Wavelet Frames and Image Restorations

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Abstract. One of the major driven forces in the area of applied and computational harmonic analysis over the last decade or longer is the development of redundant systems that have sparse approximations of various classes of functions. Such redundant systems include framelet (tight wavelet frame), ridgelet, curvelet, shearlet and so on. This paper mainly focuses on a special class of such redundant systems: tight wavelet frames, especially, those tight wavelet frames generated via a multiresolution analysis. In particular, we will survey the development of the unitary extension principle and its generalizations. A few examples of tight wavelet frame systems generated by the unitary extension principle are given. The unitary extension principle makes constructions of tight wavelet frame systems straightforward and painless which, in turn, makes a wide usage of the tight wavelet frames possible. Applications of wavelet frame, especially frame based image restorations, are also discussed in details.

Mathematics Subject Classification (2000). Primary 42C15; 42C40; 94A08 Secondary 42C30; 65T60; 90C90.

Keywords. Tight wavelet frames, Unitary extension principle, Image restorations.

1. Introduction

Since the publication of [35, 69] on compactly supported orthonormal wavelet generated by the multiresolution analysis (MRA), wavelet analysis and its applications lead the area of applied and computational harmonic analysis over the last two decades and wavelet methods become powerful tools in various applications in image and signal analysis and processing. One of the well known successful examples of applications of wavelets is image compression using orthonormal or bi-orthogonal wavelet bases generated by the MRA as given in [32, 35]. Another successful example of applications of wavelets is noise removal using redundant wavelet systems by [33, 44].

Theory of frames, especially theory of the Gabor frames (see e.g. [36, 58, 70]) and wavelet frames (see e.g. [36, 70]), has a long history of the development even before the discovery of the multiresolution analysis of [69] and the systematic construction of compactly supported orthonormal wavelets of [35]. The concept of frame can be traced back to [47]. The wide scope of applications of frames can be found in the early literature on applications of Gabor and wavelet frames (see e.g. [36, 58, 70]). Such applications include time frequency analysis for signal processing, coherent state in quantum mechanics, filter bank design in electrical
engineering, edge and singularity detection in image processing, and etc. It is not the goal of this paper to give a survey on all of these and the interested reader should consult [36, 58, 70, 71, 72] and references therein for details.

The publication of the unitary extension principle of [79] generates wide interests in tight wavelet frame systems derived by multiresolution analysis. One can find the rich literature by consulting [31, 40] and the references in these papers. Having tight wavelet frames with a multiresolution structure is very important in order to make any use of them in applications, since this guarantees the existence of the fast decomposition and reconstruction algorithms. Recently, tight wavelet frames derived by the multiresolution analysis are used to open a few new areas of applications of frames. The application of tight wavelet frames in image restorations is one of them that includes image inpainting, image denoising, image deblurring and blind deburring, and image decompositions (see e.g. [8, 9, 10, 13, 14, 20, 23, 24, 25, 28]). In particular, the unitary extension principle is used in [8, 20, 23, 25, 28] to design a tight wavelet frame system adaptive to the real life problems in hand. Frame based algorithms for image and surface segmentation, 3D surface reconstruction, and CT image reconstruction are currently being explored.

In this paper, we start with a brief survey of the theory of tight wavelet frames. A characterization of the tight wavelet frame of [54, 59, 79] is given. We then focus on the tight wavelet frames and their constructions via the multiresolution analysis (MRA). In particular, the unitary extension principle of [79] and the construction of tight wavelet frame from it will be given. We will also give an overview of the generalizations of the unitary extension principle. The second part of this paper focuses on the recent applications of tight wavelet frames in image restorations. In particular, the balanced approach of [8, 9, 10, 20, 23, 24, 25, 28] and the corresponding algorithms for image denoising, deblurring, inpainting and decomposition will be discussed in details.

Finally we remark that there are a few redundant wavelet systems other than the tight wavelet systems discussed here that are developed fast and used widely in image and signal analysis and processing. Such redundant systems include, for example, bi-frames of [31, 40, 59, 80], ridgelets of [46], curvelets of [21, 22], and shearlets of [60, 67]. We forgo discussing all of these in this paper in order to have a well focus of this paper and the interested reader should consult the relevant references for the details.

2. Tight wavelet frame

We introduce the notion of tight wavelet frame in space $L_2(\mathbb{R})$, together with some other basic concepts and notations. The space $L_2(\mathbb{R})$ is the set of all the functions $f(x)$ satisfying $\|f\|_{L_2(\mathbb{R})} := \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} < \infty$ and, similar, $\ell_2(\mathbb{Z})$ is the set of all sequences defined on $\mathbb{Z}$ satisfying $\|h\|_{\ell_2(\mathbb{Z})} := \left( \sum_{k \in \mathbb{Z}} |h(k)|^2 \right)^{1/2} < \infty$.

For any function $f \in L_2(\mathbb{R})$, the dyadic dilation operator $D$ is defined by

$$ D f(x) = 2^{-n} f(2^n x) $$

for $n = 1, 2, 3, \ldots$.
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\( Df(x) := \sqrt{2}f(2x) \) and the translation operator \( T \) is defined by \( T_a f(x) := f(x-a) \) for \( a \in \mathbb{R} \). Given \( j \in \mathbb{Z} \), we have \( T_{2^j}D^j = D^jT_{2^j} \).

For given \( \Psi := \{ \psi_1, \ldots, \psi_r \} \subset L_2(\mathbb{R}) \), define the wavelet system

\[
X(\Psi) := \{ \psi_{\ell,j,k} : 1 \leq \ell \leq r; j,k \in \mathbb{Z} \},
\]

where \( \psi_{\ell,j,k} = D^jT_k\psi_\ell = 2^{j/2}\psi_\ell(2^j \cdot -k) \). The system \( X(\Psi) \subset L_2(\mathbb{R}) \) is called a tight wavelet frame of \( L_2(\mathbb{R}) \) if

\[
\| f \|_{L_2(\mathbb{R})}^2 = \sum_{g \in X(\Psi)} |\langle f, g \rangle|^2,
\]

holds for all \( f \in L_2(\mathbb{R}) \), where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L_2(\mathbb{R}) \) and \( \| \cdot \|_{L_2(\mathbb{R})} = \sqrt{\langle \cdot, \cdot \rangle} \). This is equivalent to \( f = \sum_{g \in X(\Psi)} \langle f, g \rangle g \), for all \( f \in L_2(\mathbb{R}) \).

It is clear that an orthonormal basis is a tight frame. When \( X(\Psi) \) forms an orthonormal basis of \( L_2(\mathbb{R}) \), then \( X(\Psi) \) is called an orthonormal wavelet basis. When \( X(\Psi) \) forms a tight frame of \( L_2(\mathbb{R}) \), then \( X(\Psi) \) is called a tight wavelet frame. We note that in some literature, the definition of tight frame here is called the tight frame with bound one or Parseval frame.

Finally, the Fourier transform of a function \( f \in L_1(\mathbb{R}) \) is defined as usual by:

\[
\hat{f}(\omega) := \int_\mathbb{R} f(x)e^{-i\omega x} \, dx, \quad \omega \in \mathbb{R},
\]

and its inverse is

\[
f(x) = \frac{1}{2\pi} \int_\mathbb{R} \hat{f}(\omega)e^{i\omega x} \, d\omega, \quad x \in \mathbb{R}.
\]

They can be extended to more general functions, e.g. the functions in \( L_2(\mathbb{R}) \).

Similarly, we can define the Fourier series for a sequence \( h \in \ell_2(\mathbb{Z}) \) by

\[
\hat{h}(\omega) := \sum_{k \in \mathbb{Z}} h[k]e^{-ik\omega}, \quad \omega \in \mathbb{R}.
\]

2.1. A characterization. To characterize the wavelet system \( X(\Psi) \) to be a tight frame or even an orthonormal basis for \( L_2(\mathbb{R}) \) in terms of its generators \( \Psi \), the dual Gramian analysis of [78] is used in [79].

