Tight Wavelet Frames in Low Dimensions with Canonical Filters

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Abstract

This paper is to construct tight wavelet frame systems containing a set of canonical filters by applying the unitary extension principle of [20]. A set of filters are canonical if the filters in this set are generated by flipping, adding a conjugation with a proper sign adjusting from one filter. The simplest way to construct wavelets of s-variables is to use the $2^s - 1$ canonical filters generated by the refinement mask of a box spline. However, almost all wavelets (except Haar or the tensor product of Haar) defined by the canonical filters associated with box splines do not form a tight wavelet frame system. We consider how to build a filter bank by adding filters to a canonical filter set generated from the refinement mask of a box spline in low dimension, so that the wavelet system generated by this filter bank forms a tight frame system. We first prove that for a given low dimension box spline of s-variables, one needs at least additional 2^s filters to be added to the canonical filters from the refinement mask (that leads to the total number of highpass filters in the filter bank to be $2^{s+1}-1$) to have a tight wavelet frame system. We then provide several methods with many interesting examples of constructing tight wavelet systems with the minimal number of framelets that contain canonical filters generated by the refinement masks of box splines. The supports of the resulting framelets are not bigger than that of the corresponding box spline whose refinement mask is used to generate the first $2^s - 1$ canonical filters in the filter bank. In many of our examples, the tight frame filter bank has the double-canonical property, meaning it is generated by adding another set of canonical filters generated from a highpass filter to the canonical filters generated by the refinement mask to make a tight frame system.

Key words and phrases: Wavelet tight frame, symmetry, B-spline, box spline, canonical filters, semi-canonical tight frame filter bank, double-canonical tight frame filter bank.

1 Introduction

This paper is on the construction of wavelet tight frame systems with canonical filters in low dimension by applying the unitary extension principle (UEP) of [20]. As a consequence, we can extend some wavelet Riesz basis systems derived from splines in [11, 12, 13] to wavelet tight frame systems by adding a few new frame generators (framelets). This improves the conditional number of spline wavelet Riesz systems to one and changes a Riesz system to a self dual tight frame system. As indicated by applications in image restorations, the redundancy introduced by changing a Riesz system to a tight frame system is desirable in many applications (see e.g. [9, 23] for details).

Recall that a set $X = \{g_j : j \in \mathbb{Z}\} \subset L_2(\mathbb{R}^s)$ is called a frame of $L_2(\mathbb{R}^s)$ if

$$A\|f\|^2_{L_2(\mathbb{R}^s)} \le \sum_{j\in\mathbb{Z}} |\langle f, g_j\rangle|^2 \le B\|f\|^2_{L_2(\mathbb{R}^s)}, \quad \forall f \in L_2(\mathbb{R}^s),$$

where $\langle\cdot,\cdot\rangle$ is the inner product of $L_2(\mathbb{R}^s)$. We call X a tight frame if it is a frame with $A = B = 1$. If for some functions $\psi^{(\ell)}, 1 \le \ell \le L$ on $\mathbb{R}^s$, $X(\Psi) = \{2^{j/2}\psi^{(\ell)}(2^j \cdot -k) : \ j \in \mathbb{Z}, k \in \mathbb{Z}^s, 1 \le \ell \le L\}$ is a frame of $L_2(\mathbb{R}^s)$, then $X(\Psi)$ is called a wavelet frame (or an affine frame) and $\psi^{(\ell)}, 1 \le \ell \le L$ are called framelets or frame generators. Wavelet frames have been studied in many articles, see e.g. [5, 6, 20, 21] for theories of frames, and wavelet frames, especially, frames generated by extension principles. We also refer [23] for a short survey on the wavelet tight frame theory and applications, and [9] for a more detailed note. Comparing to the (bi)orthogonal wavelets, redundant systems like wavelet frames, give a flexibility in image restorations and are more desirable in various applications in image process (see e.g. [3, 9, 23]). Some new applications of tight wavelet frames can be also found in [14, 24]. Furthermore, the connections of wavelet frame based, especially spline tight wavelet frames based, approach for image restoration to PDE based methods are established in [1] for the total variational method and extension, in [7] for the nonlinear diffusion partial differential equation based methods, and in [2] for variational models on the space of piecewise smooth functions.

To apply the unitary extension principle of [20] to construct wavelet tight frame system, we start with the concept of a refinable function. Let $p = \{p_k\}_{k\in\mathbb{Z}^s}$ be a sequence of real numbers p_k with $\sum_k p_k = 2^s$. A distribution ϕ on $\mathbb{R}^s$ is called a refinable function (distribution) associated with p if ϕ satisfies the refinement equation

$$\phi(x) = 2^s \sum_{k\in\mathbb{Z}^s} p_k \phi(2x - k), \quad x \in \mathbb{R}^s.$$

In the Fourier domain, the above refinement equation can be written as

$$\widehat{\phi}(\omega) = p(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2}), \quad \omega \in \mathbb{R}^s,$$

where

$$p(\omega) = \sum_{k\in\mathbb{Z}^s} p_k e^{-ik\omega}, \quad \omega \in \mathbb{R}^s,$$

with $k\omega = \sum_{j=1}^s k_j\omega_j$ denoting the dot product of k and ω. p and $p(\omega)$ are called the refinement mask and the two-scale symbol of ϕ. $p(\omega)$ is also called a lowpass filter. When $p = \{p_k\}_{k\in\mathbb{Z}^s}$ has finitely many of p_k nonzero, we say p to have a compact support and call $p(\omega)$ a finite impulse response (FIR) filter.

The UEP in [20] says that if $p, q^{(1)}, \cdots, q^{(L)}$ satisfy, with $q^{(0)} = p$,

$$\sum_{\ell=0}^{L} \overline{q^{(\ell)}(\omega)}\; q^{(\ell)}(\omega + \pi\eta_k) = \delta(k), \quad 0 \le k < 2^s,\ \omega \in \mathbb{R}^s, \tag{1.1}$$

where δ is the Delta sequence and $\{\eta_k, 0 \le k < 2^s\}$ is a representation of $\mathbb{Z}^s/(2\mathbb{Z}^s)$, then $X(\Psi)$ with $\psi^{(\ell)}$, $1 \le \ell \le L$, defined by $\widehat{\psi}^{(\ell)}(\omega) = q^{(\ell)}(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2})$, is a tight frame of $L_2(\mathbb{R}^s)$, provided that the refinable function ϕ associated with p has certain smoothness. If $\{p, q^{(1)}, \cdots, q^{(L)}\}$ satisfies (1.1), then we call it a tight frame filter bank.

In the univariate case, i.e. $s = 1$, for an FIR filter $p(\omega)$, let $q^{(1)}(\omega)$ be the FIR filter defined by

$$q^{(1)}(\omega) = e^{-i\omega}\overline{p(\omega + \pi)}, \quad \omega \in \mathbb{R}. \tag{1.2}$$

Then

$$\overline{p(\omega)}\, q^{(1)}(\omega) + \overline{p(\omega+\pi)}\, q^{(1)}(\omega+\pi) = 0,\ \omega \in \mathbb{R}, \tag{1.3}$$

and it is well known that $\psi^{(1)}$ defined by

$$\widehat{\psi}^{(1)}(\omega) = q^{(1)}(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2})$$

is an orthonormal wavelet for $L_2(\mathbb{R})$ provided that the integer shifts of the refinable function ϕ associated with p form an orthonormal system.

For $s = 2$ or 3, let $\{\eta_k :\ 0 \le k < 2^s\}$ with $\eta_0 = 0$, be a representation of $\mathbb{Z}^s/(2\mathbb{Z}^s)$, and let ρ be a map: $\mathbb{Z}^s/(2\mathbb{Z}^s) \to \mathbb{Z}^s/(2\mathbb{Z}^s)$ such that

$$\rho(0) = 0 \text{ and } \big(\rho(\eta_1) + \rho(\eta_2)\big)(\eta_1 + \eta_2) \text{ is odd for any } \eta_1 \ne \eta_2, \eta_1, \eta_2 \in \mathbb{Z}^s/(2\mathbb{Z}^s). \tag{1.4}$$

Such a map ρ was defined in [18, 19] and will be given again in the next section. With this map and suppose that p is symmetric around c (which must be in $\frac{1}{2}\mathbb{Z}^s$), that is,

$$\overline{p(\omega)}\ \big(= p(-\omega)\big) = e^{i2c\omega} p(\omega),$$

define

$$q^{(\ell)}(\omega) = \begin{cases} e^{i\rho(\eta_\ell)\omega} p(\omega + \pi\eta_\ell), & \text{if } 2c\eta_\ell \text{ is even;} \\ e^{i\rho(\eta_\ell)\omega} \overline{p(\omega + \pi\eta_\ell)}, & \text{if } 2c\eta_\ell \text{ is odd,} \end{cases} \tag{1.5}$$

for $1 \le \ell < 2^s$. Let ϕ be the refinable function associated with p, and $\psi^{(\ell)}, 1 \le \ell < 2^s$, be the functions defined by

$$\widehat{\psi}^{(\ell)}(\omega) = q^{(\ell)}(\frac{\omega}{2})\widehat{\phi}(\frac{\omega}{2}),\ 1 \le \ell < 2^s. \tag{1.6}$$

Applying the UEP of [20], one can prove that if $p(\omega)$ is a QMF, i.e., $\sum_{0\le k<2^s} |p(\omega + \pi\eta_k)|^2 = 1, \omega \in \mathbb{R}^s$, and symmetric for $s = 2, 3$, then the wavelet system $X(\Psi) = \{2^{j/2}\psi^{(\ell)}(2^j \cdot -k) :\ j \in \mathbb{Z}, k \in \mathbb{Z}^s, 1 \le \ell < 2^s\}$ forms a tight frame of $L_2(\mathbb{R}^s)$. This result is still valid, when ϕ and p do not have finite supports. Furthermore, if ϕ and its shifts form an orthonormal system, then $X(\Psi)$ is an orthonormal basis of $L_2(\mathbb{R}^s)$ (see e.g. [18, 19, 20]).

It is well known that in order to apply the UEP to derive wavelet tight frame from a given refinable function, the corresponding refinement mask must satisfy

$$\sum_{0\le k<2^s} |p(\omega + \pi\eta_k)|^2 \le 1,\ \omega \in \mathbb{R}^s. \tag{1.7}$$

For many good refinable functions ϕ such as B-splines and box splines, their refinement masks are not QMFs but do satisfy this inequality. We assume that the refinement masks considered in this paper always satisfy (1.7) and they are not a QMF.

For a refinable spline ϕ, the wavelets $\psi^{(\ell)}$ defined by (1.6) with $q^{(\ell)}$ given by (1.5) have been used in several applications, due to the fact that many refinable splines have nice properties including symmetry, high order of smoothness, good approximation orders and short supports. Furthermore, wavelet masks $q^{(\ell)}$ are directly related to the refinement mask p. For example, $\phi = B_{222}$ (the C^2 3-directional box spline) and the related $\psi^{(\ell)}$ defined by (1.6) have been used in surface multiresolution processing in [16]. For the case $s = 2$, when $q^{(\ell)}, \ell = 1, 2, 3$ are used for hexagonal image multiresolution processing, $q^{(\ell)}, \ell = 1, 2, 3$ are the ideal highpass filters to separate high frequency components of an image in 3 different directions of the hexagonal array (see [15]) and they were called the idealized highpass filters associated with p. Here, for a given filter p, we call the filters defined by (1.2) for $s = 1$ and by (1.5) for $s = 2, 3$ the canonical filters (associated with p).

For a given refinable function ϕ with the refinement mask p, let $\psi^{(\ell)}$ be defined by the canonical highpass filters $q^{(\ell)}$ in (1.2)/(1.5), it is not clear whether the wavelet system $X(\Psi)$ forms an orthonormal basis, or a Riesz basis or a frame of $L_2(\mathbb{R}^s)$ in general. As shown in an example given in [10], the wavelet system $X(\Psi)$ may not form Riesz basis of $L_2(\mathbb{R}^s)$, even when ϕ and its integer shifts form a Riesz basis. However, when ϕ is a spline, this problem is carefully studied in [12, 13] and most recently in [11]. It was shown in [12], for an arbitrary given refinable univariate B-spline ϕ with the refinement mask p , let the wavelet ψ be defined by the mask q given in (1.2), then the wavelet system $X(\psi)$ form a Riesz basis in $L_2(\mathbb{R})$. For a given box spline ϕ in bivariate or trivariate with the refinement mask p, $\psi^{(\ell)}, 1 \le \ell < 2^s$ defined by (1.6) do generate a Riesz basis of $L_2(\mathbb{R}^s)$, namely, $X(\Psi)$ is a Riesz basis of $L_2(\mathbb{R}^s)$ in many cases. In addition, the Sobolev Riesz basis property of $X(\Psi)$ for $s = 1, 2, 3$ have also been studied in [11, 13].

