basis in $L_2(\mathbb{R})$ for all positive integers r.

Similarly, the following example generalizes [28, Theorem 5.3].

Example 3.2. *For a positive integer n, define*

$$P_n(x) := \sum_{j=0}^{n-1} \frac{(n+j-1)!}{j!(n-1)!} x^j, \qquad x \in \mathbb{R}$$

and $\hat{a}(\xi) := \cos^{2n}(\xi/2) P_n(\sin^2(\xi/2))$. For any positive integer r, define $\widehat{a^r}(\xi) := (\hat{a}(\xi))^r$ and $\widehat{\hat{a}^r}, \mathbf{q}^r$ as in (2.3). Define $\hat{\phi}$ and $\hat{\psi}$ as in (3.4). Then $WS^0(\phi;\psi)$ is a Riesz wavelet basis in $L_2(\mathbb{R})$ for all positive integers r and n.

Proof. Note that $\hat{a}(\xi) + \hat{a}(\xi+\pi) = 1$. Therefore, (3.3) holds. It is known ([14]) that $\nu_1(\hat{a}, 2) \geqslant \nu_2(\hat{a}, 2) > 2$ for all $n > 1$. For $n = 1$, we have $\nu_1(\hat{a}, 2) = 2$ and $\nu_2(\hat{a}, 2) = 3/2$. By the above argument before Example 3.1, we have

$$\mu_2(\widehat{a^r}, 2) \geqslant (r-1)[\nu_1(\hat{a}, 2) - 1] + \nu_2(\hat{a}, 2) + r\log_2 2^{-1} \geqslant \nu_2(\hat{a}, 2) - 1 \geqslant 1/2 > 0.$$

Now by Corollary 2.1, $\mathrm{WS}^0(\phi;\psi)$ is a Riesz wavelet basis in $L_2(\mathbb{R})$ for all positive integers r and n.

Next, we come back to our main focus of this section to discuss the Riesz property in $L_2(\mathbb{R}^2)$. To do so, we need several auxiliary results. The first one given below is used in our analysis whose proof is given in section 5.

Lemma 3.3. *Let $\mathbf{q}^{r,2d}$ be defined in (1.26). Then*

$$\frac{3^{2r+1}+1}{2^{6r}} \leqslant \mathbf{q}^{r,2d}(\xi) \leqslant 1 \tag{3.5}$$

for $r = 1, 2, 3$ with the lower and upper bounds sharp.

The next result is a special case of item (4) of Theorem 4.4, which is needed as well.

Corollary 3.4. *Let $\mathbf{p}, \mathbf{q}$ be 2π-periodic trigonometric polynomials such that $\mathbf{q}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$. Then*

$$\nu_2(\mathbf{pq}, 2I_d) = \nu_2(\mathbf{pq}(2\cdot), 2I_d). \tag{3.6}$$

We start our discussion of Riesz wavelets in $L_2(\mathbb{R}^2)$ derived from bivariate three-directional box splines with three special examples.

Example 3.5. For $r = 1$, let $\mathbf{p}(\xi) := \widehat{a^{1,2d}}$ as in (1.16) and $\mathbf{q}(\xi) := \mathbf{q}^{1,2d}(\xi)$ as in (1.26). By Lemma 3.3, $\mathbf{q}_{min} = 7/16 \leqslant \mathbf{q}(\xi) \leqslant 1 = \mathbf{q}_{max}$. Applying Proposition 4.2 with $n_1 = 16$ and $n_2 = 1$, we have 2π-periodic trigonometric polynomials $\mathbf{q}^1, \mathbf{q}^2$ such that

$$\mathbf{q}^1(0) = \mathbf{q}^2(0) = \mathbf{q}(0) = 1 \quad \text{and} \quad 0 < \mathbf{q}^1(\xi) \leqslant 1/\mathbf{q}(\xi) \leqslant \mathbf{q}^2(\xi), \qquad \xi \in \mathbb{R}^2 \tag{3.7}$$

and the time domain coefficient sequences of both masks $\mathbf{pq}^1$ and $\mathbf{pq}^2$ are supported inside $[-18, 18]^2$. Using [20, Algorithm 2.1] with the symmetry group D_6, we have

$$\nu_2(\mathbf{pq}^2, 2I_2) > 0.9902360540 \quad \text{and} \quad \nu_2(\mathbf{pq}^1, 2I_2) < 0.9902360544.$$

By Corollary 3.4, we have

$$\nu_2(\mathbf{pq}^1(2\cdot), 2I_2) = \nu_2(\mathbf{pq}^1, 2I_2) \quad \text{and} \quad \nu_2(\mathbf{pq}^2(2\cdot), 2I_2) = \nu_2(\mathbf{pq}^2, 2I_2).$$

On the other hand, by (3.7), we have $\mathbf{p}(0)\mathbf{q}^1(0) = \mathbf{p}(0)\mathbf{q}^2(0) = \mathbf{p}(0)/\mathbf{q}(0) = 1$ and $|\mathbf{p}(\xi)\mathbf{q}^1(2\xi)| \leqslant |\mathbf{p}(\xi)/\mathbf{q}(2\xi)| \leqslant |\mathbf{p}(\xi)\mathbf{q}^2(2\xi)|$ for all $\xi \in \mathbb{R}^2$. Consequently, noting $\widehat{\tilde{a}^{1,2d}} = \mathbf{p}/\mathbf{q}(2\cdot)$, we conclude that

$$0.9902360540 < \nu_2(\mathbf{pq}^2(2\cdot), 2I_2) \leqslant \mu_2(\widehat{\tilde{a}^{1,2d}}, 2I_2) \leqslant \nu_2(\mathbf{pq}^1(2\cdot), 2I_2) < 0.9902360544.$$

Then the open interval $(-\mu_2(\widehat{\tilde{a}^{1,2d}}, 2I_d), \mu_2(\widehat{a^{1,2d}}, 2I_d))$ contains 0. Hence, $\mathrm{WS}^0(\phi;\, \psi^1,\, \psi^2,\, \psi^3)$ is a Riesz basis of $L_2(\mathbb{R}^2)$.

Example 3.6. For $r = 2$, let $\mathbf{p}(\xi) := \widehat{a^{2,2d}}$ as in (1.16) and $\mathbf{q}(\xi) := \mathbf{q}^{2,2d}(\xi)$ as in (1.26). By Lemma 3.3, $\mathbf{q}_{min} = \frac{61}{1024} \leqslant \mathbf{q}(\xi) \leqslant 1 = \mathbf{q}_{max}$. Applying Proposition 4.2 with $n_1 = 10$ and $n_2 = 1$, we have 2π-periodic trigonometric polynomials $\mathbf{q}^1, \mathbf{q}^2$ such that (3.7) holds and the time domain coefficient sequences of both masks $\mathbf{pq}^1$ and $\mathbf{pq}^2$ are supported inside $[-32, 32]^2$. Using [20, Algorithm 2.1] with the symmetry group D_6, we have

$$\nu_2(\mathbf{pq}^2, 2I_2) > 1.9173353357 \quad \text{and} \quad \nu_2(\mathbf{pq}^1, 2I_2) < 2.1294230728.$$

By the same argument as in Example 3.5, we have

$$1.9173353357 < \nu_2(\mathbf{pq}^2(2\cdot), 2I_2) \leqslant \mu_2(\widehat{\tilde{a}^{2,2d}}, 2I_2) \leqslant \nu_2(\mathbf{pq}^1(2\cdot), 2I_2) < 2.1294230728.$$

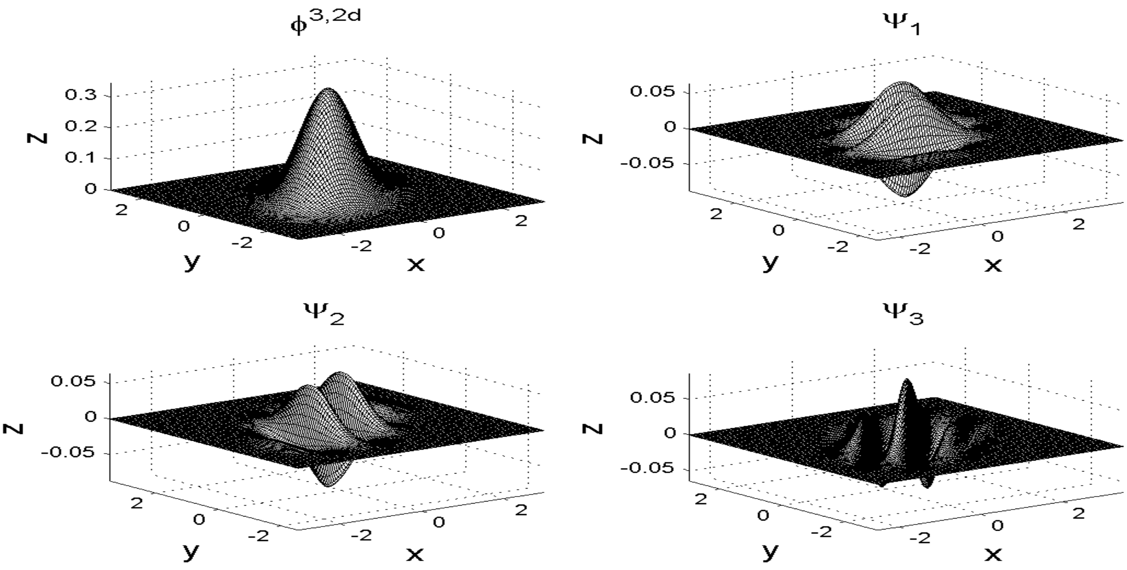

FIGURE 1. The refinable box spline function $\phi^{3,2d}$ defined in (1.15) and the associated three wavelet functions ψ^1, ψ^2, ψ^3 in Example 3.7.

