CHARACTERIZATION OF SOBOLEV SPACES OF ARBITRARY SMOOTHNESS USING NONSTATIONARY TIGHT WAVELET FRAMES

BIN HAN AND ZUOWEI SHEN

Abstract. In this paper we shall characterize Sobolev spaces of an arbitrary order of smoothness using nonstationary tight wavelet frames for $L_2(\mathbb{R})$. In particular, we show that a Sobolev space of an arbitrary fixed order of smoothness can be characterized in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequences using a fixed compactly supported nonstationary tight wavelet frame in $L_2(\mathbb{R})$ derived from masks of pseudo-splines in [15]. This implies that any compactly supported nonstationary tight wavelet frame of $L_2(\mathbb{R})$ in [15] can be properly normalized into a pair of dual frames in the corresponding pair of dual Sobolev spaces of an arbitrary fixed order of smoothness.

1. Introduction

This paper is to characterize Sobolev spaces of an arbitrary order of smoothness using the nonstationary tight wavelet frames in [15]. As a consequence, a compactly supported wavelet frame in all Sobolev spaces can be obtained by properly normalizing a fixed nonstationary tight wavelet frame for $L_2(\mathbb{R})$ and the corresponding Sobolev norm can be characterized in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequences of this fixed tight wavelet frame for $L_2(\mathbb{R})$.

Nonstationary wavelet systems are generally obtained from a sequence of nonstationary refinable functions. Let $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ be a sequence of functions in $L_2(\mathbb{R})$. We say that $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ is a sequence of *nonstationary refinable functions* if

$$\widehat{\phi_{j-1}}(\xi) = \widehat{a_j}(\xi/2)\widehat{\phi_j}(\xi/2), \qquad a.e.\ \xi \in \mathbb{R},\ j \in \mathbb{N}, \tag{1.1}$$

where $\{\widehat{a_j}\}_{j\in\mathbb{N}}$ is a sequence of 2π-periodic measurable functions, called the *refinement masks*, or simply called *masks*. Here, the Fourier transform $\hat{f}$ of a function $f \in L_1(\mathbb{R})$ used in this paper is defined to be

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx, \qquad \xi \in \mathbb{R}$$

and can be naturally extended to square integrable functions and tempered distributions. Wavelet functions ψ^{ℓ}_{j-1}, $j \in \mathbb{N}$ and $\ell = 1, \dots, \mathcal{J}_j$, are generally obtained from nonstationary refinable functions ϕ_j, $j \in \mathbb{N}$, via

$$\widehat{\psi^{\ell}_{j-1}}(\xi) := \widehat{b^{\ell}_j}(\xi/2)\widehat{\phi_j}(\xi/2), \qquad j \in \mathbb{N},\ \ell = 1, \dots, \mathcal{J}_j, \tag{1.2}$$

where $\mathcal{J}_j$ are positive integers depending on j (in this paper quite often $\mathcal{J}_j = 3$) and $\widehat{b^{\ell}_j}$ are 2π-periodic measurable functions called *wavelet masks*, or simply called *masks*. We start with

---

Date: October 11, 2007.

2000 *Mathematics Subject Classification.* 42C40, 46C15, 46E35.

Key words and phrases. characterization of Sobolev spaces of arbitrary smoothness, nonstationary tight wavelet frames, wavelet frames in Sobolev spaces, spectral frame approximation order.

Research supported in part by NSERC Canada under Grant RGP 228051.

Research supported in part by Grant R-146-000-060-112 at the National University of Singapore.

a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ generated by $\{\phi_0\} \cup \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\ldots,\mathfrak{J}_{j+1}\}}$, where $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. We say that the following wavelet system

$$\begin{aligned} X(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\ldots,\mathfrak{J}_{j+1}\}}) &:= \\ &\{\phi_0(\cdot-k)\ :\ k\in\mathbb{Z}\}\cup\{\psi_{j;j,k}^\ell := 2^{j/2}\psi_j^\ell(2^j\cdot -k)\ :\ j\in\mathbb{N}_0, \ell=1,\ldots,\mathfrak{J}_{j+1}, k\in\mathbb{Z}\} \end{aligned} \tag{1.3}$$

is a *nonstationary tight wavelet frame* for $L_2(\mathbb{R})$ if

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot-k)\rangle|^2 + \sum_{j=0}^\infty \sum_{\ell=1}^{\mathfrak{J}_{j+1}} \sum_{k\in\mathbb{Z}} |\langle f, \psi_{j;j,k}^\ell\rangle|^2 \qquad \text{for all} \quad f\in L_2(\mathbb{R}). \tag{1.4}$$

This is equivalent to

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi_0(\cdot-k)\rangle\phi_0(\cdot-k) + \sum_{j=0}^\infty \sum_{\ell=1}^{\mathfrak{J}_{j+1}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j;j,k}^\ell\rangle \psi_{j;j,k}^\ell, \qquad f\in L_2(\mathbb{R}). \tag{1.5}$$

For a 2π-periodic trigonometric polynomial $\hat{a}$, we denote $\deg(\hat{a})$ the smallest nonnegative integer such that the Fourier coefficients of $\hat{a}$ vanish outside $[-\deg(\hat{a}), \deg(\hat{a})]$.

The following unitary extension principle in the nonstationary setting, generalizing that in [18], has been proved in [15, Theorem 1.1] for constructing nonstationary tight wavelet frames in $L_2(\mathbb{R})$ from nonstationary refinable functions.

Theorem 1.1. *Let $\widehat{a_j}, j\in\mathbb{N}$, be 2π-periodic trigonometric polynomials satisfying $\widehat{a_j}(0)=1$ for all $j\in\mathbb{N}$ and*

$$\sum_{j=1}^\infty 2^{-j}\deg(\widehat{a_j}) < \infty.$$

Define a sequence of nonstationary refinable functions $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ by

$$\widehat{\phi_{j-1}}(\xi) := \widehat{a_j}(\xi/2)\widehat{\phi_j}(\xi/2) = \prod_{n=1}^\infty \widehat{a_{n+j-1}}(2^{-n}\xi), \qquad \xi\in\mathbb{R},\ j\in\mathbb{N}. \tag{1.6}$$

Suppose that there exist 2π-periodic trigonometric polynomials $\widehat{b_j^\ell}$, $j\in\mathbb{N}$ and $\ell=1,\ldots,\mathfrak{J}_j$ with each $\mathfrak{J}_j$ being a positive integer depending on j, satisfying

$$|\widehat{a_j}(\xi)|^2 + \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b_j^\ell}(\xi)|^2 = 1 \quad \textit{and} \quad \widehat{a_j}(\xi)\overline{\widehat{a_j}(\xi+\pi)} + \sum_{\ell=1}^{\mathfrak{J}_j} \widehat{b_j^\ell}(\xi)\overline{\widehat{b_j^\ell}(\xi+\pi)} = 0. \tag{1.7}$$

Define wavelet functions ψ_{j-1}^ℓ, $j\in\mathbb{N}$ and $\ell=1,\ldots,\mathfrak{J}_j$, as in (1.2). *Then all functions ϕ_{j-1} and ψ_{j-1}^ℓ, $j\in\mathbb{N}$ and $\ell=1,\ldots,\mathfrak{J}_j$, are well-defined compactly supported functions in $L_2(\mathbb{R})$, and the wavelet system $X(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\ldots,\mathfrak{J}_{j+1}\}})$ defined in* (1.3) *is a nonstationary tight wavelet frame of $L_2(\mathbb{R})$.*

Note that if (1.7) holds, then it is necessary that the inequality

$$|\widehat{a_j}(\xi)|^2 + |\widehat{a_j}(\xi+\pi)|^2 \leqslant 1 \tag{1.8}$$

holds for all $\xi\in\mathbb{R}$. For any given $\widehat{a_j}$ with real coefficients such that (1.8) holds, there are many ways to derive $\widehat{b_j^\ell}$, $\ell=1,\ldots,\mathfrak{J}_j$, so that (1.7) is satisfied. One simple way to derive the wavelet masks $\widehat{b_j^1}$, $\widehat{b_j^2}$ and $\widehat{b_j^3}$ from $\widehat{a_j}$ (e.g. [8, 15]) with $\mathfrak{J}_j = 3$ is

$$\begin{aligned} &\widehat{b_j^1}(\xi) := e^{-i\xi}\overline{\widehat{a_j}(\xi+\pi)}, \\ &\widehat{b_j^2}(\xi) := 2^{-1}[A_j(\xi) + e^{-i\xi}\overline{A_j(\xi)}], \qquad \widehat{b_j^3}(\xi) := 2^{-1}[A_j(\xi) - e^{-i\xi}\overline{A_j(\xi)}], \end{aligned} \tag{1.9}$$

where A_j is a π-periodic trigonometric polynomial with real coefficients such that

$$|A_j(\xi)|^2 = 1 - |\widehat{a_j}(\xi)|^2 - |\widehat{a_j}(\xi+\pi)|^2. \tag{1.10}$$

Then, it is easy to check that the masks $\widehat{a_j}$, $\widehat{b_j^1}$, $\widehat{b_j^2}$ and $\widehat{b_j^3}$ satisfies (1.7) for every $j \in \mathbb{N}$ and $\mathfrak{J}_j = 3$. Furthermore, the wavelet functions ψ_j^ℓ, $j \in \mathbb{N}_0$ and $\ell = 1, 2, 3$, defined in (1.2), will be symmetric or antisymmetric, if the refinable functions ϕ_{j-1} in (1.6), $j \in \mathbb{N}$, are symmetric (see e.g. [8, 15]).

In this paper, we are interested in characterizing the Sobolev norm of a function in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequence of the function by using a nonstationary tight wavelet frame in $L_2(\mathbb{R})$. Recall that for a real number s, we denote by $H^s(\mathbb{R})$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|^2_{H^s(\mathbb{R})} := \frac{1}{2\pi}\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty.$$

Note that $H^0(\mathbb{R}) = L_2(\mathbb{R})$ and $\|\cdot\|_{H^0(\mathbb{R})} = \|\cdot\|_{L_2(\mathbb{R})}$ by the Plancherel's theorem.