This paper is to consider the following problems: for a given (non-orthogonal) ϕ on $\mathbb{R}^s$ with the refinement mask p, whose shifts may not necessary form a Riesz basis, with p being symmetric for $s = 2, 3$, let $\psi^{(\ell)}$ be defined by the canonical highpass filters $q^{(\ell)}$ in (1.2)/(1.5), is it possible to add wavelets (conventionally called wavelet framelets) given by some FIR filters $q^{(\ell)}, \ell = 2^s, 2^s+1, \cdots$ into the system so that it can form a tight wavelet frame in $L_2(\mathbb{R}^s)$ by applying UEP, and what is the minimum number of wavelets which are needed to add and how? We say a frame filter bank $\{p, q^{(1)}, \cdots, q^{(L)}\}$ (with $L \ge 2^s$) a semi-canonical frame filter bank if the first $2^s - 1$ highpass filters $q^{(1)}, \cdots, q^{(2^s-1)}$ are given by (1.2) for $s = 1$ and by (1.5) for $s = 2, 3$. We say the corresponding frame system to be a semi-canonical frame system.

We will show that for $1 \le s \le 3$, if $\{p, q^{(1)}, \cdots, q^{(L)}\}$ of s-variable FIR filters with a symmetric FIR lowpass filter p for $s = 2, 3$ is a semi-canonical tight frame filter bank, then it has at least $2^{s+1} - 1$ highpass filters (including the canonical highpass filters), namely $L \ge 2^{s+1} - 1$. This coincides the result in [15] for the case that $s = 2$ and p has the symmetric center $c = (0, 0)$. We will also consider the construction of semi-canonical tight frame filter banks with exact $2^{s+1} - 1$ highpass filters. In the case $s = 1$, for an FIR filter p satisfying (1.7), there always exists an FIR filter $q^{(2)}$ such that with $q^{(1)}$ given by (1.2) and with $q^{(3)}$ defined by

$$q^{(3)}(\omega) = e^{-i\omega}\overline{q^{(2)}(\omega + \pi)}, \tag{1.8}$$

$\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$ is a semi-canonical frame filter bank associated with p with the minimal number of highpass filters. Such a filter bank will be called a double-canonical tight frame filter bank since $q^{(3)}$ is generated by $q^{(2)}$ as the canonical filter associated with $q^{(2)}$. For $s = 2, 3$, two sets of constructive conditions for the existences of semi-canonical frame filter banks with exact $2^{s+1} - 1$ highpass filters are provided. One set of the constructive conditions lead to double-canonical frame filter banks in the sense that the last $2^s - 1$ highpass filters $q^{(\ell)}, 2^s < \ell \le 2^{s+1} - 1$ are given in terms of $q^{(2^s)}$ by a similar formula to (1.5):

$$\begin{aligned} q^{(2^s+k)}(\omega) &= e^{\pm i\rho(\eta_k)\omega} q^{(2^s)}(\omega + \pi\eta_k), \text{ or} \\ q^{(2^s+k)}(\omega) &= e^{\pm i\rho(\eta_k)\omega} \overline{q^{(2^s)}(\omega + \pi\eta_k)}, \end{aligned} \qquad k = 1, \cdots, 2^s - 1. \tag{1.9}$$

2 Semi-canonical tight frames

In this section, we show that for a given FIR lowpass filter p of s-variables which satisfies (1.7), one needs at least additional 2^s filters to be added to the canonical filters generated by p to have a tight frame filter bank. Thus a semi-canonical tight frame filter bank has at least $2^{s+1} - 1$ highpass filters.

First, let us look at the map ρ from $\mathbb{Z}^s/(2\mathbb{Z}^s) \to \mathbb{Z}^s/(2\mathbb{Z}^s)$ which satisfies (1.4). We may choose η_k and $\rho(\eta_k)$ as follows: for $s = 2$,

$$\begin{aligned} &\eta_0 = (0,0),\ \eta_1 = (1,0),\ \eta_2 = (1,1),\ \eta_3 = (0,1); \\ &\rho(\eta_0) = (0,0),\ \rho(\eta_1) = (1,1),\ \rho(\eta_2) = (1,0),\ \rho(\eta_3) = (0,1); \end{aligned} \tag{2.1}$$

and for $s = 3$,

$$\begin{aligned} &\eta_0 = (0,0,0),\ \eta_1 = (1,0,0),\ \eta_2 = (1,1,0),\ \eta_3 = (0,1,0), \\ &\qquad \eta_4 = (0,0,1),\ \eta_5 = (1,0,1),\ \eta_6 = (1,1,1),\ \eta_7 = (0,1,1); \\ &\rho(\eta_0) = (0,0,0),\ \rho(\eta_1) = (1,1,0),\ \rho(\eta_2) = (1,0,0),\ \rho(\eta_3) = (0,1,1), \\ &\qquad \rho(\eta_4) = (1,0,1),\ \rho(\eta_5) = (0,0,1),\ \rho(\eta_6) = (1,1,1),\ \rho(\eta_7) = (0,1,0). \end{aligned} \tag{2.2}$$

Note that there is no such a map ρ for the case $s > 3$. See the detailed discussions in [18].

For $s = 2, 3$, suppose the scaling function $\phi(x)$ on $\mathbb{R}^s$ is symmetric around $c \in \mathbb{R}^s$. Then the lowpass filter p is symmetric around c and as shown in [19], c must be in $\frac{1}{2}\mathbb{Z}^s$. By making suitable integer shifts for p (and ϕ), we may assume $c = (c_1, \cdots, c_s)$ with $c_j = 0$ or $c_j = \frac{1}{2}$. The following fundamental result for the construction of low-dimensional symmetric orthogonal wavelets was established in [19].

Theorem 2.1. [19] *For a symmetric $p(\omega), \omega \in \mathbb{R}^s$ with $s = 2$ or $s = 3$, let $q^{(\ell)}(\omega)$, $1 \le \ell < 2^s$, be the canonical filters defined by* (1.5)*. Then, with $q^{(0)} = p$,*

$$\sum_{k=0}^{2^s-1} \overline{q^{(\ell)}(\omega + \pi\eta_k)}\, q^{(\ell')}(\omega + \pi\eta_k) = 0, \quad \text{for any } \ell \ne \ell', 0 \le \ell, \ell' < 2^s,\ \omega \in \mathbb{R}^s. \tag{2.3}$$

In this paper, for $s = 2, 3$, we will use the canonical highpass filter $q^{(\ell)}$, $1 \le \ell < 2^s$, defined by

$$q^{(\ell)}(\omega) = \begin{cases} e^{i\rho(\eta_\ell)\omega} p(\omega + \pi\eta_\ell), & \text{if } 2c\eta_\ell \text{ is even;} \\ e^{-i\rho(\eta_\ell)\omega} \overline{p(\omega + \pi\eta_\ell)}, & \text{if } 2c\eta_\ell \text{ is odd.} \end{cases} \tag{2.4}$$

Observe that the only difference between the definitions of the canonical highpass filters given by (1.5) and (2.4) is the factor $e^{-i\rho(\eta_\ell)\omega}$ for such ℓ that $2c\eta_\ell$ is odd. Since $e^{-i2\rho(\eta_\ell)(\omega+\pi\eta_k)} = e^{-i2\rho(\eta_\ell)\omega}$ for any $0 \le k < 2^s$, $q^{(\ell)}(\omega), 1 \le \ell < 2^s$, defined by (2.4), also satisfy (2.3).

Remark 2.1. *For $s = 2, 3$, let $q^{(\ell)}, 1 \le \ell < 2^s$ be the canonical highpass defined by* (2.4)*. By following the proof of Theorem 2.1 in* [19]*, one can show that if $p(\omega)$ is antisymmetric, then* (2.3) *still holds.* □

Let $h(\omega), \omega \in \mathbb{R}^s$ be an FIR filter, a trigonometric polynomial of s-variables. Write

$$h(\omega) = 2^{-s/2} \sum_{k=0}^{2^s-1} e^{-i\eta_k\omega} h_k(2\omega), \tag{2.5}$$

where $h_0(\omega), \cdots, h_{2^s-1}(\omega)$ are trigonometric polynomials, and we call them polyphase filters associated with $h(\omega)$. For an FIR filter bank $\{p(\omega), q^{(1)}(\omega), \cdots, q^{(L)}(\omega)\}$, with $q^{(0)} = p$, let $M_{q^{(0)},\cdots,q^{(L)}}(\omega)$ be its modulation matrix (of the size $L \times 2^s$) defined by

$$\begin{aligned} M_{q^{(0)},\cdots,q^{(L)}}(\omega) &= \left[q^{(\ell)}(\omega + \pi\eta_j)\right]_{1\le\ell\le L,\, 0\le j<2^s} \\ &= \begin{bmatrix} q^{(0)}(\omega) & q^{(0)}(\omega + \pi\eta_1) & \cdots & q^{(0)}(\omega + \pi\eta_{2^s-1}) \\ q^{(1)}(\omega) & q^{(1)}(\omega + \pi\eta_1) & \cdots & q^{(1)}(\omega + \pi\eta_{2^s-1}) \\ \vdots & \vdots & \vdots & \vdots \\ q^{(L)}(\omega) & q^{(L)}(\omega + \pi\eta_1) & \cdots & q^{(L)}(\omega + \pi\eta_{2^s-1}) \end{bmatrix}. \end{aligned} \tag{2.6}$$

Let $q_0^{(\ell)}(\omega), \cdots, q_{2^s-1}^{(\ell)}(\omega)$ be the polyphase filters associated with $q^{(\ell)}(\omega)$. Then $M_{q^{(0)},\cdots,q^{(L)}}$ can be written as

$$M_{q^{(0)},\cdots,q^{(L)}}(\omega) = W_{q^{(0)},\cdots,q^{(L)}}(2\omega)U(\omega),$$

where

$$U(\omega) = 2^{-s/2}\Big[e^{-i\eta_k(\omega+\pi\eta_j)}\Big]_{0\le k<2^s,\ 0\le j<2^s},$$

and

$$W_{q^{(0)},\cdots,q^{(L)}}(\omega) = \Big[q_k^{(\ell)}(\omega)\Big]_{1\le \ell\le L,\ 0\le j<2^s}$$

$$= \begin{bmatrix} q_0^{(0)}(\omega) & q_1^{(0)}(\omega) & \cdots & q_{2^s-1}^{(0)}(\omega) \\ q_0^{(1)}(\omega) & q_1^{(1)}(\omega) & \cdots & q_{2^s-1}^{(1)}(\omega) \\ \vdots & \vdots & \vdots & \vdots \\ q_0^{(L)}(\omega) & q_1^{(L)}(\omega) & \cdots & q_{2^s-1}^{(L)}(\omega) \end{bmatrix}. \tag{2.7}$$

The matrix $W_{q^{(0)},\cdots,q^{(L)}}(\omega)$ is called the polyphase matrix of $\{q^{(0)}(\omega), \cdots, q^{(L)}(\omega)\}$.