Then the open interval $(-\mu_2(\widehat{\tilde{a}^{2,2d}}, 2I_d), \mu_2(\widehat{a^{2,2d}}, 2I_d))$ contains 0. Hence, by Corollary 2.1, $\mathrm{WS}^0(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz basis of $L_2(\mathbb{R}^2)$. The graphs of $\phi^{2,2d}$ and the wavelet functions ψ^1, ψ^2, ψ^3 are given in [29, Figure 1].

This result was first proven in [29]. The wavelets are successfully used in [39] in a very efficient geometric mesh compression algorithm based on the Loop scheme. Comparing with [29, page 626], Example 3.6 greatly improves the estimate of $\mu_2(\widehat{\tilde{a}^{2,2d}}, 2I_2)$ by using Corollary 3.4 and Proposition 4.2.

Example 3.7. For $r = 3$, let $\mathbf{p}(\xi) := \widehat{a^{3,2d}}$ as in (1.16) and $\mathbf{q}(\xi) := \mathbf{q}^{3,2d}(\xi)$ as in (1.26). By Lemma 3.3, $\mathbf{q}_{min} = \frac{547}{65536} \leqslant \mathbf{q}(\xi) \leqslant 1 = \mathbf{q}_{max}$. Applying Proposition 4.2 with $n_1 = 8$ and $n_2 = 1$, we have 2π-periodic trigonometric polynomials $\mathbf{q}^1, \mathbf{q}^2$ such that (3.7) holds and the time domain coefficient sequences of both masks $\mathbf{pq}^1$ and $\mathbf{pq}^2$ are supported inside $[-30, 30]^2$. Using [20, Algorithm 2.1] with the symmetry group D_6, we have

$$\nu_2(\mathbf{pq}^2, 2I_2) > 2.0942071410 \quad \text{and} \quad \nu_2(\mathbf{pq}^1, 2I_2) < 3.4201343730.$$

By the same argument as in Example 3.5, we have

$$2.0942071410 < \nu_2(\mathbf{pq}^2(2\cdot), 2I_2) \leqslant \mu_2(\widehat{\tilde{a}^{3,2d}}, 2I_2) \leqslant \nu_2(\mathbf{pq}^1(2\cdot), 2I_2) < 3.4201343730.$$

Then the open interval $(-\mu_2(\widehat{\tilde{a}^{3,2d}}, 2I_d), \mu_2(\widehat{a^{3,2d}}, 2I_d))$ contains 0. Hence, by Corollary 2.1, $\mathrm{WS}^0(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz basis of $L_2(\mathbb{R}^2)$. Please see Figure 1 for the graphs.

Motivated by the above examples, we now discuss the general case of Riesz wavelets derived from bivariate three directional box splines with any multiplicity r. To do so, we need the following general result on estimating $\mu_2(\widehat{\tilde{a}^r}, 2I_d)$, which we shall prove at the end of section 4.

Theorem 3.8. *Let mask $\hat{a}$ be a 2π-periodic trigonometric polynomial in d-variables with $1 \leqslant d \leqslant 3$ such that* (1.10) *holds when $1 \leqslant d \leqslant 3$ and $\sum_{\omega \in \Omega_{2I_d}} |\hat{a}(\xi/2 + 2\pi\omega)| \neq 0$ for all $\xi \in \mathbb{R}^d$. For* *every positive integer r, define*

$$\widehat{a^r}(\xi) := (\hat{a}(\xi))^r \quad \text{and} \quad \widehat{\tilde{a}^r}(\xi) := \frac{(\hat{a}(\xi))^r}{\mathbf{q}^r(2\xi)} \tag{3.8}$$

with

$$\mathbf{q}^r(\xi) := \sum_{\omega\in\Omega_{2I_d}} |\widehat{a^r}(\xi/2 + 2\pi\omega)|^2 = \sum_{\omega\in\Omega_{2I^d}} |\hat{a}(\xi/2 + 2\pi\omega)|^{2r}. \tag{3.9}$$

Let the high-pass wavelet masks $\widehat{b^1}, \dots, \widehat{b^{2^d-1}}$ be derived via (1.9), (1.11), or (1.12) by replacing $\hat{a}$ by $\widehat{a^r}$. Let ϕ be the standard refinable function defined by $\hat{\phi}(\xi) := \prod_{j=1}^\infty \widehat{a^r}(2^{-j}\xi)$. Define the wavelet functions $\psi^1, \dots, \psi^{2^d-1}$ as in (1.6). Then for every $\tau \in (-\mu_2(\widehat{\tilde{a}^r}, 2I_d), \mu_2(\widehat{a^r}, 2I_d))$, $WS^\tau(\phi; \psi^1, \dots, \psi^{2^d-1})$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^d)$. Let $s \in \mathbb{N}$ and $\widehat{\mathring{\tilde{a}}^\ell}, 0 < \ell \leqslant s$ be 2π-periodic trigonometric polynomials such that

$$\widehat{\mathring{\tilde{a}}^\ell}(0) = \widehat{\tilde{a}^\ell}(0) \quad \text{and} \quad |\widehat{\tilde{a}^\ell}(\xi)| \leqslant |\widehat{\mathring{\tilde{a}}^\ell}(\xi)| \qquad \forall\, \xi \in \mathbb{R}^d,\ \ell = 1, \dots, s. \tag{3.10}$$

Then for any $r = n_r s + \ell$ with $n_r \in \mathbb{N}$ and $0 < \ell \leqslant s$,

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant n_r[\nu_1(\widehat{\mathring{\tilde{a}}^s}, 2I_d) - d] + \nu_2(\widehat{\mathring{\tilde{a}}^\ell}, 2I_d) \geqslant n_r[\nu_2(\widehat{\mathring{\tilde{a}}^s}, 2I_d) - d] + \nu_2(\widehat{\mathring{\tilde{a}}^\ell}, 2I_d). \tag{3.11}$$

In particular, we have

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant n_r[\mu_2(\widehat{\tilde{a}^s}, 2I_d) - d] + \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) \tag{3.12}$$

and

$$\nu_2(\widehat{a^r}, 2I_d) \geqslant n_r\nu_1(\widehat{a^s}, 2I_d) + \nu_2(\widehat{a^\ell}, 2I_d). \tag{3.13}$$

If for some positive integer $s \in \mathbb{N}$ such that

$$\nu_1(\widehat{a^s}, 2I_d) + \mu_2(\widehat{\tilde{a}^s}, 2I_d) \geqslant d \quad \text{and} \quad \nu_2(\widehat{a^\ell}, 2I_d) + \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) > 0, \qquad \ell = 1, \dots, s, \tag{3.14}$$

then the open interval $(-\mu_2(\widehat{\tilde{a}^r}, 2I_d), \mu_2(\widehat{a^r}, 2I_d))$ is nonempty for all $r \in \mathbb{N}$. If

$$\nu_2(\hat{a}, 2I_d) > 0 \quad \text{and} \quad \mu_2(\widehat{\tilde{a}^s}, 2I_d) \geqslant d, \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) > 0,\ \ell = 1, \dots, s-1, \tag{3.15}$$

then the open interval $(-\mu_2(\widehat{\tilde{a}^r}, 2I_d), \mu_2(\widehat{a^r}, 2I_d))$ contains 0 for all $r \in \mathbb{N}$.

For bivariate three directional box splines, we are able to show that the Riesz wavelets in Sobolev spaces in Theorem 1.1 are in fact also Riesz wavelets in $L_2(\mathbb{R}^2)$ for all $r \in \mathbb{N}$.