As one of our main results of this paper, the following result characterizes a Sobolev space $H^s(\mathbb{R})$ using a nonstationary tight wavelet frame obtained via Theorem 1.1.

Theorem 1.2. *Let $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0, \ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ be a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ obtained in Theorem 1.1. Assume that for $\alpha > 0$ there exist a positive number C and a positive integer J such that*

$$1 - |\widehat{a_j}(\xi)|^2 \leqslant C|\xi|^{2\alpha}, \qquad \xi \in \mathbb{R}, j \geqslant J, \tag{1.11}$$

and

$$\sum_{k\in\mathbb{Z}} |\widehat{\phi_j}(\xi + 2\pi k)|^2 (1 + |\xi + 2\pi k|^2)^\alpha \leqslant C, \qquad \xi \in \mathbb{R},\ j \in \mathbb{N}_0, \tag{1.12}$$

then for every $-\alpha < s < \alpha$, there exist two positive constants C_1 and C_2 such that

$$C_1\|f\|^2_{H^s(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f, \psi^\ell_{j;j,k}\rangle|^2 \leqslant C_2\|f\|^2_{H^s(\mathbb{R})}, \tag{1.13}$$

for all $f \in H^s(\mathbb{R})$.

Theorem 1.2 basically says that the weighted ℓ_2-norm of the analysis wavelet coefficient sequence $\{\langle f, \phi_0(\cdot - k)\rangle\}_{k\in\mathbb{Z}} \cup \{\langle f, \psi^\ell_{j;j,k}\rangle, \}_{k\in\mathbb{Z}, j\in\mathbb{N}_0, \ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}$ of a given function $f \in H^s(\mathbb{R})$ decomposed under the tight wavelet frame system $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ in (1.3) is equivalent to its Sobolev norm in $H^s(\mathbb{R})$. The right hand side inequality is called the upper bound of the characterization in (1.13) and the left hand side inequality is called the lower bound. In [15], we have applied Theorem 1.1 to obtain a special family of nonstationary tight wavelet frames derived from masks of pseudo-splines. Here we shall apply Theorem 1.2 to this particular family of nonstationary tight wavelet frames in [15]. For positive integers $m, l \in \mathbb{N}$, throughout the paper we denote

$$P_{m,l}(x) := \sum_{j=0}^{l-1} \binom{m+j-1}{j} x^j = \sum_{j=0}^{l-1} \frac{(m+j-1)!}{j!(m-1)!} x^j, \qquad x \in \mathbb{R}.$$

The masks for pseudo-splines of type II introduced in [8] with order (m, l) are given by

$$\widehat{a_{m,l}}(\xi) := \cos^{2m}(\xi/2) P_{m,l}(\sin^2(\xi/2)), \qquad m \in \mathbb{N},\ l = 1, \dots, m. \tag{1.14}$$

It is evident that $\widehat{a_{m,l}}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$. The (trigonometric polynomial) mask $\widehat{a^I_{m,l}}$, for the pseudo-spline of type I with order (m, l) introduced in [7] and [20], is obtained by taking the square root of the mask $\widehat{a_{m,l}}$ in (1.14) for the pseudo-spline of type II with order (m,l) using the Fejér-Riesz lemma such that

$$|\widehat{a^I_{m,l}}(\xi)|^2 = \widehat{a_{m,l}}(\xi), \qquad \xi \in \mathbb{R}. \tag{1.15}$$

While the pseudo-splines of type II and their masks in (1.14) are symmetric, their type I counterparts usually do not have symmetry. For the case $l = 1$, since $P_{m,1} \equiv 1$, we have

$$\widehat{a_{m,1}}(\xi) = \cos^{2m}(\xi/2)P_{m,1}(\sin^2(\xi/2)) = \cos^{2m}(\xi/2) = 2^{-2m}|1 + e^{-i\xi}|^{2m}.$$

Consequently, up to a multiplicative factor $e^{-ik\xi}$ for some integer k, we have $\widehat{a^I_{m,1}}(\xi) = 2^{-m}(1 + e^{-i\xi})^m$. So, for the case $l = 1$, the corresponding refinable pseudo-splines are B-splines for both types. For the case $l = m$, the corresponding refinable pseudo-spline ϕ of type I with mask $\widehat{a^I_{m,m}}$ in (1.15) has orthonormal integer shifts (i.e., $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L_2(\mathbb{R})$), and the corresponding refinable pseudo-spline ϕ of type II with mask $\widehat{a_{m,m}}$ in (1.14) is interpolatory (i.e., ϕ is a continuous function satisfying $\phi(0) = 1$ and $\phi(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$). The extensive studies of pseudo-splines, especially its smoothness, linear independence, and Riesz wavelets and tight frames from pseudo-splines are given in [7, 8, 9, 10]. Interested readers can find details there.

Let $\widehat{a_j} := \widehat{a_{m_j,l_j}}$ or $\widehat{a_j} := \widehat{a^I_{m_j,l_j}}$, $j \in \mathbb{N}$, where l_j and m_j are positive integers satisfying $1 \leqslant l_j \leqslant m_j$, $\lim_{j\to\infty} m_j = \infty$, and $\sum_{j=1}^\infty 2^{-j}m_j < \infty$. For $j \in \mathbb{N}$, define ϕ_{j-1} as in (1.6) and ψ^1_{j-1}, ψ^2_{j-1}, and ψ^3_{j-1} by

$$\widehat{\psi^\ell_{j-1}}(\xi) := \widehat{b^\ell_j}(\xi/2)\widehat{\phi_j}(\xi/2), \qquad \xi \in \mathbb{R},\ \ell = 1, 2, 3, \tag{1.16}$$

where the wavelet masks $\widehat{b^1_j}$, $\widehat{b^2_j}$ and $\widehat{b^3_j}$ are derived from $\widehat{a_j}$ as in (1.9). Then by [16, Theorem 1.2], all nonstationary refinable functions $\{\phi_{j-1}\}_{j\in\mathbb{N}}$ and all wavelet functions ψ^ℓ_{j-1}, $j \in \mathbb{N}$ and $\ell = 1, 2, 3$, are compactly supported C^∞ real-valued (symmetric or antisymmetric for type II) functions. The corresponding system $X(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,2,3\}})$ is a tight wavelet frame in $L_2(\mathbb{R})$ with the spectral frame approximation order provided that $\liminf_{j\to\infty} l_j/m_j > 0$ (see Section 3 for the concept of the spectral frame approximation order). The interested reader can check [16, Theorem 1.2] for more details.

As an application of Theorem 1.2, we show that the system derived from masks of pseudo-splines can be used to characterize Sobolev spaces of an arbitrary order of smoothness.

Theorem 1.3. *Let $\widehat{a_j} := \widehat{a_{m_j,l_j}}$ (or $\widehat{a_j} := \widehat{a^I_{m_j,l_j}}$) be defined in* (1.14) *(or in* (1.15)*), where $1 \leqslant l_j \leqslant m_j$ and m_j ($j \in \mathbb{N}$) are positive integers satisfying*

$$\lim_{j\to\infty} m_j = \infty, \quad \liminf_{j\to\infty} l_j/m_j > 0, \quad \text{and} \quad \sum_{j=1}^\infty 2^{-j}m_j < \infty. \tag{1.17}$$

For $j \in \mathbb{N}$, let ϕ_{j-1} be defined in (1.6) *and ψ^1_{j-1}, ψ^2_{j-1}, and ψ^3_{j-1} in* (1.16) *with masks $\widehat{b^1_j}$, $\widehat{b^2_j}$ and $\widehat{b^3_j}$ being defined in* (1.9) *from $\widehat{a_j} := \widehat{a_{m_j,l_j}}$ (or $\widehat{a_j} := \widehat{a^I_{m_j,l_j}}$). Then , for any $s \in \mathbb{R}$, there exist two positive constants C_1 and C_2 such that for every $f \in H^s(\mathbb{R})$,*

$$C_1\|f\|^2_{H^s(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^\infty \sum_{\ell=1}^3 \sum_{k\in\mathbb{Z}} 2^{2js}|\langle f, \psi^\ell_{j;j,k}\rangle|^2 \leqslant C_2\|f\|^2_{H^s(\mathbb{R})}. \tag{1.18}$$

This result covers a wide range of examples generated by masks of pseudo-splines in both types. For example, let $\widehat{a_j} = \widehat{a^I_{m_j,l_j}}$ and $l_j = m_j$ for all $j \in \mathbb{N}$. Since the corresponding function A_j in (1.10) for this special case is identically zero, we see that ψ^2_{j-1} and ψ^3_{j-1} in (1.16) are identically 0 for all $j \in \mathbb{N}$. Now one can easily see that for this case the nonstationary tight wavelet frame system generated by ϕ_0 and ψ_j^1 in (1.16), for all $j \in \mathbb{N}$, becomes the nonstationary orthonormal wavelet basis derived in [4]. Theorem 1.3 asserts that the nonstationary orthonormal wavelet basis of [4] can be used to characterize Sobolev spaces of arbitrary smoothness.

Characterization of the Sobolev norm as well as Besov norm of a function in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequence of the function is already given in [1, 2, 17] using a pair of stationary dual (or tight) frames in $L_2(\mathbb{R})$ with a different approach. In our recent work [16], we have systematically studied the construction of a pair of dual wavelet frames in a pair of dual Sobolev spaces, which is not necessarily a pair of dual frames in $L_2(\mathbb{R})$. We also used them to characterize the Sobolev norm of functions in terms of the weighted ℓ_2-norm of the analysis wavelet coefficient sequences in [16]. However, since there is no stationary compactly supported wavelet frame with the infinite smoothness order, it is impossible to characterize all Sobolev spaces with one fixed stationary compactly supported wavelet frame. When nonstationary wavelet frames are used, Theorem 1.3 asserts that one can use one fixed nonstationary compactly supported wavelet frame to characterize all Sobolev spaces with all orders of smoothness using the corresponding weighted ℓ_2-norm of the analysis wavelet coefficient sequences. This is particularly important in some applications. When using a wavelet to detect the singularity of an underlying function from a given set of data, one normally does not know the smoothness of the underlying function. Hence, it is difficult to select a proper wavelet system that can characterize its smoothness. It is very much helpful to have one fixed nonstationary compactly supported wavelet frame that can characterize all Sobolev spaces. We further remark that the analysis and synthesis algorithms derived from nonstationary compactly supported wavelet frames needed in applications have the same fashion and complexity of their stationary tight frame counterparts.