Observe that $\{p(\omega), q^{(1)}(\omega), \cdots, q^{(L)}(\omega)\}$ is a tight frame filter bank, that is, its filters satisfy (1.1), if and only if its modulation matrix $M_{q^{(0)},\cdots,q^{(L)}}$ satisfies

$$M_{q^{(0)},\cdots,q^{(L)}}(\omega)^*\ M_{q^{(0)},\cdots,q^{(L)}}(\omega) = I_{2^s}, \quad \omega\in\mathbb{R}^s.$$

This, together with the fact that $U(\omega)$ is a unitary matrix, leads to that $\{p(\omega), q^{(1)}(\omega), \cdots, q^{(L)}(\omega)\}$ is a tight frame filter bank if and only if its polyphase matrix $W_{q^{(0)},\cdots,q^{(L)}}(\omega)$ satisfies

$$W_{q^{(0)},\cdots,q^{(L)}}(\omega)^*\ W_{q^{(0)},\cdots,q^{(L)}}(\omega) = I_{2^s}, \quad \omega\in\mathbb{R}^s. \tag{2.8}$$

As mentioning in the introduction, throughout this paper, we assume the lowpass filter p always satisfies (1.7). This condition holds for the refinement mask p of B-splines and 2-D and 3-D box splines with high symmetry as shown in [13] and [11]. Next we show that a semi-canonical tight frame filter bank has at least $2^{s+1}-1$ highpass filters.

Theorem 2.2. *Let p be an FIR lowpass filter of s-variables. Suppose p satisfies* (1.7) *and it is not a QMF. In addition, p is symmetric around c for $s = 2, 3$. If $\{p(\omega), q^{(1)}(\omega), \cdots, q^{(L)}(\omega)\}$ is a semi-canonical FIR tight frame filter bank with the first 2^s-1 highpass filters $q^{(1)}(\omega), \cdots, q^{2^s-1}(\omega)$ defined by* (1.2)/(2.4)*, then $L \ge 2^{s+1}-1$.*

From Theorem 2.2, we know that a semi-canonical tight frame filter bank has at least 3, 7, 15 highpass filters for the 1-D, 2-D and 3-D cases, respectively.

Proof of Theorem 2.2 Let $W_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)$ (with $q^{(0)}=p$) and $W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)$ be the polyphase matrices of $\{q^{(0)}, \cdots, q^{(2^s-1)}\}$ and $\{q^{(2^s)}, \cdots, q^{(L)}\}$, respectively, as defined by (2.7). Let $M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)$ be the modulation matrix of $\{q^{(0)}, \cdots, q^{(2^s-1)}\}$ as defined by (2.6). Then (1.3) and (2.3) lead to that

$$M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)\ M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)^* = \Big(\sum_{0\le k<2^s}\big|p(\omega+\pi\eta_k)\big|^2\Big)I_{2^s},\ \omega\in\mathbb{R}^s.$$

Since $\sum_{0\le k<2^s} \left|p(\omega+\pi\eta_k)\right|^2$ is an analytic function of ω, it is not zero for $a.e\ \omega\in\mathbb{R}^s$. Thus $M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)$ is nonsingular for $a.e\ \omega\in\mathbb{R}^s$. Therefore, we have from the above equation that

$$M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)^*\ M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega) = \big(\sum_{0\le k<2^s} \left|p(\omega+\pi\eta_k)\right|^2\big) I_{2^s},\ \omega\in\mathbb{R}^s.$$

This and $M_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega) = W_{q^{(0)},\cdots,q^{(2^s-1)}}(2\omega)U(\omega)$ imply

$$W_{q^{(0)},\cdots,q^{(2^s-1)}}(2\omega)^*\ W_{q^{(0)},\cdots,q^{(2^s-1)}}(2\omega) = \big(\sum_{0\le k<2^s} \left|p(\omega+\pi\eta_k)\right|^2\big) I_{2^s},\ \omega\in\mathbb{R}^s.$$

In addition, by (2.8), that is,

$$\begin{bmatrix} W_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega) \\ W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)\end{bmatrix}^* \begin{bmatrix} W_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega) \\ W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)\end{bmatrix} = I_{2^s}, \quad \omega\in\mathbb{R}^s,$$

we have

$$W_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega)^*\ W_{q^{(0)},\cdots,q^{(2^s-1)}}(\omega) + W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)^*\ W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)^* = I_{2^s}, \quad \omega\in\mathbb{R}^s.$$

Thus,

$$W_{q^{(2^s)},\cdots,q^{(L)}}(2\omega)^*\ W_{q^{(2^s)},\cdots,q^{(L)}}(2\omega) = \big(1-\sum_{0\le k<2^s} \left|p(\omega+\pi\eta_k)\right|^2\big) I_{2^s}, \quad \omega\in\mathbb{R}^s. \tag{2.9}$$

Since p is not a QMF,

$$\mathrm{rank}\big(W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)\big) = \mathrm{rank}\big(W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)^*\ W_{q^{(2^s)},\cdots,q^{(L)}}(\omega)\big) = 2^s,\ \text{a.e. } \omega\in\mathbb{R}^s.$$

Therefore,

$$L-2^s+1 = \#\text{of rows of } W_{q^{(2^s)},\cdots,q^{(L)}} \ge \mathrm{rank}\big(W_{q^{(2^s)},\cdots,q^{(L)}}\big) = 2^s,$$

or $L\ge 2^{s+1}-1$, as desired. □

Remark 2.2. *Note that there is no restriction on s in the proof of Theorem 2.2. Thus Theorem 2.2 would hold for $s>3$ if there are $q^{(1)},\cdots,q^{(2^s-1)}$ such that* (2.3) *holds.* □

3 Construction of minimal semi-canonical tight frame filter banks

In this section we consider the construction of semi-canonical tight frame filter banks with the minimal number of highpass filters. That is for a given symmetric p, we construct $q^{(2^s)},\cdots,q^{(2^{s+1}-1)}$ such that $\{p,q^{(1)},\cdots,q^{(2^{s+1}-1)}\}$ is a tight frame filter bank with the first 2^s-1 highpass filters $q^{(1)},\cdots,q^{(2^s-1)}$ being the canonical highpass filters given by (1.2)/(2.4). We also consider the construction of semi-canonical tight frame filter banks with double-canonical property. Recall that we say a tight frame filter bank $\{p,q^{(1)},\cdots,q^{(2^{s+1}-1)}\}$ to be double-canonical if the first 2^s-1 highpass filters $q^{(1)},\cdots,q^{(2^s-1)}$ are given by (1.2)/(2.4) and the last 2^s-1 highpass filters $q^{(2^s+1)},\cdots,q^{(2^{s+1}-1)}$ are generated from a highpass filter $q^{(2^s)}$ by (1.8)/(1.9). Note that a double-canonical tight frame filter bank is a semi-canonical tight frame filter bank with the minimal number of highpass filters.

To construct semi-canonical tight frame filter banks with the minimal number of highpass filters, we need only to construct such $q^{(2^s)},\cdots,q^{(2^{s+1}-1)}$ that their polyphase matrix satisfies (2.9) (with

$L = 2^{s+1}-1$). In other words, the main point is to find $2^s \times 2^s$ matrix $W(\omega)$ of trigonometric polynomials such that

$$W(2\omega)^* \, W(2\omega) = \big(1 - \sum_{0\le k<2^s} \big|p(\omega+\pi\eta_k)\big|^2\big) I_{2^s}, \quad \omega \in \mathbb{R}^s. \tag{3.1}$$

In case we find such a $W(\omega)$, then $q^{(2^s)}, q^{(2^s+1)}, \cdots, q^{(2^{s+1}-1)}$, defined by

$$\begin{bmatrix} q^{(2^s)}(\omega) \\ q^{(2^s+1)}(\omega) \\ \vdots \\ q^{(2^{s+1}-1)}(\omega) \end{bmatrix} = 2^{-s/2} W(2\omega) \begin{bmatrix} e^{-i\eta_0\omega} \\ e^{-i\eta_1\omega} \\ \vdots \\ e^{-i\eta_{2^s-1}\omega} \end{bmatrix}, \tag{3.2}$$

together with $p, q^{(1)}, q^{(2)}, \cdots, q^{(2^s-1)}$, form a semi-canonical tight frame filter bank with the minimal number of highpass filters. In the first subsection of this section we will provide a constructive method to construct $q^{(2^s)}, \cdots, q^{(2^{s+1}-1)}$ for the cases $s=1, s=2$ and $s=3$. In the second subsection, we present a constructive method to construct symmetric double-canonical tight frame filter banks. We will present the highpass filters obtained by our approaches for the lowpass filters to be the two-scale symbols of the B-spline and box splines.

Let $\{v_j\}_{j=1}^n \subset \mathbb{Z}^s$ be a set of vectors in $\mathbb{Z}^s$ ($s=2$ or $s=3$) with multiplicity m_j for each v_j. Denote

$$\overline{v} = \lfloor \sum_{j=1}^n m_j v_j/2 \rfloor,$$

where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer not greater than x, and for $u=(u_1, \cdots, u_s) \in \mathbb{Z}^s$, $\lfloor u/2 \rfloor$ denotes $(\lfloor u_1/2 \rfloor, \cdots, \lfloor u_s/2 \rfloor) \in \mathbb{Z}^s$. The (centralized) box spline ϕ associated with $\{v_j\}_{j=1}^n$ is given by

$$\widehat{\phi}(\omega) = \prod_{j=1}^n \Big(\frac{1-e^{-iv_j\omega}}{iv_j\omega}\Big)^{m_j} e^{i\overline{v}\omega}.$$

ϕ is refinable with the two-scale symbol given by

$$p(\omega) = \prod_{j=1}^n \Big(\frac{1+e^{-iv_j\omega}}{2}\Big)^{m_j} e^{i\overline{v}\omega}.$$

For $s=2$, we let $B_{m_1m_2m_3}$ denote the 3-directional box spline of

$$v_1 = [1,0], \; v_2 = [0,1], \; v_3 = [1,1]$$

with multiplicity m_j for v_j. The two-scale symbol of $B_{m_1m_2m_3}$ is

$$p(\omega) = \big(\frac{1+e^{-i\omega_1}}{2}\big)^{m_1} \big(\frac{1+e^{-i\omega_2}}{2}\big)^{m_2} \big(\frac{1+e^{-i(\omega_1+\omega_2)}}{2}\big)^{m_3} e^{i(\lfloor \frac{m_1+m_3}{2} \rfloor \omega_1 + \lfloor \frac{m_2+m_3}{2} \rfloor \omega_2)}.$$

We use $B_{m_1m_2m_3m_4}$ to denote the 4-directional box spline of

$$v_1 = [1,0], \; v_2 = [0,1], \; v_3 = [1,1], \; v_4 = [1,-1]$$

with multiplicity m_j for v_j. The two-scale symbol of $B_{m_1 m_2 m_3 m_4}$ is

$$p(\omega) = \\ (\frac{1+e^{-i\omega_1}}{2})^{m_1}(\frac{1+e^{-i\omega_2}}{2})^{m_2}(\frac{1+e^{-i(\omega_1+\omega_2)}}{2})^{m_3}(\frac{1+e^{-i(\omega_1-\omega_2)}}{2})^{m_3} e^{i(\lfloor\frac{m_1+m_3+m_4}{2}\rfloor\omega_1+\lfloor\frac{m_2+m_3-m_4}{2}\rfloor\omega_2)}.$$

Before we move on to the first subsection, we first show that for an FIR filter bank $p(\omega)$, $R(\omega)$ defined by

$$R(\omega) = 1 - \sum_{0\le k<2^s} \left|p(\frac{\omega}{2}+\pi\eta_k)\right|^2,$$

is a trigonometric polynomial though $p(\frac{\omega}{2}+\pi\eta_k)$ may not. Indeed, let $p_0(\omega), \cdots, p_{2^s-1}(\omega)$ be the polyphase filters associated with $p(\omega)$ defined by (2.5). With

$$\begin{bmatrix} p(\frac{\omega}{2}) & p(\frac{\omega}{2}+\pi\eta_1) & \cdots & p(\frac{\omega}{2}+\pi\eta_{2^s-1}) \end{bmatrix} = \begin{bmatrix} p_0(\omega) & p_1(\omega) & \cdots & p_{2^s-1}(\omega) \end{bmatrix} U(\frac{\omega}{2})$$

and the unitariness of $U(\frac{\omega}{2})$, we have

$$\sum_{0\le k<2^s} \left|p(\frac{\omega}{2}+\pi\eta_k)\right|^2 = \sum_{0\le k<2^s} \left|p_k(\omega)\right|^2.$$

Thus $R(\omega)$ is a trigonometric polynomial.