The dual Gramian analysis identifies the frame operator corresponding to the wavelet system \( X(\Psi) \) as the dual Gramian matrix with each entry being written in term of the Fourier transform of the generators \( \Psi \). Recall that for a given system \( X(\Psi) \), the corresponding frame operator is defined by

\[
Sf = \sum_{g \in X(\Psi)} \langle f, g \rangle g, \quad f \in L_2(\mathbb{R}).
\]

It is clear that \( X(\Psi) \) is a tight frame of \( L_2(\mathbb{R}) \) if and only if \( S \) is the identity. The dual Gramian analysis decomposes the operator \( S \) into a collection of simpler operators which is called fibers in [78] in Fourier domain. The operator \( S \) is
the identity if and only if each fiber operator is the identity. This leads to the conclusion that wavelet system $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R})$ if and only if the dual Gramian matrix corresponding to the wavelet system $X(\Psi)$ is the identity almost everywhere. Writing each entry of the dual Gramian explicitly, one obtains the following theorem (see, e.g. Corollary 1.3 of [79]):

**Theorem 1.** The wavelet system $X(\Psi)$ is a tight frame of $L_2(\mathbb{R})$ if and only if the identities

$$
\sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^k \omega)|^2 = 1; \quad \sum_{\psi \in \Psi} \sum_{k = 0}^{\infty} \overline{\widehat{\psi}(2^k \omega)}\psi(2^k(\omega + (2j + 1)2\pi)) = 0 \quad j \in \mathbb{Z}
$$

(1)

hold for a.e. $\omega \in \mathbb{R}$. Furthermore, $X(\Psi)$ is an orthonormal basis of $L_2(\mathbb{R})$ if and only if (1) holds and $\|\psi\| = 1$ for all $\psi \in \Psi$.

Note that the key part of this theorem is the tight frame part. The orthonormal basis part follows from the fact that a tight frame with each generator having norm one is an orthonormal basis. The details about the dual Gramian analysis can be found in [78, 81]. The dual Gramian analysis is also applied to the Gabor frame analysis in [82] to derive the duality principle for the Gabor frames.

There were many contributions, during the last two decades, to the study of the Bessel, frame and other related properties of wavelet systems. Examples of univariate wavelet frames can be found in [37]; necessary and sufficient conditions for mother wavelets to generate frames were discussed (implicitly) in [36, 71]. Characterizations of univariate orthonormal basis associated with integer dilation were established independently in [57] and [64], with the multivariate counterparts of these results appearing in [54] for the dyadic dilation. Characterization of bi-frame (tight frame is a special case) in multivariate case for an integer dilation matrix was given in [59]. Independently of all these, a general characterization of all wavelet frames whose dilation matrix is an integral (via dual Gramian analysis) were provided in [79] and derived from it a special characterization of tight wavelet frames in [79] and bi-frame in [80].

Although this theorem gives a complete characterization of the wavelet system $X(\Psi)$ being tight frame of $L_2(\mathbb{R})$, it provides little help for construction of such wavelet system with compactly supported generators, although it may help to obtain tight wavelet frame systems with bandlimited generators. Furthermore, a tight wavelet frame from MRA is handy to use, since it has fast decomposition and reconstruction algorithms. This motivates the study of multiresolution analysis generated tight wavelet frames in [79] as we shall present next. The MRA based bandlimited tight wavelet frames are also constructed in [2].

### 2.2. Tight wavelet frame generated from MRA

The starting element of a multiresolution analysis is the concept of refinable function. Since we are interested here to construct compactly supported wavelets with finitely supported masks, for the simplicity, we start with a compactly supported refinable function $\phi$, although the unitary extension principle can be stated for general refinable function in $L_2(\mathbb{R})$ (see [40, 79]). A compactly supported function $\phi \in L_2(\mathbb{R})$ is refinable...
if it satisfies the following refinement equation
\[
\phi(x) = 2 \sum_{k \in \mathbb{Z}} h_0[k] \phi(2x - k),
\]
for some finite supported sequence \( h_0 \in \ell_2(\mathbb{Z}) \). By taking the Fourier transform, equation (2) becomes
\[
\widehat{\phi}(2\omega) = \widehat{h_0} \widehat{\phi}, \quad \text{a.e.} \quad \omega \in \mathbb{R}.
\]
We call the sequence \( h_0 \) the refinement mask of \( \phi \) and \( \widehat{h_0} \) the refinement symbol of \( \phi \).

For a compactly supported refinable function \( \phi \in L_2(\mathbb{R}) \), let \( V_0 \) be the closed shift invariant space generated by \( \{ \phi(\cdot - k) : k \in \mathbb{Z} \} \) and \( V_j := \{ f(2^j \cdot) : f \in V_0 \} \), \( j \in \mathbb{Z} \). It is known that when \( \phi \) is compactly supported, then \( \{ V_j \}_{j \in \mathbb{Z}} \) forms a multiresolution analysis (MRA). Here a multiresolution analysis is defined to be a family of closed subspaces \( \{ V_j \}_{j \in \mathbb{Z}} \) of \( L_2(\mathbb{R}) \) that satisfies: (i) \( V_j \subset V_{j+1} \), (ii) \( \bigcup_j V_j \) is dense in \( L_2(\mathbb{R}) \), and (iii) \( \bigcap_j V_j = \{0\} \) (see [4, 66]). The unitary extension principle is a principle of construction of MRA based tight wavelet frame.

For a given \( \phi \), define the quasi-interpolatory operator as
\[
P_j : f \mapsto \sum_{k \in \mathbb{Z}} \langle f, \phi_{j,k} \rangle \phi_{j,k},
\]
for an arbitrary \( f \in L_2(\mathbb{R}) \), where \( \phi_{j,k} = 2^{j/2} \phi(2^j \cdot - k) \). It is clear that \( P_j f \in V_j \). Since \( \phi \in L_2(\mathbb{R}) \) is refinable, one has
\[
\sum_{k \in \mathbb{Z}} \phi(\cdot + k) = 1.
\]
A standard proof from approximation theory shows that \( \lim_{j \to \infty} P_j f = f \), (see e.g. [40]).

2.2.1. Unitary extension principle. Let \( V_j, j \in \mathbb{Z} \) be the MRA generated by the refinable function \( \phi \) and the refinement mask \( h_0 \). Let \( \Psi := \{ \psi_1, \ldots, \psi_r \} \subset V_1 \) be of the form
\[
\psi_\ell(x) = 2 \sum_{k \in \mathbb{Z}} h_\ell[k] \phi(2x - k).
\]
The finitely supported sequences \( h_1, \ldots, h_r \) are called wavelet masks, or the high pass filters of the system, and the refinement mask \( h_0 \) is called the low pass filter. In the Fourier domain, (4) can be written as
\[
\widehat{\psi_\ell}(2\omega) = \widehat{h_\ell} \widehat{\phi}, \quad \ell = 1, \ldots, r,
\]
where \( \widehat{h_1}, \ldots, \widehat{h_r} \) are \( 2\pi \) periodic functions and are called wavelet symbols.

The Unitary extension principle of [79] for this simple case can be stated as following. For the unitary extension principle in the most general setting, the interested reader should consult [40, 79] for the details.
Theorem 2 (Unitary Extension Principle, (UEP) [79]). Let $\phi \in L_2(\mathbb{R})$ be the compactly supported refinable function with its finitely supported refinement mask $h_0$ satisfying $\hat{h}_0(0) = 1$. Let $(h_1, \ldots, h_r)$ be a set of finitely supported sequences. Then the system $X(\Psi)$ where $\Psi = \{\psi_1, \ldots, \psi_r\}$ defined in (4) forms a tight frame in $L_2(\mathbb{R})$ provided the equalities

$$\sum_{\ell=0}^r |\hat{h}_\ell(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\ell=0}^r \hat{h}_\ell(\xi)\hat{h}_\ell(\xi + \pi) = 0$$

(6)

hold for almost all $\xi \in [-\pi, \pi]$. Furthermore, assuming $r = 1$ and $\|\phi\| = 1$, then $X(\Psi)$ is an orthonormal wavelet bases of $L_2(\mathbb{R})$.

Conditions in (6) can be written in terms of sequences $h_0, \ldots, h_r$. The first condition becomes

$$\sum_{\ell=0}^r \sum_{k \in \mathbb{Z}} h_\ell[k]h_\ell[k - p] = \delta_{0,p}, \quad p \in \mathbb{Z},$$

(7)

where $\delta_{0,p} = 1$ when $p = 0$ and 0 otherwise and the second condition can be written as

$$\sum_{\ell=0}^r \sum_{k \in \mathbb{Z}} (-1)^{k-p} h_\ell[k]h_\ell[k - p] = 0, \quad p \in \mathbb{Z}.$$  

(8)

Proof. Let $\{V_j\}$, $j \in \mathbb{Z}$ be a given MRA with underlying refinable function $\phi$; $P_j$ be the quasi-interpolatory operator defined in (3); and $\Psi = \{\psi_1, \ldots, \psi_r\}$ be the set of corresponding tight framelets derived from the UEP. A simple calculation, which is the standard decomposition and reconstruction algorithms for the tight wavelet frame given in [40], shows that condition (7) and (8) imply that

$$P_j f = P_{j-1} f + \sum_{\ell=1}^r \sum_{k \in \mathbb{Z}} \langle f, \psi_{\ell,j-1,k} \rangle \psi_{\ell,j-1,k}.$$  

(9)