Nonstationary orthonormal compactly supported wavelets of this form are first introduced in [4] and the corresponding complex symmetric nonstationary orthonormal wavelets are given in [15]. It is shown in [4] that by using the class of masks for orthonormal refinable functions of [5] whose integer shifts form an orthonormal system, one can obtain a family of nonstationary compactly supported refinable functions such that every nonstationary refinable function belongs to $C^\infty(\mathbb{R})$ and its integer shifts still form an orthonormal system in $L_2(\mathbb{R})$. For this family of nonstationary refinable functions, a C^∞ compactly supported nonstationary orthonormal wavelet basis in $L_2(\mathbb{R})$ is derived in [4]. Motivated by this piece of interesting work of [4] and equipped with the pseudo-splines of [7, 8], together with the idea of unitary extension principle of [15, 18], we construct nonstationary $C^\infty(\mathbb{R})$ compactly supported tight wavelet frames in $L_2(\mathbb{R})$ with desirable properties, especially, the symmetry property in [15], which cannot be achieved by real-valued orthonormal dyadic refinable functions. We further remark that ideas of generating a class of nonstationary refinable functions in $C^\infty(\mathbb{R})$ from masks for stationary refinable functions have already appeared in [11, 19], where B-spline masks are used to generate the up-function.

The structure of this paper is as follows. In Section 2, we shall prove Theorems 1.2 and 1.3. In Section 3, we discuss the connections between the characterization and frames of Sobolev spaces. We will also link it to the frame approximation order of nonstationary tight wavelet frames.

2. Characterization of Sobolev Spaces

In this section, we first prove the upper bound of the characterization in (1.13) and then the lower bound to complete the proof of Theorems 1.2 and 1.3.

2.1. **Sufficient Conditions for the Upper Bounds.** For functions f and g on $\mathbb{R}$, we define

$$[f, g]_s(\xi) := \sum_{k\in\mathbb{Z}} f(\xi + 2\pi k)\overline{g(\xi + 2\pi k)}(1 + |\xi + 2\pi k|^2)^s, \qquad \xi \in \mathbb{R},\ s \in \mathbb{R}.$$

When $s > 0$, then $[\hat{\phi}, \hat{\phi}]_s \leqslant C$ is guaranteed by the decay of the Fourier transform of ϕ, i.e., the smoothness of ϕ. In fact, the next proposition roughly says that when $s > 0$, the upper bound of the characterization of a function in the Sobolev space $H^s(\mathbb{R})$ is ensured by the smoothness of the refinable functions, while for $s \leqslant 0$ it is guaranteed by the vanishing moments of the wavelet functions. Since we use only one tight frame in $L_2(\mathbb{R})$ for the upper bound in (1.13), the system should have both the required smoothness and vanishing moments simultaneously. In [16], a pair of stationary dual wavelet frames were used to characterize a pair of dual Sobolev spaces. One system can have the required smoothness while the other has the desired vanishing moments. This makes the construction (e.g. multivariate wavelet frames) and analysis (e.g. interpolatory Riesz wavelets) of certain systems simple. The advantage of moving from stationary to nonstationary lies in that here we can use one nonstationary system derived from masks of pseudo-splines with almost all desirable properties to characterize all Sobolev spaces.

Proposition 2.1. *Let s be a real number. Let ϕ_j, $j \in \mathbb{N}_0$, be functions in $H^s(\mathbb{R})$ and $\widehat{b_j^\ell}$, $j \in \mathbb{N}$ and $\ell = 1, \ldots, \mathfrak{J}_j$, be 2π-periodic measurable functions. Suppose that there exist a real number $t > s$, a nonnegative number $\alpha > -s$ (note that when $s > 0$, we may choose $\alpha = 0$), a positive constant C and a positive integer J such that*

$$[\widehat{\phi_j}, \widehat{\phi_j}]_t(\xi) \leqslant C \qquad a.e.\ \xi \in \mathbb{R},\ j \in \mathbb{N}_0 \tag{2.1}$$

and

$$\sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b_j^\ell}(\xi)|^2 \leqslant C \quad \forall\, j \in \mathbb{N} \quad and \quad \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b_j^\ell}(\xi)|^2 \leqslant C|\xi|^{2\alpha}, \qquad a.e.\ \xi \in \mathbb{R},\ j > J. \tag{2.2}$$

Define ψ_j^ℓ, $j \in \mathbb{N}$ and $\ell = 1, \ldots, \mathfrak{J}_{j+1}$, as in (1.2). Then there exists a positive constant C_1 such that

$$\sum_{k\in\mathbb{Z}} |\langle g, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\mathfrak{J}_{j+1}} \sum_{k\in\mathbb{Z}} 2^{-2js} |\langle g, \psi_{j;j,k}^\ell\rangle|^2 \leqslant C_1 \|g\|_{H^{-s}(\mathbb{R})}^2 \qquad \forall\, g \in H^{-s}(\mathbb{R}), \tag{2.3}$$

where $\psi_{j;j,k}^\ell := 2^{j/2}\psi_j^\ell(2^j \cdot - k)$.

Proof. Since $t > s$, it is straightforward to see from (2.1) that $[\widehat{\phi_j}, \widehat{\phi_j}]_s(\xi) \leqslant [\widehat{\phi_j}, \widehat{\phi_j}]_t(\xi) \leqslant C$ for almost every $\xi \in \mathbb{R}$ and for all $j \in \mathbb{N}_0$. By the Plancherel's theorem and Parseval's identity, it is not difficult to verify that

$$\sum_{k\in\mathbb{Z}} |\langle g, \phi_0(\cdot - k)\rangle|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |[\hat{g}, \widehat{\phi_0}]_0(\xi)|^2\, d\xi. \tag{2.4}$$

Applying the Cauchy-Schwarz inequality, we have $|[\hat{g}, \widehat{\phi_0}]_0(\xi)|^2 \leqslant [\hat{g}, \hat{g}]_{-s}(\xi)[\widehat{\phi_0}, \widehat{\phi_0}]_s(\xi)$. Consequently, we deduce from (2.4) that

$$\begin{aligned} \sum_{k\in\mathbb{Z}} |\langle g, \phi_0(\cdot - k)\rangle|^2 &\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} [\hat{g}, \hat{g}]_{-s}(\xi)[\widehat{\phi_0}, \widehat{\phi_0}]_s(\xi)\, d\xi \\ &\leqslant \|[\widehat{\phi_0}, \widehat{\phi_0}]_s\|_{L_\infty(\mathbb{R})} \frac{1}{2\pi} \int_{-\pi}^{\pi} [\hat{g}, \hat{g}]_{-s}(\xi)\, d\xi \\ &\leqslant C \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{g}(\xi)|^2 (1 + |\xi|^2)^{-s}\, d\xi = C\|g\|_{H^{-s}(\mathbb{R})}^2. \end{aligned}$$

According to the definition of ψ_j^ℓ, we have $\widehat{\psi_j^\ell}(\xi) = \widehat{b_{j+1}^\ell}(\xi/2)\widehat{\phi_{j+1}}(\xi/2)$ and

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\langle g, \psi_{j;j,k}^\ell\rangle|^2 &= \frac{2^j}{2\pi}\int_{-\pi}^{\pi}|[\hat{g}(2^j\cdot), \widehat{\psi_j^\ell}]_0(\xi)|^2\,d\xi\\ &= \frac{2^j}{2\pi}\int_{-\pi}^{\pi}\left|\overline{\widehat{b_{j+1}^\ell}(\xi/2)}[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi/2) + \overline{\widehat{b_{j+1}^\ell}(\xi/2+\pi)}[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi/2+\pi)\right|^2 d\xi\\ &\leqslant \frac{2^{j+1}}{\pi}\int_{-\pi}^{\pi}|\widehat{b_{j+1}^\ell}(\xi)|^2|[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi)|^2\,d\xi.\end{aligned}$$

That is, we have

$$\sum_{k\in\mathbb{Z}}|\langle g, \psi_{j;j,k}^\ell\rangle|^2 \leqslant \frac{2^{j+1}}{\pi}\int_{-\pi}^{\pi}|\widehat{b_{j+1}^\ell}(\xi)|^2|[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi)|^2\,d\xi. \tag{2.5}$$

Now by (2.1) and (2.2), we have

$$\begin{aligned}\sum_{j=0}^{J-1}\sum_{\ell=1}^{\mathfrak{I}_{j+1}}\sum_{k\in\mathbb{Z}}2^{-2js}|\langle g, \psi_{j;j,k}^\ell\rangle|^2 &\leqslant \sum_{j=0}^{J-1}\frac{2^{-2js}2^{j+1}}{\pi}\int_{-\pi}^{\pi}|[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi)|^2\sum_{\ell=1}^{\mathfrak{I}_{j+1}}|\widehat{b_{j+1}^\ell}(\xi)|^2\,d\xi\\ &\leqslant 2^{2J|s|}C\pi^{-1}\sum_{j=1}^{J}2^j\int_{-\pi}^{\pi}[\hat{g}(2^j\cdot), \hat{g}(2^j\cdot)]_{-s}(\xi)[\widehat{\phi_j}, \widehat{\phi_j}]_s(\xi)\,d\xi\\ &\leqslant 2^{2J|s|}C^2\pi^{-1}\sum_{j=1}^{J}2^j\int_{-\pi}^{\pi}[\hat{g}(2^j\cdot), \hat{g}(2^j\cdot)]_{-s}(\xi)\,d\xi\\ &= 2^{2J|s|+1}C^2\sum_{j=1}^{J}\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}(1+|2^{-j}\xi|^2)^{-s}(1+|\xi|^2)^s\,d\xi\\ &\leqslant 2^{2J|s|+1}C^2\sum_{j=1}^{J}\max(1, 2^{2js})\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi\\ &\leqslant C_1'\|g\|_{H^{-s}(\mathbb{R})}^2,\end{aligned}$$

where $C_1' := 2^{2J|s|+1}C^2\sum_{j=1}^{J}\max(1, 2^{2js}) < \infty$ and in the second last inequality we used the fact

$$(1+|2^{-j}\xi|^2)^{-s}(1+|\xi|^2)^s = \left(\frac{1+|\xi|^2}{1+|2^{-j}\xi|^2}\right)^s \qquad \text{and} \qquad 1 \leqslant \frac{1+|\xi|^2}{1+|2^{-j}\xi|^2} \leqslant 2^{2j}, \qquad \xi\in\mathbb{R},\ j\in\mathbb{N}.$$

