Construction of compactly supported biorthogonal wavelets: I

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Abstract: This paper presents a construction of compactly supported dual functions of a given box spline in $L_2(\mathbb{R}^s)$. In particular, a concrete method for the construction of compactly supported dual functions of bivariate box splines of increasing smoothness is provided.

Key-Words:- multivariate biorthogonal wavelets, multivariate wavelets, box splines, matrix extension

1 Introduction

This paper deals with constructions of compactly supported refinable dual functions of a given box spline. The corresponding construction of biorthogonal spline wavelets, whose dilations and shifts form a Riesz basis for $L_2(\mathbb{R}^s)$ and the dual basis is an affine system generated by compactly supported functions with required smoothness, from a given pair of dual refinable functions can be found in [19]. These two papers, in some sense, are a continuation of our earlier work [16, 17]. In [16], we constructed exponentially decaying orthogonal wavelets from box spline for the case $s = 2, 3$. In [17], we constructed compactly supported prewavelets from box splines for the case $s = 2, 3$. The prewavelets constructed are orthogonal between the different dilation levels and all dilations and shifts form a Riesz basis of $L_2(\mathbb{R}^S)$. However, the dual basis which is an affine set is not generated by compactly supported functions.

Since refinable splines are simple and symmetric, spline wavelets are desirable in applications. Univariate spline wavelet constructions were carried out in [1] and [12] for exponentially decay orthogonal spline wavelets, in [5] and [15] for compactly supported spline pre-wavelets and in [6] for biorthogonal spline wavelets.

As in the many areas of mathematics and its applications when moving from univariate to multivariate theory, the study of wavelets did not and should not stop at tensor product wavelets. The construction of spline wavelets in multivariate setting is more challenging than its univariate counterpart. For example, it is more difficult to construct compactly supported dual refinable functions for a given box spline.

There are many papers on multivariate spline wavelet theory and constructions in the literature; in particular on fast decay orthogonal spline wavelets and compactly supported spline pre-wavelets. We list here a few of them for the interested reader: [2], [4], [11], [15], [14], [16], [17], and references cited in these papers for the further references. Compactly supported tight affine spline frames, which are very close to the compactly supported orthogonal wavelets, were constructed from box splines in [21], [22] and [8]. The construction of frames was based on the theory established in [21].

An example of bivariate biorthogonal spline wavelets from piecewise linear polynomial with its dual in $L_2(\mathbb{R}^2)$ is constructed in [7]. However, there are few constructions of compactly supported biorthogonal spline wavelets for several dimensions. The recent work of [10] provides a general method to construct biorthogonal wavelets for several dimensions. As we will see in Section 2, this method combining with the numerical method provided in [18] for constructions of compactly supported interpolatory refinable functions is applied to lead to a construction of compactly supported dual refinable functions with required smoothness order for an arbitrary given box spline.

2 Constructions of dual refinable functions

2.1 General Ideas

Let $\mathcal{H}$ be a Hilbert space and $X \subset \mathcal{H}$ be a sequence from $\mathcal{H}$. Define

$$T : \mathcal{D} \subseteq \ell_2(X) \to \mathcal{H} \quad \text{by} \quad Ta = \sum_{x \in X} a(x)x,$$

where T is defined at least on the linear space of finitely supported sequences, $\ell_0(X)$. If T is bounded on $\ell_2(X)$, then X is a Bessel sequence of $\mathcal{H}$. If T bounded and has a bounded inverse, then X is a Riesz sequence of $\mathcal{H}$. If $\operatorname{ran}(T)$ is dense in $\mathcal{H}$, then X is fundamental. A fundamental Riesz sequence of $\mathcal{H}$ is called a Riesz basis of $\mathcal{H}$. The adjoint map of the map T is the mapping

$$T^* : \mathcal{H} \to \ell_2(X) \quad \text{by} \quad T^*h = (\langle x, h \rangle)_{x \in X}.$$

The sequence X is a Riesz basis of $\mathcal{H}$ if and only if T^* is a bounded operator from $\mathcal{H}$ onto $\ell_2(X)$ with a bounded inverse (see e.g. [19]). A function ϕ is called stable, when its shifts, $\{\phi(\cdot - \alpha)_{\alpha \in \mathbb{Z}^s}\}$, form a Riesz basis of the closed shift invariant subspace of $L_2(\mathbb{R}^s)$ generated by ϕ.

