Compression with Time-Frequency Localization Filters

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Abstract. The wavelet transformations combined with the zerotree algorithm have proven to be an efficient compression scheme in many applications. This scheme keeps filters fixed at the all resolution levels. We propose here a bi-filter method that uses different filters at different resolution levels. At high levels, good frequency localization filters are used. They are replaced by good time localization ones at lower levels. We choose to use filters derived from interpolatory refinable functions. The detailed design and analysis are included. The numerical empirical results show that this bi-filter compression scheme keeps more texture details of the original images.

§1. Introduction

This paper proposes a bi-filter compression scheme that balances localizations in the both time and frequency domains. The detailed derivation, filter design and numerical implementation are given. The numerical empirical results coincide with the analytic ones.

A signal f (or an image) is normally given as a set of data (i.e. a finitely supported sequence) $\{h(\alpha)\}$, $\alpha \in \mathbb{Z}^s$ by sampling. It can be represented (or approximated) by a function f_k in a sampling space $S_k(\phi)$. The space $S_k(\phi)$ is chosen to be the 2^kth dilation of a shift invariant (integer translate invariant) space

$$S(\phi) := \{f : f = \sum_{\alpha\in\mathbb{Z}^s} h(\alpha)\phi(\cdot - \alpha), a \in \ell_2(\mathbb{Z}^s)\},$$

where $\phi \in C(\mathbb{R}^s) \cap L_2(\mathbb{R}^s)$. That is

$$S_k(\phi) = \{f(2^k\cdot) : f \in S(\phi)\},$$

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and f_k has the form

$$f_k(x) := \sum_{\alpha\in\mathbb{Z}^s} h_\alpha 2^{ks/2}\phi(2^k x-\alpha).$$

Assume that the function ϕ is interpolatory, i.e., $\phi(\alpha)=\delta_\alpha$, $\alpha\in\mathbb{Z}^s$ ($\delta(0)=1$ and $\delta(\alpha)=0$, $0\neq\alpha\in\mathbb{Z}^s$). Then, the function f_k interpolates f at the lattice $\frac{1}{2^k}\mathbb{Z}^s$ if the sequence h_α is given as the values of f on the lattice $\frac{1}{2^k}\mathbb{Z}^s$. When f_k is a non-interpolatory projection of f into the subspace $S_k(\phi)$, we only have $f_k(\alpha/2^k)=h_\alpha$. Normally, the function ϕ is chosen to satisfy sufficiently high order of Strang-Fix condition so that f_k provides an accurate approximation of f.

To carry out a simple algorithm, the space $S_k(\phi)$ has to be a super space of $S_{k-1}(\phi)$, i.e., the function ϕ must be refinable. Recall that a function $\phi\in L_2(\mathbb{R}^s)$ is refinable, whenever there is a sequence $a\in\ell_2(\mathbb{Z}^s)$ such that the function ϕ satisfies the following refinement equation:

$$\phi = 2^s\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(2\cdot-\alpha). \tag{1}$$

The sequence a is called the refinement mask of ϕ. It is also called a low pass filter.

It was shown in [1] (also see [7]) that if $\phi\in L_2(\mathbb{R}^s)$ and $\widehat{\phi}$ (the Fourier transform of ϕ) is continuous at origin with $\widehat{\phi}(0)\neq 0$, then

$$\overline{\cup_{k\in\mathbb{Z}}S_k(\phi)} = L_2(\mathbb{R}^s);\quad \cap_{k\in\mathbb{Z}}S_k(\phi)=\{0\}.$$

If ϕ and its shifts form a Riesz basis of $S(\phi)$, then $S_k(\phi)$, $k\in\mathbb{Z}$ form a multiresolution analysis. Recall that a sequence $S_k(\phi)$, $k\in\mathbb{Z}$ forms a multiresolution if the following conditions are satisfied: (i) $S_k(\phi)\subset S_{k+1}(\phi)$, (ii) $\overline{\cup_{k\in\mathbb{Z}}S_k(\phi)}=L_2(\mathbb{R}^s)$ and $\cap_{k\in\mathbb{Z}}S_k(\phi)=\{0\}$, (iii) ϕ and its shifts form a Riesz basis of $S(\phi)$.

