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ON B-SPLINE FRAMELETS DERIVED FROM THE UNITARY EXTENSION PRINCIPLE*

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Abstract. The spline wavelet tight frames considered in [A. Ron and Z. Shen, *J. Funct. Anal.*, 148 (1997), pp. 408–447] have been used widely in frame based image analysis and restorations (see, e.g., survey articles [B. Dong and Z. Shen, *MRA-based wavelet frames and applications*, IAS Lecture Notes Series, Summer Program on The Mathematics of Image Processing, Park City Mathematics Institute, 2010.; Z. Shen, in *Proceedings of the International Congress of Mathematicians*, Vol. IV, Hindustan Book Agency, Hyderabad, India, 2010, pp. 2834–2863]). However, except for the properties of the tight frame and the approximation order of the truncated frame series (see Ron and Shen and I. Daubechies et al., *Appl. Comput. Harmon. Anal.*, 14 (2003), pp. 1–46]), there are few other properties of this family of tight frames that are currently known. The aim of this paper is to present a few new properties of this family that will provide some reasons why it is efficient in image analysis and restorations. In particular, we first present a recurrence formula for computing the generators of higher order spline wavelet tight frames from lower order ones. This simplifies the computations of the exact values of the functions in applications. Second, we represent each generator of spline wavelet tight frames as a certain order of derivative of some univariate box spline that satisfies a few additional properties. This is a crucial property used in [J.-F. Cai et al., *J. Amer. Math. Soc.*, 25 (2012), pp. 1033–1089], where connections between total variational and wavelet frame based approaches for image restorations are established. Finally, we further show that each generator of sufficiently high order spline wavelet tight frames is close to a derivative of a properly scaled Gaussian function. This leads to the result that the wavelet system generated by finitely many consecutive derivatives of a properly scaled Gaussian function forms a frame whose frame bounds can be almost tight.

Key words. frames, B-spline framelets, unitary extension principle

AMS subject classifications. 42C15, 42C40, 41A15

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1. Introduction.

The aim of this paper is to investigate the family of the spline wavelet tight frames derived from [20]. We start with basic notions. For given $\Psi := \{\psi_1, \ldots, \psi_m\} \subset L_2(\mathbb{R})$, the *wavelet system* generated by Ψ is defined as

$$X(\Psi) := \{\psi_{\ell,n,k} := 2^{n/2}\psi_\ell(2^n \cdot -k) : 1 \le \ell \le m;\ n, k \in \mathbb{Z}\}.$$

The system $X(\Psi) \subset L_2(\mathbb{R})$ is called a *tight frame* if

$$f = \sum_{g \in X(\Psi)} \langle f, g\rangle g$$

holds for all $f \in L_2(\mathbb{R})$. If $X(\Psi) \subset L_2(\mathbb{R})$ is a tight frame system of $L_2(\mathbb{R})$ generated by a multiresolution analysis (MRA), then its generators Ψ are called *framelets*.

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The MRA starts from a refinable function φ. A compactly supported function φ is refinable if it satisfies a refinement equation

$$\varphi(x) = 2\sum_{j\in\mathbb{Z}} a_j \varphi(2x-j) \tag{1}$$

for some sequence $a \in \ell_2(\mathbb{Z})$. The refinement equation (1) can be written via its Fourier transform as

$$\widehat{\varphi}(\omega) = \widehat{a}\left(\frac{\omega}{2}\right)\cdot\widehat{\varphi}\left(\frac{\omega}{2}\right) \quad \text{a.e.} \quad \omega\in\mathbb{R}.$$

We call the sequence a the refinement mask of φ and $\widehat{a}(\cdot)$ the refinement symbol of φ. Here, we use $\widehat{f}$ to denote the Fourier transform of $f \in L_1(\mathbb{R})$, which is defined as

$$\widehat{f}(\omega) := \int_{-\infty}^{\infty} f(x)\exp(-i\omega x)dx.$$

For a refinable function $\varphi \in L_2(\mathbb{R})$, let V_0 be the closed shift invariant space generated by $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ and $V_j := \{f(2^j\cdot) : f \in V_0\}$, $j \in \mathbb{Z}$. It is known that when φ is compactly supported, $\{V_j\}_{j\in\mathbb{Z}}$ forms an MRA. Recall that an MRA is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ that satisfies (i) $V_j \subset V_{j+1}$, (ii) $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$, and (iii) $\bigcap_j V_j = \{0\}$ (see [2] and [19]).

A special family of refinable functions is B-splines. Let $\varphi^{(m)}$ be the centered B-spline of order m, which is defined in the Fourier domain by

$$\hat{\varphi}^{(m)}(\omega) = e^{-\frac{i\omega j_m}{2}}\operatorname{sinc}\left(\frac{\omega}{2}\right)^m, \tag{2}$$

where

$$\operatorname{sinc}(x) := \begin{cases}\sin(x)/x & \text{for } x\neq 0,\\ 1 & \text{for } x = 0,\end{cases} \quad \text{and} \quad j_m := \begin{cases} 0, & m \text{ is even},\\ 1, & m \text{ is odd}.\end{cases} \tag{3}$$

Then $\varphi^{(m)}$ is refinable with refinement symbol

$$\hat{a}^{(m)}(\omega) = e^{-\frac{i\omega j_m}{2}}\cos^m\left(\frac{\omega}{2}\right).$$

Tight framelets can be constructed by the unitary extension principle (UEP) of [20] from a given multiresolution analysis. For a given B-spline $\varphi^{(m)}$ of order m, it was shown in [20] that the m functions $\Psi^{(m)} = \{\psi_\ell^{(m)} : \ell = 1,\ldots,m\}$ defined in the Fourier domain by

$$\hat{\psi}_\ell^{(m)}(\omega) := i^\ell e^{-\frac{i\omega j_m}{2}}\sqrt{\binom{m}{\ell}}\frac{\cos^{m-\ell}(\omega/4)\sin^{m+\ell}(\omega/4)}{(\omega/4)^m} \tag{4}$$

form a tight wavelet frame in $L_2(\mathbb{R})$, i.e., $\Psi^{(m)}$ is a framelet set. We call $\Psi^{(m)}$ the *B-spline framelet of order m*. The B-spline framelet is either symmetric or antisymmetric and has small support for a given smoothness order. Similarly with B-splines, each B-spline framelet has an analytic form.

Since the publication of the UEP of [20], there are many theoretical developments and applications of MRA based wavelet frames. In particular, the B-spline framelets

$\Psi^{(m)}$ derived from the UEP in [20] are widely used in various applications, which include image inpainting in [5]; image denoising in [8]; high and super resolution image reconstruction in [10]; deblurring and blind debluring in [7, 8, 6, 9]; and image segmentation in [14]. In all these applications, the *tensor products* of univariate B-spline framelets constructed in [20] were used. For simplicity of notation, we introduce the case $d = 2$, which is used in image restoration. Let

$$\psi^{(m)}_{\ell_1,\ell_2}(x,y) := \psi^{(m)}_{\ell_1}(x)\psi^{(m)}_{\ell_2}(y), \quad 0 \le \ell_1, \ell_2 \le m; (x,y) \in \mathbb{R}^2,$$

where $\psi^{(m)}_0 := \phi^{(m)}$ for convenience. We denote

$$\Psi^{(m)}_{\mathbf{2}} = \{\psi^{(m)}_{\ell_1,\ell_2} : \quad 0 \le \ell_1, \ell_2 \le m; (\ell_1,\ell_2) \ne (0,0)\}.$$

Then, $X(\Psi^{(m)}_{\mathbf{2}})$ is a tight wavelet frame for $L_2(\mathbb{R}^2)$. The interested reader should consult the survey articles [15, 22] for details.

There are a few theoretical and applied issues in our mind that motivate our adventure here. One of them is that while the function can be sampled by its inner product with proper dilation and shifts of underlying refinable functions, the question of how to sample its derivative properly so that the corresponding wavelet coefficients can be viewed as proper samples of its derivative is not completely answered. This is the crucial fact used in [4], where a theory is developed to connect the total variational method and the framelets based approach for image restorations. Results in section 2.2 and section 4 are motivated by this. Another motivation is to consider whether there are similar recurrence formulas for the spline framelets of [20], so that we can compute values fast when needed. This leads to some results of section 2.

The paper is organized as follows. Section 2 develops some basic properties of B-spline framelets. In particular, in subsection 2.1, we present recurrence formulas for B-spline framelets $\psi^{(m)}_\ell$, in which the well-known recurrence formula of B-splines can be viewed as a special form of recurrence formulas of B-spline framelets. This gives a fast algorithm for computing them. We further show that the B-spline framelets can be derived from the ℓth derivative of some univariate box splines in subsection 2.2. This fact was used in [4], where the approximation of derivatives of a given function via its wavelet frame representation is needed. In section 3, we investigate the asymptotic property of B-spline framelets $\psi^{(m)}_\ell$, $\ell = 1, \ldots, m$. We first prove that the univariate box splines defined in section 2 uniformly converge to a scaled Gaussian function under a mild condition, and we further show that

$$\max_{1\le\ell\le m} \max_{x\in\mathbb{R}} |\psi^{(m)}_\ell(x) - G^{(m)}_\ell(x)| \lesssim \frac{(\ln m)^{5/2}}{m^{3/2}},$$

where $G^{(m)}_\ell$ is the ℓth derivative of some scaled Gaussian function $G(x)$. (See section 3.2 for the detailed definition.)

This leads to the discovery that a wavelet system generated by a finite number of consecutive directives of scaled Gaussian function forms a frame whose bounds are almost tight.

2. Properties of B-spline framelets. In this section, we give a recurrence formula for the B-spline framelets which computes higher order framelets from lower order ones. We also show that one can represent the derivatives of higher order framelets by lower order ones. Furthermore, we derive another set of formulas that represents each framelet as a derivative of a univariate box spline.