On the other hand, (2.5) says that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}|\langle g, \psi_{j;j,k}^\ell\rangle|^2 &\leqslant \frac{2^{j+1}}{\pi}\int_{-\pi}^{\pi}|\widehat{b_{j+1}^\ell}(\xi)|^2|[\hat{g}(2^{j+1}\cdot), \widehat{\phi_{j+1}}]_0(\xi)|^2\,d\xi\\ &\leqslant \frac{2^{j+1}}{\pi}\int_{-\pi}^{\pi}|\widehat{b_{j+1}^\ell}(\xi)|^2[\hat{g}(2^{j+1}\cdot), \hat{g}(2^{j+1}\cdot)]_{-t}(\xi)[\widehat{\phi_{j+1}}, \widehat{\phi_{j+1}}]_t(\xi)\,d\xi\\ &\leqslant \frac{1}{\pi}\|[\widehat{\phi_{j+1}}, \widehat{\phi_{j+1}}]_t\|_{L_\infty(\mathbb{R})}2^{j+1}\int_{-\pi}^{\pi}|\widehat{b_{j+1}^\ell}(\xi)|^2[\hat{g}(2^{j+1}\cdot), \hat{g}(2^{j+1}\cdot)]_{-t}(\xi)\,d\xi\\ &\leqslant \frac{C}{\pi}2^{j+1}\int_{\mathbb{R}}|\widehat{b_{j+1}^\ell}(\xi)|^2|\hat{g}(2^{j+1}\xi)|^2(1+|\xi|^2)^{-t}\,d\xi\\ &= \frac{C}{\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|2^{-j-1}\xi|^2)^{-t}|\widehat{b_{j+1}^\ell}(2^{-j-1}\xi)|^2\,d\xi.\end{aligned}$$

Hence, we conclude that

$$\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle g, \psi^\ell_{j;j,k}\rangle|^2 \leqslant \frac{C}{\pi}\int_{\mathbb{R}} |\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j-1}\xi|^2)^t}|\widehat{b^\ell_{j+1}}(2^{-j-1}\xi)|^2\,d\xi. \tag{2.6}$$

Now we deduce from the above inequality that

$$\sum_{j=0}^{J-1}\sum_{\ell=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle g, \psi^\ell_{j;j,k}\rangle|^2 \leqslant \frac{C}{\pi}\int_{\mathbb{R}} |\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\sum_{j=0}^{J-1}\frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j-1}\xi|^2)^t}\sum_{\ell=1}^{\mathtt{J}_{j+1}}|\widehat{b^\ell_{j+1}}(2^{-j-1}\xi)|^2\,d\xi.$$

By (2.2) and noting that

$$A_{s,t,J}(\xi) := \sum_{j=0}^{J-1}\frac{2^{-2js}(1+|\cdot|^2)^s}{(1+|2^{-j-1}\cdot|^2)^t} \in L_\infty(\mathbb{R}),$$

we deduce that

$$\begin{aligned}\sum_{j=0}^{J-1}\sum_{\ell=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle g, \psi^\ell_{j;j,k}\rangle|^2 &\leqslant 2C^2\|A_{s,t,J}\|_{L_\infty(\mathbb{R})}\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi\\ &= 2C^2\|A_{s,t,J}\|_{L_\infty(\mathbb{R})}\|g\|^2_{H^{-s}(\mathbb{R})}.\end{aligned}$$

Denote

$$B_{s,t,J}(\xi) := \sum_{j=J+1}^{\infty}\frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j}\xi|^2)^t}\sum_{\ell=1}^{\mathtt{J}_j}|\widehat{b^\ell_j}(2^{-j}\xi)|^2, \qquad \xi\in\mathbb{R}. \tag{2.7}$$

Assume that $B_{s,t,J}\in L_\infty(\mathbb{R})$, which will be proved later. Then it follows from (2.6) that

$$\begin{aligned}&\sum_{j=J}^{\infty}\sum_{\ell=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle g, \psi^\ell_{j;j,k}\rangle|^2\\ &\leqslant \frac{C}{\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\sum_{j=J}^{\infty}\frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j-1}\xi|^2)^t}\sum_{\ell=1}^{\mathtt{J}_{j+1}}|\widehat{b^\ell_{j+1}}(2^{-j-1}\xi)|^2\,d\xi\\ &= \frac{2^{2s}C}{\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}B_{s,t,J}(\xi)\,d\xi\\ &\leqslant 2^{2s+1}C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})}\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{g}(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi\\ &= 2^{2s+1}C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})}\|g\|^2_{H^{-s}(\mathbb{R})}.\end{aligned}$$

Consequently, (2.3) holds with

$$C_1 = C + C_1' + 2C^2\|A_{s,t,J}\|_{L_\infty(\mathbb{R})} + 2^{2s+1}C\|B_{s,t,J}\|_{L_\infty(\mathbb{R})} < \infty.$$

To complete the proof, it remains to show that $B_{s,t,J}\in L_\infty(\mathbb{R})$. Without loss of any generality, let us assume $J=0$ and the assumption in (2.2) holds for all $j\in\mathbb{N}$. Furthermore, (2.2) implies that $\sum_{\ell=1}^{\mathtt{J}_j}|\widehat{b^\ell_j}(\xi)|^2 \leqslant C$ for almost every $\xi\in\mathbb{R}$ and for all $j\in\mathbb{N}$.

We prove $B_{s,t,0}\in L_\infty(\mathbb{R})$ in two separate cases: $s>0$ and $s\leqslant 0$. Suppose that $s>0$. For $|\xi|\leqslant 1$, by $t>s>0$, we have $(1+|\xi|^2)^s\leqslant 2^s$ and $(1+|2^{-j}\xi|^2)^t\geqslant 1$. Therefore, for $|\xi|\leqslant 1$, by the definition of $B_{s,t,0}$ in (2.7) with $J=0$, we deduce that

$$B_{s,t,0}(\xi) = \sum_{j=1}^{\infty}\frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j}\xi|^2)^t}\sum_{\ell=1}^{\mathtt{J}_j}|\widehat{b^\ell_j}(2^{-j}\xi)|^2 \leqslant 2^sC\sum_{j=1}^{\infty}2^{-2js} = \frac{2^{-s}C}{1-2^{-2s}} < \infty.$$

For $|\xi| > 1$, there exists a positive integer j_0 such that $2^{j_0-1} < |\xi| \leqslant 2^{j_0}$. Write $B_{s,t,0}(\xi) = B^1_{s,t}(\xi) + B^2_{s,t}(\xi)$, where

$$\begin{aligned} B^1_{s,t}(\xi) &:= \sum_{j=1}^{j_0} \frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j}\xi|^2)^t} \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2, \\ B^2_{s,t}(\xi) &:= \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j}\xi|^2)^t} \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2. \end{aligned} \tag{2.8}$$

Using the facts that $t > s > 0$ and $2^{j_0-1} < |\xi| \leqslant 2^{j_0}$ with $j_0 \geqslant 1$, we have

$$\begin{aligned} B^1_{s,t}(\xi) &\leqslant C \sum_{j=1}^{j_0} \frac{2^{-2js}(1+2^{2j_0})^s}{2^{2(j_0-j-1)t}} \\ &= C 2^{2t}(2^{-2j_0}+1)^s \sum_{j=1}^{j_0} 2^{2(j_0-j)(s-t)} \\ &\leqslant C 2^{2t+s} \sum_{j=0}^{\infty} 2^{2j(s-t)} \\ &= C 2^{2t+s}/(1-2^{2(s-t)}) < \infty. \end{aligned}$$

On the other hand, since $t > s > 0$ and $j_0 \geqslant 1$, we have

$$\begin{aligned} B^2_{s,t}(\xi) &\leqslant C \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}(1+|\xi|^2)^s}{(1+|2^{-j}\xi|^2)^t} \leqslant C \sum_{j=j_0+1}^{\infty} 2^{-2js}(1+2^{2j_0})^s \\ &= C(2^{-2j_0}+1)^s \sum_{j=j_0+1}^{\infty} 2^{-2(j-j_0)s} \leqslant C 2^s \sum_{j=1}^{\infty} 2^{-2js} \\ &= C 2^{-s}/(1-2^{-2s}) < \infty. \end{aligned}$$

Hence,

$$B_{s,t,0}(\xi) \leqslant C\big[2^{-s}/(1-2^{-2s}) + 2^{2t+s}/(1-2^{2(s-t)}) + 2^{-s}/(1-2^{-2s})\big] < \infty \qquad \forall\, \xi \in \mathbb{R}.$$

That is, for the case $s > 0$, we proved that $B_{s,t,0} \in L_\infty(\mathbb{R})$.