Theorem 3.9. *The system $WS^0(\phi; \psi^1, \psi^2, \psi^3)$ in Theorem 1.1 with $\tau = 0$ is a Riesz wavelet basis for $L_2(\mathbb{R}^2)$. More precisely, $WS^\tau(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz wavelet basis in $H^\tau(\mathbb{R}^2)$ for all $\tau \in O_2 := (-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), 2r - 1/2)$ with $\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) \geqslant 0.03140238034r + 0.9588336737 > 0$.*

Proof. It suffices to show that the interval $(-\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2), \nu_2(\widehat{a^{r,2d}}, 2I_2))$ contains the point 0. Note that $\nu_2(\widehat{a^{r,2d}}, 2I_2) = 2r - 1 > 0$ for all $r \in \mathbb{N}$. Now we prove $\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) > 0$, for which we shall show (3.15) in Theorem 3.8 with $s = 3$. By our example, we have $\mu_2(\widehat{\tilde{a}^{3,2d}}, 2I_2) \geqslant 2.09420714102 > 2$. By Example 3.5, $\mu_2(\widehat{\tilde{a}^{1,2d}}, 2I_2) > 0.99023605401$. By Example 3.6, we have

$\mu_2(\widehat{\tilde{a}^{2,2d}}, 2I_2) > 1.91733533578$. Consequently, for all $r = 3n + \ell$ with $n \geqslant 1$ and $1 \leqslant \ell \leqslant 3$, it follows from (3.12) that

$$\begin{aligned}\mu_2(\widehat{\tilde{a}^{r,2d}}, 2I_2) &\geqslant \frac{r-\ell}{3}[\mu_2(\widehat{\tilde{a}^{3,2d}}, 2I_2) - 2] + \mu_2(\widehat{\tilde{a}^{\ell,2d}}, 2I_2)\\ &\geqslant 0.03140238034r + \mu_2(\widehat{\tilde{a}^{\ell,2d}}, 2I_2) - 0.03140238034\ell\\ &\geqslant 0.03140238034r + 0.9588336737 > 0.\end{aligned}$$

By Corollary 2.1, $\mathrm{WS}^0(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz wavelet basis in $L_2(\mathbb{R}^2)$.

Let $\hat{a} := \widehat{a^{r,2d}}$ in (1.16) be the mask for the three directional bivariate box spline $\phi := \phi^{r,2d}$ in (1.15). Let $\widehat{b^1}, \widehat{b^2}, \widehat{b^3}$ be defined in (1.11) and ψ^1, ψ^2, ψ^3 be defined in (1.6) with $d = 2$. Next, we consider the homogeneous wavelet system

$$\mathrm{WS}(\psi^1, \psi^2, \psi^3) := \{\psi^\ell_{j,k} := 2^j \psi^\ell(2^j \cdot -k) \ : \ j \in \mathbb{Z},\ k \in \mathbb{Z}^2, \ell = 1, 2, 3\} \tag{3.16}$$

and we have the following result:

Corollary 3.10. *The system WS(ψ^1, ψ^2, ψ^3) in* (3.16) *is a Riesz basis in* $L_2(\mathbb{R}^2)$.

Proof. We have proved that $\mathrm{WS}^0(\phi; \psi^1, \psi^2, \psi^3)$ is a Riesz wavelet basis for $L_2(\mathbb{R}^2)$. By [22, Proposition 3], $\mathrm{WS}(\psi^1, \psi^2, \psi^3)$ in (3.16) is a Riesz basis in $L_2(\mathbb{R}^2)$.

4. General Results on Riesz Wavelets in Sobolev Spaces

In this section, we shall develop some general results on Riesz wavelets in Sobolev spaces and some results on estimating the quantity $\mu_2(\hat{a}, 2I_d)$. The general results developed in this section include the auxiliary results stated in section 2 as special cases.

Let us recall the definition of a pair of Riesz wavelets in a pair of Sobolev spaces. Let ϕ, ψ^1, …, ψ^s belong to $H^\tau(\mathbb{R}^d)$ and let $\tilde{\phi}, \tilde{\psi}^1, \dots, \tilde{\psi}^s$ belong to $H^{-\tau}(\mathbb{R}^d)$. We say that

$$(\mathrm{WS}^\tau(\phi; \psi^1, \dots, \psi^s), \mathrm{WS}^{-\tau}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^s))$$

is *a pair of dual Riesz wavelet bases* in the pair of dual Sobolev spaces $(H^\tau(\mathbb{R}^d), H^{-\tau}(\mathbb{R}^d))$ if

1. $\mathrm{WS}^\tau(\phi; \psi^1, \dots, \psi^s)$ is a Riesz basis of the Sobolev space $H^\tau(\mathbb{R}^d)$.
2. $\mathrm{WS}^{-\tau}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^s)$ is a Riesz basis of the Sobolev space $H^{-\tau}(\mathbb{R}^d)$.
3. $\mathrm{WS}^\tau(\phi; \psi^1, \dots, \psi^s)$ and $\mathrm{WS}^{-\tau}(\tilde{\phi}; \tilde{\psi}^1, \dots, \tilde{\psi}^s)$ are biorthogonal to each other: for all $k, k' \in \mathbb{Z}^d$, $j, j' \in \mathbb{N}_0$, and $\ell, \ell' = 1, \dots, s$,

$$\begin{aligned}&\langle \phi_{0,k}, \tilde{\phi}_{0,k'}\rangle = \delta(k-k'), \quad \langle \psi^{\ell,\tau}_{j,k}, \tilde{\psi}^{\ell',-\tau}_{j',k'}\rangle = \delta(j-j')\delta(k-k')\delta(\ell-\ell'),\\ &\langle \phi_{0,k}, \tilde{\psi}^{\ell',-\tau}_{j',k'}\rangle = 0, \quad \langle \psi^{\ell,\tau}_{j,k}, \tilde{\phi}_{0,k'}\rangle = 0,\end{aligned} \tag{4.1}$$

where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$.

We obtain the following result on Riesz wavelets in Sobolev spaces from [30, Theorem 3.1].

Theorem 4.1. *Let $\hat{a}, \widehat{b^1}, \dots, \widehat{b^{2^d-1}}$ be 2π-periodic measurable functions in d-variables such that* (1.7) *is satisfied. Define $\hat{\tilde{a}}(\xi)$ to be the* $(1,1)$*-entry of the matrix $\overline{\mathcal{P}_{[\hat{a}, \widehat{b^1}, \dots, \widehat{b^{2^d-1}}]}(\xi)}^{-1}$. Suppose that there exist positive numbers ε, λ and C such that*

$$|1 - \hat{a}(\xi)| \leqslant C|\xi|^\lambda \quad \textit{and} \quad |1 - \hat{\tilde{a}}(\xi)| \leqslant C|\xi|^\lambda, \qquad \textit{a.e. } \xi \in (-\varepsilon, \varepsilon). \tag{4.2}$$

Define ϕ and $\psi^1, \dots, \psi^{2^d-1}$ in the frequency domain as in (2.1)*. Then for any real number τ such that $\tau \in (-\mu_2(\hat{\tilde{a}}, 2I_d), \mu_2(\hat{a}, 2I_d))$, $WS^\tau(\phi; \psi^1, \dots, \psi^{2^d-1})$ is a Riesz wavelet basis for $H^\tau(\mathbb{R}^d)$.*

Proof. Let $\widehat{\tilde{b}^\ell}(\xi)$ denote the $(1,\ell+1)$-entry of the matrix $\overline{\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi)}^{-1}$, $\ell=1,\ldots,2^d-1$. Then $\mathcal{P}_{[\hat{\tilde{a}},\widehat{\tilde{b}^1},\ldots,\widehat{\tilde{b}^{2^d-1}}]}(\xi) = \overline{\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi)}^{-T}$. That is, we have $\overline{\mathcal{P}_{[\hat{\tilde{a}},\widehat{\tilde{b}^1},\ldots,\widehat{\tilde{b}^{2^d-1}}]}(\xi)}^{T}\mathcal{P}_{[\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}]}(\xi) = I_{2^d}$. Define

$$\hat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty}\hat{\tilde{a}}(2^{-j}\xi) \quad \text{and} \quad \widehat{\tilde{\psi}^\ell}(2\xi) := \widehat{\tilde{b}^\ell}(\xi)\hat{\tilde{\phi}}(\xi),\ \xi\in\mathbb{R}^d,\ \ell=1,\ldots,2^d-1. \tag{4.3}$$

By our assumption in (4.2), the two infinite products in (2.1) and (4.3) are well-defined for almost every $\xi\in\mathbb{R}^d$. Hence, all $\hat{\phi},\widehat{\psi^1},\ldots,\widehat{\psi^{2^d-1}}$ in (2.1) and $\hat{\tilde{\phi}},\widehat{\tilde{\psi}^1},\ldots,\widehat{\tilde{\psi}^{2^d-1}}$ in (4.3) are well-defined measurable functions on $\mathbb{R}^d$.