2.1. Recurrence formulas for B-spline framelets. While the recurrence formulas for B-splines and their derivatives are well-known (see [1]), the corresponding formulas for B-spline framelets are not available yet. This section is to establish such formulas. Let $B_m := \varphi^{(m)}(\cdot + j_m/2)$, where $\varphi^{(m)}$ is given in (2) and j_m is defined (3). Recall the following well-known recurrence formula of B-splines:

$$B_{m+1}(x) = \frac{2x+m+1}{2m} B_m\left(x+\frac{1}{2}\right) + \frac{m+1-2x}{2m} B_m\left(x-\frac{1}{2}\right). \tag{5}$$

Based on (5), one can compute B-splines fast and easily, which makes B-splines useful. The derivative of B-splines can be computed in terms of lower order splines as follows:

$$\frac{d}{dx} B_{m+1}(x) = B_m\left(x+\frac{1}{2}\right) - B_m\left(x-\frac{1}{2}\right). \tag{6}$$

The aim of this section is to give corresponding formulas for the B-spline framelets $\psi_\ell^{(m)}, \ell = 1, \ldots, m$. To state the formulas conveniently, we present the formulas for the function $\tilde{\psi}_\ell^{(m)}(\cdot) := \psi_\ell^{(m)}(\cdot + \frac{j_m}{2})$. Note that the Fourier transform of $\tilde{\psi}_\ell^{(m)}$ is

$$\widehat{\tilde{\psi}_\ell^{(m)}}(\omega) = i^\ell \sqrt{\binom{m}{\ell}} \frac{\cos^{m-\ell}(\frac{\omega}{4}) \sin^{m+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^m}. \tag{7}$$

We note that the formulas presented in this subsection are used to calculate the function value and the derivative of $\tilde{\psi}_\ell^{(m)}$. When m is even, $\psi_\ell^{(m)} \equiv \tilde{\psi}_\ell^{(m)}$. When m is odd, one can obtain those of the function $\psi_\ell^{(m)}$ by the half-translation of $\tilde{\psi}_\ell^{(m)}$. Hence, the formulas given in this subsection also work for $\psi_\ell^{(m)}$ with a proper shift.

Next, we present the recurrence relations of framelets $\tilde{\psi}_\ell^{(m)}$.

THEOREM 1. *Let $m \in \mathbb{N}$ be given, $1 \le \ell \le m$, and the framelet $\tilde{\psi}_\ell^{(m)}$ derived from B-spline of order m be given via its Fourier transform as* (7). *Then, we have the following recurrence formula between $\tilde{\psi}_\ell^{(m+1)}$ and $\tilde{\psi}_\ell^{(m)}$: for $1 \le \ell \le m$*

$$\begin{aligned} \tilde{\psi}_\ell^{(m+1)}(x) = \sqrt{\frac{m+1}{m+1-\ell}} \Bigg(& \frac{2x+m+1}{2m} \tilde{\psi}_\ell^{(m)}\left(x+\frac{1}{2}\right) \\ & + \frac{m+1-2x}{2m} \tilde{\psi}_\ell^{(m)}\left(x-\frac{1}{2}\right) + \frac{\ell}{m}\tilde{\psi}_\ell^{(m)}(x) \Bigg); \end{aligned} \tag{8}$$

the recurrence formula between $\tilde{\psi}_{m+1}^{(m+1)}$ and $\tilde{\psi}_m^{(m)}$ is

$$\tilde{\psi}_{m+1}^{(m+1)}(x) = \frac{2x+m+1}{2m} \tilde{\psi}_m^{(m)}\left(x+\frac{1}{2}\right) + \frac{2x-m-1}{2m} \tilde{\psi}_m^{(m)}\left(x-\frac{1}{2}\right) - \frac{2x}{m}\tilde{\psi}_m^{(m)}(x). \tag{9}$$

Proof. We first prove (8), working in the Fourier domain. Note that

$$\frac{d}{d\omega} \widehat{\tilde{\psi}_\ell^{(m)}}(\omega) = -i \int_{-\infty}^{\infty} x \tilde{\psi}_\ell^{(m)}(x) e^{-i\omega x} dx,$$

which implies that the Fourier transform of function $g_\ell(x) := x\tilde{\psi}_\ell^{(m)}(x)$ is

$$\hat{g}_\ell(\omega) = i^{\ell+1} \cdot 4^{m-1} \cdot \sqrt{\binom{m}{\ell}} \cos\left(\frac{\omega}{4}\right)^{m-\ell-1} \sin\left(\frac{\omega}{4}\right)^{m+\ell-1} \frac{m\omega\cos(\frac{\omega}{2}) - 2m\sin(\frac{\omega}{2}) + \ell\cdot\omega}{\omega^{m+1}}. \tag{10}$$

Note that

$$\frac{2x+m+1}{2m}\tilde{\psi}_\ell^{(m)}\left(x+\frac{1}{2}\right)=\frac{1}{m}g_\ell\left(x+\frac{1}{2}\right)+\frac{1}{2}\tilde{\psi}_\ell^{(m)}\left(x+\frac{1}{2}\right)$$

and

$$\frac{m+1-2x}{2m}\tilde{\psi}_\ell^{(m)}\left(x-\frac{1}{2}\right)=\frac{1}{2}\tilde{\psi}_\ell^{(m)}\left(x-\frac{1}{2}\right)-\frac{1}{m}g_\ell\left(x-\frac{1}{2}\right).$$

A simple manipulation shows that the Fourier transform of the right-hand side of (8) becomes

$$\begin{aligned}&\sqrt{\frac{m+1}{m+1-\ell}}\left(\frac{1}{m}\exp\left(\frac{i\omega}{2}\right)\hat{g}_\ell(\omega)+\frac{1}{2}\cdot\left(\exp\left(\frac{i\omega}{2}\right)+\exp\left(-\frac{i\omega}{2}\right)\right)\widehat{\tilde{\psi}}_\ell^{(m)}(\omega)\right.\\&\qquad\qquad\left.-\frac{1}{m}\exp\left(-\frac{i\omega}{2}\right)\hat{g}_\ell(\omega)+\frac{\ell}{m}\widehat{\tilde{\psi}}_\ell^{(m)}(\omega)\right)\\&=i^\ell\sqrt{\binom{m+1}{\ell}}\frac{\cos^{m+1-\ell}(\frac{\omega}{4})\sin^{m+1+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^{m+1}}=\widehat{\tilde{\psi}}_\ell^{(m+1)}(\omega).\end{aligned}$$

This proves (8). Similarly, the Fourier transform of the right side of (9) is

$$\begin{aligned}&\frac{1}{m}\left(\exp\left(\frac{i\omega}{2}\right)\left(\hat{g}_m(\omega)+\frac{m}{2}\widehat{\tilde{\psi}}_m^{(m)}(\omega)\right)+\exp\left(-\frac{i\omega}{2}\right)\left(\hat{g}_m(\omega)-\frac{m}{2}\widehat{\tilde{\psi}}_m^{(m)}(\omega)\right)-2\hat{g}_m(\omega)\right)\\&=i^{m+1}\frac{\sin^{2m+2}(\frac{\omega}{4})}{(\frac{\omega}{4})^{m+1}}=\widehat{\tilde{\psi}}_{m+1}^{(m+1)}(\omega),\end{aligned}$$

which proves (9). $\square$

Furthermore, combining (8) and (9), we have a recurrence algorithm for efficiently computing $\tilde{\psi}_\ell^{(m)}, \ell=1,\dots,m$. When $\ell<m$, we can use (8) to compute $\tilde{\psi}_\ell^{(m)}$ by $\tilde{\psi}_\ell^{(m-1)}$; we can use (9) to compute $\tilde{\psi}_\ell^{(\ell)}$ by $\tilde{\psi}_{\ell-1}^{(\ell-1)}$. Hence, we finally can reduce the computation of $\tilde{\psi}_\ell^{(m)}$ to that of $\tilde{\psi}_1^{(1)}$. Note that the function $\tilde{\psi}_1^{(1)}$ is a Haar wavelet with

$$\tilde{\psi}_1^{(1)}(x)=\begin{cases}1 & \text{if } x\in[-1/2,0),\\ -1 & \text{if } x\in[0,1/2],\\ 0 & \text{if } |x|>1/2.\end{cases}$$

We next show the method for computing $\tilde{\psi}_2^{(4)}$ by a table. In the following table, for the notation $\rightarrow$, we use the formula (8), while for the notation $\searrow$, we use (9):

$$\begin{array}{ccccccc}B_1 & & B_2 & & B_3 & & B_4\\ \tilde{\psi}_1^{(1)} & & \tilde{\psi}_1^{(2)} & & \tilde{\psi}_1^{(3)} & & \tilde{\psi}_1^{(4)}\\ & \searrow & \tilde{\psi}_2^{(2)} & \rightarrow & \tilde{\psi}_2^{(3)} & \rightarrow & \tilde{\psi}_2^{(4)}\\ & & & & \tilde{\psi}_3^{(3)} & & \tilde{\psi}_3^{(4)}\\ & & & & & & \tilde{\psi}_4^{(4)}\end{array}.$$

Finally, we compute the B-spline B_5 and corresponding framelets (see Figure 1) by applying the method here.

Next, we give the recurrence formula for computing the derivatives of $\tilde{\psi}_\ell^{(m)}$.

