Construction of compactly supported biorthogonal wavelets II

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ABSTRACT

This paper deals with constructions of compactly supported biorthogonal wavelets from a pair of dual refinable functions in $L_2(\mathbb{R}^s)$. In particular, an algorithmic method to construct wavelet systems and the corresponding dual systems from a given pair of dual refinable functions is given.

Keywords: multivariate biorthogonal wavelets, multivariate wavelets, box splines, matrix extension

1. INTRODUCTION

This paper deals with constructions of compactly supported biorthogonal wavelets, whose dilations and shifts form a Riesz basis for $L_2(\mathbb{R}^s)$ and the dual basis is an affine system generated by compactly supported functions with required order of the smoothness, from a given pair of dual refinable functions.

Constructions of compactly supported refinable dual pairs can be found in Ref. 6 and Ref. 3. With a pair of compactly supported refinable functions constructed, the key step to construct biorthogonal wavelets from a given pair of multiresolutions can be reduced to the following matrix extension problem: Let $\mathcal{P}$ be the set of all finite order matrices with Laurent polynomial entries. If $P := (p_1, \ldots, p_n)$ and $Q := (q_1, \ldots, q_n)$ are $1 \times n$ matrices in $\mathcal{P}$ satisfing $P(z)Q^T(z) = 1$, for all $z \in (\mathbb{C}\backslash\{0\})^s$, how to find matrices X and Y in $\mathcal{P}$ whose first row is P and Q respectively, such that

$$X(z)Y^T(z) = I_n, \quad z \in (\mathbb{C}\backslash\{0\})^s?$$

This paper presents an algorithmic method to solve the above matrix extension problem which in turn leads to a method to construct compactly supported biorthogonal wavelet systems and the corresponding dual systems from given pairs of dual refinable functions. Several examples are given to illustrate the method.

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In the rest of this section, we introduce some basic notations. Let $\mathcal{H}$ be a Hilbert space and $X \subset \mathcal{H}$ be a sequence from $\mathcal{H}$. Define

$$T : \mathcal{D} \subseteq \ell_2(X) \to \mathcal{H} \quad \text{by} \quad Ta = \sum_{x \in X} a(x)x,$$

where T is defined at least on the linear space of finitely supported sequences, $\ell_0(X)$. If T is bounded on $\ell_2(X)$, then X is a Bessel sequence of $\mathcal{H}$. If T bounded and has a bounded inverse, then X is a Riesz sequence of $\mathcal{H}$. If ran(T) is dense in $\mathcal{H}$, then X is fundamental. A fundamental Riesz sequence of $\mathcal{H}$ is called a Riesz basis of $\mathcal{H}$. The adjoint map of the map T is the mapping

$$T^* : \mathcal{H} \to \ell_2(X) \quad \text{by} \quad T^*h = (< x, h >)_{x \in X}.$$

The sequence X is a Riesz basis of $\mathcal{H}$ if and only if T^* is a bounded operator from $\mathcal{H}$ onto $\ell_2(X)$ with a bounded inverse (see e.g. Ref. 7). A function ϕ is called stable, when its shifts, $\{\phi(\cdot - \alpha)_{\alpha \in \mathbb{Z}^s}\}$, form a Riesz basis of the closed shift invariant subspace of $L_2(\mathbb{R}^s)$ generated by ϕ.

2. CONSTRUCTIONS OF WAVELETS

The construction of wavelets normally is done through the multiresolution analysis. Suppose that we have stable refinable functions ϕ and ϕ^d in $L_2(\mathbb{R}^s)$ which satisfy

$$< \phi, \phi^d(\cdot - \alpha) >= \delta_\alpha, \quad \alpha \in \mathbb{Z}^s, \tag{1}$$

where δ_α is the usual delta sequence, $\delta_\alpha = 1$ if $\alpha = 0$ and $\delta_\alpha = 0$ otherwise. Define

$$S^k(\phi) := \{f(2^k\cdot) : f \in S(\phi)\}, \quad S^k(\phi^d) := \{f(2^k\cdot) : f \in S(\phi)\}$$