We apply [30, Theorem 3.1] in our proof. Note that the assumptions that $\hat{a}(0)=\hat{\tilde{a}}(0)=1$ and all $\hat{a},\widehat{b^1},\ldots,\widehat{b^{2^d-1}}$ and $\hat{\tilde{a}},\widehat{\tilde{b}^1},\ldots,\widehat{\tilde{b}^{2^d-1}}$ have exponential decay in [30, Theorem 3.1] are only used in the proof of [30, Theorem 3.1] to guarantee the convergence of the two infinite products in the definition of $\hat{\phi}$ and $\hat{\tilde{\phi}}$. In other words, as long as the two infinite products in (2.1) and (4.3) converge for almost every $\xi\in\mathbb{R}^d$, without the exponential decay assumption on the masks, the conclusion in Theorem 3.1 of [30] still holds.

For any $\tau\in(-\mu_2(\hat{\tilde{a}},2I_d),\mu_2(\hat{a},2I_d))$, we certainly have $\mu_2(\hat{\tilde{a}},2I_d)>-\tau$ and $\mu_2(\hat{a},2I_d)>\tau$, that is, [30, (1.26)] is satisfied. Now by [30, Theorem 3.1], we see that

$$(\mathrm{WS}^\tau(\phi;\psi^1,\ldots,\psi^{2^d-1}),\mathrm{WS}^{-\tau}(\tilde{\phi};\tilde{\psi}^1,\ldots,\psi^{2^d-1}))$$

is a pair of dual Riesz wavelet bases in the pair of Sobolev spaces $(H^\tau(\mathbb{R}^d),H^{-\tau}(\mathbb{R}^d))$ for any $\tau\in\mathbb{R}$ such that $\tau\in(-\mu_2(\hat{\tilde{a}},2I_d),\mu_2(\hat{a},2I_d))$. In particular, a wavelet system $\mathrm{WS}^\tau(\phi;\psi^1,\ldots,\psi^{2^d-1})$ is a Riesz wavelet basis for $H^\tau(\mathbb{R}^d)$.

Now it is easy to see that Corollary 2.1 is a special case of Theorem 4.1.

In order use Theorem 4.1, the key step is to estimate $\mu_2(\hat{a},2I_d)$. For a 2π-periodic trigonometric polynomial mask $\hat{a}$, one can use [20, Algorithm 2.1] to efficiently compute $\nu_2(\hat{a},2I_d)$ by taking into account symmetry and finding the spectral radius of a finite matrix. The following result is essentially known in [27, Theorem 3.2] to estimate $\mu_2(\hat{a},2)$ for a univariate matrix mask $\hat{a}$. For the purpose of completeness, we state and prove a slightly generalized version of [27, Theorem 3.2] in any dimension. We shall also need the following result in the proof of Theorem 4.4, which is one of the main results in this section.

Proposition 4.2. *Suppose that* $\mathbf{p}$ *and* $\mathbf{q}$ *are* 2π*-periodic trigonometric polynomials in* d*-variables such that there exist two positive numbers* $\mathbf{q}_{min}$ *and* $\mathbf{q}_{max}$ *satisfying*

$$0<\mathbf{q}_{min}\leqslant\mathbf{q}(\xi)\leqslant\mathbf{q}_{max} \qquad \forall\,\xi\in\mathbb{R}^d. \tag{4.4}$$

For all nonnegative integers n_1,n_2*, define*

$$\begin{aligned}\mathbf{q}^1_{n_1,n_2}(\xi) &:= \frac{2}{\mathbf{q}_{max}+\mathbf{q}_{min}}\sum_{j=0}^{n_1-1}\left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^j\\ &+\left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^{n_1}\left(\frac{(1-\mathbf{q}(\xi)/\mathbf{q}(0))^{n_2}}{\mathbf{q}_{max}}+\frac{1}{\mathbf{q}(0)}\sum_{\ell=0}^{n_2-1}\left(1-\frac{\mathbf{q}(\xi)}{\mathbf{q}(0)}\right)^\ell\right)\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\mathbf{q}^2_{n_1,n_2}(\xi) &:= \frac{2}{\mathbf{q}_{max}+\mathbf{q}_{min}} \sum_{j=0}^{n_1-1} \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^j \\ &+ \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^{n_1} \left(\frac{(1-\mathbf{q}(\xi)/\mathbf{q}(0))^{n_2}}{\mathbf{q}_{min}} + \frac{1}{\mathbf{q}(0)} \sum_{\ell=0}^{n_2-1} \left(1-\frac{\mathbf{q}(\xi)}{\mathbf{q}(0)}\right)^\ell\right).\end{aligned} \tag{4.6}$$

Then both $\mathbf{q}^1_{n_1,n_2}$ *and* $\mathbf{q}^2_{n_1,n_2}$ *converge to* $1/\mathbf{q}$ *exponentially fast in* $L_\infty(\mathbb{R}^d)$ *as* $n_1 \to \infty$*. More precisely,*

$$\begin{aligned}&\max(|\mathbf{q}^1_{n_1,n_2}(\xi) - 1/\mathbf{q}(\xi)|, |\mathbf{q}^2_{n_1,n_2}(\xi) - 1/\mathbf{q}(\xi)|) \leqslant \\ &\max\left(\left|1-\frac{\mathbf{q}_{max}}{\mathbf{q}(0)}\right|^{n_2}, \left|1-\frac{\mathbf{q}_{min}}{\mathbf{q}(0)}\right|^{n_2}\right) \left(\frac{1}{\mathbf{q}_{min}} - \frac{1}{\mathbf{q}_{max}}\right) \left(\frac{\mathbf{q}_{max}-\mathbf{q}_{min}}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^{n_1}.\end{aligned} \tag{4.7}$$

In addition, for all even integers $n_1 \geqslant 0$ *and* $n_2 > 0$ *(when* $\mathbf{q}_{max} = \mathbf{q}(0)$*,* n_2 *can be any positive integer), then*

$$\mathbf{q}^1_{n_1,n_2}(0) = \mathbf{q}^2_{n_1,n_2}(0) = \mathbf{q}(0) \text{ and } 0 \leqslant \mathbf{q}^1_{n_1,n_2}(\xi) \leqslant \frac{1}{\mathbf{q}(\xi)} \leqslant \mathbf{q}^2_{n_1,n_2}(\xi) \quad \forall\, \xi \in \mathbb{R}^d. \tag{4.8}$$

and

$$\nu_2(\mathbf{p}\mathbf{q}^2_{n_1,n_2}, 2I_d) \leqslant \mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \leqslant \nu_2(\mathbf{p}\mathbf{q}^2_{n_1,n_2}, 2I_d). \tag{4.9}$$

Proof. The proof follows ideas from [27, Theorem 3.2] which works for any dimension. We give details here. By calculation, we have the following identity

$$\frac{1}{x} = \frac{(1-x)^{n_1}}{x} + \sum_{j=0}^{n_1-1}(1-x)^j \qquad \forall\, x > 0,\, n_1 \in \mathbb{N}. \tag{4.10}$$

Setting $x = \frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}$ in the above identity, we have

$$\frac{1}{\mathbf{q}(\xi)} = \frac{2}{\mathbf{q}_{max}+\mathbf{q}_{min}} \sum_{j=0}^{n_1-1} \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^j + \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^{n_1} \frac{1}{\mathbf{q}(\xi)}. \tag{4.11}$$

Next, applying the identity in (4.10) with x and n_1 being replaced by $\mathbf{q}(\xi)/\mathbf{q}(0)$ and n_2, we have

$$\frac{1}{\mathbf{q}(\xi)} = \frac{(1-\mathbf{q}(\xi)/\mathbf{q}(0))^{n_2}}{\mathbf{q}(\xi)} + \frac{1}{\mathbf{q}(0)} \sum_{j=0}^{n_2-1} (1-\mathbf{q}(\xi)/\mathbf{q}(0))^j.$$

Using the above identity to replace the last fraction $\frac{1}{\mathbf{q}(\xi)}$ at the end of (4.11), we conclude that

$$\begin{aligned}\frac{1}{\mathbf{q}(\xi)} &= \frac{2}{\mathbf{q}_{max}+\mathbf{q}_{min}} \sum_{j=0}^{n_1-1} \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^j + \left(1-\frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max}+\mathbf{q}_{min}}\right)^{n_1} \times \\ &\left(\frac{1}{\mathbf{q}(0)} \sum_{\ell=0}^{n_2-1} (1-\mathbf{q}(\xi)/\mathbf{q}(0))^\ell + \frac{(1-\mathbf{q}(\xi)/\mathbf{q}(0))^{n_2}}{\mathbf{q}(\xi)}\right).\end{aligned} \tag{4.12}$$

When n_1 and $n_2 > 0$ are nonnegative even integers, replacing $\mathbf{q}(\xi)$ in the denominator of the last fraction in the above identity by $\mathbf{q}_{min}$ or $\mathbf{q}_{max}$, we conclude that (4.8) holds, because that for all

$\xi \in \mathbb{R}^d$, we have

$$\mathbf{q}^2_{n_1,n_2}(\xi) - \frac{1}{\mathbf{q}(\xi)} = \left(1 - \frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max} + \mathbf{q}_{min}}\right)^{n_1} (1 - \mathbf{q}(\xi)/\mathbf{q}(0))^{n_2} \left(\frac{1}{\mathbf{q}_{min}} - \frac{1}{\mathbf{q}(\xi)}\right) \geqslant 0.$$

Since

$$0 \leqslant 1 - \frac{2\mathbf{q}(\xi)}{\mathbf{q}_{max} + \mathbf{q}_{min}} \leqslant \frac{\mathbf{q}_{max} - \mathbf{q}_{min}}{\mathbf{q}_{max} + \mathbf{q}_{min}} < 1 \qquad \forall\, \xi \in \mathbb{R}^d,$$

both $\mathbf{q}^1_{n_1,n_2}$ and $\mathbf{q}^2_{n_1,n_2}$ converge to $1/\mathbf{q}$ exponentially fast in the space $L_\infty(\mathbb{R}^d)$ as $n_1 \to \infty$.