Suppose that $s \leqslant 0$. For $|\xi| \leqslant 1$ and $j \in \mathbb{N}$, since $t > s$ and $s \leqslant 0$, by (2.2) and $\alpha + s > 0$, we have $(1+|2^{-j}\xi|^2)^{-t} \leqslant \max(2^{-t}, 1) \leqslant 2^{|t|}$ and

$$\begin{aligned} B_{s,t,0}(\xi) &= \sum_{j=1}^{\infty} \frac{2^{-2js}(1+|2^{-j}\xi|^2)^{-t}}{(1+|\xi|^2)^{-s}} \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2 \\ &\leqslant 2^{|t|} \sum_{j=1}^{\infty} 2^{-2js} \sum_{\ell=1}^{\mathfrak{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2 \leqslant 2^{|t|} C \sum_{j=1}^{\infty} |2^{-j}\xi|^{2\alpha} 2^{-2js} \\ &\leqslant 2^{|t|} C \sum_{j=1}^{\infty} 2^{-2j(\alpha+s)} = C 2^{|t|-2(\alpha+s)}/(1-2^{-2(\alpha+s)}) < \infty. \end{aligned}$$

For $|\xi| > 1$, there exists a positive integer j_0 such that $2^{j_0-1} < |\xi| \leqslant 2^{j_0}$. Then for $j = 1, \dots, j_0$, if $t \geqslant 0$, we have

$$(1+|2^{-j}\xi|^2)^{-t} \leqslant (1+2^{2(j_0-j-1)})^{-t} = 2^{-2(j_0-j)t}(2^{-2(j_0-j)}+2^{-2})^{-t} \leqslant 2^{2|t|} 2^{-2(j_0-j)t}$$

and if $t < 0$,

$$(1+|2^{-j}\xi|^2)^{-t} \leqslant (1+2^{2(j_0-j)})^{-t} = 2^{-2(j_0-j)t}(2^{-2(j_0-j)}+1)^{-t} \leqslant 2^{2|t|} 2^{-2(j_0-j)t}.$$

Write $B_{s,t,0}(\xi) = B^1_{s,t}(\xi) + B^2_{s,t}(\xi)$, where $B^1_{s,t}$ and $B^2_{s,t}$ are defined in (2.8). Then by $2^{j_0-1} < |\xi| \leqslant 2^{j_0}$ and $j_0 \geqslant 1$, it follows from $s \leqslant 0$ and $t > s$ that

$$
\begin{aligned}
B^1_{s,t}(\xi) &= \sum_{j=0}^{j_0} \frac{2^{-2js}(1+|2^{-j}\xi|^2)^{-t}}{(1+|\xi|^2)^{-s}} \sum_{\ell=1}^{\mathcal{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2 \\
&\leqslant C 2^{2|t|} \sum_{j=1}^{j_0} \frac{2^{-2js}2^{-2(j_0-j)t}}{(1+2^{2j_0-2})^{-s}} \\
&= C 2^{2|t|}(2^{-2j_0}+2^{-2})^s \sum_{j=1}^{j_0} 2^{-2(j_0-j)(t-s)} \\
&\leqslant C 2^{2|t|+2|s|} \sum_{j=0}^{\infty} 2^{-2j(t-s)} \\
&= C 2^{2|t|+2|s|}/(1-2^{-2(t-s)}) < \infty.
\end{aligned}
$$

Similarly, by $2^{j_0-1} < |\xi| \leqslant 2^{j_0}$, we have $(1+|2^{-j}\xi|^2)^{-t} \leqslant 2^{|t|}$ for all $j \geqslant j_0+1$ and

$$
B^2_{s,t}(\xi) = \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}(1+|2^{-j}\xi|^2)^{-t}}{(1+|\xi|^2)^{-s}} \sum_{\ell=1}^{\mathcal{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2 \leqslant 2^{|t|} \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}}{(1+2^{2j_0-2})^{-s}} \sum_{\ell=1}^{\mathcal{J}_j} |\widehat{b^\ell_j}(2^{-j}\xi)|^2.
$$

By (2.2) and $\alpha > -s \geqslant 0$, we have

$$
\begin{aligned}
B^2_{s,t}(\xi) &\leqslant 2^{|t|} C \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}|2^{-j}\xi|^{2\alpha}}{(1+2^{2j_0-2})^{-s}} \\
&\leqslant 2^{|t|} C \sum_{j=j_0+1}^{\infty} \frac{2^{-2js}2^{2(j_0-j)\alpha}}{(1+2^{2j_0-2})^{-s}} \\
&= 2^{|t|} C (2^{-2j_0}+2^{-2})^s \sum_{j=j_0+1}^{\infty} 2^{2(j_0-j)(\alpha+s)} \\
&\leqslant 2^{|t|+2|s|} C \sum_{j=1}^{\infty} 2^{-2j(\alpha+s)} \\
&= C 2^{|t|+2|s|-2(\alpha+s)}/(1-2^{-2(\alpha+s)}) < \infty.
\end{aligned}
$$

Therefore, for the case $s \leqslant 0$, we conclude that $B_{s,t,0} \in L_\infty(\mathbb{R})$. The proof now is complete. ∎

2.2. **Proof of Theorems 1.2 and 1.3.** Equipped with Proposition 2.1, the proof of Theorems 1.2 and 1.3 is essentially to show the lower bound of the characterization in (1.13).

Proof of Theorem 1.2. It follows from (1.7) and (1.11) that

$$
\sum_{\ell=1}^{\mathcal{J}_j} |\widehat{b^\ell_j}(\xi)|^2 = 1 - |\widehat{a_j}(\xi)|^2 \leqslant C|\xi|^{2\alpha}, \qquad \xi \in \mathbb{R},\ j \geqslant J. \tag{2.9}
$$

Since $-\alpha < s < \alpha$, applying Proposition 2.1 for both s and $-s$, it follows from (1.12) and (2.9) that there exists a positive constant C_1 such that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} |\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f, \psi^{\ell}_{j;j,k}\rangle|^2 \leqslant C_1\|f\|^2_{H^s(\mathbb{R})} \qquad \forall\, f \in H^s(\mathbb{R}),\\
&\sum_{k\in\mathbb{Z}} |\langle g, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}} 2^{-2js}|\langle g, \psi^{\ell}_{j;j,k}\rangle|^2 \leqslant C_1\|g\|^2_{H^{-s}(\mathbb{R})} \qquad \forall\, g \in H^{-s}(\mathbb{R}).
\end{aligned} \tag{2.10}$$

Let $\mathcal{B}(\mathbb{R})$ denote the set of all functions h such that $\hat{h} \in L_\infty(\mathbb{R})$ and $\hat{h}$ is compactly supported. Clearly, $\mathcal{B}(\mathbb{R}) \subseteq H^\nu(\mathbb{R})$ for all $\nu \in \mathbb{R}$. In particular, $\mathcal{B}(\mathbb{R}) \subset L_2(\mathbb{R})$. By Theorem 1.1, we see that (1.4) holds. So, by (1.4) we deduce that

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}}\langle f, \phi_0(\cdot - k)\rangle\langle \phi_0(\cdot - k), g\rangle + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}\langle f, \psi^{\ell}_{j;j,k}\rangle\langle \psi^{\ell}_{j;j,k}, g\rangle, \qquad f, g \in \mathcal{B}(\mathbb{R}). \tag{2.11}$$

Now by (2.10) and (2.11), using Cauchy-Schwarz inequality, we deduce that for all $f, g \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
|\langle f, g\rangle|^2 &\leqslant \left[\sum_{k\in\mathbb{Z}}|\langle f, \phi_0(\cdot - k)\rangle\langle \phi_0(\cdot - k), g\rangle| + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}|\langle f, \psi^{\ell}_{j;j,k}\rangle\langle \psi^{\ell}_{j;j,k}, g\rangle|\right]^2\\
&\leqslant \left(\sum_{k\in\mathbb{Z}}|\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{2js}|\langle f, \psi^{\ell}_{j;j,k}\rangle|^2\right)\\
&\qquad\times\left(\sum_{k\in\mathbb{Z}}|\langle \phi_0(\cdot - k), g\rangle|^2 + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{-2js}|\langle \psi^{\ell}_{j;j,k}, g\rangle|^2\right)\\
&\leqslant C_1\|g\|^2_{H^{-s}(\mathbb{R})}\left(\sum_{k\in\mathbb{Z}}|\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{2js}|\langle f, \psi^{\ell}_{j;j,k}\rangle|^2\right).
\end{aligned}$$

That is, we have

$$\frac{1}{C_1}\sup_{g\in\mathcal{B}(\mathbb{R})\setminus\{0\}}\frac{|\langle f, g\rangle|^2}{\|g\|^2_{H^{-s}(\mathbb{R})}} \leqslant \sum_{k\in\mathbb{Z}}|\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{2js}|\langle f, \psi^{\ell}_{j;j,k}\rangle|^2, \quad f \in \mathcal{B}(\mathbb{R}). \tag{2.12}$$

Note that $\mathcal{B}(\mathbb{R})$ is dense in $H^{-s}(\mathbb{R})$ and the space $H^{-s}(\mathbb{R})$ is a dual space of $H^s(\mathbb{R})$. We deduce from (2.12) that for all $f \in \mathcal{B}(\mathbb{R})$,

$$\frac{1}{C_1}\|f\|^2_{H^s(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}}|\langle f, \phi_0(\cdot - k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{j=1}^{\mathtt{J}_{j+1}}\sum_{k\in\mathbb{Z}}2^{2js}|\langle f, \psi^{\ell}_{j;j,k}\rangle|^2. \tag{2.13}$$

Since $\mathcal{B}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$, noting (2.10), we see that (2.13) holds for all $f \in H^s(\mathbb{R})$. Therefore, by (2.10), we conclude that (1.13) holds with C_1 and C_2 in (1.13) being replaced by $1/C_1$ and C_1, respectively. ∎

In the rest of this section, we shall prove Theorem 1.3. In order to do so, let us recall a quantity $\nu_2(\hat{a})$ from [12]. Let $\hat{a}$ be a 2π-periodic trigonometric polynomial with $\hat{a}(0) = 1$. Write $\hat{a}(\xi) = (1 + e^{-i\xi})^m\hat{c}(\xi)$ for some nonnegative integer m and some 2π-periodic trigonometric polynomial $\hat{c}(\xi)$ with $\hat{c}(\pi) \neq 0$. Write $|\hat{c}(\xi)|^2 = \sum_{k=-K}^{K} c_k e^{-ik\xi}$, where K is some nonnegative integer. Denote $\rho(\hat{a})$ the spectral radius of the square matrix $(c_{2j-k})_{-K\leqslant j,k\leqslant K}$ and define $\nu_2(\hat{a}) :=$