Then each of the sequences $(S^k(\phi))_{k\in\mathbb{Z}}$ and $(S^k(\phi^d))_{k\in\mathbb{Z}}$ forms a multiresolution of $L_2(\mathbb{R}^s)$ (cf: Refs 1, 4). We recall that a sequence of spaces $(S^k(\varphi))_{k\in\mathbb{Z}}$ forms a multiresolution if the following conditions hold: (i) $S^k(\varphi) \subset S^{k+1}(\varphi)$, $k \in \mathbb{Z}$, (ii) $\overline{\cup_{k\in\mathbb{Z}} S^k(\varphi)} = L_2(\mathbb{R}^s)$ and $\cap_{k\in\mathbb{Z}} S^k(\varphi) = \{0\}$, (iii) φ and its shifts comprise a Riesz basis of $S(\varphi)$.

As we will show that, under a mild decay condition of the Fourier transform of ϕ and ϕ^d, the construction of biorthogonal wavelets from a given pair of dual multiresolutions $S^k(\phi)$, and $S^k(\phi^d)$, $k \in \mathbb{Z}$ can be reduced to the problem of finding a set of compactly supported functions ψ_ν, $\nu \in \mathbb{Z}_2^s\backslash\{0\}$ in $S^1(\phi)$ and ψ_ν^d, $\nu \in \mathbb{Z}_2^s\backslash\{0\}$ in $S^1(\phi^d)$ respectively, so that the functions

$$\{\phi(\cdot - j), \psi_\nu(\cdot - j), \nu \in \mathbb{Z}_2^s\backslash\{0\}, j \in \mathbb{Z}^s\} \quad \textit{and} \quad \{\phi^d(\cdot - j), \psi_\nu^d(\cdot - j)\, \nu \in \mathbb{Z}_2^s\backslash\{0\}, j \in \mathbb{Z}^s\}$$

form a dual Riesz basis of the dual pair $S^1(\phi)$ and $S^1(\phi^d)$, respectively. The key step to the construction of such compactly supported functions is to solve the following matrix extension problem:

Matrix extension problem *Let $\mathcal{P}$ be the set of all finite order matrices with Laurent polynomial entries. If $P := (p_1 \dots p_n)$ and $Q := (q_1, \dots, q_n)$ are $1 \times n$ matrices in $\mathcal{P}$ with $P(z)Q^T(z) = 1$, for all $z \in (\mathbb{C}\backslash\{0\})^s$, then how does one find $(n-1) \times n$ matrices G and H in $\mathcal{P}$, such that the matrices $X := \begin{bmatrix} P \\ G \end{bmatrix}$ and $Y := \begin{bmatrix} Q \\ H \end{bmatrix}$ satisfy*

$$X(z)Y^T(z) = I_n, \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

For the case $s = 1$, Refs. 9 provides a simple algorithmic method to solve this matrix extension problem for the case when P and Q are $r \times n$ matrices. Here, we give an algorithm for an arbitrary s, which is implementable when n is not too large. This is in particular helpful in biorthogonal wavelet constructions in low dimensions, since $n = 4, 8$, when $s = 2, 3$.

Algorithm to matrix extension problem *For given $1 \times n$ vectors P and Q with $P(z)Q^T(z) = 1$ for all $z \in (\mathbb{C}\backslash\{0\})^s$, do:*

- *Step 1. Using the Quillen-Suslin Theorem, extend the row P to an $n \times n$ matrix K in $\mathcal{P}$ which is nonsingular on $(\mathbb{C}\backslash\{0\})^s$ and has P as its first row.*
- *Step 2. Alter the last $n-1$ rows to be orthogonal to Q: Let K_j, $j = 2, \ldots, n$, be the last $n-1$ rows of the matrix K and define*
$$G_j := K_j - (K_j Q^T)P, \quad j = 2, \ldots n.$$
- *Step 3. Define $X := \left[P^T, G_2^T, \ldots, G_n^T\right]^T$. Then $X \in \mathcal{P}$ and is nonsingular on $(\mathbb{C}\backslash\{0\})^s$.*
- *Step 4. Find $X^{-1} := \left[H_1^T, H_2^T, \ldots, H_n^T\right]$ and set $Y^T := \left[Q^T, H_2^T, \ldots, H_n^T\right]$ Then $Y \in \mathcal{P}$ and*
$$X(z)Y^T(z) = I_n, \quad z \in (\mathbb{C}\backslash\{0\})^s.$$