By the definition of $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d)$, we clearly have $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \geqslant \nu_2(\mathbf{p}\mathbf{q}^2_{n_1,n_2}, 2I_d)$. For any 2π-periodic trigonometric polynomial $\mathring{\hat{a}}$ such that $\mathring{\hat{a}}(0) = \mathbf{p}(0)/\mathbf{q}(0)$ and $|\mathbf{p}(\xi)/\mathbf{q}(\xi)| \leqslant |\mathring{\hat{a}}(\xi)|$ for all $\xi \in \mathbb{R}^d$, we have $|\mathbf{p}(\xi)\mathbf{q}^1_{n_1,n_2}(\xi)| \leqslant |\mathbf{p}(\xi)/\mathbf{q}(\xi)| \leqslant |\mathring{\hat{a}}(\xi)|$ for all $\xi \in \mathbb{R}^d$. Consequently, we have $\nu_2(\mathring{\hat{a}}, 2I_d) \leqslant \nu_2(\mathbf{p}\mathbf{q}^1_{n_1,n_2})$. Since $\mathring{\hat{a}}$ is arbitrary, by the definition of $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d)$, we have $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \leqslant \nu_2(\mathbf{p}\mathbf{q}^1_{n_1,n_2}, 2I_d)$.

Since $\mu_2(\hat{a}, 2I_d)$ is defined via $\nu_2(\mathring{\hat{a}}, 2I_d)$, we need to efficiently estimate $\nu_2(\mathring{\hat{a}}, 2I_d)$. For a 2π-periodic trigonometric polynomial $\hat{a}$, though $\nu_2(\hat{a}, 2I_d)$ can be theoretically computed efficiently through [20, Algorithm 2.1] by finding the spectral radius of a certain finite matrix, when the degree or size of the mask $\hat{a}$ is relatively large, the size of the finite matrix is often too large for us to find its spectral radius numerically. So, [20, Algorithm 2.1] only practically works for a mask with a reasonably small degree in high dimensions, see [20] for more detail on these issues of computing $\nu_2(\hat{a}, 2I_d)$. Since in this paper, we are interested in a family of bivariate and trivariate masks with their degrees going to infinity, we need some new ideas to efficiently estimate $\nu_2(\hat{a}, 2I_d)$ when the degree of $\hat{a}$ is large which are discussing in the rest of this section. The main idea here is to use the convolution method, which is useful in wavelet analysis, for example, see [16, Proposition 3.7] and [17, Theorem 5.2].

Proposition 4.3. *Let $\widehat{a^1}$ and $\widehat{a^2}$ be 2π-periodic trigonometric polynomials in d-variables. Then for any $1 \leqslant p, p_1, p_2 \leqslant \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$,*

$$\nu_p(\widehat{a^1}\widehat{a^2}, 2I_d) \geqslant \nu_{p_1}(\widehat{a^1}, 2I_d) + \nu_{p_2}(\widehat{a^2}, 2I_d). \tag{4.13}$$

In particular, we have

$$\nu_p(\widehat{a^1}\widehat{a^2}, 2I_d) \geqslant \nu_p(\widehat{a^1}, 2I_d) + \nu_1(\widehat{a^2}, 2I_d), \qquad 1 \leqslant p \leqslant \infty. \tag{4.14}$$

Proof. Denote $\hat{a}(\xi) := \widehat{a^1}(\xi)\widehat{a^2}(\xi)$. Suppose that $\hat{a}, \widehat{a^1}, \widehat{a^2}$ have $\kappa, \kappa_1, \kappa_2$ but not higher sum rules, respectively. By the definition of sum rules, it is easy to see that $\kappa \geqslant \kappa_1 + \kappa_2$. Let $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = \kappa$. Then we can find $\beta, \gamma \in \mathbb{N}_0^d$ such that $\alpha = \beta + \gamma$ and $|\beta| = \kappa_1, |\gamma| = \kappa - \kappa_1 \geqslant \kappa_2$. Therefore, by (1.22), we have

$$\widehat{\nabla^\alpha \delta}(\xi) = \widehat{\nabla^\beta \delta}(\xi)\widehat{\nabla^\gamma \delta}(\xi) \quad \text{and} \quad \widehat{a_n}(\xi) = \widehat{a_n^1}(\xi)\widehat{a_n^2}(\xi),$$

where $\widehat{a_n^1}(\xi) := \Pi_{j=0}^{n-1}\widehat{a^1}(2^j\xi)$ and $\widehat{a_n^2}(\xi) := \Pi_{j=0}^{n-1}\widehat{a^2}(2^j\xi)$. That is, we have

$$a_n * \nabla^\alpha \delta = (a_n^1 * \nabla^\beta \delta) * (a_n^2 * \nabla^\gamma \delta).$$

Since $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$, by Young's inequality, we have

$$\|a_n * \nabla^\alpha \delta\|_{\ell_p(\mathbb{Z}^d)} \leqslant \|a_n^1 * \nabla^\beta \delta\|_{\ell_{p_1}(\mathbb{Z}^d)} \|a_n^2 * \nabla^\gamma \delta\|_{\ell_{p_2}(\mathbb{Z}^d)}.$$

Consequently,

$$\limsup_{n\to\infty} \|a_n * \nabla^\alpha \delta\|_{\ell_p(\mathbb{Z}^d)}^{1/n} \leqslant \limsup_{n\to\infty} \|a_n^1 * \nabla^\beta \delta\|_{\ell_{p_1}(\mathbb{Z}^d)}^{1/n} \limsup_{n\to\infty} \|a_n^2 * \nabla^\gamma \delta\|_{\ell_{p_2}(\mathbb{Z}^d)}^{1/n}.$$

By the definition of $\nu_p(\hat{a}, 2I_d)$ in (1.21) and the relation $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$, now it is easy to see that (4.13) holds. (4.14) is a special case of (4.13) with $p_1 = p$ and $p_2 = 1$.

As an application of Proposition 4.3, we have the following result.

Theorem 4.4. *Let* $\mathbf{p}$ *be a* 2π*-periodic measurable function such that* $\mathbf{p}(0) = \lim_{j\to\infty} \mathbf{p}(2^{-j}\xi)$ *for almost every* $\xi \in \mathbb{R}^d$. *Let* $\mathbf{q}$ *be a* 2π*-periodic trigonometric polynomial such that* $\mathbf{q}(\xi) > 0$ *for all* $\xi \in \mathbb{R}^d$. *Then*

(1) *For every positive real number* ρ, *then* $\mu_2(\rho\mathbf{p}, 2I_d) = \mu_2(\mathbf{p}, 2I_d) - \log_2 \rho$;
(2) *For any* 2π*-periodic measurable function* $\tilde{\mathbf{p}}$ *such that* $\lim_{j\to\infty} \tilde{\mathbf{p}}(2^{-j}\xi) = \mathbf{p}(0)$ *and* $|\tilde{\mathbf{p}}(\xi)| \leqslant |\mathbf{p}(\xi)|$ *for almost every* $\xi \in \mathbb{R}^d$, *then* $\mu_2(\tilde{\mathbf{p}}, 2I_d) \geqslant \mu_2(\mathbf{p}, 2I_d)$;
(3) *For any* 2π*-periodic continuous function* $\tilde{\mathbf{q}}$ *(without assuming* $\tilde{\mathbf{q}}(0) = 1/\mathbf{q}(0)$*) such that* $1/\mathbf{q}(\xi) \leqslant \tilde{\mathbf{q}}(\xi)$ *for all* $\xi \in \mathbb{R}^d$, *then* $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \geqslant \mu_2(\mathbf{p}\tilde{\mathbf{q}}, 2I_d)$.
(4) *If in addition* $\mathbf{p}$ *is a* 2π*-periodic trigonometric polynomial, then*

$$\nu_2(\mathbf{pq}, 2I_d) = \nu_2(\mathbf{pq}(2\cdot), 2I_d).$$

Proof. In the proof of items (1) and (2), we denote $\mathring{\mathbf{p}}$ by any arbitrary 2π-periodic trigonometric polynomial such that $\mathring{\mathbf{p}}(0) = \mathbf{p}(0)$ and $|\mathbf{p}(\xi)| \leqslant |\mathring{\mathbf{p}}(\xi)|$ for almost every $\xi \in \mathbb{R}^d$.