Iterating (9) and applying the fact $\lim_{j \to -\infty} V_j = \{0\}$ which follows from $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, one derives that this quasi-interpolatory operator $P_j$ is the same as truncated representation

$$Q_j : f \mapsto \sum_{\ell=1}^r \sum_{j' < j, k \in \mathbb{Z}} \langle f, \psi_{\ell,j',k} \rangle \psi_{\ell,j',k},$$

(10)

i.e. $P_j f = Q_j f$. Since $\lim_{j \to -\infty} P_j f = f$ for all $f \in L_2(\mathbb{R})$, one concludes that

$$f = \lim_{j \to -\infty} P_j f = \lim_{j \to -\infty} Q_j f = \sum_{\ell=1}^r \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}} \langle f, \psi_{\ell,j',k} \rangle \psi_{\ell,j',k}.$$  

Hence, $X(\Psi)$ is a tight frame of $L_2(\mathbb{R})$. The orthonormal basis part follows from the fact that if $r = 1$ and $\|\phi\| = 1$, then the $\psi$ constructed from the UEP has the norm one as well.
The generators $\Psi$ via the UEP is called framelet in [40]. For the special case $r = 1$, the above theorem is given in [68]. The freedom of the choice of the number of the generators $r$ in the UEP, makes the construction of tight framelets become painless. For example, one can construct tight framelets from spline easily. In fact, [79] gives a systematic construction of tight wavelet frame system from B-splines by using the UEP. Next, we give two examples of spline tight framelets of [79].

Example 1. Let $h_0 = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$ be the refinement mask of the piecewise linear function $\phi(x) = \max(1 - |x|, 0)$. Define $h_1 = [-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}]$ and $h_2 = [\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}]$. Then $h_0$, $h_1$ and $h_2$ satisfy (7) and (8). Hence, the system $X(\Psi)$ where $\Psi = \{\psi_1, \psi_2\}$ defined in (4) by using $h_1$, $h_2$ and $\phi$ is a tight frame of $L_2(\mathbb{R})$ (see Figure 1).

Example 2. Let $h_0 = [\frac{1}{16}, \frac{5}{8}, \frac{1}{16}, \frac{1}{16}]$ be the refinement mask of $\phi$. Then $\phi$ is the piecewise cubic B-spline. Define $h_1$, $h_2$, $h_3$, $h_4$ as follows:

$$h_1 = [\frac{1}{16}, -\frac{1}{4}, \frac{3}{8}, -\frac{1}{4}, \frac{1}{16}], \quad h_2 = [-\frac{1}{4}, \frac{1}{2}, 0, -\frac{1}{4}, \frac{1}{8}],$$

$$h_3 = [\frac{\sqrt{6}}{16}, 0, -\frac{\sqrt{6}}{8}, 0, \frac{\sqrt{6}}{16}], \quad h_4 = [-\frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}].$$

Then $h_0$, $h_1$, $h_2$, $h_3$, $h_4$ satisfy (7) and (6) and hence the system $X(\Psi)$ where $\Psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ defined in (4) by $h_1$, $h_2$, $h_3$, $h_4$ and $\phi$ is a tight frame of $L_2(\mathbb{R})$ (see Figure 2).

An advantage of the tight wavelet frames derived from the UEP is that those systems have fast decomposition and reconstruction algorithms, just as the orthonormal wavelet bases of [35]. The detailed discussions of decomposition and reconstruction algorithms are given in [40].
2.2.2. Pseudo-spline tight wavelet frames. The approximation order of the truncated tight wavelet frame generated by the UEP depends on the flatness of the Fourier transform of the underlying refinable function \( \phi \) with refinement mask \( h_0 \) at the origin. More precisely, the order of the approximation of \( Q_j f \), where the operator \( Q_j \) is the truncation operator defined in (10), to a sufficient smooth function \( f \) cannot exceed the order of the zero of \( 1 - |\hat{\phi}|^2 \) at the origin (see [40] for details) which is the same as the order of zeros of \( 1 - |\hat{h}_0|^2 \) at the origin.

Recall that the operator \( Q_j \) provides approximation order \( m_1 \), if for all \( f \) in the Sobolev space \( W^{m_1}_2(\mathbb{R}) \)

\[
\| f - Q_j f \|_{L^2(\mathbb{R})} = O(2^{-nm_1}).
\]

As shown in [40], the approximation order of \( Q_j f \) depends on the order of the zero of \( 1 - |\hat{\phi}|^2 \) at the origin. In fact, if \( 1 - |\hat{h}_0|^2 = O(|\cdot|^m_2) \) at the origin, then \( m_1 = \min\{m_0, m_2\} \) (see [40] for details), where \( m_0 \) is the order of Strang-Fix condition that \( \phi \) satisfies. Recall that a function \( \phi \) satisfies the Strang-Fix condition of order \( m_0 \) if \( \hat{\phi}(0) \neq 0 \), \( \hat{\phi}^{(j)}(2\pi k) = 0 \), \( j = 0, 1, 2, ..., m_0 - 1 \), \( k \in \mathbb{Z} \setminus \{0\} \).

Furthermore, it is easy to see from the UEP condition (6) and the definition of \( \Psi \) that there is at least one of framelets in \( \Psi \) that has the vanishing moment of the half of the order of zero of \( 1 - |\hat{\phi}|^2 \) at the origin. Recall that the order of the vanishing moment of a function is the order of the zero of its Fourier transform at the origin. However, for an arbitrary refinable spline \( \phi \), the order of the zero of \( 1 - |\hat{\phi}|^2 \) at the origin cannot exceed 2. This means that \( Q_j f \) cannot have approximation order more than 2. (In fact, the approximation order \( Q_j f \) for any refinable spline is exact 2.) Furthermore, there is at least one framelet among \( \Psi \) constructed via the UEP from a refinable spline only has the vanishing moment of order 1. It is clear that the high order of the approximation of \( Q_j \) gives good approximations for smooth functions and the high order of vanishing moment of the framelets gives good sparse approximations for piecewise smooth functions. Hence, in order to have a good tight wavelet system, we need to have refinable functions whose Fourier transform are very flat at the origin. This leads to the introduction of the pseudo-splines in [40, 42]. The results given here are mainly from [42].

Pseudo-splines are defined in terms of their refinement masks. It starts with the simple identity, for given nonnegative integers \( l \) and \( m \) with \( l \leq m - 1 \),

\[
1 = \left( \cos^2(\xi/2) + \sin^2(\xi/2) \right)^{m+l}. \tag{11}
\]

The refinement masks of pseudo-splines are defined by the summation of the first \( l+1 \) terms of the binomial expansion of (11). In particular, the refinement mask of a pseudo-spline of Type I with order \((m, l)\) is given by, for \( \xi \in [-\pi, \pi] \),

\[
|\tilde{a}(\xi)|^2 := |\tilde{a}_{(m,l)}(\xi)|^2 := \cos^{2m}(\xi/2) \sum_{j=0}^{l} \binom{m+l}{j} \sin^{2j}(\xi/2) \cos^{2(l-j)}(\xi/2) \tag{12}
\]

and the refinement mask of a pseudo-spline of Type II with order \((m, l)\) is given
by, for $\xi \in [-\pi, \pi]$,

$$2\widehat{a}(\xi) := 2\widehat{a}_{(m,l)}(\xi) := \cos^2(m/2) \sum_{j=0}^{l} \left(\frac{m+l}{j}\right) \sin^2(j/2) \cos^2(l-j)(\xi/2). \quad (13)$$

We note that the mask of Type I is obtained by taking the square root of the mask of Type II using the Fejér-Riesz lemma (see e.g. [36]), i.e. $2\widehat{a}(\xi) = \sqrt{|1\widehat{b}(\xi)|^2}$. Type I and Type II were introduced and used in [40] and [42] respectively in their constructions of tight framelets. Furthermore, it was shown in [40, 42] (see e.g. Theorem 3.10 [42]), when $\phi$ is a pseudo-spline of an arbitrary type with order $(m, l)$ the order of zero of $1 - |h|^2$ at the the origin is $2l + 2$.

The corresponding pseudo-splines can be defined in terms of their Fourier transforms, i.e.

$$k\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} k\widehat{a}(2^{-j}\xi), \quad k = 1, 2. \quad (14)$$

The pseudo-splines with order $(m, 0)$ for both types are B-splines. Recall that a B-spline with order $m$ and its refinement mask are defined by

$$B_m(\xi) = e^{-ij\frac{2}{\xi}} \left(\frac{\sin(\xi/2)}{\xi/2}\right)^m \quad \text{and} \quad \widehat{a}(\xi) = e^{-ij\frac{2}{\xi}} \cos^m(\xi/2),$$

where $j = 0$ when $m$ is even, $j = 1$ when $m$ is odd. The pseudo-splines of Type I with order $(m, m-1)$ are the refinable functions with orthonormal shifts (called orthogonal refinable functions) given in [35]. The key step of construction of orthonormal wavelet systems is to derive orthogonal refinable functions. The pseudo-splines of Type II with order $(m, m-1)$ are the interpolatory refinable functions (which were first introduced in [43] and a systematic construction was given in [35]). Recall that a continuous function $\phi \in L^2(\mathbb{R})$ is interpolatory if $\phi(j) = \delta(j), j \in \mathbb{Z}$, i.e. $\phi(0) = 1$, and $\phi(j) = 0$, for $j \neq 0$ (see e.g. [43]). The other pseudo-splines fill in the gap between the B-splines and orthogonal or interpolatory refinable functions.