We construct two compactly supported refinable functions ϕ and ϕ^d in $L_2(\mathbb{R}^s)$ such that the set of functions $\{\phi(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^s}$ forms a Riesz basis of

$$S(\phi) = \overline{\operatorname{span}\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}}$$

and the set of functions $\{\phi^d(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^s}$ forms a Riesz basis of

$$S(\phi^d) := \overline{\operatorname{span}\{\phi^d(\cdot - \alpha) : \alpha \in \mathbb{Z}^s\}}.$$

Furthermore, we require

$$\langle \phi, \phi^d(\cdot - \alpha) \rangle = \delta_\alpha, \quad \alpha \in \mathbb{Z}^s, \tag{1}$$

where δ_α is the usual delta sequence, $\delta_\alpha = 1$ if $\alpha = 0$ and $\delta_\alpha = 0$ otherwise. Then ϕ and ϕ^d are said to form a dual Riesz basis pair.

The $L_2(\mathbb{R}^s)$ functions ϕ and ϕ^d are refinable means that there are sequences m and m^d such that

$$\phi = 2^s \sum_{\alpha \in \mathbb{Z}^s} m(\alpha)\phi(2\,\cdot - \alpha)$$

and

$$\phi^d = 2^s \sum_{\alpha \in \mathbb{Z}^s} m^d(\alpha)\phi(2\,\cdot - \alpha).$$

The sequences m and m^d are the refinement masks for ϕ and ϕ^d respectively. In terms of the Fourier transform, we have

$$\widehat{\phi}(2\cdot) = M\widehat{\phi}, \quad \widehat{\phi}^d(2\cdot) = M^d\widehat{\phi}^d,$$

where M and M^d are the symbols of the refinement masks

$$M = \sum_{\alpha \in \mathbb{Z}^s} m(\alpha)\exp(-i\alpha\,\cdot)$$

and

$$M^d = \sum_{\alpha \in \mathbb{Z}^s} m^d(\alpha)\exp(-i\alpha\,\cdot).$$

Suppose ϕ and ϕ^d are compactly supported and in $L_2(\mathbb{R}^s)$, then the conditions that ϕ and ϕ^d

are stable and that they satisfy (1) (see e.g. [19]) are equivalent to

$$\sum_{\alpha\in 2\pi\mathbb{Z}^s} \widehat{\phi}(\omega+\alpha)\overline{\widehat{\phi^d}(\omega+\alpha)} = 1, \quad \omega \in \mathbb{T}^s. \tag{2}$$

Let $\varphi = \phi * \phi^d(-\cdot)$. Then, (2) is equivalent to

$$\varphi(\alpha) = \delta_\alpha, \quad \alpha \in \mathbb{Z}^s.$$

When (1) holds, the mask symbols M and M^d must satisfy the following necessary condition:

$$\sum_{\nu\in\mathbb{Z}_2^s} M(\omega+\pi\nu)\overline{M^d(\omega+\pi\nu)} = 1, \tag{3}$$

where $\mathbb{Z}_2^s = \{0,1\}^s$ is the set of all s-tuples of 0's and 1's, which can also be identified with $\mathbb{Z}^s/2\mathbb{Z}^s$.

In our construction, we pick ϕ to be a stable box spline and ϕ^d be a stable box spline convolved with a distribution. The box spline part of ϕ^d will provide a certain smoothness and the distribution part will be chosen so that (3) holds.