3.1 Semi-canonical tight frame filter banks

In this subsection we consider the construction of semi-canonical tight frame filter banks with the minimal number of highpass filters. For a given FIR p, we will provide a constructive method to construct $q^{(2^s)}, \cdots, q^{(2^{s+1}-1)}$ such that they, together with p and the canonical filters $q^{(1)}, q^{(2)}, \cdots, q^{(2^s-1)}$, form a tight frame filter bank. All the three cases $s=1, s=2$ and $s=3$ are considered.

First let us look at the case $s=1$. By assumption

$$|p(\omega)|^2 + |p(\omega+\pi)|^2 \le 1, \tag{3.3}$$

we know $R(\omega) = 1 - |p(\omega/2)|^2 - |p(\omega/2+\pi)|^2$ is a nonnegative trigonometric polynomial. By the Fejér-Riesz lemma, there is a trigonometric polynomial $g_1(\omega)$ such that $R(\omega) = |g_1(\omega)|^2$. Choose

$$W(\omega) = \mathrm{diag}\{g_1(\omega), g_1(\omega)\}.$$

Then $W(\omega)$ satisfies (3.1) with $s=1$, and $q^{(2)}(\omega)$ and $q^{(3)}(\omega)$ given by (3.2) with $s=1$ are

$$q^{(2)}(\omega) = \frac{\sqrt{2}}{2} g_1(2\omega),\ q^{(3)}(\omega) = \frac{\sqrt{2}}{2} e^{-i\omega} g_1(2\omega) = e^{-i\omega} q^{(2)}(\omega+\pi). \tag{3.4}$$

If we choose,

$$W(\omega) = \mathrm{diag}\{g_1(\omega), \overline{g_1(\omega)}\},$$

then

$$q^{(2)}(\omega) = \frac{\sqrt{2}}{2} g_1(2\omega),\ q^{(3)}(\omega) = e^{-i\omega} \overline{q^{(2)}(\omega+\pi)}. \tag{3.5}$$

Thus, in the 1-D case, for an FIR lowpass filter, there always exists an associated double-canonical tight frame filter bank.

Theorem 3.1. *Let $p(\omega), \omega \in \mathbb{R}$ be an FIR filter satisfying (3.3). Then there is an FIR filter $q^{(2)}(\omega)$ such that $p(\omega)$, $q^{(2)}(\omega)$ and their associated canonical filters $q^{(1)}(\omega)$ and $q^{(3)}(\omega)$ defined by (1.2) and (3.4)/(3.5) form a double-canonical tight frame filter bank.*

Example 3.1. *Let $p(\omega) = \frac{1}{4}e^{i\omega}(1+e^{-i\omega})^2$ be the two-scale symbol of the continuous linear B-spline $B_2(x)$ supported on $[-1,1]$. Then $R(\omega) = 1-|p(\omega/2)|^2-|p(\omega/2+\pi)|^2$ can be written as $R(\omega) = |g_1(\omega)|^2$ with*

$$g_1(\omega) = \frac{\sqrt{2}}{4}(1-e^{-i\omega}).$$

Thus the corresponding double-canonical highpass filters are

$$q^{(1)}(\omega) = -\frac{1}{4}(1-e^{-i\omega})^2,\ q^{(2)}(\omega) = \frac{1}{4}(1-e^{-i2\omega}),\ q^{(3)}(\omega) = \frac{1}{4}(e^{-i\omega}-e^{i\omega}).$$

□

Example 3.2. *Let $p(\omega) = \frac{1}{16}e^{i2\omega}(1+e^{-i\omega})^4$ be the two-scale symbol of the C^2 cubic B-spline $B_4(x)$ supported on $[-2,2]$. $q^{(1)}(\omega)$ defined by (1.2) is*

$$q^{(1)}(\omega) = e^{-i\omega}\overline{p(\omega+\pi)} = \frac{1}{16}e^{i\omega}(1-e^{-i\omega})^4.$$

Then $R(\omega) = 1-|p(\omega/2)|^2-|p(\omega/2+\pi)|^2 = |g_1(\omega)|^2$, where

$$g_1(\omega) = a_{-1}e^{i\omega} + a_1 + a_1 e^{-i\omega},$$

with

$$a_{-1} = \frac{1}{4} - \frac{\sqrt{14}}{16},\ a_0 = \frac{\sqrt{14}}{8},\ a_1 = -\frac{1}{4} - \frac{\sqrt{14}}{16}.$$

Thus the corresponding highpass filters $q^{(2)}(\omega)$ and $q^{(3)}(\omega)$ are

$$\begin{aligned} q^{(2)}(\omega) &= \frac{\sqrt{2}}{2}\big(a_{-1}e^{i2\omega} + a_0 + a_1 e^{-i2\omega}\big),\\ q^{(3)}(\omega) &= e^{-i\omega}\overline{q^{(2)}(\omega+\pi)} = \frac{\sqrt{2}}{2}\big(a_{-1}e^{-i3\omega} + a_0 e^{-i\omega} + a_1 e^{i\omega}\big). \end{aligned}$$

□

Next, we consider the cases $s = 2, 3$. In this and the next subsection, for the simplicity of presentation, we will use the notations

$$z_1 = e^{-i\omega_1},\ z_2 = e^{-i\omega_2},\ z_3 = e^{-i\omega_3}.$$

Also for a 2-D FIR filter $h(\omega) = \sum_{k_1,k_2\in\mathbb{Z}} h_{k_1,k_2}e^{-i(k_1\omega_1+k_2\omega_2)}$, we use the following matrix to display its (nonzero) coefficients with a box □ to highlight the coefficient $h_{0,0}$ with index $(0,0)$:

$$h(\omega) \,\hat{=} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & h_{-2,1} & h_{-1,1} & h_{0,1} & h_{1,1} & h_{2,1} & \cdots \\ \cdots & h_{-2,0} & h_{-1,0} & \boxed{h_{0,0}} & h_{1,0} & h_{2,0} & \cdots \\ \cdots & h_{-2,-1} & h_{-1,-1} & h_{0,-1} & h_{1,-1} & h_{2,-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

In addition, zero coefficients are in general not displayed in the above matrix.

To construct semi-canonical tight frame filter banks with the minimal number of highpass filters, the following approach was considered in [15].

Theorem 3.2. *Let p be a symmetric 2-D FIR lowpass filter and $q^{(1)}, q^{(2)}, q^{(3)}$ be the highpass filters defined by (2.4) with $s = 2$. Suppose $R(\omega) = 1 - \sum_{j=0}^{3} |p(\omega/2 + \pi\eta_j)|^2$ can be written as*

$$R(\omega) = |g_1(\omega)|^2 + |g_2(\omega)|^2 + |g_3(\omega)|^2, \quad \omega \in \mathbb{R}^2, \tag{3.6}$$

for some trigonometric polynomials $g_k(\omega)$. Let $q^{(4)}, \cdots, q^{(7)}$ be the FIR filters defined by

$$[q^{(4)}(\omega), \cdots, q^{(7)}(\omega)]^T = \frac{1}{2} W(2\omega)[1, e^{-i\omega_1}, e^{-i(\omega_1+\omega_2)}, e^{-i\omega_2}]^T, \tag{3.7}$$

where

$$W(\omega) = \begin{bmatrix} 0 & g_1(\omega) & e^{i(\omega_1+\omega_2)} g_2(\omega) & \overline{g_3(\omega)} \\ -g_1(\omega) & 0 & -e^{i(\omega_1+\omega_2)} \overline{g_3(\omega)} & \overline{g_2(\omega)} \\ -g_2(\omega) & \overline{g_3(\omega)} & 0 & -\overline{g_1(\omega)} \\ -g_3(\omega) & -\overline{g_2(\omega)} & e^{i(\omega_1+\omega_2)} \overline{g_1(\omega)} & 0 \end{bmatrix}. \tag{3.8}$$

Then $\{p, q^{(1)}, \cdots, q^{(7)}\}$ is a tight frame filter bank.

One can easily verify that $W(\omega)$ defined by (3.8) satisfies (3.1) with $s = 2$. Thus $q^{(4)}, \cdots, q^{(7)}$, whose polyphase matrix is $W(\omega)$, together with $p, q^{(1)}, q^{(2)}, q^{(3)}$, form a semi-canonical tight frame filter bank. Multiplying the factor $e^{i(\omega_1+\omega_2)}$ in the third column of $W(\omega)$ is for the purpose that resulting $q^{(j)}$ may have smaller supports.

Let

$$p(\omega) = \frac{1}{8z_1z_2}(1 + z_1)(1 + z_2)(1 + z_1z_2) = \frac{1}{8}(2 + z_1 + z_1z_2 + z_2 + z_1^{-1} + z_1^{-1}z_2^{-1} + z_2^{-1})$$

be the refinement mask (two-scale symbol) of the Courant element B_{111} on the 3-directional mesh of $\mathbb{Z}^2$. Then $q^{(1)}, q^{(2)}, q^{(3)}$ defined by (2.4) with $s = 2$ are

$$\begin{aligned} q^{(1)}(\omega) &= \frac{1}{8z_1z_2}(2 - z_1 - z_1z_2 + z_2 - z_1^{-1} - z_1^{-1}z_2^{-1} + z_2^{-1}), \\ q^{(2)}(\omega) &= \frac{1}{8z_1}(2 - z_1 + z_1z_2 - z_2 - z_1^{-1} + z_1^{-1}z_2^{-1} - z_2^{-1}), \\ q^{(3)}(\omega) &= \frac{1}{8z_2}(2 + z_1 - z_1z_2 - z_2 + z_1^{-1} - z_1^{-1}z_2^{-1} - z_2^{-1}). \end{aligned}$$

One can obtain that (refer to [15])

$$1 - \sum_{j=0}^{3} |p(\omega + \pi\eta_j)|^2 = \sum_{k=1}^{3} |g_k(2\omega)|^2,$$

with

$$g_1(\omega) = \frac{1}{4}(1 - z_1^{-1}), \; g_2(\omega) = \frac{1}{4}(1 - z_1z_2), \; g_3(\omega) = \frac{1}{4}(1 - z_2^{-1}).$$

The filters $q^{(4)}, \cdots, q^{(7)}$ defined by (3.7) are

$$\begin{aligned}
q^{(4)}(\omega) &= \frac{1}{8}(z_1^{-1}z_2^{-1} - z_1z_2 + z_1 - z_1^{-1} + z_2 - z_2^{-1}),\\
q^{(5)}(\omega) &= \frac{1}{8}(z_1^{-2} - 1 + z_1^{-1}z_2 - z_1^{-1}z_2^{-1} + z_2 - z_1^{-2}z_2^{-1}),\\
q^{(6)}(\omega) &= \frac{1}{8}(z_1^2z_2^2 - 1 + z_1 - z_1z_2^2 - z_2 + z_1^2z_2),\\
q^{(7)}(\omega) &= \frac{1}{8}(z_2^{-2} - 1 + z_1^{-1}z_2^{-1} - z_1z_2^{-1} - z_1 + z_1^{-1}z_2^{-2}).
\end{aligned}$$

Next we generalize Theorem 3.2 from the 2-D case to the 3-D case.

Theorem 3.3. *Let p be a symmetric 3-D FIR lowpass filter and $q^{(1)}, \cdots, q^{(7)}$ be the highpass filters defined by (2.4) with $s = 3$, where η_j and $\rho(\eta_j)$ are defined by (2.2). Suppose $R(\omega) = 1 - \sum_{j=0}^{7} |p(\omega/2 + \pi\eta_j)|^2$ can be written as*

$$R(\omega) = \sum_{k=1}^{6} |g_k(\omega)|^2, \quad \omega \in \mathbb{R}^2, \tag{3.9}$$

for some trigonometric polynomials $g_k(\omega)$. Let $q^{(8)}, \cdots, q^{(15)}$ be the FIR filters defined by

$$[q^{(8)}(\omega), \cdots, q^{(15)}(\omega)]^T = \frac{\sqrt{2}}{4} W(2\omega)[1, e^{-i\eta_1\omega_1}, \cdots, e^{-i\eta_7\omega}]^T, \tag{3.10}$$

where

$$W(\omega) = diag\Big\{ \begin{bmatrix} 0 & g_1(\omega) & g_2(\omega) & g_3(\omega) \\ -g_1(\omega) & 0 & -\overline{g_3(\omega)} & \overline{g_2(\omega)} \\ -g_2(\omega) & \overline{g_3(\omega)} & 0 & -\overline{g_1(\omega)} \\ -g_3(\omega) & -\overline{g_2(\omega)} & \overline{g_1(\omega)} & 0 \end{bmatrix}, \begin{bmatrix} 0 & g_4(\omega) & g_5(\omega) & g_6(\omega) \\ -g_4(\omega) & 0 & -\overline{g_6(\omega)} & \overline{g_5(\omega)} \\ -g_5(\omega) & \overline{g_6(\omega)} & 0 & -\overline{g_4(\omega)} \\ -g_6(\omega) & -\overline{g_5(\omega)} & \overline{g_4(\omega)} & 0 \end{bmatrix} \Big\}. \tag{3.11}$$

Then $\{p, q^{(1)}, \cdots, q^{(15)}\}$ is a tight frame filter bank.