Proof and explanation. For the first step, there is an algorithm for the Quillen-Suslin theorem in Ref. 5 that utilizes Gröbner bases. However, for the examples we give, a matrix extension can be found by inspection. Once the first step is done, the second step can be implemented easily.

For the third step, we must show that X is nonsingular on $(\mathbb{C}\backslash\{0\})^s$. We show that the rows of X are linearly independent for each $z \in (\mathbb{C}\backslash\{0\})^s$. If there are constants a_j, $j = 1, \ldots n$, and $z \in (\mathbb{C}\backslash\{0\})^s$, such that $a_1 P + \sum_{j=2}^n a_j G_j(z) = 0$, then

$$\sum_{j=2}^n a_j G_j(z) = a_1 P + \sum_{j=2}^n a_j K_j - (a_1 + \sum_{j=2}^n a_j (K_j Q^T))P = 0.$$

Hence $a_j = 0$, $j = 1, \ldots n$, since the vectors $\{P(z),\ K_j(z)\ (j = 2, \ldots n)\}$ are linearly independent for each $z \in (\mathbb{C}\backslash\{0\})^s$.

For step 4, we observe that the nonsingularity of $X(z)$ on $(\mathbb{C}\backslash\{0\})^s$ implies that $\det X(z) = \text{const}\, z^\alpha$, for some $\alpha \in \mathbb{Z}^s$. Therefore, X^{-1} can be calculated directly and, most importantly $X^{-1} \in \mathcal{P}$. It is clear by the definition of X and Y that $XY^T = I_n$. $\square$

Remark. Once step 1 is available in Algorithm 2, the other steps follow easily. Hence, we will provide the details of the first step for the examples in the next section, and leave the remaining steps of the algorithm to the reader. In fact, the interested reader can derive their own computer programs for this algorithm given step 1. $\square$

Remark. Algorithm 2 can be also applied to constructions of fast decay biorthogonal wavelets, or even more generally, biorthogonal wavelets in a Hilbert space from a given pair of multiresolutions. In those general cases, one needs to extend P to a square matrix K with certain properties in the first step. In fact, such extensions normally are simpler than the extension with polynomial entries. The details of constructions of such extensions can be found in Ref. 4. □

Remark. Algorithm 2 also can be applied to obtain an algorithm for the construction of multiwavelets from a multiresolution generated by several refinable functions. In particular, for the univariate case, the algorithm provided by Ref. 9. □

The matrix extension problem is used in the construction of biorthogonal wavelets in the following way:

Algorithm for Biorthogonal Wavelet Construction *For a given dual Riesz basis pair ϕ, ϕ^d with mask symbols $M = \sum_{\alpha\in\mathbb{Z}^s} m(\alpha)\exp(-i\alpha\cdot)$ and $M^d = \sum_{\alpha\in\mathbb{Z}^s} m^d(\alpha)\exp(-i\alpha\cdot)$ respectively, do:*

- *Step 1. Define polynomials corresponding to the masks restricted to the cosets of $\mathbb{Z}_2^s := \mathbb{Z}^s/2\mathbb{Z}^2$ by*

$$A_{0,\nu}(z) := \sum_{\alpha\in\mathbb{Z}^s} m(\nu+2\alpha)z^{2\alpha}, \nu \in \mathbb{Z}_2^s,$$

and

$$A_{0,\nu}^d(z) := \sum_{\alpha\in\mathbb{Z}^s} \overline{m^d(\nu+2\alpha)}z^{-2\alpha}, \nu \in \mathbb{Z}_2^s.$$