$-1/2 - \log_2 \sqrt{\rho(\hat{a})}$. The quantity $\nu_2(\hat{a})$ plays an important role in wavelet analysis (see [12, 14] for detail). For a (stationary) refinable function ϕ with a refinement mask $\hat{a}$, we define

$$\nu_2(\phi) := \sup\{n + \alpha \ : \ n \in \mathbb{N} \cup \{0\}, 0 < \alpha \leqslant 1, \phi^{(n)} \in \text{Lip}(\alpha, L_2(\mathbb{R}))\}. \tag{2.14}$$

Then, $\nu_2(\hat{a}) = \nu_2(\phi)$ provided that $\{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\}$ is a Riesz system in $L_2(\mathbb{R})$. By [6, Lemmas 7.1.7 and 7.1.8], we have $\lim_{m\to\infty} \nu_2(\widehat{\phi^I_{m,m}}) = \infty$, where $\widehat{\phi^I_{m,m}}$ is the (stationary) refinable function with the refinement mask $\widehat{a^I_{m,m}}$. Therefore, we conclude that $\lim_{m\to\infty} \nu_2(\widehat{a^I_{m,m}}) = \infty$.

Using results in [15, Lemmas 2.7 and 3.3], as an application of Theorem 1.2, now we prove Theorem 1.3 as follows.

Proof of Theorem 1.3. To prove Theorem 1.3, by [15, Theorem 1.2], it suffices to prove (1.18).

Let α be a fixed positive number. Since $\lim_{m\to\infty} \nu_2(\widehat{a^I_{m,m}}) = \infty$, there is a large enough m such that $\nu_2(\widehat{a^I_{m,m}}) > \alpha + 3$. Let $\widehat{b_m}(\xi)$ be a 2π-periodic trigonometric polynomial such that $\widehat{a^I_{m,m}}(\xi) = 2^{-1}(1 + e^{-i\xi})\widehat{b_m}(\xi)$. Consequently, we have $\nu_2(\widehat{b_m}) = \nu_2(\widehat{a^I_{m,m}}) - 1 > \alpha + 2$. Let η denote the refinable function associated with mask $\widehat{b_m}$, that is, $\hat{\eta}(\xi) := \prod_{j=1}^{\infty} \widehat{b_m}(2^{-j}\xi)$ for $\xi \in \mathbb{R}$. Then by $\nu_2(\widehat{b_m}) > \alpha + 1$, we see that $\eta^{(n)}$ is a compactly supported function in $L_2(\mathbb{R})$ (see [12, 13, 14]), where n is the unique integer such that $\alpha + 1 < n \leqslant \alpha + 2$. Consequently, $\widehat{\eta^{(n)}}(\xi) = (-i\xi)^n \hat{\eta}(\xi)$. Note that $\eta^{(n)} \in L_1(\mathbb{R})$ since $\eta \in L_2(\mathbb{R})$ is compactly supported. Therefore, $|\hat{\eta}(\xi)| \leqslant |\xi|^{-n} \|\widehat{\eta^{(n)}}\|_{L_\infty(\mathbb{R})}$ for all $|\xi| \geqslant 1$. Now by $\alpha + 1 < n$, we see that $[\hat{\eta}, \hat{\eta}]_\alpha \in L_\infty(\mathbb{R})$.

On the other hand, since $\lim_{j\to\infty} m_j = \infty$, there exists an integer J_1 such that $m_j \geqslant m$ for all $j \geqslant J_1$. Now by [15, Lemma 2.7], we have

$$|\widehat{a_j}(\xi)| = |\widehat{a_{m_j,l_j}}(\xi)| \leqslant |\widehat{a^I_{m_j,l_j}}(\xi)| \leqslant |\widehat{b_m}(\xi)|, \qquad \xi \in \mathbb{R},\ j \geqslant J_1. \tag{2.15}$$

By the definition of ϕ_j in (1.6), it follows from the above inequality in (2.15) that

$$|\widehat{\phi_j}(\xi)| = \prod_{n=1}^{\infty} |\widehat{a_{n+j}}(2^{-n}\xi)| \leqslant \prod_{n=1}^{\infty} |\widehat{b_m}(2^{-n}\xi)| = |\hat{\eta}(\xi)|, \qquad \xi \in \mathbb{R}\ j \geqslant J_1.$$

Consequently, it follows from the above inequality that

$$[\widehat{\phi_j}, \widehat{\phi_j}]_\alpha(\xi) = \sum_{k\in\mathbb{Z}} |\widehat{\phi_j}(\xi + 2\pi k)|^2 (1 + |\xi + 2\pi k|^2)^\alpha \leqslant \|[\hat{\eta}, \hat{\eta}]_\alpha\|_{L_\infty(\mathbb{R})}, \quad \xi \in \mathbb{R},\ j \geqslant J_1. \tag{2.16}$$

On the other hand, since $\liminf_{j\to\infty} l_j/m_j > 0$, by [15, Lemma 3.3], there exist a positive constant C and a positive integer J_2 such that

$$0 \leqslant 1 - |\widehat{a_j}(\xi)|^2 \leqslant 1 - |\widehat{a_{m_j,l_j}}(\xi)|^2 \leqslant C|\xi|^{2\alpha} \qquad \forall\, \xi \in \mathbb{R},\ j \geqslant J_2. \tag{2.17}$$

Set $J := \max(J_1, J_2)$. It is easy to see that all the conditions in Theorem 1.2 are satisfied. Therefore, for any s such that $-\alpha < s < \alpha$, there exist two positive constants C_1 and C_2 such that (1.13) holds. Since α is arbitrary, this completes the proof. ∎

3. Connections to Frames in Sobolev Spaces and Frame Approximation Order

In this section, we first show the equivalence between the characterization of Sobolev spaces using a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ and the frame property of its properly normalized system in Sobolev spaces. Then we shall connect the characterization of Sobolev spaces to the approximation order of the truncated frame series.

3.1. **Characterization and Frames in Sobolev Spaces.** In this subsection, we link the characterization of a Sobolev space to the frame property of the system

$$X^s(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})=\{\phi_0(\cdot-k)\,:\,k\in\mathbb{Z}\}\cup\{\psi_{j;j,k}^{\ell,s}\,:\,j\in\mathbb{N}_0,\ell=1,\dots,\jmath_{j+1},k\in\mathbb{Z}\},\tag{3.1}$$

where $\psi_{j;j,k}^{\ell,s}:=2^{j(1/2-s)}\psi_j^\ell(2^j\cdot-k)$, in the Hilbert space $H^s(\mathbb{R})$ with the inner product defined by:

$$\langle f,g\rangle_{H^s(\mathbb{R})}:=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{g}(\xi)}(1+|\xi|^2)^s\,d\xi,\qquad f,\,g\in H^s(\mathbb{R}).$$

Note that $X^0(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})=X(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$.

We say that $X^s(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ is a *frame* in the Sobolev space $H^s(\mathbb{R})$ if there exist two positive constants C_1 and C_2 such that for every $g\in H^s(\mathbb{R})$,

$$C_1\|g\|^2_{H^s(\mathbb{R})}\leqslant\sum_{k\in\mathbb{Z}}|\langle g,\phi_0(\cdot-k)\rangle_{H^s(\mathbb{R})}|^2+\sum_{j=0}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}}|\langle g,\psi_{j;j,k}^{\ell,s}\rangle_{H^s(\mathbb{R})}|^2\leqslant C_2\|g\|^2_{H^s(\mathbb{R})}.\tag{3.2}$$

Proposition 3.1. *Let $X(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ be a tight wavelet frame in $L_2(\mathbb{R})$. Then, for any $s\in\mathbb{R}$,* (1.13) *holds for all $f\in H^s(\mathbb{R})$ if and only if $X^{-s}(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ is a frame in $H^{-s}(\mathbb{R})$, that is, for all $g\in H^{-s}(\mathbb{R})$,*

$$C_1\|g\|^2_{H^{-s}(\mathbb{R})}\leqslant\sum_{k\in\mathbb{Z}}|\langle g,\phi_0(\cdot-k)\rangle_{H^{-s}(\mathbb{R})}|^2+\sum_{j=0}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}}|\langle g,\psi_{j;j,k}^{\ell,-s}\rangle_{H^{-s}(\mathbb{R})}|^2\leqslant C_2\|g\|^2_{H^{-s}(\mathbb{R})}.\tag{3.3}$$

Proof. Define an operator $\theta_s:H^s(\mathbb{R})\mapsto H^{-s}(\mathbb{R})$ by $\widehat{\theta_s(f)}(\xi):=\hat{f}(\xi)(1+|\xi|^2)^s$. Then

$$\|\theta_s(f)\|^2_{H^{-s}(\mathbb{R})}:=\frac{1}{2\pi}\int_{\mathbb{R}}|\widehat{\theta_s(f)}(\xi)|^2(1+|\xi|^2)^{-s}\,d\xi=\frac{1}{2\pi}\int_{\mathbb{R}}|\hat{f}(\xi)|^2(1+|\xi|^2)^s\,d\xi=\|f\|^2_{H^s(\mathbb{R})}.$$

It is easy to see that θ_s is an isometric and onto mapping between $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$. On the other hand, for $f,h\in H^s(\mathbb{R})$, we have

$$\begin{aligned}\langle f,h\rangle&=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)(1+|\xi|^2)^s(1+|\xi|^2)^{-s}\overline{\hat{h}(\xi)}\,d\xi\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\theta_s(f)}(\xi)\overline{\hat{h}(\xi)}(1+|\xi|^2)^{-s}\,d\xi\\&=\langle\theta_s(f),h\rangle_{H^{-s}(\mathbb{R})}.\end{aligned}$$

Since $\psi_{j;j,k}^{\ell,-s}=2^{js}\psi_{j;j,k}^{\ell}$, it is straightforward to conclude that (1.13) is equivalent to (3.3) by taking $g=\theta_s(f)$ in (3.3). ∎

This proposition says that if the weighted ℓ_2-norm of the analysis wavelet coefficient sequences under the tight frame $X(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ in $L_2(\mathbb{R})$ can be used to characterize functions in $H^s(\mathbb{R})$, then the normalized system $X^{-s}(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ must be also a frame in $H^{-s}(\mathbb{R})$. Furthermore, the systems $X^s(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ and $X^{-s}(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ satisfy the duality relations as we shall discuss next.