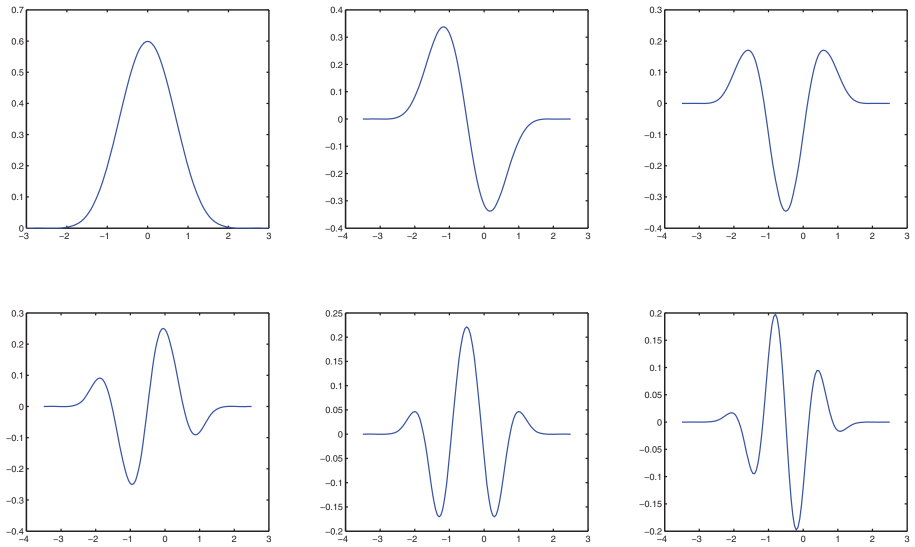


FIG. 1. *The B_5 and corresponding framelets.*

THEOREM 2. *Let $m \in \mathbb{N}$ be given, $1 \le \ell \le m$, and the framelet $\tilde{\psi}_\ell^{(m)}$ derived from B-spline of order m be defined by its Fourier transform as (7). When $1 \le \ell \le m-1$, we have*

$$\frac{d}{dx}\tilde{\psi}_\ell^{(m)}(x) = \sqrt{\frac{m}{m-\ell}}\left(\tilde{\psi}_\ell^{(m-1)}\left(x+\frac{1}{2}\right) - \tilde{\psi}_\ell^{(m-1)}\left(x-\frac{1}{2}\right)\right). \tag{11}$$

When $\ell = m$, we have

$$\frac{d}{dx}\tilde{\psi}_m^{(m)}(x) = \tilde{\psi}_{m-1}^{(m-1)}\left(x+\frac{1}{2}\right) + \tilde{\psi}_{m-1}^{(m-1)}\left(x-\frac{1}{2}\right) - 2\tilde{\psi}_{m-1}^{(m-1)}(x). \tag{12}$$

Proof. We prove (11) here while (12) can be proved similarly. A simple calculation shows that the Fourier transform of the right side of (11) is

$$\begin{aligned}&\sqrt{\frac{m}{m-\ell}}\cdot\sqrt{\binom{m-1}{\ell}}\cdot i^\ell\cdot\frac{\cos^{m-1-\ell}(\frac{\omega}{4})\sin^{m-1+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^{m-1}}(e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}})\\ &= 4i^{\ell+1}\sqrt{\binom{m}{\ell}}\frac{\cos^{m-\ell}(\frac{\omega}{4})\sin^{m+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^{m-1}}.\end{aligned} \tag{13}$$

Note that the Fourier transform of $\frac{d}{dx}\tilde{\psi}_\ell^{(m)}(x)$ is

$$4\cdot i^{\ell+1}\cdot\sqrt{\binom{m}{\ell}}\frac{\cos^{m-\ell}(\frac{\omega}{4})\sin^{m+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^{m-1}}. \tag{14}$$

Combining (13) and (14), we conclude (11). □

Remark 1. Note that $\tilde{\psi}_0^{(m)} = B_m$. If we take $\ell = 0$ in (8), the recurrence relation (8) is reduced to (5), which is the recurrence formula for B-splines. Similarly, if we take $\ell = 0$ in (11), then (11) is reduced to the derivative formula of B-splines (6).

2.2. Representing $\psi_\ell^{(m)}$ as the ℓth derivative of a univariate box spline. We first recall the definition of box splines. The univariate *box spline* $B(\cdot|\Xi)$ associated with a matrix $\Xi \in \mathbb{R}^{1\times m}$ is the distribution given by the rule (see [3])

$$\int_{\mathbb{R}} B(x|\Xi)\varphi(x)dx = \int_{[-\frac{1}{2},\frac{1}{2})^m} \varphi(\Xi u)du \quad \text{for all } \varphi \in \mathscr{D}(\mathbb{R}), \tag{15}$$

where $\mathscr{D}(\mathbb{R})$ is the test function space. The box spline can be considered as a volume function of the section of unit cubes (see [3, 25, 24]). If we take $\Xi = (1, 1, \ldots, 1) \in \mathbb{R}^{1\times m}$, then the box spline $B(\cdot|\Xi)$ is reduced to a B-spline of order m. In the following theorem, we show that the B-spline framelet can be considered as the higher order derivative of a box spline up to a constant.

THEOREM 3. *Let $m \in \mathbb{N}$ be given and $1 \le \ell \le m$. Suppose that the framelet $\psi_\ell^{(m)}$ is defined by its Fourier transform in* (4). *Set*

$$\Xi_{m,\ell} := \Big[\underbrace{1, \ldots, 1}_{m-\ell}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{2\ell}\Big].$$

Then

$$\psi_\ell^{(m)}(x) = \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \frac{d^\ell}{dx^\ell} B\left(x - \frac{j_m}{2}|\Xi_{m,\ell}\right), \tag{16}$$

where j_m is defined in (3). *The spline $B(x - \frac{j_m}{2}|\Xi_{m,\ell})$ is compactly supported, and its Fourier transform does not vanish at the origin. In particular, $\psi_m^{(m)}$ is the m-order derivative of $B_{2m}(2\cdot -j_m)/4^m$, where B_{2m} is the B-spline of order $2m$, which is compactly supported and whose Fourier transform does not vanish at the origin.*

Proof. This again is proved in the Fourier domain. It follows from the definition of box splines (15) that the Fourier transform of the box spline $B(\cdot|\Xi_{m,\ell})$ is

$$\hat{B}(\omega|\Xi_{m,\ell}) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{m-\ell} \left(\frac{\sin\frac{\omega}{4}}{\frac{\omega}{4}}\right)^{2\ell}.$$

Then the Fourier transform of

$$\sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \frac{d^\ell}{dx^\ell} B\left(x - \frac{j_m}{2}|\Xi_{m,\ell}\right)$$

can be computed as

$$\begin{aligned}
&\sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot e^{-ij_m\omega/2}(i\omega)^\ell \hat{B}(\omega|\Xi_{m,\ell}) \\
&= i^\ell \sqrt{\binom{m}{\ell}} e^{-ij_m\omega/2} \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{m-\ell} \frac{\left(\sin\frac{\omega}{4}\right)^{2\ell}}{\left(\frac{\omega}{4}\right)^\ell} \\
&= i^\ell \sqrt{\binom{m}{\ell}} e^{-ij_m\omega/2} \left(\frac{\sin\frac{\omega}{4}\cos\frac{\omega}{4}}{\frac{\omega}{4}}\right)^{m-\ell} \frac{(\sin\frac{\omega}{4})^{2\ell}}{(\frac{\omega}{4})^\ell} \\
&= i^\ell \sqrt{\binom{m}{\ell}} e^{-ij_m\omega/2} \frac{\cos^{m-\ell}(\frac{\omega}{4})\sin^{m+\ell}(\frac{\omega}{4})}{(\frac{\omega}{4})^m} \\
&= \hat{\psi}_\ell^{(m)}(\omega),
\end{aligned}$$

which proves (16). According to the definition of box splines, we have

$$B_{2m}(2x - j_m) = B\left(x - \frac{j_m}{2}|\Xi_{m,m}\right).$$

And hence, $\psi_m^{(m)}$ is the m-order derivative of $B_{2m}(2\cdot - j_m)/4^m$. □

Remark 2. Theorem 3 shows that one can obtain the B-spline framelet by calculating the derivative of box splines, which provides a new path to construct spline framelets. We hope to construct multivariate spline framelets by calculating the derivative of some relevant box splines in future work.

Theorem 3 shows that each spline framelet defined in (4) with vanishing moments of order up to ℓ is the ℓth derivative of a univariate box spline whose support is the same as the framelet and whose Fourier transform dose not vanish at the origin. More importantly, a general formula for such a function is given here. In [4], a general formula is absent, although it shows the existence of such functions and gives the explicit form of such functions for the spline framelets defined in (4) with $m = 1, 2$. Such functions are used in [4] to discretize differential operators by using spline framelets. Indeed, let $\varphi_{m,\ell}(x) := \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} B(x - \frac{j_m}{2}|\Xi_{m,\ell})$. According to Theorem 3 and the integration by parts, we can obtain that

$$\langle f, \psi_{\ell,n,k}^{(m)}\rangle = (-1)^\ell 2^{n/2} \cdot \frac{1}{2^{n\ell}} \cdot \left\langle f^{(\ell)}\left(\frac{\cdot + k}{2^n}\right), \varphi_{m.\ell}\right\rangle, \tag{17}$$

which is used in [4] to approximate the ℓ_1 norm of the differential operator by the weighed ℓ_1 norm of the wavelet coefficients.

3. The asymptotic property of B-spline framelets.

3.1. The asymptotic convergence of univariate box splines. It is known that up to a normalization, B-splines are close to Gaussian function pointwise, as well as in L_p sense with $2 \le p < +\infty$ as the order of the spline close to infinity (see [23]). Motivated by this, in this subsection we investigate the asymptotic convergence of univariate box splines, which is helpful to understand the convergence of $\psi_\ell^{(m)}$ with $\psi_\ell^{(m)}$ being the ℓ-order derivative of a box spline up to a constant.

To state the results conveniently, throughout the rest of this paper we shall use the notation $X \lesssim_{a,b,\ldots} Y$ to refer to the inequality $X \le C \cdot Y$, where the constant C may depend on $a, b, \ldots$, but no other variable. In the next theorem, we show that the normalized box splines converge uniformly to a Gaussian function.

THEOREM 4. *For each $k \in \mathbb{N}$, let*

$$\Xi_k := [a_1^{(k)}, \ldots, a_k^{(k)}] \in \mathbb{R}^{1\times k},$$

where $a_j^{(k)} > 0, j = 1, \ldots, k$. Let $B(\cdot|\Xi_k)$ be the box spline associated with Ξ_k. Assume that

$$\|\Xi_k\|_2^2 = \sigma^2 + \epsilon_k \tag{18}$$

with $\sigma \in \mathbb{R}$ a fixed constant and $\lim_{k\to\infty} \epsilon_k = 0$, and assume that

$$c_1 \le \frac{\max_{1\le j\le k} a_j^{(k)}}{\min_{1\le j\le k} a_j^{(k)}} \le c_2, \tag{19}$$

where c_1 and c_2 are fixed positive constants independent of k. Then,

$$\max_x \left| \sqrt{\frac{6}{\pi\sigma^2}} \exp\left(-\frac{6x^2}{\sigma^2}\right) - B(x|\Xi_k) \right| \lesssim_{c_1,c_2} \frac{(\ln k)^3}{k} + |\epsilon_k| \cdot |\ln(|\epsilon_k|)| \cdot \ln(k). \tag{20}$$

In order to prove Theorem 4, we need the following lemma of the box spline $B(\cdot|\Xi_k)$.