- *Step 2. Take $P(z)$ as the 1×2^s vector $P(z) := A_0(z) := (A_{0,\nu}(z))_{\nu\in\mathbb{Z}_2^s}$, and take $Q(z)$ as the 1×2^s vector $Q(z) := A_0^d := (A_{0,\mu}^d(z))_{\nu\in\mathbb{Z}_2^s}$.*

- *Step 3. Apply Algorithm 2 to $P = A_0$ and $Q = A_0^d$ to obtain matrices X and Y.*

- *Step 4. Label the rows and columns of X and Y by $\mathbb{Z}_2^s$ with the first row labeled by 0 and the remaining rows in some fixed order (e.g. by the lexicographic order). For the μ-th rows $A_\mu = (A_{\mu\nu})_{\nu\in\mathbb{Z}_2^s}$ and $A_\mu^d := (A_{\mu\nu}^d)_{\nu\in\mathbb{Z}_2^s}$ respectively of X and Y, define*

$$A_\mu(\omega) := \sum_{\nu\in\mathbb{Z}_2^s} \exp(-i\nu\omega)A_{\mu\nu}(\exp(-2i\omega)),$$

and

$$A_\mu^d(\omega) := \overline{\sum_{\nu\in\mathbb{Z}_2^s} \exp(-i\nu\omega)A_{\mu\nu}^d(\exp(-2i\omega))}.$$

- *Step 5. The matrices*

$$\mathbb{A} := (A_\mu(\cdot+\pi\nu))_{\mu,\nu\in\mathbb{Z}_2^s}, \quad \text{and} \quad \mathbb{A}^d := (A_\mu^d(\cdot+\pi\nu))_{\mu\in\mathbb{Z}_2^s}$$

are unitarily equivalent to X and Y respectively (cf. Ref. 2, Chapter 5); therefore,

$$\mathbb{A}(\omega)(\mathbb{A}^d)^*(\omega) = I, \quad \omega \in \mathbb{R}^s.$$

- *Step 6. Define two sets of functions*

$$\widehat{\psi}_\mu(2\omega) := A_\mu(\omega)\widehat{\phi}(\omega), \quad \textit{and} \quad \widehat{\psi}^d_\mu(2\omega) := A^d_\mu(\omega)\widehat{\phi}^d(\omega), \qquad \mu \in \mathbb{Z}^s_2.$$

Then

  - *(i)* $\psi_0 = \phi$ *and* $\psi^d_0 = \phi^d$.
  - *(ii) The functions* ψ_μ, $\mu \in \mathbb{Z}^s_2\backslash\{0\}$, *are called the wavelets for the scaling function* ϕ.
  - *(iii) The functions* ψ^d_μ, $\mu \in \mathbb{Z}^s_2\backslash\{0\}$, *are called the dual wavelets, that is, the wavelets of the dual function* ϕ^d.

In what follows, we adopt the notation

$$\Psi := \{\psi_\mu : \mu \in \mathbb{Z}^s_2\} \quad \textit{and} \quad \Psi^d := \{\psi_\mu : \mu \in \mathbb{Z}^s_2\}$$

for the two sets of functions just constructed.

PROPOSITION 1. *The functions* ψ_μ, $\mu \in \mathbb{Z}^s_2$, *together with their shifts form a Riesz basis of* $S^1(\phi)$ *and the functions* ψ^d_μ, $\mu \in \mathbb{Z}^s_2$, *together with their shifts form a Riesz basis of* $S^1(\phi^d)$. *Moreover,*

$$(2) \qquad < \psi_\nu, \psi^d_\mu(\cdot - \alpha) >= \delta_\alpha\, \delta_{\nu,\mu}, \quad \textit{for all } \alpha \in \mathbb{Z}^s \textit{ and } \nu, \mu \in \mathbb{Z}^s_2.$$