A complete regularity analysis of the pseudo-splines is given in [42] through the analysis of the decay of the Fourier transform of pseudo-splines. For fixed $m$, since the value of the mask $|k\widehat{a}(\xi)|$, for $k = 1, 2$ and $\xi \in \mathbb{R}$, increases with $l$ and the length of the mask $k\widehat{a}$ also increases with $l$, we conclude that the decay rate of the Fourier transform of a pseudo-spline decreases with $l$ and the support of the corresponding pseudo-spline increases with $l$. In particular, for fixed $m$, the pseudo-spline with order $(m, 0)$ has the highest order of smoothness with the shortest support, the pseudo-spline with order $(m, m-1)$ has the lowest order of smoothness with the largest support in the family. As mentioned above, when we move from B-splines to orthogonal or interpolatory refinable functions, we sacrifice the smoothness and short support of the B-splines to gain some other desirable properties, such as orthogonality or interpolatory property. What do we get from the pseudo-splines of the other orders? When we move from B-splines to pseudo-splines, we gain the sparse approximation power of the corresponding tight wavelet
frame $X(\Psi)$ derived by the UEP, since the Fourier transform of the corresponding refinable functions becomes flat at the origin.

Next, we give a genetic construction of tight wavelet frame system from pseudo-splines. The construction is from [42] which is motivated from [30] and one of the constructions of [40]. The construction can be applied to any refinable function whose mask is a trigonometric polynomial and satisfies

$$|\hat{h}_0|^2 + |\hat{h}_0(\cdot + \pi)|^2 \leq 1. \quad (15)$$

Note that when (6) holds, (15) must holds. Hence, (15) is the necessary condition to apply the UEP.

Let $\phi \in L_2(\mathbb{R})$ be a compactly supported refinable function with its trigonometric polynomial refinement mask $\hat{h}_0$ satisfying $\hat{h}_0(0) = 1$ and (15). Let

$$A = \frac{1}{2} \sqrt{1 - |\hat{h}_0|^2 - |\hat{h}_0(\cdot + \pi)|^2}. $$

Here the square root is derived via the Fejér-Riesz lemma. Hence, $A$ is a trigonometric polynomial. Define

$$\hat{h}_1(\xi) := e^{-i\xi \hat{h}_0(\xi + \pi)}, \quad \hat{h}_2(\xi) := A(\xi) + e^{-i\xi}A(-\xi) \quad \text{and} \quad \hat{h}_3(\xi) := e^{-i\xi}\hat{h}_2(\xi + \pi).$$

Let $\Psi := \{\psi_1, \psi_2, \psi_3\}$, where

$$\hat{\psi}_j(\xi) := \hat{h}_j(\xi/2)\hat{\phi}(\xi/2), \quad j = 1, 2, 3. \quad (16)$$

Then $X(\Psi)$ is a tight frame for $L_2(\mathbb{R})$. Each generator in $\Psi$ is compactly supported. Moreover, if the refinement masks $\hat{h}_0$ is symmetry to the origin, which leads to the refinable function $\phi$ is symmetric to the original, $\psi_1$ is symmetric about $\frac{1}{2}$, $\psi_2$ is symmetric about $\frac{1}{4}$ and $\psi_3$ is antisymmetric about $\frac{1}{4}$. Furthermore, it was shown in [42, 61] that $X(\psi_1)$ forms a Riesz basis for $L_2(\mathbb{R})$ when $\phi$ is a pseudo-spline. On the other hand, since $\hat{h}_2$ and $\hat{h}_3$ have zeros at both 0 and $\pi$, one can check easily that neither the shifts of $\psi_2$ nor those of $\psi_3$ can form a Riesz system. Hence, $X(\psi_2)$ and $X(\psi_3)$ cannot form a Riesz basis for $L_2(\mathbb{R})$. In this case, the redundancy provided by the systems $X(\psi_2)$ and $X(\psi_3)$ moves Riesz system $X(\psi_1)$ to a self dual tight frame system. When $\phi$ is the pseudo-spline of Type I with order $(m, m-1)$, $\phi$ and its integer shifts form an orthonormal system and its masks satisfies $|\hat{h}_0|^2 + |\hat{h}_0(\cdot + \pi)|^2 = 1$. In this case, the above construction leads to that $\psi_2 = \psi_3 = 0$ and $X(\psi_1)$ is an orthonormal basis of $L_2(\mathbb{R})$ which is the compactly supported orthonormal wavelet construction of [35].

Since the order of zero of $1 - |\hat{h}_0|^2$ at the the origin is $2l + 2$ when $\phi$ is a pseudo-spline of an arbitrary type with order $(m, l)$, the approximation order of $Q_jf$ corresponding to $X(\Psi)$ constructed above is $2l + 2$ and the order of vanishing moments is $l + 1$. Next, we give an example from [42].

**Example 3.** Let $\widehat{\phi}$ to be the mask of pseudo-spline of Type II with order $(3, 1)$ i.e.

$$\widehat{\phi}(\xi) = \cos^6(\xi/2)(1 + 3 \sin^2(\xi/2)).$$
Figure 3. (a) is the pseudo-spline of Type II with order (3, 1) and (b)-(d) are the corresponding (anti)symmetric tight framelets.

We define

\[
\tilde{b}_1(\xi) := e^{-i\xi}a(\xi + \pi) = e^{-i\xi}\sin^6(\xi/2)(1 + 3\cos^2(\xi/2)),
\]

\[
\tilde{b}_2(\xi) := A(\xi) + e^{-i\xi}A(-\xi) \quad \text{and} \quad \tilde{b}_3(\xi) := e^{-i\xi}A(-\xi) - A(\xi),
\]

where

\[
A = \frac{1}{2}\left(0.00123930398199e^{-4i\xi} + 0.00139868605052e^{-2i\xi} - 0.22813823298962 + 0.44712319189971e^{2i\xi} - 0.22162294894260e^{4i\xi}\right).
\]

The graphs of \(\Psi\) are given by (b)-(d) in Figure 3. The tight frame system has approximation order 4.

2.3. Other extension principles. Since the publication of [79] in 1997, there are many generalizations of the unitary extension principle. Here, we briefly review some of them. The interested reader should consult the references mentioned below for the details.

We start with the oblique extension principle of [31, 40]. As mentioned before, when the unitary extension principle is applied to construct tight wavelet frames from refinable spline functions, the approximation order of the corresponding truncated wavelet system cannot exceed 2; and there is at least one framelet that has its vanishing moment to be 1. To obtain spline tight wavelet systems with better approximation power, the unitary extension principle was extended to oblique extension principle in [31, 40] by introducing a \(2\pi\) periodic function \(\Theta\). The oblique extension principle says that in order to find tight wavelet frame system \(X(\Psi)\) from a given refinable function with its refinement mask \(h_0\), one needs to find the \(2\pi\) periodic function \(\Theta\) which is non-negative, essentially bounded, continues at the origin with \(\Theta(0) = 1\), and the wavelet masks \(h_1, \ldots, h_r\), such that the following two equalities hold a.e. \(\omega \in \mathbb{R}\):

\[
|h_0(\omega)|^2\Theta(2\omega) + \sum_{\ell=1}^r |\hat{h}_\ell(\omega)|^2 = \Theta(\omega); \quad h_0(\omega)\overline{h_0(\omega + \pi)}\Theta(2\omega) + \sum_{\ell=1}^r \hat{h}_\ell(\omega)\overline{\hat{h}_\ell(\omega + \pi)} = 0.
\]

(17)
We note that the unitary extension principle can be viewed as a special case of the oblique extension principle by taking $\Theta$ to be 1. When the Fourier transform of the refinable function used is not flat at the origin, one can chose a proper $\Theta$ which is flat at the origin, so that the resulting framelets have high order of the vanishing moment and the truncated tight wavelet system has a high approximation order. The detailed discussions can be found in [40]. This leads to many nice examples of spline tight wavelet frames with high order of vanishing moment and approximation power in [31, 39, 40].