LEMMA 5. *Under the conditions of Theorem* 4,

$$\max_\omega \left| f_k(\omega) - \exp\left(-\frac{(\sigma\omega)^2}{24}\right) \right| \lesssim_{c_1,c_2} \frac{(\ln k)^2}{k} + |\epsilon_k| \cdot |\ln|\epsilon_k||, \tag{21}$$

where

$$f_k(\omega) := \hat{B}(\omega|\Xi_k) = \prod_{j=1}^{k} \operatorname{sinc}\left(\frac{a_j^{(k)}\omega}{2}\right).$$

Proof. Without loss of generality, we suppose that for each fixed k

$$0 < a_1^{(k)} \leq a_2^{(k)} \leq \cdots \leq a_k^{(k)}.$$

Then (18) and (19) imply that

$$\frac{1}{\sqrt{k}} \lesssim_{c_1,c_2} a_1^{(k)} \leq a_k^{(k)} \lesssim_{c_1,c_2} \frac{1}{\sqrt{k}}.$$

We first consider the case $|\omega| \geq \pi/a_k^{(k)}$. Note that $\operatorname{sinc}(\cdot)$ is a monotone decreasing function in $[0,\pi]$ and

$$|\operatorname{sinc}(\omega)| \leq \frac{1}{\pi} \qquad \text{for } |\omega| \geq \pi.$$

Then, we have

$$\begin{aligned}
\max_{|\omega|\geq\pi/a_k^{(k)}} |f_k(\omega)| &= \max_{|\omega|\geq\pi/a_k^{(k)}} \prod_{j=1}^{k} \left|\operatorname{sinc}\left(\frac{a_j^{(k)}\omega}{2}\right)\right| \\
&\leq \max\left\{\frac{1}{\pi^k}, \left(\operatorname{sinc}\left(\frac{\pi}{2}\frac{a_1^{(k)}}{a_k^{(k)}}\right)\right)^k\right\} \\
&\lesssim_{c_1,c_2} \beta^k,
\end{aligned}$$

where $\beta < 1$ is a positive constant. And hence, when $|\omega| \geq \pi/a_k^{(k)}$,

$$\left| f_k(\omega) - \exp\left(-\frac{(\sigma\omega)^2}{24}\right) \right| \leq |f_k(\omega)| + \exp\left(-\frac{(\sigma\omega)^2}{24}\right) \lesssim_{c_1,c_2} \frac{1}{k}, \tag{22}$$

which implies (21).

We next consider the case where $|\omega| \leq \pi/a_k^{(k)}$. Taylor expansion shows that when $|\omega| \leq \pi/a_k^{(k)}$,

$$\ln f_k(\omega) = \sum_{j=1}^{k} \ln\left(\operatorname{sinc}\left(\frac{a_j^{(k)}\omega}{2}\right)\right) = -\left(\|\Xi_k\|_2^2 \cdot \frac{\omega^2}{24} + S(\omega)\right), \tag{23}$$

where

$$S(\omega) = \frac{\|\Xi_k\|_4^4 \cdot \omega^4}{2880} + \frac{\|\Xi_k\|_6^6 \cdot \omega^6}{181440} + \cdots$$

is a uniformly convergent series on $|\omega| \le \pi/a_k^{(k)}$. By (23), we now obtain that when $|\omega| \le \pi/a_k^{(k)}$,

$$f_k(\omega) = \prod_{j=1}^{k} \operatorname{sinc}\left(\frac{a_j^{(k)}\omega}{2}\right) = \exp\left(-\frac{(\sigma\omega)^2}{24}\right) \cdot \exp\left(-\frac{\epsilon_k\omega^2}{24}\right) \cdot \exp(-S(\omega)).$$

Hence,

$$\begin{aligned}\left|f_k(\omega) - \exp\left(-\frac{\sigma^2\omega^2}{24}\right)\right| &\le \exp(-S(\omega)) \exp\left(-\frac{\sigma^2\omega^2}{24}\right) \cdot \left|\left(\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right)\right| \\ &\quad + \exp\left(-\frac{\sigma^2\omega^2}{24}\right) \cdot |\exp(-S(\omega)) - 1|. \end{aligned} \tag{24}$$

First, we prove that

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \cdot \left|\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right| \lesssim |\epsilon_k| \cdot |\ln|\epsilon_k|| \tag{25}$$

and

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \cdot |\exp(-S(\omega)) - 1| \lesssim \frac{(\ln k)^2}{k}. \tag{26}$$

Then, combining (24), (25) and (26), we obtain (21).

Equations (25) and (26) remain to be proved. We first prove that

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \cdot \left|\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right| \ \lesssim\ |\epsilon_k| \cdot |\ln|\epsilon_k||.$$

By Taylor expansion, when $\omega^2 \le 24 \cdot |\ln|\epsilon_k||/\sigma^2$,

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \left|\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right| \le \left|\left(\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right)\right| \lesssim |\epsilon_k| \cdot |\ln|\epsilon_k||;$$

when $\omega^2 \ge 24 \cdot |\ln|\epsilon_k||/\sigma^2$,

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \left|\exp\left(-\frac{\epsilon_k\omega^2}{24}\right) - 1\right| \le 2\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \lesssim |\epsilon_k|.$$

This gives (25).

We next prove (26). Note that when $|\omega| \le \sqrt{24\ln k}/\sigma$,

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \big|1 - \exp(-S(\omega))\big| \lesssim |1 - \exp(-S(\omega))| \lesssim_{c_1,c_2} \frac{(\ln k)^2}{k}. \tag{27}$$

When $\sqrt{24\ln k}/\sigma \le |\omega| \le \pi/a_k^{(k)}$, we have

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right) \le \frac{1}{k},$$

which implies that

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right)|\exp(-S(\omega))-1| \le \frac{2}{k} \lesssim \frac{(\ln k)^2}{k}. \tag{28}$$

Combining (27) and (28), one derives (26). □

Proof of Theorem 4. Note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\sigma^2\omega^2}{24}\right)\exp(i\omega x)d\omega = \sqrt{\frac{6}{\pi\sigma^2}}\exp\left(-\frac{6x^2}{\sigma^2}\right),$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f_k(\omega)\exp(i\omega x)d\omega = B(x|\Xi_k).$$

Then

$$\begin{aligned}
&\max_x\left|\sqrt{\frac{6}{\pi\sigma^2}}\exp\left(-\frac{6x^2}{\sigma^2}\right)-B(x|\Xi_k)\right| \\
&\lesssim \int_{-\infty}^{\infty}\left|\exp\left(-\frac{\sigma^2\omega^2}{24}\right)-f_k(\omega)\right|d\omega \\
&= \int_{|\omega|\le\frac{\sqrt{24}\ln k}{\sigma}}\left|\exp\left(-\frac{\sigma^2\omega^2}{24}\right)-f_k(\omega)\right|d\omega \\
&\quad+\int_{\frac{\sqrt{24}\ln k}{\sigma}\le|\omega|\le\frac{\pi}{a_k^{(k)}}}\left|\exp\left(-\frac{\sigma^2\omega^2}{24}\right)-f_k(\omega)\right|d\omega \\
&\quad+\int_{\frac{\pi}{a_k^{(k)}}\le|\omega|\le\frac{\pi}{a_1^{(k)}}}\left|\exp\left(-\frac{\sigma^2\omega^2}{24}\right)-f_k(\omega)\right|d\omega \\
&\quad+\int_{|\omega|\ge\frac{\pi}{a_1^{(k)}}}\left|\exp\left(-\frac{\sigma^2\omega^2}{24}\right)-f_k(\omega)\right|d\omega \\
&\lesssim \frac{(\ln k)^3}{k}+|\epsilon_k|\cdot|\ln|\epsilon_k||\cdot\ln k+\frac{\sqrt{k}}{k^{\ln k}}+\beta^k \lesssim \frac{(\ln k)^3}{k}+|\epsilon_k|\cdot|\ln|\epsilon_k||\cdot\ln k,
\end{aligned}$$

where $\beta=\max\{\frac{1}{\pi},\mathrm{sinc}(\frac{\pi a_1^{(k)}}{2a_k^{(k)}})\}<1$. Here, we use (21) to obtain that

$$\int_{|\omega|\le\frac{\sqrt{24}\ln k}{\sigma}}\left|f_k(\omega)-\exp\left(-\frac{\sigma^2\omega^2}{24}\right)\right|d\omega \lesssim \frac{(\ln k)^3}{k}+|\epsilon_k|\cdot|\ln|\epsilon_k||\cdot\ln k.$$

Note that $a_k^{(k)}/\sqrt{k}, k=1,2,\ldots,$ is a bounded sequence and

$$\exp\left(-\frac{\sigma^2\omega^2}{24}\right)\lesssim\frac{1}{k^{\ln k}} \qquad \text{for } \omega\ge\frac{\sqrt{24}\ln k}{\sigma}.$$

Using a similar method as in the proof of (28), we have that

$$\int_{\frac{\sqrt{24}\ln k}{\sigma}\le|\omega|\le\frac{\pi}{a_k^{(k)}}}\left|f_k(\omega)-\exp\left(-\frac{\sigma^2\omega^2}{24}\right)\right|d\omega \lesssim \frac{\sqrt{k}}{k^{\ln k}}.$$

To estimate

$$\int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} |f_k(\omega)-\exp(-\omega^2/24)|d\omega \leq \int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} |f_k(\omega)|d\omega + \int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} \exp(-\omega^2/24)d\omega,$$

we use the facts of

$$\int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} |f_k(\omega)|d\omega \leq \int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} \left(\frac{2}{a_1^{(k)}\omega}\right)^k d\omega \lesssim \frac{1}{k}$$

and

$$\int_{|\omega|\geq \frac{\pi}{a_1^{(k)}}} \exp\left(-\frac{\omega^2}{24}\right)d\omega \ \lesssim\ \frac{1}{k}. \qquad \square$$

Theorem 4 implies that the normalized box spline $B(\cdot|\Xi_{m,\ell})$ converges uniformly to a Gaussian function.