For a frame $X^s(\phi_0;\{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ in $H^s(\mathbb{R})$ and a frame $X^{-s}(\tilde{\phi}_0;\{\tilde{\psi}_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ in $H^{-s}(\mathbb{R})$, we say that they form *a pair of dual wavelet frames in* $(H^s(\mathbb{R}),H^{-s}(\mathbb{R}))$, under the linear functional

$$\langle f,g\rangle:=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi,\quad f\in H^s(\mathbb{R}),\quad g\in H^{-s}(\mathbb{R}),$$

if the identity

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{0,k}\rangle\langle \phi_{0,k}, g\rangle + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}} \langle f, \tilde{\psi}^{\ell,-s}_{j;j,k}\rangle\langle \psi^{\ell,s}_{j;j,k}, g\rangle \tag{3.4}$$

holds for all $f \in H^s(\mathbb{R})$ and $g \in H^{-s}(\mathbb{R})$.

For a pair of dual wavelet frames $(X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}), X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}))$ in a pair of Sobolev spaces $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$, we have the wavelet frame representations in the Sobolev spaces $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$ as follows:

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{0,k}\rangle \phi_{0,k} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}} \langle f, \tilde{\psi}^{\ell,-s}_{j;j,k}\rangle \psi^{\ell,s}_{j,k}, \qquad f \in H^s(\mathbb{R}),\\ g &= \sum_{k\in\mathbb{Z}} \langle g, \phi_{0,k}\rangle \tilde{\phi}_{0,k} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}} \langle g, \psi^{\ell,s}_{j,k}\rangle \tilde{\psi}^{\ell,-s}_{j;j,k}, \qquad g \in H^{-s}(\mathbb{R}) \end{aligned} \tag{3.5}$$

with the series converging unconditionally in the norms of $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$, respectively.

When $X(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ is a tight wavelet frame in $L_2(\mathbb{R})$, then (3.4) holds for all $f, g \in L_2(\mathbb{R})$. If $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ and $X^{-s}(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ are frames in $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$, respectively, then (3.4) holds for all $f \in H^s(\mathbb{R})$ and $g \in H^{-s}(\mathbb{R})$, because for $s \geqslant 0$, $L_2(\mathbb{R})$ contains $H^s(\mathbb{R})$ and $L_2(\mathbb{R})$ is dense in $H^{-s}(\mathbb{R})$.

As a direct consequence of Theorem 1.2, Proposition 3.1 and the above observation, we obtain the following corollary:

Corollary 3.2. *Let $X(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ be a tight wavelet frame in $L_2(\mathbb{R})$ obtained via Theorem 1.1 such that all the assumptions in Theorem 1.2 are satisfied. Then for any $-\alpha < s < \alpha$, the normalized systems $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ and $X^{-s}(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ form a pair of dual wavelet frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$. So, let $X(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,2,3\}})$ be a tight wavelet frame derived in Theorem 1.3. Then $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,2,3\}})$ and $X^{-s}(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,2,3\}})$ form a pair of dual wavelet frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ for every $s \in \mathbb{R}$.*

Let ϕ_0 and ψ^ℓ_j, $j \in \mathbb{N}_0$ and $\ell = 1, \dots, \mathfrak{J}_{j+1}$, belong to the Sobolev space $H^s(\mathbb{R})$. We say that $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ is *a Riesz basis* in the Sobolev space $H^s(\mathbb{R})$ if

(1) the set of all linear combinations of elements in $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ is dense in $H^s(\mathbb{R})$.
(2) $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ is a *Riesz sequence* in $H^s(\mathbb{R})$: there exist positive constants C_1 and C_2 such that

$$\begin{aligned} C_1\left[\sum_{k\in\mathbb{Z}}|c_k|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}|d^\ell_{j,k}|^2\right] &\leqslant \left\|\sum_{k\in\mathbb{Z}} c_k\phi_{0,k} + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}} d^\ell_{j,k}\psi^{\ell,s}_{j;j,k}\right\|^2_{H^s(\mathbb{R})}\\ &\leqslant C_2\left[\sum_{k\in\mathbb{Z}}|c_k|^2 + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\sum_{\ell=1}^{\mathfrak{J}_{j+1}}|d^\ell_{j,k}|^2\right] \end{aligned} \tag{3.6}$$

holds for all finitely supported sequences $\{c_k\}_{k\in\mathbb{Z}}$ and $\{d^\ell_{j,k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z},\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}$.

Let ϕ_0 and ψ^ℓ_j, $j \in \mathbb{N}_0$ and $\ell = 1, \dots, \mathfrak{J}_{j+1}$, belong to the Sobolev space $H^s(\mathbb{R})$. Let $\tilde{\phi}_0$ and $\tilde{\psi}^\ell_j$, $j \in \mathbb{N}_0$ and $\ell = 1, \dots, \mathfrak{J}_{j+1}$, belong to the Sobolev space $H^{-s}(\mathbb{R})$. We say that $(X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}), X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}}))$ is *a pair of dual Riesz wavelet bases* in a pair of Sobolev spaces $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ if

(1) $X^s(\phi_0; \{\psi^\ell_j\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\mathfrak{J}_{j+1}\}})$ is a Riesz basis of the Sobolev space $H^s(\mathbb{R})$.

(2) $X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ is a Riesz basis of the Sobolev space $H^{-s}(\mathbb{R})$.

(3) $X^s(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ and $X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ are biorthogonal: for all $k, k' \in \mathbb{Z}$, $j, j' \in \mathbb{N}_0$ and $\ell = 1, \dots, \jmath_{j+1}, \ell' = 1, \dots, \jmath_{j'+1}$,

$$\begin{aligned} &\langle \phi_{0,k}, \tilde{\phi}_{0,k'} \rangle = \delta_{k-k'}, \quad \langle \psi^{\ell,s}_{j;j,k}, \tilde{\psi}^{\ell',-s}_{j';j',k'} \rangle = \delta_{j-j'}\delta_{k-k'}\delta_{\ell-\ell'}, \\ &\langle \phi_{0,k}, \tilde{\psi}^{\ell',-s}_{j';j',k'} \rangle = 0, \quad \langle \psi^{\ell,s}_{j;j,k}, \tilde{\phi}_{0,k'} \rangle = 0, \end{aligned} \tag{3.7}$$

where δ denotes the *Dirac sequence* such that $\delta_0 = 1$ and $\delta_k = 0$ for all $k \neq 0$.

It is easy to check that $(X^s(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}}), X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ if and only if the biorthogonality relation in (3.7) holds and $X^s(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ and $X^{-s}(\tilde{\phi}_0; \{\tilde{\psi}_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ form a pair of dual wavelet frames in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$.

Since $\psi^{\ell,s}_{j;j,k} = 2^{-sj}\psi^\ell_{j;j,k}$ and $\tilde{\psi}^{\ell,-s}_{j;j,k} = 2^{sj}\tilde{\psi}^\ell_{j;j,k}$, we observe that (3.7) is equivalent to that for all $k, k' \in \mathbb{Z}$, $j, j' \in \mathbb{N}_0$ and $\ell \in \{1, \dots, \jmath_{j+1}\}, \ell' \in \{1, \dots, \jmath_{j'+1}\}$,

$$\langle \phi_{0,k}, \tilde{\phi}_{0,k'} \rangle = \delta_{k-k'}, \quad \langle \psi^\ell_{j;j,k}, \tilde{\psi}^{\ell'}_{j';j',k'} \rangle = \delta_{j-j'}\delta_{k-k'}\delta_{\ell-\ell'}, \quad \langle \phi_{0,k}, \tilde{\psi}^{\ell'}_{j';j',k'} \rangle = 0, \quad \langle \psi^\ell_{j;j,k}, \tilde{\phi}_{0,k'} \rangle = 0.$$

Let $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ be a tight wavelet frame of $L_2(\mathbb{R})$. Let $s \in \mathbb{R}$ and assume that $(X^s(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}}), X^{-s}(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}}))$ is a pair of dual wavelet frames in the pair of Sobolev spaces $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$. Then this pair becomes a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$ if and only if $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ is an orthonormal basis of $L_2(\mathbb{R})$.