In order to get an arbitrary high approximation order of the truncated tight frame system, one has to use the non-stationary wavelets, i.e. the masks used at the different level are different, as suggested by [62]. It starts with a non-stationary multiresolution analysis that has the different refinable function and refinement mask at the different level. The non-stationary version of the unitary extension principle is established and the corresponding wavelet masks are obtained in [62]. The different level has a different set of wavelet masks, since the refinement mask at the different level is different. By a proper choice of the masks, symmetric $C^\infty$ real-valued tight wavelet frames in $L^2(\mathbb{R})$ with compact support and the spectral frame approximation order are obtained in [62].

More recently, in order to get a fast flexible decomposition strategy adapted to the data that give a sparse approximation of the underlying function, a concept of an adaptive MRA (AMRA) structure which is a variant of the classical MRA structure is introduced in [60]. For this general case of affine systems, a unitary extension principle for filter design is derived and then applied to the directional representation system of shearlets. This, in turn, leads to the unitary extension principle for shearlets which further leads to a comprehensive theory for fast decomposition algorithms associated with 2D as well as 3D-shearlet systems which encompasses tight shearlet frame with spatially compactly supported generators within such an AMRA structure. Furthermore, shearlet-like systems associated with parabolic scaling and unimodular matrices optimally close to rotation are studied within the framework in [60].

Finally, both the unitary extension principle and the oblique extension principle can be generalized to a bi-frame setting which is called the mixed extension principle. The interested reader should consult [31, 40, 80] where the mixed extension principle is given in the multivariate setting with arbitrary integer dilation matrix.

Furthermore, the mixed extension principle for $L^2(\mathbb{R}^d)$ of [80] is generalized to a pair of dual Sobolev spaces $H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d)$ in [63]. Here we briefly discuss the univariate case and encourage the reader to consult [63] for details of multivariate case. The mixed extension principle is given to ensure that a pair of systems $X^s(\phi; \psi_1, \ldots, \psi_r)$ and $X^{-s}(\phi; \psi_1, \ldots, \psi_r)$ forms a dual wavelet frame pair in the corresponding dual Sobolev spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$. Recall that the system $X^s(\psi, \Psi) := X^s(\phi; \psi_1, \ldots, \psi_r)$ is the homogenous wavelet system generated by $\phi$ and $\Psi := \{\psi_1, \ldots, \psi_r\}$, i.e.,

$$X^s(\psi, \Psi) := \{\phi(-k) : k \in \mathbb{Z}^d\} \cup \{2^{(d/2-s)}\psi_\ell(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}^d, 1 \leq \ell \leq r\}.$$ 

In this general mixed extension principle, the regularity and vanishing moment are
shared by two different systems in the dual pair separately instead of requiring both systems in the dual pair to have certain order of regularity and vanishing moment. For \( s > 0 \), the regularity of \( \phi, \psi_1, \ldots, \psi_r \), and the vanishing moments of \( \tilde{\psi}_1, \ldots, \tilde{\psi}_r \) are required, while allowing \( \tilde{\phi}, \tilde{\psi}_1, \ldots, \tilde{\psi}_r \) to be tempered distributions instead of in \( L^2(\mathbb{R}) \) and \( \psi_1, \ldots, \psi_r \) to have no vanishing moments. This implies that the systems \( X^s(\phi; \psi_1, \ldots, \psi_r) \) and \( X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_r) \) are not necessary to be able to be normalized into a frame of \( L^2(\mathbb{R}) \). This leads to simple constructions of frames in an arbitrary given Sobolev space. For example, it was shown in [63] that \( \{2^{j(1/2-s)}B_m(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z} \} \) is a wavelet frame in \( H^s(\mathbb{R}) \) for any \( 0 < s < m - 1/2 \), where \( B_m \) is the \( B \)-spline of order \( m \). This construction is also applied to multivariate box splines to obtain wavelet frames with small supports while it is well known that it is hard to construct nonseparable multivariate wavelet frames with small supports if the traditional approach is taken, i.e. normalizing a frame in \( L^2(\mathbb{R}) \) to a frame in Sobolev space, since it is hard to construct small sport wavelet frames in \( L^2(\mathbb{R}^d) \) in general. This general mixed extension principle also naturally leads to a characterization of the Sobolev norm of a function in terms of weighted norm of its wavelet coefficient sequence (decomposition sequence) without requiring that dual wavelet frames should be in \( L^2(\mathbb{R}) \), which is quite different to other approaches in the literature (see e.g. [5, 6, 65, 71]). Furthermore, by applying this general mixed extension principle, a characterization for a pair of systems \( X^s(\phi; \psi_1, \ldots, \psi_r) \) and \( X^{-s}(\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_r) \) in Sobolev spaces \( H^s(\mathbb{R}) \) and \( H^{-s}(\mathbb{R}) \) that forms a pair of dual Riesz bases is obtained. This characterization, for example, leads to a proof of the fact that all interpolatory wavelet systems defined in [45] generated by an interpolatory refinable function \( \phi \in H^s(\mathbb{R}) \) with \( s > 1/2 \) are Riesz bases of the Sobolev space \( H^s(\mathbb{R}) \).

### 3. Frame based image restoration

Image restoration is often formulated as an inverse problem. For the simplicity of the notation, we denote images as vectors in \( \mathbb{R}^n \) by concatenating their columns. The objective is to find the unknown true image \( u \in \mathbb{R}^n \) from an observed image (or measurements) \( b \in \mathbb{R}^\ell \) defined by

\[
    b = Au + \eta, \quad (18)
\]

where \( \eta \) is a white Gaussian noise with variance \( \sigma^2 \), and \( A \in \mathbb{R}^{\ell \times n} \) is a linear operator, typically a convolution operator in image deconvolution, a projection in image inpainting and the identity in image denoising.

This section is devoted to frame based image restorations. The frame, especially tight wavelet frame, based image restoration has been developed very fast in the past decade, since the redundancy makes algorithms robust and stable.

Tight frames are redundant system in \( \mathbb{R}^n \) generated by tight wavelet frames. In particular, for given \( W \in \mathbb{R}^{m \times n} \) (with \( m \geq n \)), the rows of \( W \) form a tight frame in \( \mathbb{R}^n \) if \( W \) satisfies \( W^TW = I \), where \( I \) is the identity matrix. Thus, for every
vector \( u \in \mathbb{R}^n \),

\[
u = W^T(Wu).
\] (19)

The components of the vector \( Wu \) are called the canonical coefficients representing \( u \). The matrix \( W \) normally generated from the decomposition algorithm of a tight wavelet frame system by using the corresponding masks. The details in the construction of \( W \) from a given wavelet tight frame system can be found in, for example, [8, 9, 10, 11, 20, 23, 24, 25, 28].

Since tight frame systems are redundant systems, the mapping from the image \( u \) to its coefficients is not one-to-one, i.e., the representation of \( u \) in the frame domain is not unique. Therefore, there are three formulations for the sparse approximation of the underlying images, namely analysis based approach, synthesis based approach and balanced approach. The analysis based approach was first proposed in [49, 85]. In that approach, we assume that the analyzed coefficient vector \( Wu \) can be sparsely approximated, and it is usually formulated as a minimization problem involving a penalty on the term \( \| Wu \|_1 \). The synthesis based approach was first introduced in [41, 50, 51, 52, 53]. In that approach, the underlying image \( u \) is assumed to be synthesized from a sparse coefficient vector \( \alpha \) with \( u = W^T \alpha \), and it is usually formulated as a minimization problem involving a penalty on the term \( \| \alpha \|_1 \) and the distance of the \( \alpha \) to the range of \( W \). Although the synthesis based, analysis based and balanced approaches are developed independently in the literature, the balanced approach can be motivated from our desire to balance the analysis and synthesis based approaches.

Next, we give the exact models of the above three approaches. Before that, we set up some notation. For any \( x \in \mathbb{R}^n \), \( \| x \|_p = \left( \sum_{j=1}^{n} |x_j|^p \right)^{1/p}, \) \( 1 \leq p < \infty \). For simplicity, we write \( \| x \| = \| x \|_2 \). Let \( \| x \|_D \) denote the \( D \)-norm, where \( D \) is a symmetric positive definite matrix, defined by \( \| x \|_D = \sqrt{x^T Dx} \). For any real symmetric matrix \( H_1 \), \( \lambda_{\text{max}}(H_1) \) denotes the maximum eigenvalue of \( H_1 \). For any \( m \times n \) real matrices \( A \), \( \| A \|_2 = \sqrt{\lambda_{\text{max}}(A^T A)} \).

These three approaches can be formulated as the following minimization problem:

\[
\min_{\alpha \in \mathbb{R}^n} \frac{1}{2} \| AW^T \alpha - b \|_D^2 + \frac{\kappa}{2} \| (I - WW^T) \alpha \|_2^2 + \| \text{diag}(\lambda) \alpha \|_1,
\] (20)

where \( 0 \leq \kappa \leq \infty \), \( \lambda \) is a given positively weighted vector, and \( D \) is a given symmetric positive definite matrix.