COROLLARY 6. *Suppose that*

$$\Xi_{m,\ell} = [\underbrace{1,\ldots,1}_{m-\ell},\underbrace{1/2,\ldots,1/2}_{2\ell}].$$

Then, for each fixed ℓ, the function $\sqrt{m-\frac{\ell}{2}}\cdot B(\sqrt{m-\frac{\ell}{2}}x|\Xi_{m,\ell})$ converges uniformly to $\sqrt{\frac{6}{\pi}}\exp(-6x^2)$, as $m\to\infty$.

Proof. By the definition of box splines, we have

$$\sqrt{m-\frac{\ell}{2}}\ B\left(\sqrt{m-\frac{\ell}{2}}x \mid \Xi_{m,\ell}\right) = B\left(x \mid \frac{\Xi_{m,\ell}}{\sqrt{m-\ell/2}}\right).$$

Note that for each fixed ℓ, $\|\frac{\Xi_{m,\ell}}{\sqrt{m-\ell/2}}\|_2^2 = 1$. Then, Theorem 4 shows that the box spline $B(x|\frac{\Xi_{m,\ell}}{\sqrt{m-\ell/2}})$, and hence $\sqrt{m-\frac{\ell}{2}}\ B(\sqrt{m-\frac{\ell}{2}}x|\Xi_{m,\ell})$, converges uniformly to the Gaussian function $\sqrt{\frac{6}{\pi}}\exp(-6x^2)$. $\square$

Remark 3. A well-known result is that $\sqrt{m}B_m(\sqrt{m}x)$ converges uniformly to $\sqrt{\frac{6}{\pi}}\exp(-6x^2)$ with $m\to\infty$ (see [23, 11]). In fact, the result can be considered as a particular case of Corollary 6. Note that

$$\sqrt{m}\cdot B(\sqrt{m}x|\Xi_{m,0}) = \sqrt{m}B_m(\sqrt{m}x).$$

If we take $\ell=0$ in Corollary 6, then we have that $\sqrt{m}\cdot B(\sqrt{m}x|\Xi_{m,0})$, and hence $\sqrt{m}B_m(\sqrt{m}x)$, converges uniformly to $\sqrt{\frac{6}{\pi}}\exp(-6x^2)$ with $m\to\infty$.

3.2. The asymptotic property of B-spline framelets. We observe from Corollary 6 that by changing variables, $B(x|\Xi_{m,\ell})$ is close to

$$\sqrt{\frac{6}{\pi}}\frac{1}{\sqrt{m-\ell/2}}\exp\left(-\frac{12\cdot x^2}{2m-\ell}\right).$$

Recall that Theorem 3 says that

$$\psi_\ell^{(m)}(x) = \sqrt{\binom{m}{\ell}}\cdot\frac{1}{4^\ell}\cdot\frac{d^\ell}{dx^\ell}B\left(x-\frac{j_m}{2}|\Xi_{m,\ell}\right).$$

These two observations lead us to consider the relation between $\psi_\ell^{(m)}(x)$ and the ℓth derivative of a Gaussian function $G_{m,\ell}(x)$, which is defined as

$$G_{m,\ell}(x) := C_{m,\ell} \cdot \exp\left(-\frac{12 \cdot x^2}{2m-\ell}\right),$$

where

$$C_{m,\ell} = \sqrt{\frac{6}{\pi}} \frac{\sqrt{\binom{m}{\ell}}}{\sqrt{m-\ell/2} \cdot 4^\ell}.$$

Let

$$G_\ell^{(m)}(x) := \frac{d^\ell}{dx^\ell} G_{m,\ell}\left(x - \frac{j_m}{2}\right), \qquad \ell = 1, \dots, m, \tag{29}$$

where j_m is given in (3), and

$$G^{(m)} := \{G_1^{(m)}, \dots, G_m^{(m)}\}.$$

THEOREM 7. *Let $m \in \mathbb{N}$ be given, $1 \le \ell \le m$, and the framelet $\psi_\ell^{(m)}$ be defined by its Fourier transform in* (4) *derived from B-splines of order m. Then,*

$$\max_{1\le \ell \le m} \max_{x \in \mathbb{R}} |\psi_\ell^{(m)}(x) - G_\ell^{(m)}(x)| \lesssim \frac{(\ln m)^{5/2}}{m^{3/2}}.$$

In order to prove Theorem 7, we need the following two lemmas.

LEMMA 8. *Let $|\omega| \ge 20\sqrt{\frac{\ln m}{m}}$; then the inequality*

$$\max_{1\le \ell \le m} \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \left|\omega^\ell \cdot \exp\left(-\left(m - \frac{\ell}{2}\right)\frac{\omega^2}{24}\right)\right| \lesssim \frac{1}{m^4}$$

holds.

Proof. For convenience, denote

$$F_\ell(\omega) := \sqrt{\binom{m}{\ell}} \cdot \frac{1}{4^\ell} \cdot \omega^\ell \cdot \exp\left(-\left(m - \frac{\ell}{2}\right)\frac{\omega^2}{24}\right)$$

and

$$\omega_\ell := \sqrt{\frac{24 \cdot \ell}{2m-\ell}}.$$

For each fixed $\ell \in [1, m] \cap \mathbb{Z}$, the function F_ℓ is increasing on the interval $[0, \omega_\ell]$, while F_ℓ is decreasing on $[\omega_\ell, \infty)$. And hence F_ℓ arrives at the maximum value at ω_ℓ. According to the inequality

$$\binom{m}{\ell} \le \left(\frac{m \cdot e}{\ell}\right)^\ell,$$

we have

$$\ln F_\ell(\omega_\ell) \le -\frac{\ell}{2} \cdot \ln \frac{2(2m-\ell)}{3m}.$$

Then, when

$$\frac{2\ln m}{\ln 16-\ln 15}\leq \ell \leq \frac{2}{5}m,$$

we have that

$$\ln F_\ell(\omega_\ell)\leq -\frac{\ell}{2}\cdot \ln\frac{2(2m-\ell)}{3m}\leq -\frac{\ln\frac{4}{3}}{\ln\frac{16}{15}}\ln m\leq -4\ln m.$$

This implies that whenever

$$\frac{2\ln m}{\ln 16-\ln 15}\leq \ell \leq \frac{2}{5}m$$

holds, one has

$$\max_\omega F_\ell(\omega) \ \leq\ F_\ell(\omega_\ell)\ \leq\ \frac{1}{m^4}. \tag{30}$$

We next consider the case where $\frac{2m}{5}\leq \ell\leq \frac{4m}{5}$. Using the inequality

$$\binom{m}{\ell}\leq 2^m,$$

one gets that

$$\ln F_\ell(\omega_\ell)\leq \frac{\ln 2}{2}m-\frac{\ell}{2}\ln\frac{2e(2m-\ell)}{3\ell}.$$

Therefore, when $\frac{2m}{5}\leq \ell\leq \frac{4m}{5}$, one has that

$$\ln F_\ell(\omega_\ell)\leq \frac{\ln 2}{2}m-\frac{\ell}{2}\ln\frac{2e(2m-\ell)}{3\ell}\lesssim -m\lesssim -4\ln m,$$

which implies that whenever $\frac{2m}{5}\leq \ell\leq \frac{4m}{5}$, the following holds:

$$\max_\omega F_\ell(\omega)\leq F_\ell(\omega_\ell)\lesssim \frac{1}{m^4}. \tag{31}$$

We now turn to the case $\frac{4}{5}m\leq \ell\leq m$. For this case, we apply the inequality

$$\binom{m}{\ell}\leq \left(\frac{m\cdot e}{m-\ell}\right)^{m-\ell}$$

to obtain that

$$\ln F_\ell(\omega_\ell)\leq \frac{m-\ell}{2}\ln\frac{m\cdot e}{m-\ell}-\frac{\ell}{2}\ln\frac{(4m-2\ell)\cdot e}{3\ell}\lesssim -m\lesssim -4\ln m.$$

Hence, when $\frac{4}{5}m\leq \ell\leq m$, we have that

$$\max_\omega F_\ell(\omega)\leq F_\ell(\omega_\ell)\lesssim \frac{1}{m^4}. \tag{32}$$

We finally consider the case where $1\leq \ell\leq \frac{2\ln m}{\ln 16-\ln 15}$. Note that when m is large enough, we have $\omega_\ell\leq 20\sqrt{\frac{\ln m}{m}}$. When $|\omega|\geq 20\sqrt{\frac{\ln m}{m}}$, $F_\ell(\omega)$ reach the maximum value at $\bar{\omega}_0 := 20\sqrt{\frac{\ln m}{m}}$. Then a simple calculation shows that when $1 \le \ell \le \frac{2\ln m}{\ln 16 - \ln 15}$,

$$\max_{|\omega| \ge 20\sqrt{\frac{\ln m}{m}}} F_\ell(\omega) \le F_\ell(\bar{\omega}_0) \lesssim \frac{1}{m^4}. \tag{33}$$

Combining (30), (31), (32), and (33), we conclude the proof. □

LEMMA 9. *For every $\omega \in \mathbb{R}$, the inequality*

$$\max_{1\le \ell \le m} \frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot |\omega|^\ell \cdot \left| \operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell} \operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} - \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \right| \lesssim \frac{\ln^2 m}{m}$$

holds.