A box spline is defined for a given $s\times n$ matrix Ξ of full rank with integer entries. The Fourier transform of the box spline for the matrix Ξ is

$$\widehat{B}_\Xi(\omega) := 2^{s-n}\prod_{\xi\in\Xi}\frac{1-\exp(-i\xi\omega)}{i\xi\omega}. \tag{4}$$

Here a comment on notation is in order. The direction matrix is treated as a (multi)-set of its columns. Moreover, $\xi\omega$ for two vectors from $\mathbb{R}^s$ will mean their dot product. Thus, the product runs over the columns of the matrix Ξ and is complex-valued. The basic facts and much of the notation concerning box splines are taken from [3]; the reader is referred to [3] for the appropriate references. In particular, for the bivariate case, stable box splines based on the matrix Ξ_{n_1,n_2,n_3} consists of the three columns $[1,0]^T$, $[0,1]^T$ and $[1,1]^T$ appearing with multiplicities n_1, n_2, and n_3 respectively. For simplicity of notation, we describe the box splines by a triple subscript corresponding to the multiplicities.

The (total) degree of the polynomial pieces of the box spline does not exceed $n-s$. The support of the box spline is the polyhedron

$$\Xi[0\,..1]^n := \{x : x = \sum_{\xi\in\Xi} t_\xi\xi, 0\le t_\xi\le 1\}$$

where $[0\,..1]^n$ is the n-cube and the summation runs over the columns of the matrix Ξ. Finally, the box spline belongs to $C^r(\mathbb{R}^s)$ where $r+2$ is the minimum number of columns that can be discarded from Ξ to obtain a matrix of rank $< s$.

A box spline B_Ξ satisfies the refinement equation

$$B_\Xi = 2^s\sum_{\alpha\in\mathbb{Z}^s} m_\Xi(\alpha)B_\Xi(2\,\cdot\,-\alpha),$$

where

$$\begin{aligned} M_\Xi(\omega) &:= \sum_{\alpha\in\mathbb{Z}^s} m_\Xi(\alpha)\exp(-i\alpha\omega) \\ w &= \prod_{\xi\in\Xi}\frac{1+\exp(-iy\xi)}{2}. \end{aligned} \tag{5}$$

There is an easily checked criterion for when the shifts of a box spline form a Riesz basis; namely, when the direction set Ξ is a unimodular matrix (all bases of columns from Ξ have determinant ±1). The last condition is equivalent to there being no $\omega\in\mathbb{R}^s$ at which all of the functions $\widehat{B}(\omega+2\pi j)$, $j\in\mathbb{Z}^s$, vanish. The latter fact in turn implies the following result whose proof can be found in Proposition (1.11) of [18].

Proposition6. *If the direction matrix Ξ defining the box spline $B=B_\Xi$ is unimodular, then Laurent polynomials in the set*

$$\Big\{M_\mu(z^2) := \sum_{j\in\mathbb{Z}^s} m_\Xi(\mu+2j)z^{2j} : \mu\in\mathbb{Z}_2^s\Big\}$$

have no common zeros in $(\mathbb{C}\backslash\{0\})^s$.

We shall use this Proposition to find candidates for M^d that satisfy (3). Suppose that $\phi = B_\Xi$

with the refinement mask symbol $M = M_\Xi$. Let $M^d := \overline{M_{\Xi'} Q}$, where $M_{\Xi'}$ is the refinement mask symbol of a box spline $B_{\Xi'}$ and Q is to be chosen so that (3) holds. Define $M_{\Xi\cup\Xi'} := M_\Xi M_{\Xi'} = \sum_{\alpha\in\mathbb{Z}^s} m_{\Xi\cup\Xi'}(\alpha)\exp(-i\alpha\cdot)$ and

$$M_\mu(z^2) := \sum_{j\in\mathbb{Z}^s} m_{\Xi\cup\Xi'}(\mu+2j)z^{2j} : \mu \in \mathbb{Z}_2^s.$$

Then the Laurent polynomials in the set $\{M_\mu : \mu \in \mathbb{Z}_2^s\}$ have no common zeros in $(\mathbb{C}\backslash\{0\})^s$, if B_Ξ and B'_Ξ are stable box splines.