Proof: This can be seen by using the pre-Gramian matrix introduced in Ref. 7. We review some of the relevant facts from Ref. 7. For a given finite set of functions Φ, the **pre-Gramian** matrix of Φ at a point $\omega \in \mathbb{T}^s$ is defined as a $\mathbb{Z}^s \times \Phi$ matrix by

$$J(\omega) := J_\Phi(\omega) := (\widehat{\phi}(\omega + 2\pi\alpha))_{\alpha,\phi},$$

where Φ with some definite order is also used as an index set. The adjoint matrix of $J(\omega)$ is the $\Phi \times \mathbb{Z}^s$ matrix

$$J^*(\omega) := J^*_\Phi(\omega) := (\overline{\widehat{\phi}(\omega + 2\pi\alpha)})_{\phi,\alpha}$$

Assume that the functions in $\Phi \subset L_2(\mathbb{R}^s)$ are compactly supported. Let $S(\Phi)$ denote the closed shift invariant subspace of $L_2(\mathbb{R}^s)$ generated by Φ. If the functions $\phi(\cdot - \alpha)$ form a Bessel system, then the functions $\phi(\cdot - \alpha)$, $\phi \in \Phi$, $\alpha \in \mathbb{Z}^s$, form a Riesz basis of $S(\Phi)$ if and only if $J_\Phi(\omega)$ has full rank for all $\omega \in \mathbb{R}^s$ (see e.g. Proposition 1.4.11 of Ref. 7. Furthermore, assume that there is another system of $L_2(\mathbb{R}^s)$ functions $\phi^d(\cdot - \alpha)$, $\phi^d \in \Phi^d$, $\alpha \in \mathbb{Z}^s$ which also forms a Bessel system, then this system (with a proper order) forms a dual Riesz basis to the Riesz basis of $\phi(\cdot - \alpha)$, $\phi \in \Phi$, $\phi \in \Phi$, $\alpha \in \mathbb{Z}^s$ if and only if $J^*_{\Phi^d} J_\Phi = I$.

Returning to the proof of the proposition, we write $J_\Psi(\omega)$ as a column block matrix by

$$J_\Psi(\omega) = (\widehat{\psi}(\omega + 2\pi\nu + 4\pi\alpha))_{(\nu,\alpha)\times\phi \in (\mathbb{Z}^s_2 \times \mathbb{Z}^s)\times\Psi}$$

and let A be the first column of $\mathbb{A}$, then $A(\cdot + \pi\nu)$ is the ν-th column of $\mathbb{A}$ and from (2) we have

$$\overline{J_\Psi(\omega)} = (\overline{\phi(\omega/2 + \pi\nu + 2\pi\alpha)} A^*(\omega/2 + \pi\nu))_{(\nu,\alpha)\in\mathbb{Z}^s_2\times\mathbb{Z}^s}.$$

Similarly,

$$\overline{J^*_{\Psi^d}(\omega)} = (A^d(\omega/2+\pi\nu)\widehat{\phi}^d(\omega/2+\pi\nu+2\pi\alpha))_{(\nu,\alpha)\in\mathbb{Z}_2^s\times\mathbb{Z}^s}.$$

Therefore, we have

$$\begin{aligned}
&\overline{J^*_{\Psi^d}(\omega)J_\Psi(\omega)}\\
&=\sum_{\nu\in\mathbb{Z}_2^s} A^d(\omega/2+\pi\nu)\left(\sum_{\alpha\in\mathbb{Z}^s}\widehat{\phi}^d(\omega/2+\pi\nu+2\pi\alpha)\overline{\widehat{\phi}(\omega/2+\pi\nu+2\pi\alpha)}\right)A^*(\omega/2+\pi\nu)\\
&=\sum_{\nu\in\mathbb{Z}_2^s} A^d(\omega/2+\pi\nu)A^*(\omega/2+\pi\nu)=I.
\end{aligned}$$

The criterion from Ref. 7 now implies that Ψ and Ψ^d and their shifts form a Riesz basis of $S^1(\phi)$ and $S^1(\phi^d)$ respectively and that they are a dual Riesz basis pair. □