When \( 0 < \kappa < \infty \), the problem (20) is called balanced approach.

When \( \kappa = 0 \), the problem (20) is reduced to a synthesis based approach:

\[
\min_{\alpha \in \mathbb{R}^n} \frac{1}{2} \| AW^T \alpha - b \|_D^2 + \| \text{diag}(\lambda) \alpha \|_1.
\] (21)
On the other extreme, when $\kappa = \infty$, the problem (20) is reduced to an analysis based approach. To see this, we note that the distance $\| (I - WW^T) \alpha \|$ must be 0 when $\kappa = \infty$. This implies that $\alpha$ is in the range of $W$, i.e., $\alpha = Wu$ for some $u \in \mathbb{R}^n$, so we can rewrite (20) as

$$\min_{\alpha \in \text{Range}(W)} \frac{1}{2} \| AW^T \alpha - b \|^2_D + \| \text{diag}(\lambda) \alpha \|_1 = \min_{u \in \mathbb{R}^n} \frac{1}{2} \| Au - b \|^2_D + \| \text{diag}(\lambda) Wu \|_1$$

(22)

Problem (22) is the analysis based approach. It is clear that when $0 < \kappa < \infty$, (20) balances between (21) and (22), hence is called a balanced approach.

We note that when the rows of $W$ form an orthonormal basis, instead of being a redundant tight frame, the above three approaches are exactly the same, since in this case, $WW^T = I$. However, for redundant tight frame system $W$, the analysis based, synthesis based and balanced approaches cannot be derived from one another. In fact, it was observed in, for example, [29, 48] that there is a gap between the analysis based and synthesis based approaches. Both of them have their own favorable data sets and applications. In general, it is hard to draw definitive conclusions on which approach is better without specifying the applications and data sets. We further note that the $\ell_1$-minimization problem arising from compressed sensing is akin to the synthesis based approach in nature.

On the other hand, the TV-norm minimization problem in imaging restoration is, in many cases, an analysis based approach. For frame based image restoration, numerical simulation results in [15] show that the analysis based approach tends to generate smoother images. This is because the coefficient $Wu$ is quite often linked to the smoothness of the underlying image [5, 6, 56, 63, 65]. However, the synthesis based approach tends to explore more on the sparse representation of the underlying solution in terms of the given frame system by utilizing the redundancy. This enhances and sharpens edges, although it may introduce some artifacts as shown in [10]. The balanced approach bridges the analysis based and synthesis based approaches in image restoration and it balances the smoothness and the sparsity provided by frames as shown in [8, 9, 10, 20, 23, 24, 25, 28].

For the synthesis based approach, the proximal forward and backward splitting algorithm was used in [38, 41, 50, 51, 52, 53]. The accelerated proximal gradient algorithms of [83] can be applied to get a fast algorithm for the synthesis based approach.

For the analysis approach, the coordinate dissent method is used in [49, 85]. The split Bregman iteration is used to develop a fast algorithm for the analysis based approach in frame based image restoration in [15]. The numerical simulation shows that the split Bregman is efficient for image deblurring, decomposition, denoise, and inpainting. The split Bregman iteration was first proposed in [55] which was shown to be powerful in [55, 88] when it is applied to various PDE based image restoration approaches, e.g., ROF and nonlocal PDE models. The convergence analysis of the split Bregman was given in [15].

For the balanced approach in frame based image restoration, the model and algorithm were first developed in [23, 24, 25, 28]. The balanced approach was reformulated as the proximal forward-backward splitting algorithm in [8, 9, 10, 20].
The balanced approach gives satisfactory simulation results, as shown in [8, 9, 10, 20, 23, 24, 25, 28]. Recently, fast algorithms for the balanced approach in frame based image restoration whose convergence speeds are much faster than those of the proximal forward-backward splitting algorithm are developed in [83]. The accelerated proximal gradient algorithms proposed in [83] are based on and extended from several variants of accelerated proximal gradient algorithms that were studied in [1, 73, 74, 75, 76, 86]. These accelerated proximal gradient algorithms have an attractive iteration complexity of $O(1/\sqrt{\epsilon})$ for achieving $\epsilon$-optimality. Also these accelerated proximal gradient algorithms are simple and use only the soft-thresholding operator, just like algorithms such as the linearized Bregman iteration, the split Bregman iteration and the proximal forward-backward splitting algorithm.

Recently, the linearized Bregman iteration is applied to develop a fast algorithm for frame based image deblurring in [14], which converges to the minimizer of the follows minimization problem:

$$\min_{\alpha \in \mathbb{R}^m} \left\{ \frac{\kappa}{2} \| \alpha \|^2 + \| \text{diag}(\lambda)\alpha \|_1 : AW^T\alpha = b \right\},$$

when $A$ is invertible. Furthermore, it converges to the minimizer of $\min_{\alpha \in \mathbb{R}^m} \{ \| \alpha \|_1 : AW^T\alpha = b \}$ as $\lambda \to \infty$, where, for the simplicity, the each entry of the vector $\lambda$ is set to be the same (see [14]). Hence, linearized Bregman is used here to solve a variation of the synthesis based approach. The linearized Bregman iteration was first proposed to solve the $\ell_1$-minimization problems in compressed sensing by [87] and it was made efficient in [77]. The convergence analysis of linearized Bregman iteration was given in [12, 13]. It was then used in the nuclear norm minimization in matrix completion by [7]. The linearized Bregman can be re-formulated as the Uzawa’s algorithm as shown in [7].

A simple computation of [77] shows that (23) is equivalent to

$$\min_{\alpha \in \mathbb{R}^m} \left\{ \frac{\kappa}{2} \|(I - WW^T)\alpha\|^2 + \| \text{diag}(\lambda)\alpha \|_1 : AW^T\alpha = b \right\},$$

when $A$ is invertible. This looks like a variation of balanced approach. However, when the large parameter vector $\lambda$ is chosen which happens when one applies linearized Bregman, it is more close to a variation of the synthesis based approach. For the case that $A$ is not invertible, the detailed discussions also given in [77]

The advantages of Bregman iterations (either linearized Bregman or split Bregman iterations) in frame based image restorations are that big coefficients come back at first after few iterations and stay. This is, in particular, important in image deblurring, since the big wavelet frame coefficients contain information of edges and features of images. The main goal of deblurring is to restore the blurred edges and features. Although neither the synthesis nor analysis based approach is the focus of this paper, we still give an example of blind deburring using analysis base approach with split Bregman iterations at the end of this paper. The frame based blind deblurring has been investigated extensively in [16, 17, 18, 19].

The formulation of (20) can also be extended to image restoration of two-layered images [15, 49, 85]. Real images usually have two layers, referring to cartoons (the
piecewise smooth part of the image) and textures (the oscillating pattern part of the image). Different layers usually have sparse approximations under different tight frame systems. Therefore, these two different layers should be considered separately. One natural idea is to use two tight frame systems that can sparsely represent cartoons and textures separately. The corresponding image restoration problem can be formulated as the following $\ell_1$-minimization problem:

$$\min_{\alpha_1, \alpha_2} \frac{1}{2} \| A(\sum_{i=1}^{2} W_i^T \alpha_i) - b \|^2_D + \sum_{i=1}^{2} \frac{\kappa_i}{2} \| (I - W_i W_i^T) \alpha_i \|^2 + \sum_{i=1}^{2} \| \text{diag}(\lambda_i) \alpha_i \|_1,$$

where, for $i = 1, 2, W_i^T W_i = I$, $\kappa_i > 0$, $\lambda_i$ is a given positive weight vector, and $D$ is a given symmetric positive definite matrix.

In the rest of the paper, we will focus on the balanced approach in frame based image restorations. For those who are interested in the synthesis and analysis based approach for frame based image restorations and the linearized Bregman and split Bregman iterations should consult the literature mentioned above for details.

3.1. Balanced approach for image inpainting. The balanced approach for frame based image restorations was first developed in [24, 25, 28] for the high resolution image reconstruction from a few low resolution images. The problem of high resolution image construction is converted to the problem of filling the miss wavelet frame coefficients, i.e. inpainting a wavelet frame transform domain, by designing a proper wavelet tight frame in [24, 25, 28]. The ideas of [24, 25] is used in [27] to develop balanced approach for frame based image inpainting (in pixel domain) whose complete analysis of convergence and optimal properties of the solution are given in [10]. Analysis of the convergence and optimal properties of the solutions of algorithms in [24, 25, 28] is given in [8, 20, 23]. In this section, we use image inpainting as an example to illustrate how the ideas of the balanced approach are formed and developed.