By Hilbert's Nullstellungsatz, there exist Laurent polynomials Q_μ such that

(7)
$$\sum_{\mu\in\mathbb{Z}_2^s} M_\mu(z^2)Q_\mu(z^2) = 2^{-s}z^{2\alpha}, \quad \text{for any} \quad \alpha \in \mathbb{N}^s.$$

We set $Q := \sum_{\mu\in\mathbb{Z}_2^s} z^{-\mu}Q_\mu(z^2)$ and define

$$M^d(\omega) = \overline{\exp(i2\omega\alpha)Q(\exp(-i\omega))M_{\Xi'}(\omega)}. \tag{8}$$

Then (3) is satisfied (the proof can be modified from the corresponding proof of [18]).

2.2 Numerical Method

The preceding subsection outlines a numerical construction of the mask to meet the necessary condition (3). The procedure is as follows:

- Start with a stable box spline B_Ξ with mask symbol M_Ξ. Choose another stable box spline $B_{\Xi'}$ with mask symbol $M_{\Xi'}$, and form the mask $M_{\Xi\cup\Xi'}$ of stable refinable box spline

$$B_{\Xi\cup\Xi'} = B_\Xi * B_{\Xi'}.$$

- The polynomials M_μ, $\mu \in \mathbb{Z}_2^s$, corresponding to $B_{\Xi\cup\Xi'}$ have no common zeros on $(\mathbb{C}\backslash\{0\})^s$ by Proposition 6.
- Find polynomials Q_μ, $\mu \in \mathbb{Z}_2^s$, such that (7) holds, and define M^d by (8).
- Finally, the Fourier transform of ϕ^d is given in the usual way by infinite product

$$\widehat{\phi}^d := \prod_{k=1}^{\infty} M^d(2^{-k}\cdot).$$

Since any box spline can be written as a convolution of two box splines by disjointly decomposing the columns into two sets of columns, $B_{\Xi\cup\Xi'} = B_\Xi * B_{\Xi'}$, the current construction can be reduced to the construction of multivariate interpolation subdivision schemes as carried out in [18]. In [18], we presented an algorithmic method for the two dimensional case in detail and provided several examples. Hence, this method can be carried out directly to derive dual refinable functions numerically for a given box spline.

Example 9. Let φ be the refinable function corresponding to the refinement mask

(10)
$$\widehat{a}(\omega) = \Big(\cos\big(\frac{\omega_1}{2}\big)\cos\big(\frac{\omega_2}{2}\big)\cos\big(\frac{\omega_1+\omega_2}{2}\big)\Big)^2 \times \Big(5-\cos(\omega_1)-\cos(\omega_2)-\cos(\omega_1+\omega_2)\Big)/2.$$

Then, the mask defines an interpolatory subdivision scheme (see [18]), i.e., φ satisfies

$$\varphi(\alpha) = \delta_\alpha, \quad \alpha \in \mathbb{Z}^2. \tag{11}$$

Let $\phi = B_{1,1,1}$, centered (piecewise linear) box spline, then ϕ is refinable with its mask

$$M(\omega) = \cos\big(\frac{\omega_1}{2}\big)\cos\big(\frac{\omega_2}{2}\big)\cos\big(\frac{\omega_1+\omega_2}{2}\big)$$

Let

$$\phi^d := \Pi_{j=1}^{\infty}\widehat{M}^d(\omega/2^j),$$

where

$$M^d = \Big(\cos\big(\frac{\omega_1}{2}\big)\cos\big(\frac{\omega_2}{2}\big)\cos\big(\frac{\omega_1+\omega_2}{2}\big)\Big) \times \Big(5-\cos(\omega_1)-\cos(\omega_2)-\cos(\omega_1+\omega_2)\Big)/2.$$

Then, it can be proved that $\phi^d \in L_2(\mathbb{R}^2)$ and $\varphi = \phi * \phi^d(-\cdot)$. Hence, ϕ^d is a refinable dual function of the centered box spline $B_{1,1,1}$. □

Finally, we remark that the smoothness of a refinable function can be estimated by a smoothness criterion in terms of the refinement mask, established in [18]. For example, φ is estimated of L_∞ regularity component 2 and ϕ^d of L_2 regularity component 0.4077.