It is evident that $\rho\mathring{\mathbf{p}}(0) = \rho\mathbf{p}(0)$, $\mathring{\mathbf{p}}(0) = \mathbf{p}(0)$ and $|\rho\mathbf{p}(\xi)| \leqslant |\rho\mathring{\mathbf{p}}(\xi)|$ for almost every $\xi \in \mathbb{R}^d$. By the definition of $\mu_2(\rho\mathbf{p}, 2I_d)$, we have $\mu_2(\rho\mathbf{p}, 2I_d) \geqslant \nu_2(\rho\mathring{\mathbf{p}}, 2I_d) = \nu_2(\mathring{\mathbf{p}}, 2I_d) - \log_2 \rho$. Since $\mathring{\mathbf{p}}$ is arbitrary, we conclude that $\mu_2(\rho\mathbf{p}, 2I_d) \geqslant \mu_2(\mathbf{p}, 2I_d) - \log_2 \rho$. Consequently, we also have $\mu_2(\mathbf{p}, 2I_d) = \mu_2(\rho^{-1}\rho\mathbf{p}, 2I_d) \geqslant \mu_2(\rho\mathbf{p}, 2I_d) + \log_2 \rho$. Hence, $\mu_2(\rho\mathbf{p}, 2I_d) = \mu_2(\mathbf{p}, 2I_d) - \log_2 \rho$ and item (1) is verified.

To prove item (2), we have $|\tilde{\mathbf{p}}(\xi)| \leqslant |\mathbf{p}(\xi)| \leqslant |\mathring{\mathbf{p}}(\xi)|$ for almost every $\xi \in \mathbb{R}^d$. Since $\mathring{\mathbf{p}}(0) = \tilde{\mathbf{p}}(0)$, by the definition of $\mu_2(\tilde{\mathbf{p}}, 2I_d)$, we have $\mu_2(\tilde{\mathbf{p}}, 2I_d) \geqslant \mu_2(\mathring{\mathbf{p}}, 2I_d)$. Since $\mathring{\mathbf{p}}$ is arbitrary, we now conclude that $\mu_2(\tilde{\mathbf{p}}, 2I_d) \geqslant \nu_2(\mathbf{p}, 2I_d)$. Hence, item (ii) is proved.

We now prove item (3). Since $\mathbf{q}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$, for any $\rho > 1$, we have $1/\mathbf{q}(\xi) < \rho/\mathbf{q}(\xi)$ for all $\xi \in \mathbb{R}^d$. Consequently, $\varepsilon := (\rho - 1)\|1/\mathbf{q}\|_{L_\infty(\mathbb{R}^d)} > 0$. Now by Proposition 4.2, there exists a 2π-periodic trigonometric polynomial $\mathbf{q}^2$ such that

$$\mathbf{q}^2(0) = 1/\mathbf{q}(0), \quad 1/\mathbf{q}(\xi) \leqslant \mathbf{q}^2(\xi) \leqslant \varepsilon/2 + 1/\mathbf{q}(\xi) < \rho/\mathbf{q}(\xi), \; \xi \in \mathbb{R}^d. \tag{4.15}$$

Let $\hat{u}$ be any arbitrary 2π-periodic trigonometric polynomial such that $|\hat{u}(\xi)| \geqslant |\mathbf{p}(\xi)\tilde{\mathbf{q}}(\xi)|$ for all $\xi \in \mathbb{R}^d$ and $\hat{u}(0) = \mathbf{p}(0)\tilde{\mathbf{q}}(0)$. Then it follows from (4.15) that

$$|\mathbf{p}(\xi)/\mathbf{q}(\xi)| \leqslant |\mathbf{p}(\xi)\mathbf{q}^2(\xi)| \leqslant \rho|\mathbf{p}(\xi)/\mathbf{q}(\xi)| \leqslant \rho|\mathbf{p}(\xi)\tilde{\mathbf{q}}(\xi)| \leqslant |\rho\hat{u}(\xi)|, \; \xi \in \mathbb{R}^d. \tag{4.16}$$

Since $(\mathbf{pq}^2)(0) = (\mathbf{p}/\mathbf{q})(0)$, by the definition of $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d)$, we have $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \geqslant \nu_2(\mathbf{pq}^2, 2I_d)$. On the other hand, it follows from (4.16) that

$$\nu_2(\mathbf{pq}^2, 2I_d) \geqslant \nu_2(\rho\hat{u}, 2I_d) = \nu_2(\hat{u}, 2I_d) - \log_2 \rho,$$

where in the last step we used item (1). In other words, we now have

$$\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \geqslant \nu_2(\mathbf{pq}^2, 2I_d) \geqslant \nu_2(\hat{u}, 2I_d) - \log_2 \rho.$$

Since $\hat{u}$ is arbitrary, it follows that $\mu_2(\mathbf{p}/\mathbf{q}, 2I_d) \geqslant \mu_2(\mathbf{p}\tilde{\mathbf{q}}, 2I_d) - \log_2 \rho$. Now taking $\rho \to 1^+$, we see that item (3) holds.

To prove item (4), we denote $\widehat{a^1}(\xi) := \mathbf{p}(\xi)\mathbf{q}(\xi)$ and $\widehat{a^2}(\xi) := \mathbf{p}(\xi)\mathbf{q}(2\xi)$. We observe that

$$\widehat{a^1_n}(\xi) := \Pi_{j=0}^{n-1}\widehat{a^1}(2^j\xi) = \frac{\mathbf{q}(\xi)}{\mathbf{q}(2^n\xi)}\Pi_{j=0}^{n-1}\widehat{a^2}(2^j\xi) =: \frac{\mathbf{q}(\xi)}{\mathbf{q}(2^n\xi)}\widehat{a^2_n}(\xi).$$

Since $\mathbf{q}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$, both $\mathbf{q}_{min} := \inf_{\xi\in\mathbb{R}^d}\mathbf{q}(\xi)$ and $\mathbf{q}_{max} := \sup_{\xi\in\mathbb{R}^d}\mathbf{q}(\xi)$ are positive finite numbers. Therefore,

$$C^{-1}|\widehat{a^1_n}(\xi)| \leqslant |\widehat{a^2_n}(\xi)| \leqslant C|\widehat{a^1_n}(\xi)|, \qquad \forall\, \xi \in \mathbb{R}^d, n \in \mathbb{N}, \tag{4.17}$$

where $0 < C := \mathbf{q}_{max}/\mathbf{q}_{min} < \infty$. Note that $\widehat{a^1}$ and $\widehat{a^2}$ have the same κ sum rules. For any $\beta \in \mathbb{N}_0^d$ with $|\beta| = \kappa$, we have

$$\|a^1_n * \nabla^\beta\delta\|^2_{\ell_2(\mathbb{Z}^d)} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}|\widehat{\nabla^\beta\delta}(\xi)|^2|\widehat{a^1_n}(\xi)|^2 d\xi.$$

Now combining with (4.17), we see that

$$C^{-1/n}\|a^1_n * \nabla^\beta\delta\|^{1/n}_{\ell_2(\mathbb{Z}^d)} \leqslant \|a^2_n * \nabla^\beta\delta\|^{1/n}_{\ell_2(\mathbb{Z}^d)} \leqslant C^{1/n}\|a^1_n * \nabla^\beta\delta\|^{1/n}_{\ell_2(\mathbb{Z}^d)} \qquad \forall\, n \in \mathbb{N}.$$

Now by the definition of $\nu_2(\widehat{a^1}, 2I_d)$ and $\lim_{n\to\infty} C^{1/n} = 1$, we can easily see that item (4) holds. This completes the proof.