Proof. For convenience, we only provide the proof for the case where $\omega \ge 0$. The proof of the other case is similar. By Taylor expansion, when $0 \le \omega \le \frac{3\pi}{2}$, we have

$$\begin{aligned} &\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell} \operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} \\ &= \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \exp\left(-\frac{(m-7\ell/8)\omega^4}{2880} - O(\omega^6)\right). \end{aligned}$$

Then, for $20\sqrt{\frac{\ln m}{m}} \le \omega \le \frac{3\pi}{2}$, we have

$$\begin{aligned} &\frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot \omega^\ell \cdot \left| \operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell} \operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} - \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \right| \\ &\lesssim \frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot \omega^\ell \cdot \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \left(1 - \exp\left(-\frac{(m-7\ell/8)\omega^4}{2880}\right)\right) \\ &\le 2\frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot \omega^\ell \cdot \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \lesssim \frac{1}{m^4}, \end{aligned}$$

where the last inequality is obtained by Lemma 8. Next, when $0 \le \omega \le 20\sqrt{\frac{\ln m}{m}}$, note that

$$1 - \exp\left(-\frac{(m-7\ell/8)\omega^4}{2880}\right) \lesssim \frac{\ln^2 m}{m}$$

and

$$F_\ell(\omega) = \frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot \omega^\ell \cdot \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right)$$

is a bounded function. Hence, for $0 \le \omega \le 20\sqrt{\frac{\ln m}{m}}$, we have

$$\frac{\sqrt{\binom{m}{\ell}}}{4^\ell} \cdot \omega^\ell \cdot \left| \operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell} \operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} - \exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right) \right| \lesssim \frac{\ln^2 m}{m}.$$

Finally, we consider the case when $\omega \ge \frac{3\pi}{2}$. We assert that when $\omega \ge \frac{3\pi}{2}$, the following inequality holds:

$$\max_{1\le \ell \le m} \sqrt{\binom{m}{\ell}} \cdot \left| \operatorname{sinc}\left(\frac{\omega}{2}\right) \right|^{m-\ell} \cdot \left| \operatorname{sinc}\left(\frac{\omega}{4}\right) \right|^{\ell} \le \left(\frac{8 \cdot e^{1/8}}{3\pi}\right)^m. \tag{34}$$

With this assertion, we have

$$
\begin{aligned}
&\frac{\sqrt{\binom{m}{\ell}}}{4^\ell}\cdot\omega^\ell\cdot\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell}-\exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right)\right|\\
&\le\frac{\sqrt{\binom{m}{\ell}}}{4^\ell}\cdot\omega^\ell\cdot\left(\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell}\right|+\left|\exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right)\right|\right)\\
&\le\sqrt{\binom{m}{\ell}}\left(\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{\ell}\right|+\frac{1}{4^\ell}\cdot\left|\omega^\ell\cdot\exp\left(-\frac{(m-\ell/2)\omega^2}{24}\right)\right|\right)\\
&\lesssim\frac{1}{m^4}.
\end{aligned}
$$

Here, the last inequality is followed by (34) and Lemma 8. Equation (34) remains to be proved. Note that when $\omega\ge\frac{3\pi}{2}$,

$$
\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)\right|^{m-\ell}\cdot\left|\operatorname{sinc}\left(\frac{\omega}{4}\right)\right|^{\ell}\le\frac{1}{(\omega/2)^{m-\ell}}\cdot\frac{1}{(\omega/4)^{\ell}}\le\frac{1}{(3\pi/4)^{m-\ell}}\cdot\frac{1}{(3\pi/8)^{\ell}}.
$$

Then we only need prove

$$
\max_{1\le\ell\le m}\sqrt{\binom{m}{\ell}}\cdot\frac{1}{(3\pi/4)^{m-\ell}}\cdot\frac{1}{(3\pi/8)^{\ell}}\lesssim\left(\frac{8\cdot e^{1/8}}{3\pi}\right)^m.
$$

Applying the inequality $\binom{m}{\ell}\le(\frac{m\cdot e}{m-\ell})^{m-\ell}$, we have that

$$
\max_{1\le\ell\le m}\sqrt{\binom{m}{\ell}}\cdot\frac{1}{(3\pi/4)^{m-\ell}}\cdot\frac{1}{(3\pi/8)^{\ell}}\ \le\ \left(\frac{8\cdot e^{1/8}}{3\pi}\right)^m.
$$

This proves (34). □

Proof of Theorem 7. Let

$$
M(\omega):=\max_{1\le\ell\le m}\frac{\sqrt{\binom{m}{\ell}}}{4^\ell}\cdot|\omega|^\ell\cdot\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell}-\exp\left(-\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24}\right)\right|
$$

and

$$
\begin{aligned}
&I_1:=\left\{\omega\in\mathbb{R}:|\omega|\le 20\sqrt{\frac{\ln m}{m}}\right\},\ I_2:=\left\{\omega\in\mathbb{R}:20\sqrt{\frac{\ln m}{m}}\le|\omega|\le\frac{3\pi}{2}\right\},\\
&I_3:=\left\{\omega\in\mathbb{R}:|\omega|\ge\frac{3\pi}{2}\right\}.
\end{aligned}
$$

Applying Lemma 9 and Lemma 8, we conclude that

$$
\int_{I_1}M(\omega)d\omega\lesssim\frac{(\ln m)^{5/2}}{m^{3/2}},\qquad\int_{I_2}M(\omega)d\omega\lesssim\frac{1}{m^4},
$$

respectively. By an argument similar to that leading to (34), we can obtain that there exists $0<\gamma<1$ such that

$$
\int_{I_3}M(\omega)d\omega\lesssim\gamma^m\lesssim\frac{(\ln m)^{5/2}}{m^{3/2}}.
$$

This leads to

$$
\begin{aligned}
\max_{1\le\ell\le m}\max_{x\in\mathbb{R}}|\psi_\ell^{(m)}(x)-G_\ell^{(m)}(x)| &\le \int_{-\infty}^{\infty} M(\omega)d\omega \\
&= \int_{I_1} M(\omega)d\omega + \int_{I_2} M(\omega)d\omega + \int_{I_3} M(\omega)d\omega \\
&\lesssim \frac{(\ln m)^{5/2}}{m^{3/2}}. \qquad \square
\end{aligned}
$$

Remark 4. It was proved in [17] that for each fixed ℓ, up to a normalization, a proper scaled $\psi_\ell^{(m)}$ uniformly converges to the ℓ-order derivative of a scaled Gaussian function with m tending to infinity. Our result is in a different direction. In fact, we show that for sufficiently large m framelets $\psi_1^{(m)},\dots,\psi_\ell^{(m)},\dots,\psi_m^{(m)}$ uniformly in x and ℓ close to derivatives of consecutive orders $1,\dots,m$ of a scaled Gaussian function whose scale depends on m.

4. Gaussian frame. Theorem 7 leads us to consider whether a wavelet system generated by a finite number of consecutive derivatives of a properly scaled Gaussian function forms a frame of $L_2(\mathbb{R})$. In this section, we show that the frame property of $X(\psi_1^{(m)},\dots,\psi_m^{(m)})$ can be transferred to that of $X(G_1^{(m)},\dots,G_m^{(m)})$. Here we recall the definition of $G_\ell^{(m)}$. For each fixed $m\in\mathbb{N}$, we consider the rescaled Gaussian function

$$G_{m,\ell}(x) = C_{m,\ell}\cdot\exp\left(-\frac{12\cdot x^2}{2m-\ell}\right),$$

where

$$C_{m,\ell} = \sqrt{\frac{6}{\pi}}\frac{\sqrt{\binom{m}{\ell}}}{\sqrt{m-\ell/2}\cdot 4^\ell},$$

and

$$G_\ell^{(m)}(x) = \frac{d^\ell}{dx^\ell}G_{m,\ell}\left(x-\frac{j_m}{2}\right), \qquad \ell=1,\dots,m,$$

where j_m is given in (3), and

$$G^{(m)} = \left\{G_1^{(m)},\dots,G_m^{(m)}\right\}.$$

Before stating the following main theorem of this section, we recall the definitions of the frame and Bessel sequence. A family $\{f_j\}_{j\in J}\subset L_2(\mathbb{R})$ is called a *frame* with bounds A and B if

$$A\|f\|^2 \le \sum_{j\in J}|\langle f,f_j\rangle|^2 \le B\|f\|^2$$

holds for all $f\in L_2(\mathbb{R})$. If $A=B$, then $\{f_j\}_{j\in J}$ is called an *A-tight frame.* Moreover, a family $\{f_j\}_{j\in J}\subset L_2(\mathbb{R})$ is called a *Bessel sequence* with a bound R if

$$\sum_{j\in J}|\langle f,f_j\rangle|^2 \le R\|f\|^2$$

holds for all $f\in L_2(\mathbb{R})$.

THEOREM 10. *Let $X(G^{(m)})$ be the wavelet system generated by functions $G^{(m)}$. Then $X(G^{(m)})$ is a frame system with frame bounds A_m and B_m for sufficiently large m. Furthermore, the frame is close to being tight when m is sufficiently large. In fact, asymptotically, we have*

$$\lim_{m\to\infty} A_m = \lim_{m\to\infty} B_m = 1.$$

We call $X(G^{(m)})$ a *Gaussian frame*. To prove Theorem 10, we need the following theorem, which is proved in [16], together with several lemmas.

THEOREM 11 (see [16]). *Let $\{f_j\}_{j\in J}$ be a frame of $L_2(\mathbb{R})$ with bounds A and B. Assume that $\{g_j\}_{j\in J} \subset L_2(\mathbb{R})$ is such that $\{f_j - g_j\}_{j\in J}$ is a Bessel sequence with a bound $R < A$. Then $\{g_j\}_{j\in J}$ is a frame with bounds $A(1-\sqrt{\frac{R}{A}})^2$ and $B(1+\sqrt{\frac{R}{B}})^2$.*

Let

$$\phi_\ell^{(m)} := \psi_\ell^{(m)} - G_\ell^{(m)},\ \ell = 1,\dots,m, \quad \text{and} \quad \Phi^{(m)} := \{\phi_1^{(m)},\dots,\phi_m^{(m)}\}. \tag{35}$$

Since $X(\Psi^{(m)})$ is a tight frame with frame bound 1, to prove that $X(G^{(m)})$ is a frame, according to Theorem 11, we only need to show that $X(\Phi^{(m)})$ is a Bessel sequence with a bound $R_m \to 0$. An estimate of the Bessel bound of a given a sequence is provided in [21] that enables us to estimate the Bessel bound of $X(\Phi^{(m)})$ (see also [12]). Let

$$R_m := \sup_{1\le|\omega|\le 2} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \sum_{\ell=1}^{m} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|. \tag{36}$$

Then, for arbitrary $f \in L^2(\mathbb{R})$, the following inequality holds:

$$\sum_{\phi\in X(\Phi^{(m)})} |\langle f,\phi\rangle|^2 \le R_m \|f\|_2^2,$$

i.e., R_m is the Bessel upper bound of the system $X(\Phi^{(m)})$. Next, we estimate R_m. For this, we need the following lemmas.