2.3 General Method

The numerical method discussed in the last subsection is very efficient to construct dual refinable functions with a low smoothness order for a given box spline. In this subsection, we apply the general constructions of [10] to derive dual refinable functions with an arbitrary high smoothness for a given (high order) box spline to obtain smooth wavelets. There are several constructions given in [10] for the multi-dimensional case. In this section, we use one of them to obtain a construction for the bivariate case. The interested readers should consult [10] to derive some other constructions for their applications.

Let $\widehat{a}$ be the mask given in Example 9. Define a sequence of masks $\widehat{a}_k$, $k = 0, 1, \ldots$ inductively as follows:

(i) Set $\widehat{a}_0 = \widehat{a}$.

(ii) For $\widehat{a}_k$, set $P = \widehat{a}_k$ and define

(12)
$$\begin{aligned}\widehat{a}_{k+1} =& P_0^2\big(P_0^2 + 4P_0(P_1 + P_2 + P_3) + \\ & 12(P_1P_2 + P_2P_3 + P_1P_3) \\ & + 3(P_1^2 + P_2^2 + P_3^2) + 24P_1P_2P_3\big),\end{aligned}$$

where

$$\begin{aligned}&P_0 = P, \qquad P_1(\omega) = P(\omega + \pi(0,1)),\\ &P_2(\omega) = P(\omega + \pi(1,0)),\\ &P_3(\omega) = P(\omega + \pi(1,1)).\end{aligned}$$

Applying the theory of [10], one can show that for each $\widehat{a}_k$, the corresponding refinable function φ_k satisfies

$$\varphi_k(\alpha) = \delta_\alpha. \quad \alpha \in \mathbb{Z}^2.$$

The order of the smoothness of φ_k goes to infinity as k does.

Since φ is a convolution of the centered box spline $M_{2,2,2}$ with a distribution, the function φ_k is a convolution of a box spline $B_{2^{k+1},2^{k+1},2^{k+1}}$ with a distribution. The mask $\widehat{a}_k$ of φ_k has the form

$$\widehat{a}_k = M_{2^k,2^k,2^k} Q_k,$$

where

$$M_{2^k,2^k,2^k}(\omega) = \\ \Big(\cos\big(\frac{\omega_1}{2}\big)\cos\big(\frac{\omega_2}{2}\big)\cos\big(\frac{\omega_1+\omega_2}{2}\big)\Big)^{2^k},$$

the mask of centered box spline $B_{2^k,2^k,2^k}$ and Q_k is the mask of some distribution.

We remark that the starting function φ can be any interpolatory refinable functions constructed in [18].

For a given box spline $B_{r,r,r}$, one first chooses a mask $\widehat{a}_k$ with $2^k > r$ whose corresponding refinable function φ_k has a sufficient order of the smoothness. Let ϕ_k^d be the refinable function corresponding to the mask $M_{2^k-r,2^k-r,2^k-r}Q_k$. Then, $\varphi = B_{r,r,r} * \phi^d(-\cdot)$. Hence, ϕ_k^d is a dual of $B_{r,r,r}$ whenever $\phi_k^d \in L_2(\mathbb{R}^2)$. In fact, the order of the smoothness of ϕ_k^d can reach arbitrary high by choosing sufficient large k.

The extension of the above method to the more general box spline B_{r_1,r_2,r_3} is straightforward. One merely needs to insert a proper $\exp(i\cdot/2)$ into the definition of various masks.

The construction of wavelets from a given pair of dual refinable functions are given in [19].

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