The mathematical model for image inpainting can be stated as follows. We will denote images as vectors in $\mathbb{R}^n$ by concatenating their columns. Let the original image $u$ be defined on the domain $\Omega = \{1, 2, \cdots, n\}$ and the nonempty set $\Lambda \subseteq \Omega$ be the given observed region. Then the observed (incomplete) image $b$ is

$$b(i) = \begin{cases} u(i) + \eta(i), & i \in \Lambda, \\ \text{arbitrary}, & i \in \Omega \setminus \Lambda, \end{cases}$$

(25)

where $\eta(i)$ is the noise. The goal is to find $u$ from $b$. When $\eta(i) = 0$ for all $i \in \Lambda$, we require that $u(i) = b(i)$ and $u$ is just the solution of an interpolation problem. Otherwise, we seek a smooth solution $u$ that satisfies $|u(i) - b(i)| \leq \eta(i)$ for all $i \in \Lambda$. In both cases, variational approaches will penalize some cost functionals (which normally are weighted function norms of the underlying solution) to control the roughness of the solution, see for instance [3, 26].

The image inpainting is to recover data by interpolation. There are many interpolation schemes available, e.g., spline interpolation, but majority of them are
only good for smooth functions. Images are either piecewise smooth function or texture which do not have the required globe smoothness to provide a good approximation of underlying solutions. The major challenge in image inpainting is to keep the features, e.g. edges of images which many of those available interpolation algorithms cannot preserve. Furthermore, since images are contaminated by noises, the algorithms should have a building in denoising component.

The simple idea of the balanced approach for frame based image inpainting comes as follows: one may use any simple interpolation scheme to interpolate the given data that leads to an inpainted image. The edges might be blurred in this inpainted image. One of the simplest ways to sharpen the image is to throw out small coefficients under a tight wavelet frame transform. The deletion of small wavelet frame coefficients not only sharpens edges but also removes noises. When it is reconstruct back to image domain, it will not interpolate the data anymore, the simplest way to make it interpolate the given data is to put the given data back. One may iterate this process till convergence.

To be precise, let $\mathcal{P}_{\Lambda}$ be the diagonal matrix with diagonal entries 1 for the indices in $\Lambda$ and 0 otherwise. Starting with the initial guess $u_0$, the iteration is

$$u_{k+1} = \mathcal{P}_{\Lambda} b + (I - \mathcal{P}_{\Lambda})W^T T_{\lambda}(Wu_k).$$

Here

$$T_{\lambda}([\beta_1, \beta_2, \ldots, \beta_m]^T) \equiv [t_{\lambda_1}(\beta_1), t_{\lambda_2}(\beta_2), \ldots, t_{\lambda_m}(\beta_m)]^T$$

with $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]^T$, and $t_{\lambda_i}()$ is the soft-thresholding function [44]:

$$t_{\lambda_i}(\beta_i) \equiv \begin{cases} \text{sgn}(\beta_i)(|\beta_i| - \lambda_i), & \text{if } |\beta_i| > \lambda_i, \\ 0, & \text{if } |\beta_i| \leq \lambda_i. \end{cases}$$

Note that by using the soft-thresholding instead of the hard-thresholding which is traditionally used to sharpen edges, we reduces artifacts and obtain the desire minimization property in each iteration. Besides, the thresholding operator $T_{\lambda}$ also plays two other important roles, namely, removing noises in the image and perturbing the frame coefficients $Wu_n$ so that information contained in the given region can permeate into the missing region.

Let the thresholding parameters be

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T,$$

where $\lambda_i > 0$ for $i = 1, \cdots, m$. The whole algorithm is given as follows:

**Algorithm 1.**

(i) Set an initial guess $u_0$.

(ii) Iterate on $n$ until convergence:

$$u_{k+1} = \mathcal{P}_{\Lambda} b + (I - \mathcal{P}_{\Lambda})W^T T_{\lambda}(Wu_k).$$
(iii) Let \( u^* \) to the output of Step (ii). If \( \eta(i) = 0 \) for all \( i \in \Lambda \) in (25), we set \( u^* \) to be the solution (to the interpolation problem); otherwise, since \( T_\lambda \) can remove the noise, we set \( u^\diamond = W^T T_\lambda (W u^*) \) to be the solution (to the inpainting-plus-denoising problem).

Algorithm 1 was first proposed in [27], whose complete analysis of its convergence was given in [10] by re-formulating Algorithm 1 as an iteration for minimizing a special functional. Indeed, it was shown in [10] that

\[
\alpha_n \equiv T_\lambda W u_n \text{ converges to } \alpha^\star \equiv T_\lambda W u^* \text{ which is a minimizer of }
\]

\[
\min_{\alpha} \left\{ \frac{1}{2} \| P_\lambda (W^T \alpha) - P_\lambda b \|^2 + \frac{1}{2} \| (I - WW^T) \alpha \|^2 + \| \text{diag}(\lambda) \alpha \|_1 \right\}.
\tag{30}
\]

The idea of proof is that the iteration deriving sequence \( \alpha_n \) can be written as a proximal forward-backward splitting iteration. Recall that for any proper, convex, lower semi-continuous function \( \phi \) which takes its values in \(( -\infty, +\infty ] \), its proximal operator is defined by

\[
\text{prox}_\phi(x) \equiv \arg \min_y \left\{ \frac{1}{2} \| x - y \|^2 + \phi(y) \right\},
\tag{31}
\]

The proximal forward-backward splitting iteration for \( \alpha_n \) can be derived by using Algorithm 1 as:

\[
\alpha_{k+1} = \text{prox}_{F_1}(\alpha_k - \nabla F_2(\alpha_k)),
\tag{32}
\]

where

\[
F_1(\alpha) = \| \text{diag}(\lambda) \alpha \|_1, \text{ and } F_2(\alpha) = \frac{1}{2} \| P_\lambda (W^T \alpha) - P_\lambda b \|^2 + \frac{1}{2} \| (I - WW^T) \alpha \|^2.
\tag{33}
\]

It was shown in [34] that when \( F_1 \) with range \(( -\infty, +\infty ] \) is a proper, convex, lower semi-continuous function, and \( F_2 \) with range in \( \mathbb{R} \) is a proper, convex, differentiable function with a \( L \)-Lipschitz continuous gradient, i.e.

\[
\| \nabla F_2(\alpha) - \nabla F_2(\beta) \| \leq L \| \alpha - \beta \|, \quad \forall \alpha, \beta
\tag{34}
\]

for some \( L > 0 \). Then for any initial guess \( u_0 \), the proximal forward-backward splitting iteration

\[
\alpha_{k+1} = \text{prox}_{F_1/L}(\alpha_k - \nabla F_2(\alpha_k)/L)
\]

converges to a minimizer of:

\[
\min_{\alpha} \{ F_1(\alpha) + F_2(\alpha) \},
\tag{35}
\]

It is not difficult to check that \( F_1 \) and \( F_2 \) defined in (33) satisfy the conditions needed here, and furthermore, \( F_2 \) has 1-Lipschitz continuous gradient, hence, \( \alpha_n \) converges.
3.2. Role of the redundancy. Tight frames are different from orthonormal systems because tight frames are redundant. What does the redundancy bring us here? We start with a sort of philosophical point of views on the algorithm and then give some quantitative analysis on the error being reduced at each iteration. Assume that some blocks of pixels are missing in a given image and we like to solve the inpainting problem in the wavelet frame domain as mentioned before. Since the framelets used are compactly supported, the coefficients of those framelets whose supports fall in the missing blocks are missing and the coefficients of those framelets whose supports overlap with the missing blocks are inaccurate. The main step of Algorithm 1 perturbs the frame coefficients \( W_u \) by thresholding so that information contained in the available coefficients will permeate into the missing frame coefficients. Here, the redundancy is very important, since the available coefficients and its associated atoms in the the system contain information of the missing coefficients only if the system is redundant, as the atoms in an orthonormal basis are orthogonal to each other and do not contain information of other atoms in \( L_2 \) sense.

While applying the thresholding operator on the frame coefficients is a very important step in Algorithm 1 in order to remove the noises and perturb the coefficients and sharpen the edges, it, however, also brings in new errors and artifacts. To explain how the numerical errors and artifacts introduced by the thresholding can be reduced by the redundancy of the system \( W \), we take the computed solution \( u^\star \) as an example. Similar analysis holds for the computation of each iteration. Our computed solution \( u^\star \) that interpolates the given data satisfies

\[
 u^\star = \mathcal{P}_\lambda b + (I - \mathcal{P}_\lambda)W^T\mathcal{T}_\lambda W u^\star.
\]

That is, on \( \Lambda \), \( W^T\mathcal{T}_\lambda W u^\star \) is replaced by \( b \). But since \( \mathcal{P}_\lambda b = \mathcal{P}_\lambda u^\star = \mathcal{P}_\lambda W^T W u^\star \), we are actually replacing \( \mathcal{P}_\lambda W^T\mathcal{T}_\lambda W u^\star \) by \( \mathcal{P}_\lambda W^T W u^\star \), which generates artifacts. Hence to reduce the artifacts, we require that the norm of

\[
\mathcal{P}_\lambda W^T W u^\star - \mathcal{P}_\lambda W^T\mathcal{T}_\lambda W u^\star = \mathcal{P}_\lambda W^T (W u^\star - \mathcal{T}_\lambda W u^\star)
\]

to be small.