Theorem 4.5. *Let* $\mathbf{p}$ *and* $\hat{u}$ *be* 2π*-periodic trigonometric polynomials such that*

$$\sum_{\omega\in\Omega_{2I_d}}|\hat{u}(\xi/2 + 2\pi\omega)| > 0 \qquad \forall\, \xi \in \mathbb{R}^d. \tag{4.18}$$

For every positive integer r*, define a mask* $\widehat{\tilde{a}^r}$ *and* $\mathbf{q}^r$ *as in* (2.3)*. Then*

$$\begin{aligned}\mu_2(\widehat{\tilde{a}^r}, 2I_d) &\geqslant \mu_2((\mathbf{p})^r, 2I_d) - d(r-1) - r\log_2\|1/\mathbf{q}^1\|_{L_\infty(\mathbb{R}^d)}\\ &\geqslant (r-1)[\nu_1(\mathbf{p}, 2I_d) - d] + \nu_2(\mathbf{p}, 2I_d) - r\log_2\|1/\mathbf{q}^1\|_{L_\infty(\mathbb{R}^d)}.\end{aligned} \tag{4.19}$$

For any 2π*-periodic trigonometric polynomial* $\hat{\mathring{a}}$ *satisfying*

$$\hat{\mathring{a}}(0) = \widehat{\tilde{a}^1}(0), \qquad |\widehat{\tilde{a}^1}(\xi)| \leqslant |\hat{\mathring{a}}(\xi)| \qquad \forall\, \xi \in \mathbb{R}^d, \tag{4.20}$$

then

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant (r-1)[\nu_1(\hat{\mathring{a}}, 2I_d) - d] + \nu_2(\hat{\mathring{a}}, 2I_d) \geqslant r\nu_2(\hat{\mathring{a}}, 2I_d) - d(r-1) \tag{4.21}$$

and

$$\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant r[\mu_2(\widehat{\tilde{a}^1}, 2I_d) - d] + d. \tag{4.22}$$

Proof. Using Cauchy-Schwarz inequality and noting that the cardinality of Ω_{2I_d} is 2^d, we have

$$|\mathbf{q}^1(\xi)| \leqslant \left(\sum_{\omega\in\Omega_{2I_d}}|\hat{u}(\xi/2 + 2\pi\omega)|^r\right)^{1/r} 2^{d(1-1/r)} = |\mathbf{q}^r(\xi)|^{1/r}2^{d(1-1/r)}.$$

That is, we have

$$0 < \frac{1}{\mathbf{q}^r(\xi)} = \frac{1}{\sum_{\omega\in\Omega_{2I_d}}|\hat{u}(\xi/2 + 2\pi\omega)|^r} \leqslant \frac{2^{d(r-1)}}{|\mathbf{q}^1(\xi)|^r} \qquad \forall\, \xi \in \mathbb{R}^d. \tag{4.23}$$

Now (4.19) is a direct consequence of Theorem 4.4 and Proposition 4.3. Also,

$$|\widehat{\tilde{a}^r}(\xi)| = \frac{|\mathbf{p}(\xi)|^r}{\mathbf{q}^r(\xi)} \leqslant \frac{2^{d(r-1)}|\mathbf{p}(\xi)|^r}{|\mathbf{q}^1(\xi)|^r} = 2^{d(r-1)}|\widehat{\tilde{a}^1}(\xi)|^r \leqslant 2^{d(r-1)}|\hat{\mathring{a}}(\xi)|^r \qquad \forall\, \xi \in \mathbb{R}^d.$$

Now by Theorem 4.4 and Proposition 4.3, we see that

$$\begin{aligned}\mu_2(\widehat{\tilde{a}^r}, 2I_d) &\geqslant \mu_2(2^{d(r-1)}(\mathring{\hat{a}}(\xi))^r, 2I_d) = d(1-r) + \nu_2((\mathring{\hat{a}})^r, 2I_d)\\ &\geqslant (r-1)[\nu_1(\mathring{\hat{a}}, 2I_d) - d] + \nu_2(\mathring{\hat{a}}, 2I_d).\end{aligned}$$

Since $\nu_1(\mathring{\hat{a}}, 2I_d) \geqslant \nu_2(\mathring{\hat{a}}, 2I_d)$, now we see that (4.21) holds. Since $\mathring{\hat{a}}$ is arbitrarily chosen such that (4.20) holds, by the definition of $\mu_2(\mathring{\hat{a}}, 2I_d)$, we see that (4.22) is also true.

Note that Proposition 2.2 is a direct consequence of Theorem 4.5 by taking $\mathbf{p}(\xi) = \hat{a}(\xi)$ and $\hat{u}(\xi) = |\hat{a}(\xi)|^2$.

To complete this section, we are now ready to prove Theorem 3.8.

Proof of Theorem 3.8. By Cauchy-Schwarz inequality, similar to (4.23), we have

$$\mathbf{q}^s(\xi) = \sum_{\omega \in \Omega_{2I^d}} |\hat{a}(\xi/2 + 2\pi\omega)|^{2s} \leqslant |\mathbf{q}^r(\xi)|^{s/r} 2^{d(1-s/r)} \text{ and } \mathbf{q}^\ell(\xi) \leqslant |\mathbf{q}^s(\xi)|^{\ell/s} 2^{d(1-\ell/s)}.$$

Consequently, we have

$$|\widehat{\tilde{a}^r}(\xi)| = \frac{|\hat{a}(\xi)|^r}{\mathbf{q}^r(2\xi)} \leqslant \frac{2^{d(r/s-1)}|\hat{a}(\xi)|^r}{|\mathbf{q}^s(2\xi)|^{r/s}} = |\widehat{\tilde{a}^s}(\xi)|^{n_r} \frac{2^{d(n_r+\ell/s-1)}|\hat{a}(\xi)|^\ell}{|\mathbf{q}^s(2\xi)|^{\ell/s}} \leqslant 2^{dn_r}|(\widehat{\tilde{a}^s}(\xi))^{n_r}\widehat{\tilde{a}^\ell}(\xi)|.$$

That is, for $r = n_r s + \ell$ with $n_r \in \mathbb{N}_0$, $1 \leqslant \ell \leqslant s$, we always have

$$|\widehat{\tilde{a}^r}(\xi)| \leqslant 2^{dn_r}|(\widehat{\tilde{a}^s}(\xi))^{n_r}\widehat{\tilde{a}^\ell}(\xi)| \leqslant 2^{dn_r}|(\widehat{\mathring{\tilde{a}}^s}(\xi))^{n_r}\widehat{\mathring{\tilde{a}}^\ell}(\xi)|.$$

By (3.10), we have $\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant \nu_2(2^{dn_r}(\widehat{\mathring{\tilde{a}}^s})^{n_r}\widehat{\mathring{\tilde{a}}^\ell}, 2I_d) \geqslant n_r\nu_1(\widehat{\mathring{\tilde{a}}^s}, 2I_d) + \nu_2(\widehat{\mathring{\tilde{a}}^\ell}, 2I_d) - dn_r$. Hence, (3.11) and (3.12) hold. When (3.14) holds, it follows from (3.11) and (3.13) that for $r = n_r s + \ell > s$ with $n_r \in \mathbb{N}$ and $1 \leqslant \ell \leqslant s$, the length of the interval $(-\mu_2(\widehat{\tilde{a}^r}, 2I_d), \mu_2(\widehat{a^r}, 2I_d))$ is at least

$$\mu_2(\widehat{a^r}, 2I_d) + \mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant n_r[\nu_1(\widehat{a^s}, 2I_d) + \mu_2(\widehat{\tilde{a}^s}, 2I_d) - d] + \nu_2(\widehat{a^\ell}, 2I_d) + \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) > 0.$$

Therefore, $(-\mu_2(\widehat{\tilde{a}^r}, 2I_d), \mu_2(\widehat{a^r}, 2I_d))$ is a nonempty open interval.

If (3.15) holds, by $\nu_1(\hat{a}, 2I_d) \geqslant \nu_2(\hat{a}, 2I_d) > 0$ and applying (3.13) with $s = 1$, we have $\nu_2(\widehat{a^r}, 2I_d) \geqslant (r-1)\nu_1(\hat{a}, 2I_d) + \mu_2(\hat{a}, 2I_d) > 0$. So, it suffices to show $\mu_2(\widehat{\tilde{a}^r}, 2I_d) > 0$, which follows from (3.12), since $\mu_2(\widehat{\tilde{a}^r}, 2I_d) \geqslant n_r[\mu_2(\widehat{\tilde{a}^s}, 2I_d) - d] + \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) \geqslant \mu_2(\widehat{\tilde{a}^\ell}, 2I_d) > 0$. This completes the proof.

5. Some Inequalities for Bivariate and Trivariate Trigonometric Polynomials

In this section, we shall prove some inequalities involving bivariate and trivariate trigonometric polynomials, which play a key role in our estimate of the quantity $\mu_2(\widehat{\tilde{a}^r}, 2I_d)$ in this paper and in our proofs of the main results of this paper. Let us first prove the following auxiliary inequality.

Lemma 5.1. *Define $f(x) := ax^2 + bx + c$ with some parameters a, b and c independent of x. If $a \geqslant 0$, then* $\min_{x\in[-1,1]} f(x) \geqslant \min\{a+b+c, a-b+c, c-a\}$.

Proof. Since $f(x)$ is a quadratic function of x with a non-negative leading coefficient. The minimum of f can be achieved at either $-b/(2a)$, -1, or 1. If $-1 \leqslant -b/(2a) \leqslant 1$, then $\min_{x\in[-1,1]} f(x) = f(-b/(2a)) = c - a(b/(2a))^2 \geqslant c - a$. Adding the possibilities at boundary points -1 and 1, we have $\min_{x\in[-1,1]} f(x) \geqslant \min\{a+b+c, a-b+c, c-a\}$.

To prove Lemma 2.3, let us first prove the inequalities for bivariate polynomials in Lemma 3.3.