LEMMA 12. *Let $\hat{\phi}_\ell^{(m)}$ be the Fourier transform of $\phi_\ell^{(m)}$ defined in (35). Then the following three estimates for $|\hat{\phi}_\ell^{(m)}|$ hold:*

(i)

$$|\hat{\phi}_\ell^{(m)}(\omega)| \ \le\ \sqrt{\binom{m}{\ell}} \cdot \frac{2^{m+\ell+1}}{|\omega|^m}, \qquad |\omega| \ge 20.$$

(ii)

$$|\hat{\phi}_\ell^{(m)}(\omega)| \ \lesssim\ \sqrt{\binom{m}{\ell}} \left(\frac{\omega}{4}\right)^\ell \cdot m \cdot \omega^4, \qquad |\omega| \le \sqrt{\frac{1}{m}}.$$

(iii)

$$\max_{1\le\ell\le m} \max_{\omega\in\mathbb{R}} |\hat{\phi}_\ell^{(m)}(\omega)| \ \lesssim\ \frac{\ln^2 m}{m}.$$

Proof. First, a simple computation leads to

$$\begin{aligned}|\hat{\phi}_\ell^{(m)}(\omega)| &= |\hat{\psi}_\ell^{(m)}(\omega) - \hat{G}_\ell^{(m)}(\omega)| \\ &= \left|\sqrt{\binom{m}{\ell}}\frac{\omega^\ell}{4^\ell}\left(\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} - \exp\left(-\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24}\right)\right)\right|.\end{aligned}$$

For (i), when $|\omega| \geq 20$, a simple argument shows that

$$\left|\frac{\omega^\ell}{4^\ell}\right|\exp\left(-\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24}\right) \leq \frac{2^{m+\ell}}{|\omega|^m}, \quad \left|\left(\frac{\omega}{4}\right)^\ell \cdot \operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell} \cdot \operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell}\right| \leq \frac{2^{m+\ell}}{|\omega|^m},$$

which implies that

$$|\hat{\phi}_\ell^{(m)}(\omega)| \;\leq\; \sqrt{\binom{m}{\ell}} \cdot \frac{2^{m+\ell+1}}{|\omega|^m}.$$

For (ii), the Taylor expansion shows that when $|\omega| \leq \pi$,

$$\ln\left(\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell}\right) = -\left(\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24} + \left(m-\frac{7}{8}\ell\right)\frac{\omega^4}{2880} + O(\omega^6)\right).$$

Then, when $|\omega| \leq \sqrt{\frac{1}{m}}$,

$$\begin{aligned}&\left|\operatorname{sinc}\left(\frac{\omega}{2}\right)^{m-\ell}\operatorname{sinc}\left(\frac{\omega}{4}\right)^{2\ell} - \exp\left(-\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24}\right)\right| \\ &\quad= \exp\left(-\left(m-\frac{\ell}{2}\right)\frac{\omega^2}{24}\right)\left|\exp\left(-\left(m-\frac{7}{8}\ell\right)\frac{\omega^4}{2880} - O(\omega^6)\right) - 1\right| \\ &\quad\lesssim m\cdot\omega^4,\end{aligned}$$

which implies that

$$|\hat{\phi}_\ell^{(m)}(\omega)| \;\lesssim\; \sqrt{\binom{m}{\ell}} \cdot \left(\frac{\omega}{4}\right)^\ell \cdot m \cdot \omega^4.$$

Finally, the conclusion of (iii) can be obtained by Lemma 9 directly. ∎

LEMMA 13. *Let R_m be given by* (36). *Then*

$$R_m \lesssim \frac{\ln^5 m}{m} \quad \text{and} \quad \lim_{m\to\infty} R_m = 0.$$

Proof. Let

$$\begin{aligned}R_{m,1} &:= \sup_{1\leq|\omega|\leq 2}\sum_{n\in\mathbb{Z}}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2, \\ R_{m,2} &:= \sup_{1\leq|\omega|\leq 2}\sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\sum_{\ell=1}^{m}\left(|\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega+2k\pi)|\right).\end{aligned}$$

Then, we have that

$$R_m \leq R_{m,1} + R_{m,2}. \tag{37}$$

To estimate R_m, we consider $R_{m,1}$ and $R_{m,2}$, respectively. We first estimate $R_{m,1}$. For this, we rewrite

$$R_{m,1} = \sup_{1\le|\omega|\le 2}\sum_{n\in\mathbb{Z}}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2 = \sup_{1\le|\omega|\le 2}[S_1(\omega)+S_2(\omega)+S_3(\omega)],$$

where

$$S_1(\omega) := \sum_{n\ge 5}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2, \qquad S_2(\omega) := \sum_{-\lfloor\log_2 m\rfloor<n<5}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2,$$

$$S_3(\omega) := \sum_{n\le -\lfloor\log_2 m\rfloor}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2.$$

By (i) in Lemma 12, we obtain that for $1\le|\omega|\le 2$,

$$\begin{aligned} S_1(\omega) &= \sum_{n\ge 5}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2 \\ &\le 4\sum_{n\ge 5}\sum_{\ell=1}^{m}\binom{m}{\ell}\frac{4^{m+\ell}}{(2^n\omega)^{2m}} \\ &= 4\sum_{n\ge 5}\frac{4^m}{(2^n\omega)^{2m}}\sum_{\ell=1}^{m}\binom{m}{\ell}4^\ell \\ &\lesssim \left(\frac{5}{256}\right)^m. \end{aligned}$$

Using (ii) in Lemma 12, when $1\le|\omega|\le 2$,

$$\begin{aligned} S_3(\omega) &= \sum_{n\le -\lfloor\log_2 m\rfloor}\sum_{\ell=1}^{m}|\hat{\phi}_\ell^{(m)}(2^n\omega)|^2 = \sum_{n\ge\lfloor\log_2 m\rfloor}\sum_{\ell=1}^{m}\left|\hat{\phi}_\ell^{(m)}\left(\frac{\omega}{2^n}\right)\right|^2 \\ &\lesssim \sum_{n\ge\lfloor\log_2 m\rfloor}\sum_{\ell=1}^{m}\binom{m}{\ell}\left(\frac{\omega}{4\cdot 2^n}\right)^{2\ell}\cdot m^2\cdot\left(\frac{\omega^4}{2^{4n}}\right)^2 \\ &= \sum_{n\ge\lfloor\log_2 m\rfloor} m^2\cdot\left(\frac{\omega^4}{2^{4n}}\right)^2\sum_{\ell=1}^{m}\binom{m}{\ell}\left(\frac{\omega}{4\cdot 2^n}\right)^{2\ell} \\ &\le m^2\sum_{n\ge\lfloor\log_2 m\rfloor}\left(\frac{\omega^4}{2^{4n}}\right)^2\left(1+\left(\frac{\omega}{4\cdot 2^n}\right)^2\right)^m \\ &\lesssim m^2\sum_{n\ge\lfloor\log_2 m\rfloor}\left(\frac{1}{2^{4n}}\right)^2\left(1+\left(\frac{1}{2^{n+1}}\right)^2\right)^m \\ &\lesssim \frac{1}{m^6}. \end{aligned}$$

Here, the last inequality uses the fact of $\{(1+(\frac{1}{2^{n+1}})^2)^m\}_{n\ge\lfloor\log_2 m\rfloor,\, m\in\mathbb{Z}_+}$ being a bounded sequence and

$$\sum_{n\ge\lfloor\log_2 m\rfloor}\left(\frac{1}{2^{4n}}\right)^2 = \sum_{n\ge\lfloor\log_2 m\rfloor}\frac{1}{256^n} = \frac{256}{255}\cdot\frac{1}{256^{\lfloor\log_2 m\rfloor}} \lesssim \frac{1}{m^8}.$$

Moreover, by (iii) in Lemma 12, we have

$$S_3(\omega) = \sum_{\ell=1}^{m} \sum_{-\lfloor \log_2 m \rfloor \le n \le 5} |\hat{\phi}_\ell(2^n\omega)|^2 \lesssim \frac{\ln^5 m}{m}.$$

Combining the results above, we obtain that

$$R_{m,1} = \sup_{1\le|\omega|\le 2} [S_1(\omega) + S_2(\omega) + S_3(\omega)] \lesssim \frac{\ln^5 m}{m}. \tag{38}$$

We next turn to

$$R_{m,2} = \sup_{1\le|\omega|\le 2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}} \left(|\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|\right).$$

To state conveniently, we set

$$\beta(2k\pi) := \sup_{1\le|\omega|\le 2} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|.$$

Then

$$R_{m,2} \le \sum_{k\in\mathbb{Z}\setminus\{0\}} \beta(2k\pi).$$

Set $k_0 := 10$. When $1 \le |k| \le k_0 - 1$, using the argument similar to the one in the estimation of $R_{m,1} = \beta(0)$, we can show that

$$\beta(2k\pi) \lesssim \frac{1}{m} \qquad \text{for } 1 \le |k| \le k_0 - 1.$$

We claim that when $|k| \ge k_0$,

$$\beta(2k\pi) \lesssim \frac{3^m}{|2k\pi|^{m/2}}. \tag{39}$$

And hence,

$$R_{m,2} = \left(\sum_{1\le|k|\le k_0-1} + \sum_{|k|\ge k_0}\right) \beta(2k\pi) \lesssim \frac{1}{m} + \sum_{|k|\ge k_0} \frac{3^m}{|2k\pi|^{m/2}} \lesssim \frac{1}{m}. \tag{40}$$

Combining (38) and (40), we obtain that

$$R_m = R_{m,1} + R_{m,2} \lesssim \frac{\ln^5 m}{m},$$

which implies the conclusion.