It is easy to verify that $W(\omega)$ defined by (3.11) satisfies (3.1) for $s = 3$. Thus $q^{(8)}, \cdots, q^{(15)}$, defined by (3.11), together with $p, q^{(1)}, \cdots, q^{(7)}$, form a semi-canonical tight frame filter bank. One could multiply columns of $W(\omega)$ with $e^{i(s_1\omega_1+s_2\omega_2+s_3\omega_3)}$ for some suitable integers s_1, s_2, s_3 to make $q^{(\ell)}, 8 \le \ell \le 15$ have smaller supports.

The 1-D double-canonical tight frame banks given by Theorem 3.1 may not have a symmetric property as shown in Example 3.2. For $s = 2, 3$, the highpass filters obtained by the approach in Theorems 3.2 and 3.3 may not have the double-canonical property, see e.g. Example 3 in [15], where the two-scale symbol of the 2-D cubic C^2 box spline B_{222} was discussed. In the next subsection, we consider the construction of symmetric/antisymmetric double-canonical tight frame banks.

3.2 Symmetric/antisymmetric double-canonical tight frames

In this subsection, we present a constructive method to construct symmetric double-canonical tight frame filter banks. Recall that a double-canonical tight frame filter bank is a semi-canonical tight frame filter bank with the minimal number of highpass filters. In all examples provided below, the supports of the constructed framelets are not bigger than that of the corresponding B-splines and box spline whose refinement mask is used to generate the first $2^s - 1$ canonical filters in the filter bank.

First we consider the 1-D case. The next theorem leads to a symmetric/antisymmetric double-canonical tight frame filter bank $\{p, q^{(1)}, q^{(2)}, q^{(3)}\}$.

Theorem 3.4. *Let p be a symmetric 1-D lowpass filter. Let $q^{(1)}$ be the canonical filter associated with p defined by* (1.2)*. Suppose $R(\omega) = 1 - |p(\omega/2)|^2 - |p(\omega/2+\pi)|^2$ can be written as*

$$R(\omega) = |h_0(\omega)|^2 + |h_1(\omega)|^2$$

where $h_0(\omega)$ and $h_1(\omega)$ are trigonometric polynomials satisfying

$$h_0(-\omega) = s_0 h_0(\omega),\ h_1(-\omega) = s_0 e^{-i\omega} h_1(\omega), \tag{3.12}$$

or

$$h_1(-\omega) = s_0 h_0(\omega), \tag{3.13}$$

where $s_0 = 1$ or $s_0 = -1$. Let $q^{(2)}$ be the FIR filters defined by

$$q^{(2)}(\omega) = \frac{\sqrt{2}}{2} h_0(2\omega) + \frac{\sqrt{2}}{2} e^{-i\omega} h_1(2\omega),$$

and $q^{(3)}$ be the canonical filter associated with $q^{(2)}$ defined by (1.8)*. Then $\{p(\omega), q^{(1)}(\omega), q^{(2)}(\omega), q^{(3)}(\omega)\}$ is a double-canonical FIR tight frame filter bank with symmetric/antisymmetric highpass filters.*

Proof. By the fact that

$$\overline{q^{(2)}(\omega)} q^{(3)}(\omega) + \overline{q^{(2)}(\omega+\pi)} q^{(3)}(\omega+\pi) = 0,$$

we know the modulation matrix $M_{q^{(2)},\, q^{(3)}}(\omega)$ of $q^{(2)}(\omega)$ and $q^{(3)}(\omega)$ satisfies

$$M_{q^{(2)},\, q^{(3)}}(\omega)^* \, M_{q^{(2)},\, q^{(3)}}(\omega) = \big(|q^{(2)}(\omega)|^2 + |q^{(2)}(\omega+\pi)|^2\big) I_2 = \big(|h_0(2\omega)|^2 + |h_1(2\omega)|^2\big) I_2 = R(2\omega) I_2.$$

Hence, the modulation matrix $M_{q^{(0)},\cdots,\, q^{(3)}}(\omega)$ (with $q^{(0)} = p$) of $p, q^{(1)}, q^{(2)}, q^{(3)}$ satisfies

$$\begin{aligned} M_{q^{(0)},\cdots,\, q^{(3)}}(\omega)^* \, M_{q^{(0)},\cdots,\, q^{(3)}}(\omega) &= M_{q^{(0)},\, q^{(1)}}(\omega)^* \, M_{q^{(0)},\, q^{(1)}}(\omega) + M_{q^{(2)},\, q^{(3)}}(\omega)^* \, M_{q^{(2)},\, q^{(3)}}(\omega) \\ &= \big(|p(\omega)|^2 + |p(\omega+\pi)|^2\big) I_2 + R(2\omega) I_2 = I_2. \end{aligned}$$

This shows that $p, q^{(1)}, q^{(2)}, q^{(3)}$ form a tight frame filter bank.

Clearly, $q^{(1)}$ is symmetric/antisymmetric. It is straightforward to verify that $q^{(2)}$ is symmetric/antisymmetric around 0 (when h_0, h_1 satisfy (3.12)) or around 1/2 (when h_0, h_1 satisfy (3.13)). Thus $q^{(3)}$ is also symmetric/antisymmetric. Therefore, all highpass filters are symmetric/antisymmetric, as desired. □

Remark 3.1. *If h_0 and h_1 in Theorem 3.4 satisfy* (3.12)*, then $q^{(2)}$ is symmetric/antisymmetric* (*depending $s_0 = 1$ or $s_0 = -1$*) *around 0; and if h_0 and h_1 satisfy* (3.13)*, then $q^{(2)}$ is symmetric/antisymmetric around $\frac{1}{2}$. If h_0 and h_1 satisfy* (3.13)*, then $|h_1(\omega)| = |h_0(\omega)|$ and hence,*

$$|h_0(\omega)|^2 = |h_1(\omega)|^2 = \frac{1}{2} R(\omega) = \frac{1}{2} |g_1(\omega)|^2,$$

where $g_1(\omega)$ is the "square-root" of $R(\omega)$ obtained by the Fejér-Riesz lemma. So in this case $h_0(\omega)$ and $h_1(\omega)$ are essentially $h_0(\omega) = \frac{\sqrt{2}}{2} g_1(\omega)$ and $h_1(\omega) = \pm\frac{\sqrt{2}}{2} g_1(-\omega)$. This is exactly what [4] *and* [8] *proposed to construct symmetric/antisymmetric tight frames with 3 generators:*

$$h_0(\omega) = h_1(-\omega) = \frac{\sqrt{2}}{2} g_1(\omega) \tag{3.14}$$

in [8]*, and*

$$h_0(\omega)=\frac{\sqrt{2}}{2}g_1(\omega),\ h_1(\omega)=\frac{\sqrt{2}}{2}e^{-iN\omega}g_1(-\omega) \tag{3.15}$$

in [4]*, where N is an integer. When h_0, h_1 are given by* (3.14)*, the corresponding highpass filters, denoted by $\widetilde{q}^{(2)}$ and $\widetilde{q}^{(3)}$, are*

$$\widetilde{q}^{(2)}(\omega)=\frac{1}{2}\big(g_1(2\omega)+e^{-i\omega}g_1(-2\omega)\big),\ \widetilde{q}^{(3)}(\omega)=e^{-i\omega}\overline{q^{(2)}(\omega+\pi)}=\frac{1}{2}\big(-g_1(2\omega)+g_1(-2\omega)e^{-i\omega}\big). \tag{3.16}$$

If $R(\omega)$ can be written as $R(\omega)=|h_0(\omega)|^2+|h_1(\omega)|^2$ with h_0 and h_1 satisfying (3.12) *and $|h_0(\omega)|\neq|h_1(\omega)|$, then symmetric/antisymmetric highpass filters $q^{(2)}, q^{(3)}$ may have smaller supports than the highpass filters such as $\widetilde{q}^{(2)}, \widetilde{q}^{(3)}$ in* (3.16) *constructed by* [4] *and* [8] *with h_0, h_1 given by* (3.14) *or* (3.15)*. As shown in the following examples, $R(\omega)$ corresponding to some B-splines can be decomposed as $|h_0(\omega)|^2+|h_1(\omega)|^2$ with symmetric and different h_0 and h_1 (in modulus) and the resulting highpass filters have smaller supports. In addition, such a decomposition can be generalized to the high dimensional case. Note that* [17] *showed that $R(\omega_1,\omega_2)$ corresponding to some type of 2-D box splines can always be decomposed as a finite sum of the squares of some 2-D trigonometric polynomials $h_j(\omega_1,\omega_2)$.* □

Example 3.3. *Let $p(\omega)=\frac{1}{16}e^{i2\omega}(1+e^{-i\omega})^4$ be the two-scale symbol of the C^2 cubic B-spline $B_4(x)$ considered in Example 3.2. By a direct calculation, we have*

$$R(\omega)=1-|p(\omega/2)|^2-|p(\omega/2+\pi)|^2=\frac{1}{64}(29-28\cos\omega-\cos 2\omega).$$

Thus,

$$\begin{aligned}R(\omega)&=\frac{1}{64}(1-\cos 2\omega)+\frac{28}{64}(1-\cos\omega)=\frac{1}{32}\sin^2\omega+\frac{7}{16}\sin^2\frac{\omega}{2}\\&=\frac{1}{32}\Big|\frac{e^{-i\omega}-e^{i\omega}}{2}\Big|^2+\frac{7}{16}\Big|\frac{1-e^{i\omega}}{2}\Big|^2.\end{aligned}$$

Hence $R(\omega)$ can be written as $R(\omega)=|h_0(\omega)|^2+|h_1(\omega)|^2$ with

$$h_0(\omega)=s_1\frac{\sqrt{2}}{16}(e^{-i\omega}-e^{i\omega}),\ h_1(\omega)=s_2\frac{\sqrt{14}}{8}(1-e^{i\omega}),$$

where $s_1=\pm1$ or $s_2=\pm1$. Thus the corresponding $q^{(2)}, q^{(3)}$ given in Theorem 3.4 are

$$\begin{aligned}q^{(2)}(\omega)&=s_2\frac{\sqrt{7}}{8}(e^{-i\omega}-e^{i\omega})+s_1\frac{1}{16}(e^{-i2\omega}-e^{i2\omega}),\\q^{(3)}(\omega)&=e^{-i\omega}\overline{q^{(2)}(\omega+\pi)}=s_2\frac{\sqrt{7}}{8}(e^{-i2\omega}-1)+s_1\frac{1}{16}(e^{i\omega}-e^{-i3\omega}).\end{aligned}$$

Observe that both $q^{(2)}$ and $q^{(3)}$ are antisymmetric and they have the same filter lengths as p. □

Example 3.4. *Let $p(\omega)=\frac{1}{32}e^{i2\omega}(1+e^{-i\omega})^5$ be the two-scale symbol of the C^3 quartic B-spline $B_5(x)$ supported on $[-2,3]$. In this case $q^{(1)}(\omega)$ defined by* (1.2) *is*

$$q^{(1)}(\omega)=e^{-i\omega}\overline{p(\omega+\pi)}=\frac{1}{32}e^{i2\omega}(e^{-i\omega}-1)^5.$$