Consider now the two sets of functions

$$\psi_\mu(2^k\cdot-\alpha),\quad \mu\in\mathbb{Z}_2^s\backslash\{0\}, k\in\mathbb{Z},\alpha\in\mathbb{Z}^s \tag{3}$$

and

$$\psi^d_\mu(2^k\cdot-\alpha),\quad \mu\in\mathbb{Z}_2^s\backslash\{0\}, k\in\mathbb{Z},\alpha\in\mathbb{Z}^s. \tag{4}$$

The functions in $\Psi\backslash\{\phi\}$ and their shifts are constructed to be orthogonal to $S(\phi^d)$ and the functions in $\Psi^d\backslash\{\phi^d\}$ and its shifts are constructed to be orthogonal to $S(\phi)$. Therefore, we have that

$$< 2^{k_1/2}\psi_{\mu_1}(2^{k_1}\cdot-\alpha_1), 2^{k_2/2}\psi_{\mu_2}(2^{k_2}\cdot-\alpha_2) >= \delta_{k_1,k_2}\delta_{\mu_1,\mu_2}\delta_{\alpha_1,\alpha_2},$$

where

$$\mu_1,\mu_2\in\mathbb{Z}_2^s\backslash\{0\};\quad k_1,k_2\in\mathbb{Z};\quad \alpha_1,\alpha_2\in\mathbb{Z}^s.$$

Since $S^k(\phi)$ and $S^k(\phi^d)$ form multiresolutions in $L_2(\mathbb{R}^s)$, the span (3), as well as the span of (4), is dense in $L_2(\mathbb{R}^s)$.

The following Proposition implies that in order to show that the sets of functions (3) and (4) form a biorthogonal Riesz basis of $L_2(\mathbb{R}^d)$, it is only necessary to show that they are Bessel systems.

PROPOSITION 2. *Let $\mathcal{H}$ be a Hilbert space and $X := (x_i)_{i\in\mathbb{Z}}$ and $Y := (y_i)_{i\in\mathbb{Z}}$ be two fundamental Bessel sequences in $\mathcal{H}$. Suppose that spanX, as well as spanY, is dense in $\mathcal{H}$ and the sequences x_i and y_i, $i\in\mathbb{Z}$ satisfy*

$$< x_i, y_j >= \delta_{i,j}. \tag{5}$$

Then X and Y form a biorthogonal Riesz basis of $\mathcal{H}$.

Proof. Define

$$T_Y^* : \mathcal{H} \to \ell_2(Y) \quad \text{by} \quad T_Y^* x = (< x, y >)_{y \in Y},$$

and

$$T_X : \ell_2(X) \to \mathcal{H} \quad \text{by} \quad T_X a = \sum_{x \in X} a(x)x.$$

Then both T_X and T_Y^* are bounded operators. Further, Since X and Y satisfy (5), T_X is 1-1. For an arbitrary $a \in \ell_2(\mathbb{Z})$, the function

$$f := \sum_{\alpha \in \mathbb{Z}} a_\alpha x_\alpha \in \mathcal{H},$$

since T_X is a bounded operator. Further, $< f, y_\alpha >= a_\alpha$. Hence, T_Y^* is onto. For an arbitrary $x \in \mathcal{H}$, if $T_Y^* x = 0$, then x orthogonal to all $y \in Y$. Since Y is fundamental, the element x must be zero. Hence T^* is 1-1. Since T_Y^* is 1-1 bounded and onto map, it has a bounded inverse by the open mapping theorem, hence Y is a Riesz basis of $\mathcal{H}$. A similar proof, one can show that X is a Riesz basis of $\mathcal{H}$. □

It was shown in Ref. 8 that the dilations and shifts of functions in Ψ' form a Bessel system if the functions

$$R_E := \max_{\psi \in \Psi'} \sum_{\alpha \in 2\pi\mathbb{Z}^s} |\widehat{\psi}(\cdot + \alpha)|,$$

and

$$R_D := \sum_{\psi \in \Psi', k \in \mathbb{Z}} |\widehat{\psi}(2^k \cdot)|$$

are in L_∞. The first condition merely says that ψ and its shifts form a Bessel system and the second condition will be satisfied, if the functions ψ, $\psi \in \Psi'$, have certain decay at the infinity and have a zero of certain order at origin. Both conditions are satisfied in our constructions.