Finally, we prove (39). A simple observation is that $\beta(2k\pi) = \beta(-2k\pi)$. Hence, we only need consider the case where $k \ge k_0$. For convenience, let

$$\beta_+(2k\pi) := \sup_{1\le|\omega|\le 2} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} \left(|\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|\right),$$

$$\beta_-(2k\pi) := \sup_{1\le|\omega|\le 2} \sum_{\ell=1}^{m} \sum_{n<0} \left(|\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|\right).$$

Then $\beta(2k\pi) \le \beta_+(2k\pi) + \beta_-(2k\pi)$. To estimate $\beta_+(2k\pi)$, we furthermore set

$$\beta_+^+(2k\pi) := \sup_{1\le\omega\le 2} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|,$$

$$\beta_+^-(2k\pi) := \sup_{-2\le\omega\le -1} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)|.$$

Then $\beta_+(2k\pi) = \max\{\beta_+^+(2k\pi), \beta_+^-(2k\pi)\}$. Noting that $k \ge k_0 = 10$, by Lemma 12, we have

$$\begin{aligned}
\beta_+^+(2k\pi) &= \sup_{1\le\omega\le 2} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)| \\
&\lesssim \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} \binom{m}{\ell} \frac{2^{m+\ell}}{(2^n + 2k\pi)^m} \\
&= \sum_{n\in\mathbb{Z}_+} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{2^{m+\ell}}{(2^n + 2k\pi)^m} \\
&\le \sum_{n\in\mathbb{Z}_+} \frac{6^m}{(2^n + 2k\pi)^m} \le \sum_{n\in\mathbb{Z}_+} \frac{6^m}{2^m \cdot 2^{nm/2} \cdot (2k\pi)^{m/2}} \\
&\lesssim \frac{3^m}{(2k\pi)^{m/2}}.
\end{aligned}$$

We next consider

$$\begin{aligned}
\beta_+^-(2k\pi) &= \sup_{-2\le\omega\le -1} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega + 2k\pi)| \\
&= \sup_{1\le\omega\le 2} \sum_{\ell=1}^{m} \sum_{n\in\mathbb{Z}_+} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega - 2k\pi)|.
\end{aligned}$$

A simple observation is that $\max\{|2^n\omega|, |2^n\omega - 2k\pi|\} \ge k\pi \ge k_0\pi$. Then using (i) in Lemma 12, we obtain that

$$|\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega - 2k\pi)| \lesssim \binom{m}{\ell} \cdot \frac{2^{m+\ell}}{(k\pi)^m}.$$

Set $n_0 := \lceil \log_2(2k\pi) \rceil + 5$. Then,

$$\begin{aligned}
&\sup_{1\le\omega\le 2} \sum_{\ell=1}^{m} \sum_{0\le n\le n_0} |\hat{\phi}_\ell^{(m)}(2^n\omega)| \cdot |\hat{\phi}_\ell^{(m)}(2^n\omega - 2k\pi)| \\
&\lesssim \sum_{0\le n\le n_0} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{2^{m+\ell}}{(k\pi)^m} = \sum_{0\le n\le n_0} \frac{2^m}{(k\pi)^m} \sum_{\ell=1}^{m} \binom{m}{\ell} \cdot 2^\ell \\
&\le \sum_{0\le n\le n_0} \frac{6^m}{(k\pi)^m} \lesssim \log_2(2k\pi) \cdot \frac{6^m}{(k\pi)^m}.
\end{aligned}$$

We next consider

$$\sup_{1\le\omega\le 2}\sum_{\ell=1}^{m}\sum_{n_0+1\le n}|\hat{\phi}_\ell^{(m)}(2^n\omega)|\cdot|\hat{\phi}_\ell^{(m)}(2^n\omega-2k\pi)|.$$

By (i) in Lemma 12, when $n\ge n_0+1$,

$$\sup_{1\le\omega\le 2}|\hat{\phi}_\ell^{(m)}(2^n\omega)|\le\sqrt{\binom{m}{\ell}}\frac{2^{m+\ell}}{(2k\pi)^m},$$
$$\sup_{1\le\omega\le 2}|\hat{\phi}_\ell^{(m)}(2^n\omega-2k\pi)|\le\sqrt{\binom{m}{\ell}}\frac{2^{m+\ell}}{(2^n-2k\pi)^m}.$$

Hence,

$$\begin{aligned}&\sup_{1\le\omega\le 2}\sum_{\ell=1}^{m}\sum_{n\ge n_0+1}|\hat{\phi}_\ell^{(m)}(2^n\omega)|\cdot|\hat{\phi}_\ell^{(m)}(2^n\omega-2k\pi)|\\&\le\sum_{\ell=1}^{m}\sum_{n\ge n_0+1}\binom{m}{\ell}\frac{2^{m+\ell}}{(2k\pi)^m}\cdot\frac{2^{m+\ell}}{(2^n-2k\pi)^m}\\&\le\frac{4^m\cdot 5^m}{(2k\pi)^m}\sum_{n\ge n_0+1}\frac{1}{(2^n-2k\pi)^m}\lesssim\left(\frac{10}{k\pi}\right)^m,\qquad k\ge k_0.\end{aligned}$$

Therefore,

$$\beta_+^-(2k\pi)\lesssim\log_2(2k\pi)\cdot\frac{6^m}{(k\pi)^m}+\left(\frac{10}{k\pi}\right)^m.$$

This leads to

$$\beta_+(2k\pi)=\max\{\beta_+^+(2k\pi),\beta_+^-(2k\pi)\}\lesssim\frac{3^m}{(2k\pi)^{m/2}}.$$

Using (ii) in Lemma 12 and a similar analysis as above, we can obtain that

$$\begin{aligned}\beta_-(2k\pi)&=\sup_{1\le|\omega|\le 2}\sum_{\ell=1}^{m}\sum_{n<0}|\hat{\phi}_\ell^{(m)}(2^n\omega)|\cdot|\hat{\phi}_\ell^{(m)}(2^n\omega+2k\pi)|\\&\lesssim m\cdot\frac{4^m}{(2k\pi)^m}.\end{aligned}$$

Putting everything together, we have that

$$\beta(2k\pi)=\beta_+(2k\pi)+\beta_-(2k\pi)\lesssim\frac{6^m}{(2\sqrt{2k\pi})^m}+m\cdot\frac{4^m}{(2k\pi)^m}\lesssim\frac{3^m}{(2k\pi)^{m/2}}.$$

This proves (39). □

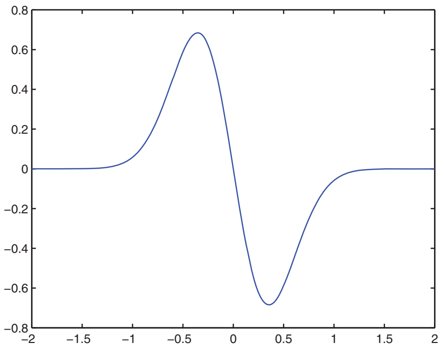
Proof of Theorem 10. Recall that

$$\Phi^{(m)}=\{\phi_1^{(m)},\dots,\phi_m^{(m)}\},\ \phi_\ell^{(m)}=\psi_\ell^{(m)}-G_\ell^{(m)},\quad\ell=1,\dots,m,$$

and that $X(\Psi^{(m)})$ is a tight frame with frame bound 1, where $\Psi^{(m)}=\{\psi_1^{(m)},\dots,\psi_m^{(m)}\}$. Lemma 13 shows that $X(\Phi^{(m)})$ is a Bessel sequence with a bound $R_m\to 0$. Then

Table 1
The numerical results of frame bounds of $X(G^{(m)})$.

m	2	3	4	5	6	7	8
A	0.3855	0.5266	0.5898	0.6407	0.6803	0.7095	0.7274
B	1.9020	1.6239	1.5179	1.4390	1.3811	1.3403	1.3159



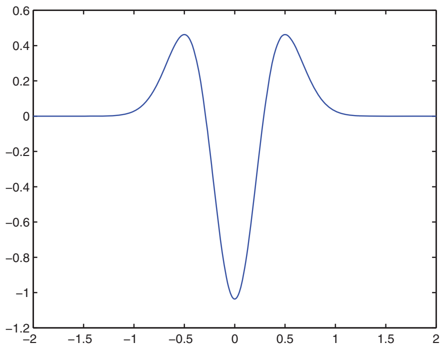


Fig. 2. *The graphs of* $G_1^{(2)}(x) = -\sqrt{32/\pi}\,x\exp(-4x^2)$ *(left) and* $G_2^{(2)}(x) = \sqrt{27/(8\pi)}(12x^2 - 1)\exp(-6x^2)$ *(right).*

Theorem 11 gives us that $X(G^{(m)})$ is a frame with frame bound $A_m = (1-\sqrt{R_m})^2$ and $B_m = (1+\sqrt{R_m})^2$ as m sufficiently large. Furthermore, it can be close to a tight frame, since $\lim_{m\to\infty} A_m = \lim_{m\to\infty} B_m = 1$, which completes the proof. ◻

As the spline framelet case, the Gaussian functions used in Theorem 10 to derive a wavelet frame system can be used to sample derivatives of functions, so that the norms of its various weighted wavelet coefficients at each dilation level can be viewed as approximations of corresponding norms of derivatives of functions.

Remark 5. Although Theorem 10 confirms the case where m is sufficiently large, the result of Theorem 10 seems to hold for small m (m can be as small as 2). For small m, combining (36) and Theorem 11, we can estimate the frame bounds of $X(G^{(m)})$ numerically. We list the frame bound estimation of $X(G^{(m)})$, $2 \le m \le 8$, in Table 1, which clearly shows the frame property of $X(G^{(m)})$ for small m. For example, for $m = 2$, $X(G^{(2)})$ is a frame with frame bounds $A \approx 0.3855$ and $B \approx 1.9020$. Figure 2 shows the graphs of the functions of $G_1^{(2)}$ and $G_2^{(2)}$, respectively. Furthermore, once a wavelet system is a frame in $L_2(\mathbb{R})$, with proper choosing of weights, the system can become a frame in various proper spaces, such as Sobolev or Besov spaces.

Remark 6. In the literature, there is only one example of a certain order of the derivative of the Gaussian function whose dilations and translations of a proper chosen lattice form a frame in $L_2(\mathbb{R})$. That is the Mexican hat function which is the second derivative of a Gaussian function (see, e.g., [12]). Here we provide a large family of wavelet frame systems derived from various order of derivative of the Gaussian function. Furthermore, the systems given here can have m consecutive derivatives of some variations of the Gaussian functions, which can be useful when the approximation of the different order of the derivatives of a function is needed. Furthermore, when m is sufficiently large, the corresponding system is almost a tight frame for $L_2(\mathbb{R})$.

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