By a direct calculation, we have

$$\begin{aligned}
&R(\omega) = 1 - |p(\omega/2)|^2 - |p(\omega/2+\pi)|^2 = \frac{1}{256}(130 - 120\cos\omega - 10\cos 2\omega)\\
&= \frac{10}{256}(1-\cos 2\omega) + \frac{120}{256}(1-\cos\omega) = \frac{5}{64}\sin^2\omega + \frac{15}{16}\sin^2\frac{\omega}{2}\\
&= \frac{5}{64}\Big|\frac{e^{-i\omega}-e^{i\omega}}{2}\Big|^2 + \frac{15}{16}\Big|\frac{1-e^{i\omega}}{2}\Big|^2.
\end{aligned}$$

Thus $R(\omega)$ can be written as $R(\omega) = |h_0(\omega)|^2 + |h_1(\omega)|^2$ with

$$h_0(\omega) = s_1\frac{\sqrt{5}}{16}(e^{-i\omega} - e^{i\omega}),\ h_1(\omega) = s_2\frac{\sqrt{15}}{8}(1 - e^{i\omega}),$$

where $s_1 = \pm 1$, $s_2 = \pm 1$. Hence, the corresponding $q^{(2)}, q^{(3)}$ given by Theorem 3.4 are

$$\begin{aligned}
q^{(2)}(\omega) &= \frac{\sqrt{10}}{32}\Big(s_2 2\sqrt{3}(e^{-i\omega} - e^{i\omega}) + s_1(e^{-i2\omega} - e^{i2\omega})\Big),\\
q^{(3)}(\omega) &= e^{-i\omega}\overline{q^{(2)}(\omega+\pi)} = \frac{\sqrt{10}}{32}\Big(s_2 2\sqrt{3}(e^{-i2\omega} - 1) + s_1(e^{i\omega} - e^{-i3\omega})\Big).
\end{aligned}$$

Observe that $q^{(2)}$ and $q^{(3)}$ are antisymmetric and they both have shorter filter lengths than p. □

Though [4] proposed to use (3.15) to construct symmetric/antisymmetric tight frames, in practice [4] also constructed $q^{(2)}, q^{(3)}$ with smaller filter lengths in Examples 3.3 and 3.4.

Example 3.5. *Let $p(\omega) = \frac{1}{64}e^{i3\omega}(1+e^{-i\omega})^6$ be the two-scale symbol of the C^4 6th-order B-spline $B_6(x)$. By a direct calculation, one can obtain that $R(\omega)$ can be written as $R(\omega) = |h_0(\omega)|^2 + |h_1(\omega)|^2$ with*

$$h_0(\omega) = a_1(e^{-i\omega} - e^{i\omega}),\ h_1(\omega) = b_1(1 - e^{i\omega}) + b_2(e^{-i\omega} - e^{i2\omega}),$$

where

$$a_1 = \frac{1}{16}\sqrt{8 - \sqrt{31}},\ b_1 = \frac{1}{64}(\sqrt{2} + 4\sqrt{62}),\ b_2 = \frac{\sqrt{2}}{64}.$$

The corresponding $q^{(2)}, q^{(3)}$ given in Theorem 3.4 are

$$\begin{aligned}
q^{(2)}(\omega) &= \frac{\sqrt{2}}{2}\Big(b_1(e^{-i\omega} - e^{i\omega}) + a_1(e^{-i2\omega} - e^{i2\omega}) + b_2(e^{-i3\omega} - e^{i3\omega})\Big),\\
q^{(3)}(\omega) &= e^{-i\omega}\overline{q^{(2)}(\omega+\pi)}.
\end{aligned}$$

Both $q^{(2)}$ and $q^{(3)}$ are antisymmetric and they have the same filter lengths as p. □

Next we consider the 2-D case. The following theorem provides an approach for the construction of $q^{(4)}, \cdots, q^{(7)}$ with the double-canonical and symmetric property.

Theorem 3.5. *Let p be a symmetric 2-D FIR lowpass filter and $q^{(1)}, q^{(2)}, q^{(3)}$ be the highpass filters defined by (2.4) with $s = 2$. Suppose $R(\omega) = 1 - \sum_{j=0}^{3} |p(\omega/2 + \pi\eta_j)|^2$ can be written as*

$$R(\omega) = |h_0(\omega)|^2 + |h_1(\omega)|^2 + |h_2(\omega)|^2 + |h_3(\omega)|^2, \quad \omega \in \mathbb{R}^2, \tag{3.17}$$

for some trigonometric polynomials $h_k(\omega)$ *satisfying*

$$\begin{aligned} &h_0(-\omega)=h_0(\omega), h_1(-\omega)=e^{-i\omega_1}h_1(\omega), h_2(-\omega)=e^{-i(\omega_1+\omega_2)}h_2(\omega), h_3(-\omega)=e^{-i\omega_2}h_3(\omega), \text{ or} \\ &h_0(-\omega)=-h_0(\omega), h_1(-\omega)=-e^{-i\omega_1}h_1(\omega), h_2(-\omega)=-e^{-i(\omega_1+\omega_2)}h_2(\omega), h_3(-\omega)=-e^{-i\omega_2}h_3(\omega). \end{aligned} \tag{3.18}$$

Let $q^{(4)},\cdots,q^{(7)}$ *be the FIR filters defined by*

$$q^{(4)}(\omega)=\frac{1}{2}\Big(h_0(2\omega)+e^{-i\omega_1}h_1(2\omega)+e^{-i(\omega_1+\omega_2)}h_2(2\omega)+e^{-i\omega_2}h_3(2\omega)\Big), \tag{3.19}$$

and

$$q^{(k+4)}(\omega)=e^{i\rho(\eta_k)\omega}q^{(4)}(\omega+\pi\eta_k), \quad k=1,2,3. \tag{3.20}$$

Then $\{p,q^{(1)},\cdots,q^{(7)}\}$ *is a double-canonical FIR tight frame filter bank. In addition,* $q^{(4)},\cdots,q^{(7)}$ *are symmetric/antisymmetric.*

Observe that if $h_0,\cdots,h_3$ satisfy (3.18), then $q^{(4)}$ in Theorem 3.5 is symmetric/antisymmetric around $(0,0)$. For simplicity of presentation, we give in Theorem 3.5 just the conditions on $h_0,\cdots,h_3$ such that $q^{(4)}$ has symmetric/antisymmetric center $(0,0)$. The statement in Theorem 3.5 still holds if $R(\omega)$ can be written as the sum of $|h_j(\omega)|^2$ with $h_0,\cdots,h_3$ satisfying similar conditions to (3.18) such that $q^{(4)}$ has a different symmetric/antisymmetric center.

Example 3.6. *Let*

$$p(\omega)=\frac{1}{16z_1}(1+z_1)(1+z_2)(1+z_1z_2)(1+\frac{z_1}{z_2})$$

be the two-scale symbol of the box spline B_{1111} *on the 4-directional mesh of* $\mathbb{Z}^2$. *Lowpass filter* $p(\omega)$ *is symmetric around* $(\frac{1}{2},\frac{1}{2})$ *and* $q^{(1)},q^{(2)},q^{(3)}$ *defined by* (2.4) *are*

$$q^{(1)}(\omega)=z_1z_2p(-\frac{1}{z_1},\frac{1}{z_2}),\ q^{(2)}(\omega)=\frac{1}{z_1}p(-z_1,-z_2),\ q^{(3)}(\omega)=z_2p(\frac{1}{z_1},-\frac{1}{z_2}).$$

$R(\omega)=1-\sum_{j=0}^{3}|p(\omega/2+\pi\eta_j)|^2$ *can be written as* (3.17) *with*

$$h_0(\omega)=0,\ h_1(\omega)=\frac{\sqrt{3}}{4}(1-\frac{1}{z_1}),\ h_2(\omega)=\frac{\sqrt{2}}{8}(1-\frac{1}{z_1z_2}+\frac{1}{z_1}-\frac{1}{z_2}),\ h_3(\omega)=\frac{\sqrt{3}}{4}(1-\frac{1}{z_2}).$$

$q^{(4)}$ *defined by* (3.19) *is antisymmetric around* $(0,0)$. *Thus* $p(\omega)$ *has an associated double-canonical tight frame filter bank (with 7 highpass filters). In addition, these highpass filters are symmetric/antisymmetric. The coefficients of* $q^{(4)}$ *and* $q^{(5)},q^{(6)},q^{(7)}$ *defined by* (3.20) *are given in the following matrices:*

$$q^{(4)}(\omega)\hat{=}\frac{1}{16}\begin{bmatrix}\sqrt{2} & 2 & \sqrt{2}\\ -2\sqrt{3} & \boxed{0} & 2\sqrt{3}\\ -\sqrt{2} & -2 & -\sqrt{2}\end{bmatrix},\ q^{(5)}(\omega)\hat{=}\frac{1}{16}\begin{bmatrix}-\sqrt{2} & 2 & \boxed{-\sqrt{2}}\\ 2\sqrt{3} & 0 & -2\sqrt{3}\\ \sqrt{2} & -2 & \sqrt{2}\end{bmatrix},$$

$$q^{(6)}(\omega)\hat{=}\frac{1}{16}\begin{bmatrix}\sqrt{2} & -2 & \sqrt{2}\\ 2\sqrt{3} & 0 & \boxed{-2\sqrt{3}}\\ -\sqrt{2} & 2 & -\sqrt{2}\end{bmatrix},\ q^{(7)}(\omega)\hat{=}\frac{1}{16}\begin{bmatrix}-\sqrt{2} & \boxed{-2} & -\sqrt{2}\\ -2\sqrt{3} & 0 & 2\sqrt{3}\\ \sqrt{2} & 2 & \sqrt{2}\end{bmatrix}.$$

Observe that all the highpass filters have the same size of support as the lowpass filter p. □

Example 3.7. *Let*

$$p(\omega) = \frac{1}{64 z_1^2 z_2^2}(1+z_1)^2(1+z_2)^2(1+z_1z_2)^2$$

be the two-scale symbol of the box spline B_{222} *on the 3-directional mesh of* $\mathbb{Z}^2$*. Lowpass filter* $p(\omega)$ *is symmetric around* $(0,0)$ *and* $q^{(1)}, q^{(2)}, q^{(3)}$ *defined by* (2.4) *are*

$$q^{(1)}(\omega) = \frac{1}{z_1 z_2} p(-z_1, z_2),\ q^{(2)}(\omega) = \frac{1}{z_1} p(-z_1, -z_2),\ q^{(3)}(\omega) = \frac{1}{z_2} p(z_1, -z_2). \tag{3.21}$$

$R(\omega) = 1 - \sum_{j=0}^{3} |p(\omega/2 + \pi\eta_j)|^2$ *can be written as* (3.17) *with*

$$h_0(\omega) = \frac{1}{32}(z_1 - \frac{1}{z_1} + z_2 - \frac{1}{z_2} + z_1z_2 - \frac{1}{z_1z_2}),\ h_1(\omega) = \frac{\sqrt{26}}{16}(1 - \frac{1}{z_1}) + \frac{\sqrt{2}}{16}(z_2 - \frac{1}{z_1z_2}),$$
$$h_2(\omega) = \frac{\sqrt{26}}{16}(1 - \frac{1}{z_1z_2}) + \frac{\sqrt{2}}{16}(\frac{1}{z_1} - \frac{1}{z_2}),\ h_3(\omega) = \frac{\sqrt{26}}{16}(1 - \frac{1}{z_2}) + \frac{\sqrt{2}}{16}(\frac{1}{z_1z_2} - z_1).$$

$q^{(4)}$ *defined by* (3.19) *is antisymmetric around* $(0,0)$*. Thus* $p(\omega)$ *has an associated double-canonical tight frame filter bank* (*with 7 highpass filters*)*. In addition, these highpass filters are symmetric/antisymmetric. In the following we provide the coefficients of* $q^{(4)}$ *and those of* $q^{(5)}$ *defined by* (3.20)*:*

$$q^{(4)}(\omega) \hat{=} \frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & 2\sqrt{2} & -1 \\ 0 & 2\sqrt{2} & 2\sqrt{26} & 2\sqrt{26} & -2\sqrt{2} \\ -1 & -2\sqrt{26} & \boxed{0} & 2\sqrt{26} & 1 \\ 2\sqrt{2} & -2\sqrt{26} & -2\sqrt{26} & -2\sqrt{2} & 0 \\ 1 & -2\sqrt{2} & -1 & 0 & 0 \end{bmatrix},$$
$$q^{(5)}(\omega) \hat{=} \frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & -2\sqrt{2} & -1 \\ 0 & -2\sqrt{2} & 2\sqrt{26} & -\boxed{2\sqrt{26}} & -2\sqrt{2} \\ -1 & 2\sqrt{26} & 0 & -2\sqrt{26} & 1 \\ 2\sqrt{2} & 2\sqrt{26} & -2\sqrt{26} & 2\sqrt{2} & 0 \\ 1 & 2\sqrt{2} & -1 & 0 & 0 \end{bmatrix}.$$