3. EXAMPLES

In this section we produce a series of two dimensional examples from pairs of multiresolutions generated by box splines and its duals refinable functions given in Ref. 6.

A box spline is defined for a given $s \times n$ matrix Ξ of full rank with integer entries. The Fourier transform of the box spline for the matrix Ξ is

$$\widehat{B}_\Xi(\omega) := 2^{s-n} \prod_{\xi \in \Xi} \frac{1 - \exp(-i\xi\omega)}{i\xi\omega}. \tag{6}$$

Here a comment on notation is in order. The direction matrix is treated as a (multi)-set of its columns. Moreover, $\xi\omega$ for two vectors from $\mathbb{R}^s$ will mean their dot product. Thus, the product runs over the columns of the matrix Ξ and is complex-valued. The basic facts and much of the notation concerning box splines are taken from Ref. 2; the reader is referred to Ref. 2 for the appropriate references. In particular, for the bivariate case, stable box splines based on the matrix Ξ_{n_1,n_2,n_3} consists of the three columns $[1,0]^T$, $[0,1]^T$ and $[1,1]^T$ appearing with multiplicities n_1, n_2, and n_3 respectively. For simplicity of notation, we describe the box splines by a triple subscript corresponding to the multiplicities.

The (total) degree of the polynomial pieces of the box spline does not exceed $n-s$. The support of the box spline is the polyhedron

$$\Xi[0,1]^n := \{x : x = \sum_{\xi\in\Xi} t_\xi \xi, 0 \le t_\xi \le 1\}$$

where $[0,1]^n$ is the n-cube and the summation runs over the columns of the matrix Ξ.

A box spline B_Ξ satisfies the refinement equation

$$B_\Xi = 2^s \sum_{\alpha\in\mathbb{Z}^s} m_\Xi(\alpha) B_\Xi(2\,\cdot\,-\alpha),$$

where

$$M_\Xi(\omega) := \sum_{\alpha\in\mathbb{Z}^s} m_\Xi(\alpha)\exp(-i\alpha\omega) \tag{7}$$

$$= \prod_{\xi\in\Xi} \frac{1+\exp(-iy\xi)}{2}. \tag{8}$$

There is an easily checked criterion for when the shifts of a box spline form a Riesz basis; namely, when the direction set Ξ is a unimodular matrix (all bases of columns from Ξ have determinant ± 1). The last condition is equivalent to there being no $\omega \in \mathbb{R}^s$ at which all of the functions $\widehat{B}(\omega + 2\pi j)$, $j \in \mathbb{Z}^s$, vanish.

For applications, two things are important: (a) the size of the support, and (b) the smoothness of the box spline, which increases the size of the support. It is therefore natural for us to choose as ϕ one of the three box splines $B_{1,1,1}$, $B_{2,2,1}$ and $B_{2,2,2}$ for which respectively the first, second and third derivatives are piecewise constant and which have the smallest support among the three-direction box splines with the given smoothness. In order to find the dual functions ϕ^d with required smoothness, one can carry out the procedure given by Ref. 6.

The only step left in the construction of biorthogonal wavelet is the first step in Algorithm 2. As we will see that these box splines have another advantage in that the matrix extension problem can be solved by inspection.