Clearly the smaller the norm of \( W^T e := W^T (W u^\star - \mathcal{T}_\lambda W u^\star) \) is, the smaller the artifact is. Note that the reconstruction operator \( W^T \) can eliminate the error components sitting in the kernel of \( W^T \). In fact, since \( W^T \) projects all sequences down to the orthogonal complement of the kernel of \( W^T \), which is the range of \( W \), the component of \( e \) in the kernel of \( W^T \) does not contribute. The redundant system reduces the errors as long as the component of \( e \) in the kernel of \( W^T \) is not zero. Therefore, the larger is the kernel of \( W^T \), the more redundant is the frame system. In other words, higher redundancy will lead to more error reduction in general. To increase the redundancy, we use undecimated tight wavelet frame system (i.e. no down sampling in the decomposition). In contrast, if \( W \) is not a redundant system but an orthonormal system, then the kernel of \( W^T \) is just \( \{0\} \). In this case, \( ||W^T e|| = ||e|| \).

3.3. Accelerated algorithm. In this section we introduce accelerated proximal gradient algorithms of [83] for solving (20), a similar algorithm for the mini-
minimization problem of (24) can be obtained easily. Let
\[ F_1(\alpha) = \|\text{diag}(\lambda)\alpha\|_1, \quad \text{and} \quad F_2(\alpha) = \frac{1}{2}\|AW^T\alpha - b\|_D^2 + \frac{\kappa}{2}\|(I - W^T)\alpha\|_2^2. \] (36)

Then, the proximal forward-backward splitting iteration
\[ \alpha_{k+1} = \text{prox}_{F_1/L}(\alpha_k - \nabla F_2(\alpha_k)/L) \]
converges to a minimizer of:
\[ \min_{\alpha} \{F_1(\alpha) + F_2(\alpha)\}, \] (37)

Here, the gradient of $F_2$ is given by
\[ \nabla F_2(\alpha) = WA^T D(\alpha W^T - b) + \kappa (I - WW^T)\alpha. \] (38)

It can be proven easily that $F_1$ and $F_2$ given here satisfy the conditions for the convergence of the the proximal forward-backward splitting iteration. This generalizes the inpainting algorithm given in Section 3.1 to algorithms for various image restoration problems. Although the original development of algorithms took a different path, this idea is used in the proof of the convergence of the balanced approach frame based algorithms given in [8, 9, 10, 11, 20, 23, 24, 25, 28]. The accelerated proximal gradient is obtained by adjusting the term $\alpha_n - \nabla F_2(\alpha_n)/L$ in the the proximal forward-backward splitting iteration. Next, we describe exactly the accelerated proximal gradient algorithm for solving (20).

Algorithm 2. For a given nonnegative vector $\lambda$, choose $\alpha_0 = \alpha_{-1} \in \mathbb{R}^m$, $t_0 = t_{-1} = 1$. For $k = 0, 1, 2, \ldots$, generate $\alpha_{k+1}$ from $\alpha_k$ according to the following iteration:

(i) Set $\beta_k = \alpha_k + \frac{t_{k-1}-1}{t_k} (\alpha_k - \alpha_{k-1})$.

(ii) Set $g_k = \beta_k - \nabla F_2(\beta_k)/L$.

(iii) Set $\alpha_{k+1} = T_{\lambda/L}(g_k)$.

(iv) Compute $t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2}$.

When $F_2(\alpha) = \frac{1}{2}\|AW^T\alpha - b\|_D^2$, Algorithm 2 leads to an efficient algorithm for the synthesis based approach for the frame based image restoration as a side produce.

When the accelerated proximal gradient algorithm with $t^k = 1$ for all $k$ is applied to the problem (20), it is the proximal forward-backward splitting algorithm developed in [8, 9, 10, 11, 20, 23, 24, 25, 28] for the balanced approach in frame based image restorations, and it is also the popular iterative shrinkage/thresholding algorithms [38, 41, 52, 53]. The iterative shrinkage/thresholding algorithms and
the proximal forward-backward splitting algorithms have been developed and analyzed independently by many researchers. These algorithms only require gradient evaluations and soft-thresholding operations, so the computation at each iteration is very cheap. But, for any \( \epsilon > 0 \), these algorithms terminate in \( O(L/\epsilon) \) iterations with an \( \epsilon \)-optimal solution \([1, 84]\). Hence the sequence \( \{\alpha_k\} \) converges slowly. On the other hand, the accelerated proximal gradient algorithm proposed here gets an \( \epsilon \)-optimal solution in \( O(\sqrt{L}/\epsilon) \) iterations. Thus the algorithm accelerates the proximal forward-backward splitting algorithms used in \([8, 9, 10, 11, 20, 23, 24, 25, 28]\) for the balanced approach in frame based image restorations. In fact, it was proven in \([83]\) (see Theorem 2.1 of \([83]\)) that for given \( \epsilon \),

\[
F_1(\alpha_k) + F_2(\alpha_k) - F_1(\alpha^*) - F_2(\alpha^*) \leq \epsilon \quad \text{whenever} \quad k \geq C \sqrt{\frac{2L}{\epsilon}}. \tag{39}
\]

where \( \alpha^* \) is a minimizer of (20). Furthermore, the constant \( C \) is explicitly given in Theorem 2.1 [83]. Numerical simulations in [83] illustrate and verify that Algorithm 2 is very effective for frame based image inpainting, decomposition, denoising and deblurring.

### 3.4. Some simulation results.

This section gives a few simulation results to show the effectiveness of the frame based image restorations. We omit the detailed discussions on the numerical simulations and the interested reader should consult the relevant references for the details.

Table 1: Numerical results for the accelerated proximal decent algorithm in solving (20) and (24) arising from image inpainting without noise (i.e., \( \sigma = 0 \) in (18)).

<table>
<thead>
<tr>
<th>inpainting</th>
<th>one system</th>
<th>two systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma = 0 )</td>
<td>( \lambda = 0.03 )</td>
<td>( \lambda_1 = \lambda_2 = 0.01 )</td>
</tr>
<tr>
<td>iter</td>
<td>psnr</td>
<td>time</td>
</tr>
<tr>
<td>----------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>peppers256</td>
<td>22</td>
<td>33.69</td>
</tr>
<tr>
<td>goldhill256</td>
<td>24</td>
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<tr>
<td>boat256</td>
<td>23</td>
<td>30.99</td>
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<tr>
<td>camera256</td>
<td>23</td>
<td>30.13</td>
</tr>
<tr>
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<tr>
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<tr>
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<td>29.12</td>
</tr>
<tr>
<td>fingerprint512</td>
<td>25</td>
<td>26.51</td>
</tr>
<tr>
<td>zebra512</td>
<td>25</td>
<td>28.47</td>
</tr>
</tbody>
</table>

Table 1 is from [83], that gives the numerical performance of the accelerated proximal gradient algorithm applied to the balanced approach (20) and (24) for the image inpainting problem. As indicated, the accelerated proximal gradient
Figure 4. (a) Real motion-blurred image with one region after zooming in; (b) deblurred image with one region after zooming in by using blind motion deblur algorithm based on the analysis-based sparsity prior of images/kernels under wavelet tight frame system.

The algorithm takes no more than 27 iterations and 25 seconds to solve the model (20) for all the images. For the model (24), the accelerated proximal gradient algorithm takes no more than 35 iterations and 40 seconds to solve all the problems. More simulation results on the balanced approach of frame based image restoration by using the accelerated proximal gradient algorithm can be found in [83].

The next two figures illustrate examples of the frame based blind deblurring via the analysis based approach by using split Bregman algorithm. The assumption is that the blurring is caused by a convolution and the convolution kernel is sparse at the space domain. The convolution kernel and the deblurring image are solved alternatively and iteratively. The interested reader should consult [19] for the details.

References

Figure 5. (a)-(h) are the intermediate results when de-blurring a synthesized motion-blurred image using the blind motion deblur algorithm based on the analysis-based sparsity prior of images/kernels under tight wavelet frame system, for the loop index $k = 0, 10, 20, 40, 80, 160, 320, 640$ respectively. De-blurred images are shown in the first row with the corresponding estimated blur kernels shown in their top right region respectively, and the zoomed regions are shown in the second row.


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