The subdivision scheme based on the two-scale symbol of B_{222} *is called Loop's scheme. It was shown in* [12] *that the wavelet system* $\{2^{j/2}\psi^{(\ell)}(2^j \cdot -k) :\ j \in \mathbb{Z}, k \in \mathbb{Z}^2, 1 \le \ell \le 3\}$ *generated by the canonical wavelets is a Riesz basis of* $L_2(\mathbb{R}^2)$*. This Riesz basis has been used in* [16] *for surface compression. Now with 4 framelets added,* $\{2^{j/2}\psi^{(\ell)}(2^j \cdot -k) :\ j \in \mathbb{Z}, k \in \mathbb{Z}^2, 1 \le \ell \le 7\}$ *is a tight frame of* $L_2(\mathbb{R}^2)$*. Tight frames associated with* B_{222} *with* 7 *framelets were constructed in* [17] *and* [15]*. All the highpass filters constructed here have the same size of support as the lowpass filter* p*. These highpass filters have a smaller size of support than those constructed in both* [17] *and* [15]*. In addition, our highpass filters have the "double-canonical" property. Note that the construction of multivariate compactly supported tight affine spline frames was first considered in* [22]*. But the tight frames constructed by the method in* [22] *also have big supports.* □

Example 3.8. *Let*

$$p(\omega) = \frac{1}{64 z_1^2 z_2}(1+z_1)^2(1+z_2)^2(1+z_1z_2)(1+\frac{z_1}{z_2})$$

be the two-scale symbol of the box spline B_{2211} *on the 4-directional mesh of* $\mathbb{Z}^2$*. Lowpass filter* $p(\omega)$ *is symmetric around* $(0,0)$ *and* $q^{(1)}, q^{(2)}, q^{(3)}$ *defined by* (2.4) *are given as in* (3.21)*.* $R(\omega) = 1 -$

$\sum_{j=0}^{3}|p(\omega/2+\pi\eta_j)|^2$ *can be written as* (3.17) *with*

$$h_0(\omega)=\frac{1}{16}(z_1-\frac{1}{z_1}+z_2-\frac{1}{z_2}),\ h_1(\omega)=a_1(1-\frac{1}{z_1})+\frac{1}{32}(z_2-\frac{1}{z_1z_2}+\frac{z_2}{z_1}-\frac{1}{z_2}),$$
$$h_2(\omega)=a_2(1-\frac{1}{z_1z_2})+b_2(\frac{1}{z_1}-\frac{1}{z_2}),\ h_3(\omega)=a_3(1-\frac{1}{z_2})+\frac{1}{32}(z_1-\frac{1}{z_1z_2}+\frac{z_1}{z_2}-\frac{1}{z_1}),$$

where a_1,a_2,a_3,b_2 *are given by*

$$a_1=\frac{\sqrt{12\mu-\mu^2}}{8(\mu-7)}+\frac{\mu-7}{16},\ a_2=\frac{\sqrt{24-2\mu}}{16},\ a_3=\frac{\mu-7}{8}-a_1,\ b_2=\frac{\sqrt{2\mu}}{16}$$

with $\mu=1.5369740543671196320$*, a root of the polynomial* $833-876x+258x^2-28x^3+x^4$*. The numerical values for* a_1,a_2,a_3,b_2 *are*

$$a_1=-0.43319600820800456763,\ a_2=0.28590626121205684060,$$
$$a_3=-0.24968223499610547836,\ b_2=0.10957923982097668360.$$

$q^{(4)}$ *defined by* (3.19) *is antisymmetric around* $(0,0)$*. Thus* $p(\omega)$ *has an associated double-canonical symmetric/antisymmetric tight frame filter bank which has 7 highpass filters. In the following we provide the coefficients of* $q^{(4)}$ *and those of* $q^{(5)}$ *defined by* (3.20)*:*

$$q^{(4)}(\omega)\hat{=}\frac{1}{64}\begin{bmatrix}0&1&2&1&0\\-1&32b_2&32a_3&32a_2&1\\-2&-32a_1&\boxed{0}&32a_1&2\\-1&-32a_2&-32a_3&-32b_2&1\\0&-1&-2&-1&0\end{bmatrix},\ q^{(5)}(\omega)\hat{=}\frac{1}{64}\begin{bmatrix}0&-1&2&-1&0\\-1&-32b_2&32a_3&\boxed{-32a_2}&1\\-2&32a_1&0&-32a_1&2\\-1&32a_2&-32a_3&32b_2&1\\0&1&-2&1&0\end{bmatrix}.$$

Observe that all the highpass filters, again, have the same size of support as the lowpass filter p. □

Finally we consider the case $s=3$. First we generalize Theorem 3.5 to the 3-D case for the construction of $q^{(8)},\cdots,q^{(15)}$ with the double-canonical and symmetric property.

Theorem 3.6. *Let* p *be a symmetric 3-D FIR lowpass filter and* $q^{(1)},\cdots,q^{(7)}$ *be the highpass filters defined by* (2.4) *with* $s=3$*, where* η_j *and* $\rho(\eta_j)$ *are defined by* (2.2)*. Suppose*

$$R(\omega)=1-\sum_{j=0}^{7}|p(\omega/2+\pi\eta_j)|^2=\sum_{k=0}^{7}|h_k(\omega)|^2,\quad \omega\in\mathbb{R}^3, \tag{3.22}$$

where $h_k(\omega), 0\le k\le 7$ *are trigonometric polynomials satisfying*

$$\begin{aligned}&h_0(-\omega)=-h_0(\omega),\ h_1(-\omega)=-e^{-i\omega_1}h_1(\omega),\ h_2(-\omega)=-e^{-i(\omega_1+\omega_2)}h_2(\omega),\\&h_3(-\omega)=e^{-i\omega_2}h_3(\omega),\ h_4(-\omega)=-e^{-i\omega_3}h_4(\omega),\ h_5(-\omega)=-e^{-i(\omega_1+\omega_3)}h_5(\omega),\\&h_6(-\omega)=-e^{-i(\omega_1+\omega_2+\omega_3)}h_6(\omega),\ h_7(-\omega)=-e^{-i(\omega_2+\omega_3)}h_7(\omega).\end{aligned} \tag{3.23}$$

Let $q^{(8)},\cdots,q^{(15)}$ *be the FIR filters defined by*

$$q^{(8)}(\omega)=\frac{\sqrt{2}}{4}\sum_{k=0}^{7}h_k(2\omega)e^{-i\eta_k\omega}, \tag{3.24}$$

and

$$q^{(k+8)}(\omega) = e^{i\rho(\eta_k)\omega} q^{(8)}(\omega + \pi\eta_k), \quad k = 1, 2, \cdots, 7. \tag{3.25}$$

Then $\{p, q^{(1)}, \cdots, q^{(15)}\}$ *is a double-canonical FIR tight frame filter bank. In addition,* $q^{(8)}, \cdots, q^{(15)}$ *are symmetric/antisymmetric.*

We present in Theorem 3.6 the conditions on $h_0, \cdots, h_7$ such that $q^{(8)}$ defined by (3.24) is antisymmetric about $(0, 0, 0)$. The statement in Theorem 3.6 still holds if $R(\omega)$ can be written as $\sum_{k=0}^{7} |h_k(\omega)|^2$ with $h_0, \cdots, h_7$ satisfying the conditions such that $q^{(8)}$ is symmetric about $(0, 0, 0)$ or has a different symmetric/antisymmetric center.

The proof of Theorems 3.5 and 3.6 is similar to that for Theorem 3.4. For example, for the proof of Theorem 3.6, by the antisymmetry property of $q^{(8)}$ and Remark 2.1, we have

$$\sum_{k=0}^{7} \overline{q^{(\ell)}(\omega + \pi\eta_k)}\, q^{(\ell')}(\omega + \pi\eta_k) = 0, \quad \text{for } \ell \neq \ell', 8 \leq \ell, \ell' \leq 15.$$

Thus, the modulation matrix $M_{q^{(8)},\cdots,\, q^{(15)}}(\omega)$ of $q^{(8)}(\omega), \cdots, q^{(15)}(\omega)$ satisfies

$$M_{q^{(8)},\cdots,\, q^{(15)}}(\omega)^* \, M_{q^{(8)},\cdots,\, q^{(15)}}(\omega) = \sum_{k=0}^{7} |q^{(8)}(\omega + \pi\eta_k)|^2 I_8 = \sum_{k=0}^{7} |h_k(2\omega)|^2 I_8 = R(2\omega) I_8.$$

Hence, the modulation matrix $M_{q^{(0)},\cdots,\, q^{(15)}}(\omega)$ (with $q^{(0)} = p$) of $p, q^{(1)}, \cdots, q^{(15)}$ satisfies

$$\begin{aligned} &M_{q^{(0)},\cdots,\, q^{(15)}}(\omega)^* \, M_{q^{(0)},\cdots,\, q^{(15)}}(\omega) \\ &= M_{q^{(0)},\cdots,\, q^{(7)}}(\omega)^* \, M_{q^{(0)},\cdots,\, q^{(7)}}(\omega) + M_{q^{(8)},\cdots,\, q^{(15)}}(\omega)^* \, M_{q^{(8)},\cdots,\, q^{(15)}}(\omega) \\ &= \sum_{k=0}^{7} |p(\omega + \pi\eta_k)|^2 I_8 + R(2\omega) I_8 = I_8, \end{aligned}$$

which means that $\{p, q^{(1)}, \cdots, q^{(15)}\}$ is a tight frame filter bank. □

Example 3.9. *Let*

$$p(\omega) = \frac{1}{16 z_1 z_2 z_3}(1 + z_1)(1 + z_2)(1 + z_3)(1 + z_1 z_2 z_3)$$

be the two-scale symbol of the 3-D box spline with vectors:

$$v_1 = [1, 0, 0],\ v_2 = [0, 1, 0],\ v_3 = [0, 0, 1],\ v_4 = [1, 1, 1].$$

Lowpass filter $p(\omega)$ *is symmetric around* $(0, 0, 0)$*. Let* $q^{(1)}, \cdots, q^{(7)}$ *be the highpass filters defined by* (2.4)*.* $R(\omega) = 1 - \sum_{j=0}^{7} |p(\omega/2 + \pi\eta_j)|^2$ *can be written as* (3.22) *with*

$$\begin{aligned} &h_0(\omega) = 0,\ h_1(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_1} - 1),\ h_2(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_1 z_2} - 1),\ h_3(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_2} - 1), \\ &h_4(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_3} - 1),\ h_5(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_1 z_3} - 1),\ h_6(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_1 z_2 z_3} - 1),\ h_7(\omega) = \frac{\sqrt{2}}{8}(\frac{1}{z_2 z_3} - 1). \end{aligned}$$

$h_j, 0 \leq j \leq 7$ *satisfy the conditions in* (3.23) *and* $q^{(8)}$ *defined by* (3.24) *is antisymmetric around* $(0, 0)$*. Thus* $p(\omega)$ *has an associated 3-D double-canonical symmetric/antisymmetric tight frame filter bank* (*with 15 highpass filters*)*.* $q^{(8)}$ *is given by*

$$q^{(8)}(\omega) = \frac{1}{16 z_1 z_2 z_3}(1 + z_1)(1 + z_2)(1 + z_3)(1 - z_1 z_2 z_3).$$

One can obtain easily other highpass filters $q^{(9)}, \cdots, q^{(15)}$ from $q^{(8)}$ by the formula (3.25). Observe that all the highpass filters have the same size of support as the lowpass filter p. $\square$