$\phi = B_{1,1,1}$: The first step of Algorithms 2 and 2 requires the row $P = [A_{0,\nu}]_{v\in\mathbb{Z}_2^2}$ and its extension to a nonsingular matrix as determined by the symbol $M_{1,1,1}(z) := (1+z_1)(1+z_2)(1+z_1z_2)/8$. In this case, a suitable extension is:

$$\begin{bmatrix} \frac{1+z_1^2z_2^2}{8} & \frac{1+z_2^2}{8} & \frac{1+z_1^2}{8} & \frac{1}{4} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$\phi = B_{2,2,1}$: Here we use an extension already appearing in Ref. 7. The first row P and the extended matrix are:

$$\begin{bmatrix} \frac{1+z_1^2+z_2^2+5z_1^2z_2^2}{32} & \frac{1+2z_2^2+z_1^2z_2^2}{16} & \frac{1+2z_1^2+z^2z_2^2}{16} & \frac{5+z_1^2+z_2^2+z_1^2z_2^2}{32} \\ 1 & 0 & 0 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \end{bmatrix}$$

$\phi = B_{2,2,2}$: The first row of the matrix to be extended is

$$\begin{bmatrix} p_{00} & p_{10} & p_{01} & p_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{64}(1 + z_1^2 + z_2^2 + 10z_1^2z_2^2 + z_2^4z_1^2 + z_1^4z_2^2 + z_1^4z_2^4) \\ \frac{1}{32}(1 + 3z_2^2 + 3z_1^2z_2^2 + z_2^4z_1^2) \\ \frac{1}{32}(1 + 3z_1^2 + 3z_1^2z_2^2 + z_1^4z_2^2) \\ \frac{1}{32}(3 + z_2^2 + 3z_1^2z_2^2 + z_1^2) \end{bmatrix}^T .$$

For the extension we use polynomials

$$q_{00} := \frac{5}{32}, \qquad q_{10} := -\frac{1+z_1^2}{64}, \qquad q_{01} := -\frac{1+z_2^2}{64}, \qquad q_{11} := -\frac{1+z_1^2z_2^2}{64}.$$

determined by $Q_{2,2,2}$ so that $p_{00}q_{00} + p_{10}q_{10} + p_{01}q_{01} + p_{11}q_{11} = z_1^2z_2^2/64$. We take the extended matrix as

$$\begin{bmatrix} p_{00} & p_{10} & p_{01} & p_{11} \\ -q_{10} & q_{00} & 0 & 0 \\ -q_{01}/q_{00} & 0 & 1 & 0 \\ -q_{11}/q_{00} & 0 & 0 & 1 \end{bmatrix}.$$

REFERENCES

1. C. de Boor, R. DeVore, and A. Ron, On the construction of multivariate (pre) wavelets, *Constr. Approx.*, **9**, pp. 123-166, 1993.
2. C. de Boor, K. Höllig and S.D. Riemenschneider, Box splines, Springer Verlag, New York, 1993.
3. Hui Ji, S. D. Riemenschneider and Zuowei Shen, Multivariate Compactly supported fundamental refinable functions, duals and biorthogonal wavelets, *Studies in Applied Mathematics*, **102**, pp. 173-204, 1999.
4. Rong-Qing Jia and Zuowei Shen, Multiresolution and wavelets, *Proc. Edinburgh Math. Soc.*, **37**, pp. 271-300, 1994.
5. A. Logar and B. Sturmfels, Algorithm for the Quillen-Suslin theorem, *J. of Algebra*, **145**, pp. 231-239, 1992.
6. S. D. Riemenschneider and Zuowei Shen, Construction of compactly supported biorthogonal wavelets: I, reprint.
7. A. Ron and Zuowei Shen, Frames and stable bases for shift invariant subspaces of $L_2(R^d)$, *Canad. J. Math.*, **47**, pp. 1051-1094, 1995.
8. A. Ron and Zuowei Shen, Gramian analysis of affine basis, in *Approximation Theory VIII-Vo12, Wavelets and Multilevel Approximation*, C.K Chui and L.L. Schumaker eds, World Scientific Publishing, New Jersey, 1995.
9. Zuowei Shen Extension of matrices with Laurent polynomial entries, Proceedings of the 15th IMACS World Congress on Scientific Computation Modeling and Applied Mathematics, Vol 1, Ashim Syclow eds., pp. 57-61, 1997.