Proof of Lemma 3.3. The estimate in (3.5) for $\mathbf{q}^{1,2d}$ has been obtained in [29, Lemma 1]. Now we provide another relatively simple proof. By definition, it is easy to see that $\mathbf{q}^{r,2d}(\xi) \leqslant 1$. Furthermore, by basic identities on trigonometric functions, we obtain

$$\begin{aligned}
&\cos^{2r}\frac{\xi_1}{2}\cos^{2r}\frac{\xi_2}{2}+\sin^{2r}\frac{\xi_1}{2}\sin^{2r}\frac{\xi_2}{2}\\
=&\frac{1}{2^{2r}}\left(\cos\frac{\xi_1-\xi_2}{2}+\cos\frac{\xi_1+\xi_2}{2}\right)^{2r}+\frac{1}{2^{2r}}\left(\cos\frac{\xi_1-\xi_2}{2}-\cos\frac{\xi_1+\xi_2}{2}\right)^{2r}\\
=&\frac{2}{2^{2r}}\sum_{n=0}^{\lfloor r/2\rfloor}\binom{r}{2n}\left(\cos^2\frac{\xi_1-\xi_2}{2}+\cos^2\frac{\xi_1+\xi_2}{2}\right)^{r-2n}\left(2\cos\frac{\xi_1-\xi_2}{2}\cos\frac{\xi_1+\xi_2}{2}\right)^{2n} \qquad (5.1)\\
=&\frac{2}{2^{3r}}\sum_{n=0}^{\lfloor r/2\rfloor}\binom{r}{2n}4^n(2+x+y)^{r-2n}(1+x)^n(1+y)^n,
\end{aligned}$$

where $x := \cos(\xi_1-\xi_2)$ and $y := \cos(\xi_1+\xi_2)$. By the above identity and the definition of $\mathbf{q}^{r,2d}$, we have

$$\mathbf{q}^{r,2d}(2\xi) = f_r(x,y)(1+y)^r/2^r + f_r(-x,-y)(1-y)^r/2^r =: g_r(x,y) \qquad (5.2)$$

with $f_r(x,y) := \frac{2}{2^{3r}}\sum_{n=0}^{\lfloor r/2\rfloor}\binom{r}{2n}4^n(2+x+y)^{r-2n}(1+x)^n(1+y)^n$. When $r=1$, by (5.2) we have $g_1(x,y) = (2+xy+y^2)/4 = 1/2+(y+x/2)^2/4-x^2/16$. Hence

$$\min_{\xi\in\mathbb{R}^2}\mathbf{q}^{1,2d}(2\xi) = \min_{(x,y)\in[-1,1]^2} g_1(x,y) = 7/16 = g_1(1,-1/2).$$

When $r=2$, by (5.2) and direct expanding,

$$g_2(x,y) = \frac{1}{64}[(y^2+1)x^2+2y(3y^2+11)x+(y^4+25y^2+8)] \qquad (5.3)$$

Applying Lemma 5.1 on x, we obtain

$$\min_{(x,y)\in[-1,1]^2} g_2(x,y) \geqslant \frac{1}{64}\min_{y\in[-1,1]}\min\{y^4+24y^2+7, y^4+6y^3+26y^2+22y+9\}.$$

By a direct calculation, $y^4+24y^2+7 \geqslant 7$ and

$$y^4+6y^3+26y^2+22y+9 = (2y+1)^2(y^2+5y+83/4)/4+61/16 \geqslant 61/16.$$

Hence

$$\min_{(x,y)\in[-1,1]^2} g_2(x,y) \geqslant 61/1024 = g_2(1,-1/2).$$

Therefore, (3.5) holds for $r=2$.

For $r=3$, by the same formula in (5.2), we obtain $g_3(x,y) = a_3(y)x^3+a_2(y)x^2+a_1(y)x+a_0(y)$, where

$$\begin{aligned}
&a_3(y) = y(y^2+1)/1024, \quad a_2(y) = (15y^4+99y^2+18)/1024,\\
&a_1(y) = y(15y^4+273y^2+204)/1024, \quad a_0(y) = (y^6+105y^4+258y^2+32)/1024.
\end{aligned}$$

We can assume $y \leqslant 0$ since $g_3(x,y) = g_3(-x,-y)$. By a simple inequality $a_3(y)x^3 \geqslant a_3(y)x$ for all $x \in [-1,1]$ since $a_3(y) \leqslant 0$ and $x^2 \leqslant 1$, we obtain

$$\min_{(x,y)\in[-1,1]^2} g_3(x,y) \geqslant \min_{(x,y)\in[-1,1]\times[-1,0]}\left(a_2(y)x^2+[a_3(y)+a_1(y)]x+a_0(y)\right).$$

Then applying Lemma 5.1 on x, we have

$$\min_{(x,y)\in[-1,1]^2} g_3(x,y) \geqslant \frac{1}{1024} \min_{y\in[-1,0]} \min\{y^6+90y^4+159y^2+14,\\ y^6+15y^5+120y^4+274y^3+357y^2+207y+50\}.$$

By direct calculation, we have $y^6+90y^4+159y^2+14 \geqslant 14$ and

$$\begin{aligned}&\min_{y\in[-1,0]}(y^6+15y^5+120y^4+274y^3+357y^2+207y+50)\\ &= 547/64 + \min_{y\in[-1,0]}(2y+1)^2(16y^4+224y^3+1692y^2+2636y+2653)/64 \geqslant 547/64.\end{aligned}$$

Hence $\min_{(x,y)\in[-1,1]^2} g_3(x,y) \geqslant 547/2^{16} = g_3(1,-1/2)$. Thus, (3.5) holds for $r=3$.

Finally We complete the paper by proving Lemma 2.3, a crucial inequality on trivariate trigonometric polynomials, which has been used in the proof of Theorem 1.2 in section 2.

Proof of Lemma 2.3. By direct calculation, we have

$$\mathbf{q}^{r,3d}(\xi) = c_r(\xi_1,\xi_2)c_r(\xi_3,\xi_1+\xi_2+\xi_3) + c_r(\xi_1,\xi_2+\pi)c_r(\xi_3,\xi_1+\xi_2+\xi_3+\pi)$$

where $c_r(\xi_1,\xi_2) := \cos^{2r}\frac{\xi_1}{2}\cos^{2r}\frac{\xi_2}{2} + \sin^{2r}\frac{\xi_1}{2}\sin^{2r}\frac{\xi_2}{2}$. It is easy to see that $\mathbf{q}^{r,3d}(\xi) \leqslant 1$. Moreover, by the above direct calculation and identity (5.1), we have

$$\mathbf{q}^{r,3d}(2\xi) = f_r(x,z)f_r(y,z) + f_r(-x,-z)f_r(-y,-z) =: g_r(x,y,z) \tag{5.4}$$

with $f_r(x,y) := \frac{2}{2^{3r}}\sum_{n=0}^{\lfloor r/2\rfloor}\binom{r}{2n}4^n(2+x+y)^{r-2n}(1+x)^n(1+y)^n$, $x := \cos(\xi_1-\xi_2)$, $y := \cos(\xi_1+\xi_2+2\xi_3)$ and $z := \cos(\xi_1+\xi_2)$. Therefore, when $r=2$, by direct calculation, we have

$$\begin{aligned}g_2(x,y,z) =&\frac{1}{512}[z^4+6(x+y)z^3+(x^2+36xy+y^2+80)z^2\\ &+(6xy+112)(x+y)z+x^2y^2+8x^2+8y^2+64xy+64].\end{aligned} \tag{5.5}$$

Applying Lemma 5.1 on x and y separately, we obtain

$$\min_{(x,y,z)\in[-1,1]^3} g_2(x,y,z) \geqslant \frac{1}{512}\min_{z\in[-1,1]}\min\{z^4+46z^2+17, z^4+78z^2+49,\\ z^4+12z^3+118z^2+236z+145, z^4+6z^3+80z^2+106z+63\}.$$

Among the above four polynomials, it is easy to see that $z^4+46z^2+17 \geqslant 17$ and $z^4+78z^2+49 \geqslant 49$. For the rest two polynomials, using basic inequalities $z^4 \geqslant 2z^2-1$ and $z^3 \geqslant z$ for all $z\in[-1,0]$, by direct calculation, we are able to prove $\min_{z\in[-1,1]}(z^4+12z^3+118z^2+236z+145) \geqslant 16$, and $\min_{z\in[-1,1]}(z^4+6z^3+80z^2+106z+63) \geqslant 17$. Summarizing the above four inequalities, we have $\min_{(x,y,z)\in[-1,1]^3} g_2(x,y,z) \geqslant 1/32 = g_2(1,1,-1)$. Therefore, $1/32 \leqslant \mathbf{q}^{2,3d}(\xi) \leqslant 1$ holds for all $\xi\in\mathbb{R}^3$. This completes the proof.

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