Example 3.10. *Let*

$$p(\omega) = \frac{1}{64 z_1 z_2 z_3^2}(1+z_1)(1+z_2)(1+z_3)(1+z_1z_2z_3)(1+z_1z_3)(1+z_2z_3)$$

be the two-scale symbol of the 3-D box spline with vectors:

$$v_1 = [1,0,0],\ v_2 = [0,1,0],\ v_3 = [0,0,1],\ v_4 = [1,1,1],\ v_5 = [1,0,1],\ v_6 = [0,1,1].$$

Lowpass filter $p(\omega)$ is symmetric around $(1/2,1/2,0)$. Let $q^{(1)}, \cdots, q^{(7)}$ be the highpass filters defined by (2.4). $R(\omega) = 1 - \sum_{j=0}^{7} |p(\omega/2 + \pi\eta_j)|^2$ can be written as (3.22) with

$$\begin{aligned}
&h_0(\omega) = 0,\ h_1(\omega) = \frac{3\sqrt{2}}{16}(1 - \frac{1}{z_1}),\ h_2(\omega) = \frac{\sqrt{2}}{16}(1 - \frac{1}{z_1z_2} + \frac{1}{z_1} - \frac{1}{z_2} + z_3 - \frac{1}{z_1z_2z_3}),\\
&h_3(\omega) = \frac{\sqrt{10}}{16}(1 - \frac{1}{z_2}),\ h_4(\omega) = \frac{3\sqrt{2}}{16}(1 - \frac{1}{z_3}),\ h_5(\omega) = \frac{3\sqrt{2}}{16}(1 - \frac{1}{z_1z_3}),\\
&h_6(\omega) = \frac{\sqrt{10}}{16}(1 - \frac{1}{z_1z_2z_3}),\ h_7(\omega) = \frac{\sqrt{10}}{16}(1 - \frac{1}{z_2z_3}).
\end{aligned}$$

$h_j, 0 \le j \le 7$ satisfy the conditions in (3.23) and $q^{(8)}$ defined by (3.24) is antisymmetric around $(0,0,0)$. Thus $p(\omega)$ has an associated 3-D double-canonical tight frame filter bank (with 15 highpass filters). In addition, these highpass filters are symmetric/antisymmetric. $q^{(8)}$ is provided below in (3.26), where for each k_3, $\{q^{(8)}_{k_1,k_2,k_3}\}_{k_1,k_2}$ is displayed as a matrix.

$$q^{(8)}(\omega) \hat{=} \frac{1}{32}\Big\{ \begin{bmatrix} 0 & \boxed{0} \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & \boxed{-3} \\ -\sqrt{5} & -\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{5} & 1 \\ -3 & \boxed{0} & 3 \\ -1 & -\sqrt{5} & -1 \end{bmatrix}, \begin{bmatrix} \sqrt{5} & \sqrt{5} \\ \boxed{3} & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \boxed{0} & 0 \end{bmatrix} \Big\}. \tag{3.26}$$

(From left to right: $q^{(8)}_{k_1,k_2,-2}, q^{(8)}_{k_1,k_2,-1}, q^{(8)}_{k_1,k_2,0}, q^{(8)}_{k_1,k_2,1}, q^{(8)}_{k_1,k_2,2}$.)

One can obtain easily other highpass filters $q^{(9)}, \cdots, q^{(15)}$ from $q^{(8)}$ by the formula (3.25). Here we provide $q^{(9)}$:

$$q^{(9)}(\omega) \hat{=} \frac{1}{32}\Big\{ \begin{bmatrix} 0 & 0 & \boxed{0} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \boxed{0} \\ 3 & -3 & 0 \\ \sqrt{5} & -\sqrt{5} & 0 \end{bmatrix}, \begin{bmatrix} -1 & \sqrt{5} & \boxed{-1} \\ 3 & 0 & -3 \\ 1 & -\sqrt{5} & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{5} & \boxed{-\sqrt{5}} \\ 3 & -3 \end{bmatrix}, \begin{bmatrix} 0 & \boxed{-1} \\ 0 & 0 \end{bmatrix} \Big\}.$$

(From left to right: $q^{(9)}_{k_1,k_2,-2}, q^{(9)}_{k_1,k_2,-1}, q^{(9)}_{k_1,k_2,0}, q^{(9)}_{k_1,k_2,1}, q^{(9)}_{k_1,k_2,2}$.)

For the purpose to compare the support of p with those of $q^{(8)}, \cdots, q^{(15)}$, we also provide the coefficients p_{k_1,k_2,k_3} of p:

$$p(\omega) \hat{=} \frac{1}{64}\Big\{ \begin{bmatrix} 1 & \boxed{1} \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 \\ 2 & \boxed{4} & 2 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 4 & 4 & 1 \\ 1 & \boxed{4} & 4 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ \boxed{1} & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \boxed{0} & 0 & 0 \end{bmatrix} \Big\}.$$

(From left to right: $p_{k_1,k_2,-2}, p_{k_1,k_2,-1}, p_{k_1,k_2,0}, p_{k_1,k_2,1}, p_{k_1,k_2,2}$.)

$\square$

Example 3.11. *Let*

$$p(\omega) = \frac{1}{128 z_1^2 z_2^2 z_3^2}(1+z_1)(1+z_2)(1+z_3)(1+z_1z_2z_3)(1+z_1z_2)(1+z_1z_3)(1+z_2z_3)$$

be the two-scale symbol of the 3-D box spline with vectors:

$$v_1 = [1,0,0],\ v_2 = [0,1,0],\ v_3 = [0,0,1],\ v_4 = [1,1,1],\ v_5 = [1,0,1],\ v_6 = [0,1,1],\ v_7 = [1,1,0].$$

Lowpass filter $p(\omega)$ is symmetric around $(0,0,0)$. Let $q^{(1)}, \cdots, q^{(7)}$ be the highpass filters defined by (2.4)*. $R(\omega) = 1 - \sum_{j=0}^{7} |p(\omega/2 + \pi\eta_j)|^2$ can be written as* (3.22) *with*

$$\begin{aligned}
h_0(\omega) &= \frac{\sqrt{2}}{64}(z_1 - \frac{1}{z_1} + z_2 - \frac{1}{z_2} + z_3 - \frac{1}{z_3} + z_1z_2z_3 - \frac{1}{z_1z_2z_3}),\\
h_1(\omega) &= a_1(1 - \frac{1}{z_1}) + \frac{\sqrt{2}}{64}(z_2 - \frac{1}{z_1z_2} + z_3 - \frac{1}{z_1z_3} + z_2z_3 - \frac{1}{z_1z_2z_3}),\\
h_2(\omega) &= a_2(1 - \frac{1}{z_1z_2}) + \frac{\sqrt{2}}{32}(\frac{2}{z_1} - \frac{2}{z_2} + z_3 - \frac{1}{z_1z_2z_3}),\\
h_3(\omega) &= a_1(1 - \frac{1}{z_2}) + \frac{\sqrt{2}}{64}(z_1 - \frac{1}{z_1z_2} + z_3 - \frac{1}{z_2z_3} + z_1z_3 - \frac{1}{z_1z_2z_3}),\\
h_4(\omega) &= a_1(1 - \frac{1}{z_3}) + \frac{\sqrt{2}}{64}(z_1 - \frac{1}{z_1z_3} + z_2 - \frac{1}{z_2z_3} + z_1z_2 - \frac{1}{z_1z_2z_3}),\\
h_5(\omega) &= a_2(1 - \frac{1}{z_1z_3}) + \frac{\sqrt{2}}{32}(\frac{2}{z_3} - \frac{2}{z_1} + z_2 - \frac{1}{z_1z_2z_3}),\\
h_6(\omega) &= a_3(1 - \frac{1}{z_1z_2z_3}) + \frac{\sqrt{2}}{64}(\frac{1}{z_1} - \frac{1}{z_2z_3} + \frac{1}{z_2} - \frac{1}{z_1z_3} + \frac{1}{z_3} - \frac{1}{z_1z_2}),\\
h_7(\omega) &= a_2(1 - \frac{1}{z_2z_3}) + \frac{\sqrt{2}}{32}(\frac{2}{z_2} - \frac{2}{z_3} + z_1 - \frac{1}{z_1z_2z_3}),
\end{aligned}$$

where

$$a_1 = -0.25635520335894088696,\ a_2 = 0.19920005127022041990,\ a_3 = 0.19817365580271475087,$$

which are a solution of

$$\begin{cases} 32a_1^2 - 2\sqrt{2}a_1 - 2\sqrt{2}a_2 - \sqrt{2}a_3 - \frac{127}{64} = 0,\\ 32a_2^2 + \sqrt{2}a_1 + \sqrt{2}a_3 - \frac{19}{16} = 0,\\ 32a_3^2 + 3\sqrt{2}a_1 + 6\sqrt{2}a_2 - \frac{119}{64} = 0. \end{cases}$$

$h_j, 0 \le j \le 7$ satisfy the conditions in (3.23) *and $q^{(8)}$ defined by* (3.24) *is antisymmetric around $(0,0,0)$. Thus $p(\omega)$ has an associated 3-D double-canonical symmetric/antisymmetric tight frame filter bank with*

15 highpass FIR filters. $q^{(8)}$ *is provided below:*

$$q^{(8)}(\omega)\hat{=}\frac{1}{128}\left\{\begin{bmatrix} & -1 & \boxed{-1} \\ -1 & -2 & -1 \\ -1 & -1 & \end{bmatrix},\begin{bmatrix} & -1 & -4 & 1 \\ -1 & -32\sqrt{2}a_2 & \boxed{-32\sqrt{2}a_1} & 4 \\ -2 & -32\sqrt{2}a_3 & -32\sqrt{2}a_2 & -1 \\ -1 & -2 & -1 & \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} & & 1 & 1 & \\ & 4 & 32\sqrt{2}a_1 & 32\sqrt{2}a_2 & 1 \\ -1 & -32\sqrt{2}a_1 & \boxed{0} & 32\sqrt{2}a_1 & 1 \\ -1 & -32\sqrt{2}a_2 & -32\sqrt{2}a_1 & -4 & \\ & -1 & -1 & & \end{bmatrix},\begin{bmatrix} & 1 & 2 & 1 \\ 1 & 32\sqrt{2}a_2 & 32\sqrt{2}a_3 & 2 \\ -4 & \boxed{32\sqrt{2}a_1} & 32\sqrt{2}a_2 & 1 \\ -1 & 4 & 1 & \end{bmatrix},\begin{bmatrix} & 1 & 1 \\ 1 & 2 & 1 \\ \boxed{1} & 1 & \end{bmatrix}\right\}.$$

(From top left to right: $q^{(8)}_{k_1,k_2,-2}, q^{(8)}_{k_1,k_2,-1}$; from bottom left to right: $q^{(8)}_{k_1,k_2,0}, q^{(8)}_{k_1,k_2,1}, q^{(8)}_{k_1,k_2,2}$.)

Again, one can obtain easily other highpass filters $q^{(9)},\cdots,q^{(15)}$ *by* (3.25). *To compare the support of* p *with those of* $q^{(8)},\cdots,q^{(15)}$*, we also provide the coefficients* p_{k_1,k_2,k_3} *of* p*:*

$$p(\omega)\hat{=}\frac{1}{128}\left\{\begin{bmatrix} & 1 & \boxed{1} \\ 1 & 2 & 1 \\ 1 & 1 & \end{bmatrix},\begin{bmatrix} & 1 & 2 & 1 \\ 1 & 4 & \boxed{5} & 2 \\ 2 & 5 & 4 & 1 \\ 1 & 2 & 1 & \end{bmatrix},\begin{bmatrix} & & 1 & 1 & \\ & 2 & 5 & 4 & 1 \\ 1 & 5 & \boxed{8} & 5 & 1 \\ 1 & 4 & 5 & 2 & \\ & 1 & 1 & & \end{bmatrix},\begin{bmatrix} & 1 & 2 & 1 \\ 1 & 4 & 5 & 2 \\ 2 & \boxed{5} & 4 & 1 \\ 1 & 1 & 1 & \end{bmatrix},\begin{bmatrix} & 1 & 1 \\ 1 & 2 & 1 \\ \boxed{1} & 1 & \end{bmatrix}\right\}.$$

(From left to right: $p_{k_1,k_2,-2}, p_{k_1,k_2,-1}, p_{k_1,k_2,0}, p_{k_1,k_2,1}, p_{k_1,k_2,2}$.)

□

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