If $\widehat{a_j}$ is picked to be $\widehat{a^I_{m_j,m_j}}$ in Theorem 1.3, i.e., the masks for the orthonormal refinable functions in [6], then the number of wavelets derived from (1.16) by masks defined in (1.9) reduced from three functions to only one function, since $|A_j(\xi)|^2 = 1 - |\widehat{a_j}(\xi)|^2 - |\widehat{a_j}(\xi+\pi)|^2 = 0$. The corresponding wavelet system is denoted by $X(\phi_0; \{\psi_j\}_{j\in\mathbb{N}_0})$ with $\psi_j := \psi_j^1$ in (1.16). Then it is proven in [4] (also c.f. [15, Theorem 1.4]) that $X(\phi_0; \{\psi_j\}_{j\in\mathbb{N}_0})$ is an orthonormal basis of $L_2(\mathbb{R})$. Hence, applying Corollary 3.2 and the above observation, we obtain:

Corollary 3.3. *Let $X(\phi_0; \{\psi_j\}_{j\in\mathbb{N}_0})$ be the orthonormal basis in $L_2(\mathbb{R})$ derived in Theorem 1.3 with $\widehat{a_j} := \widehat{a^I_{m_j,m_j}}$. Then for every $s \in \mathbb{R}$, $(X^s(\phi_0; \{\psi_j\}_{j\in\mathbb{N}_0}), X^{-s}(\phi_0; \{\psi_j\}_{j\in\mathbb{N}_0}))$ is a pair of dual Riesz wavelet bases in $(H^s(\mathbb{R}), H^{-s}(\mathbb{R}))$.*

3.2. **Characterization and Frame Approximation Order.** This subsection is to show that the upper bound in (1.13) implies the approximation order of the truncation of the tight wavelet frame expansion in (1.5). The frame approximation operators $Q_n, n \in \mathbb{N}$, associated with the truncation of the tight wavelet frame expansion in (1.5) at level n, are defined to be

$$Q_n(f) := \sum_{k\in\mathbb{Z}} \langle f, \phi_0(\cdot - k)\rangle \phi_0(\cdot - k) + \sum_{j=0}^{n-1}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{j;j,k}\rangle \psi^\ell_{j;j,k}, \qquad f \in L_2(\mathbb{R}). \tag{3.8}$$

For $s \geqslant 0$, following [7], we say that a tight wavelet frame $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ in $L_2(\mathbb{R})$ provides the *frame approximation order s* if there exist a positive constant C, independent of f and n, and a positive integer N such that

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant C2^{-ns}\|f\|_{H^s(\mathbb{R})} \qquad \forall\, f \in H^s(\mathbb{R}) \quad \text{and} \quad n \geqslant N. \tag{3.9}$$

We say that a tight wavelet frame provides *the spectral frame approximation order* if it provides frame approximation order s for any positive integer s. The next proposition links the approximation order of the frame approximation operators to the characterization of the Sobolev norm in $H^s(\mathbb{R})$ via the weighted ℓ_2-norm of the analysis wavelet coefficient sequences.

Proposition 3.4. *Let $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ be a tight wavelet frame in $L_2(\mathbb{R})$. For $s>0$, if the right side of the inequality in (1.13) holds, then we have*

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leqslant \sqrt{C_2}2^{-sn}\|f\|_{H^s(\mathbb{R})} \qquad \forall\, f \in H^s(\mathbb{R}), \tag{3.10}$$

that is, the tight wavelet frame must provide the frame approximation order s.

Proof. Since $X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}})$ is a tight wavelet frame in $L_2(\mathbb{R})$, (1.4) holds. By the definition of Q_n in (3.8), noting that $H^s(\mathbb{R}) \subset L_2(\mathbb{R})$ by $s>0$, we deduce from (1.5) that

$$f - Q_n(f) = \sum_{j=n}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} \langle f, \psi_{j;j,k}^\ell\rangle \psi_{j;j,k}^\ell, \qquad f \in H^s(\mathbb{R}). \tag{3.11}$$

On the other hand, by (1.4), it is evident that

$$\sum_{j=0}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j;j,k}^\ell\rangle|^2 \leqslant \|f\|^2 \qquad \forall\, f \in L_2(\mathbb{R}),$$

which is equivalent to (see [6, pp. 57–58])

$$\Big\|\sum_{j=0}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} c_{j,k}^\ell \psi_{j;j,k}^\ell\Big\|_{L_2(\mathbb{R})}^2 \leqslant \sum_{j=0}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} |c_{j,k}^\ell|^2$$

for all sequences $\{c_{j,k}^\ell\}_{k\in\mathbb{Z},j\in\mathbb{N}_0,\ell\in\{1,\dots,\jmath_{j+1}\}}$ such that $\sum_{j=0}^\infty \sum_{\ell=1}^{\jmath_{j+1}} \sum_{k\in\mathbb{Z}} |c_{j,k}^\ell|^2 < \infty$. Therefore, by (3.11), we have

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})}^2 \leqslant \sum_{j=n}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j;j,k}^\ell\rangle|^2, \qquad f \in H^s(\mathbb{R}).$$

Since the right side of the inequality in (1.13) holds, for every $f \in H^s(\mathbb{R})$, by $s>0$, we get

$$\begin{aligned}\|f - Q_n(f)\|_{L_2(\mathbb{R})}^2 &\leqslant \sum_{j=n}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} |\langle f, \psi_{j;j,k}^\ell\rangle|^2 = \sum_{j=n}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} 2^{-2js}2^{2js}|\langle f, \psi_{j;j,k}^\ell\rangle|^2\\ &\leqslant 2^{-2ns}\sum_{j=n}^{\infty}\sum_{\ell=1}^{\jmath_{j+1}}\sum_{k\in\mathbb{Z}} 2^{2js}|\langle f, \psi_{j;j,k}^\ell\rangle|^2\\ &\leqslant C_2 2^{-2ns}\|f\|_{H^s(\mathbb{R})}^2.\end{aligned}$$

Now it is easy to see that (3.10) holds. ■

The up function φ is defined ([4, 11, 19]) by $\hat{\varphi}(\xi) := \prod_{j=1}^\infty (\frac{1+e^{-i2^{-j}\xi}}{2})^j$, $\xi \in \mathbb{R}$. It is known that $\varphi \in C^\infty(\mathbb{R})$ and φ is supported inside $[0,2]$. As an application of Proposition 3.4, we obtain the next result showing that the $C^\infty(\mathbb{R})$ tight wavelet frame in $L_2(\mathbb{R})$ obtained from the up-function cannot be used to characterize any Sobolev space with a positive order of smoothness. Hence, it cannot be normalized into a frame in a Sobolev space with a negative order of smoothness. It also shows that the condition $\liminf_{j\to\infty} l_j/m_j > 0$ in Theorem 1.3 cannot be removed.

Theorem 3.5. *Let $\widehat{a_j}(\xi) := 2^{-j}(1+e^{-i\xi})^j$, $j \in \mathbb{N}$, be the masks for the up-function, that is, we take $m_j := j$ and $l_j := 1$ in Theorem 1.3 and ϕ_0 is the up function. For $j \in \mathbb{N}$, define ϕ_{j-1} as in (1.6) and ψ_{j-1}^1, ψ_{j-1}^2, ψ_{j-1}^3 as in (1.16) with the wavelet masks $\widehat{b_j^1}$, $\widehat{b_j^2}$ and $\widehat{b_j^3}$ being derived from $\widehat{a_j}$ in (1.9). Then*

(i) *$X(\phi_0; \{\psi_j^\ell\}_{j\in\mathbb{N}_0,\ell\in\{1,2,3\}})$ is a compactly supported symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$ and each ψ_j has one vanishing moment.*

(ii) *The nonstationary tight wavelet frame cannot characterize any Sobolev space $H^s(\mathbb{R})$ for any $s > 0$. That is, for any positive number s, there does not exist positive constants C_1 and C_2 such that* (1.18) *is satisfied.*

Proof. Item (i) is a direct consequence of [15, Theorem 1.2]. It is straightforward to check that each ψ_j has only one vanishing moment. Item (ii) can be proved via proof by contradiction. Suppose that there is some $s > 0$ such that (1.18) holds. Then by Proposition 3.4, the tight wavelet frame must provide the frame approximation order s, which is a contradiction to [15, Theorem 1.3], since [15, Theorem 1.3] says that the truncation of this frame expansion does not attain any frame approximation order. This proves Item (ii). ∎

References

[1] L. Borup, R. Gribonval and M. Nielsen, Tight wavelet frames in Lebesgue and Sobolev spaces. *J. Funct. Spaces Appl.* **2** (2004), 227–252.

[2] L. Borup, R. Gribonval and M. Nielsen, Bi-framelet systems with few vanishing moments characterize Besov spaces. *Appl. Comput. Harmon. Anal.* **17** (2004), 3–28.

[3] C. K. Chui, W. He and J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, *Appl. Comput. Harmon. Anal.* **13** (2002), 224–262.

[4] A. Cohen and N. Dyn, Nonstationary subdivision schemes and multiresolution analysis, *SIAM J. Math. Anal.* **27** (1996), 1745–1769.

[5] I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.

[6] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[7] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.* **14** (2003), 1–46.

[8] B. Dong and Z. Shen, Pseudo-splines, wavelets and framelets, *Appl. Comput. Harmon. Anal.* **22** (2007), 78–104.

[9] B. Dong and Z. Shen, Linear independence of pseudo-splines, *Proc. Amer. Math. Soc.* **134** (2006), 2685–2694.

[10] B. Dong and Z. Shen, Construction of biorthogonal wavelets from pseudo-splines, *J. Approx. Theory*, **138** (2006), 211–23.

[11] G. Derfel, N. Dyn and D. Levin, Genarlized functional equations and subdivision processes, *J. Approx. Theory*, **80** (1995), 272–297.

[12] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, *J. Approx. Theory* **124** (2003), 44–88.

[13] B. Han, Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, *J. Comput. Appl. Math.* **155** (2003), 43–67.

[14] B. Han, Solutions in Sobolev spaces of vector refinement equations with a general dilation matrix, *Adv. Comput. Math.* **24** (2006), 375–403.

[15] B. Han and Z. Shen, Compactly supported symmetric C^∞ wavelets with spectral approximation order, preprint, (2006).

[16] B. Han and Z. Shen, Dual wavelet frames and Riesz bases in Sobolev spaces, preprint, (2007).

[17] Y. Hur and A. Ron CAPlets: wavelet representations without wavelets, preprint, (2005).

[18] A. Ron and Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.* **148** (1997), 408–447.

[19] V. L. Rvachev and V. A. Rvachev, On a function with ompact support, *Dopov. Dokl. Akad. Nauk. Ukraini*, **8** (1971), 705–707.

[20] I. W. Selesnick, Smooth wavelet tight frames with zero moments, *Appl. Comp. Harmon. Anal.* **10** (2000), 163–181.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

E-mail address: `bhan@math.ualberta.ca`

URL: `http://www.ualberta.ca/∼bhan`

Department of Mathematics, National University of Singapore, Singapore

E-mail address: `matzuows@nus.edu.sg`

URL: `http://www.math.nus.edu.sg/∼matzuows`