The function ϕ is called stable if ϕ and its shifts form a Riesz basis of $S(\phi)$, which is equivalent to the fact that there are constants $0<\mathrm{const}_1\leq\mathrm{const}_2<\infty$, such that

$$\mathrm{const}_1 \leq \sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}(\omega+2\pi\alpha)|^2 \leq \mathrm{const}_2,\quad a.e.,\quad \omega\in\mathbb{R}^s$$

and ϕ is pre-stable if the left inequality above holds. We say that ϕ and its shifts form a Bessel system if the right inequality above holds. Hence, ϕ is stable if and only if ϕ is pre-stable and its shifts form a Bessel system.

One of the key steps in de-noising and compression is to identify those parts of a given set of data h_α, $\alpha\in\mathbb{Z}^s$ which are mainly contributing noise to the signal in de-noising and 'unimportant information' to the signal in compression. This can be done by decomposing the function f_k into a different resolution levels via multiresolution analysis. Mathematically, this decomposition procedure is to represent f_k by a (wavelet) basis which provides better localizations. Practically, this algorithm makes signals passing through a pair of low-high pass filters iteratively, which separates signals according to their frequencies. The low pass filter is the refinement mask of the function ϕ and the high pass filter is the corresponding wavelet mask. This procedure is done by wavelet decomposition algorithm (see [13]). To make a good separation of frequencies, it is desirable to use filters whose corresponding refinable functions and wavelets are compactly supported or decay fast in the Fourier domain. This requires that the refinable functions and the corresponding wavelets are either band limited or very smooth. On the other hand, the filters generating smaller supported refinable functions and wavelets give the better localization in the time domain. Ideal filters would be those generating highly smooth refinable functions and wavelets with small supports. However, the Heisenberg uncertainty principle asserts the contrary: higher smoothness leads to larger support.

This paper provides a bi-filter scheme using different filters at different resolution levels. At the high resolution levels, the large dilation improves the localization in the time domain. Therefore, at those levels, filters corresponding to the high smoothness or band limited basis are used to enhance the localization in the Fourier domain. On the other hand, we use the filters corresponding to smaller support basis at low resolution levels to enhance the localization in the time domain. This bi-filter scheme can be understood as a wavelet packet decomposition-reconstruction algorithm. We use filters generated from interpolatory refinable functions that make a smooth data flow.

The filter design and analysis are given in Section 2. The numerical experiment results together with comparisons with other methods are presented in Section 3. Our conclusion is drawn in Section 4.

Filter design combining with wavelet analysis has been studied by many authors (see e.g. [6] [17] [18] [19]). The interested readers should consult these references for more details.

§2. Filter Design and Analysis

We start with a low pass filter a whose corresponding refinable function ϕ is continuous and interpolatory i.e., $\phi \in C(\mathbb{R})$ and for $\alpha \in \mathbb{Z}$

$$\phi(\alpha) = \delta_\alpha = \begin{cases} 1, & \alpha = 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that if ϕ is interpolatory, then ϕ is pre-stable. Furthermore, if ϕ is interpolatory and compactly supported, then ϕ is stable.

The refinement mask a of any refinable function ϕ satisfies

$$\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 1.$$

This implies that a convolves with a constant sequence h results in the same sequence h. The sequence a, as a filter, passes lower frequency signals. Hence, refinement mask is also called a low pass filter.

If the refinable function ϕ is interpolatory, its refinement mask must satisfy

$$a(2\alpha) = (1/2)\delta_\alpha. \tag{2}$$

This condition (2) is only necessary for the corresponding refinable function ϕ to be interpolatory. However, if the refinable function ϕ whose refinement mask satisfies (2) is stable, then ϕ is interpolatory (see [11]). It is well known that if a function $\phi \in L_2(\mathbb{R})$ and its shifts form an orthonormal system, then its autocorrelation is interpolatory.

For a given sequence $h \in \ell_2(\mathbb{Z})$, $\widehat{h}$ denote the Fourier series with h as its Fourier coefficients, i.e.

$$\widehat{h}(\omega) := \sum_{\alpha\in\mathbb{Z}} h(\alpha)\exp(-i\alpha\omega).$$

The refinement equation (1) in the Fourier domain is

$$\widehat{\phi}(2\omega) = \widehat{a}(\omega)\widehat{\phi}(\omega).$$

The identity $\sum_{\alpha\in\mathbb{Z}} a(\alpha) = 1$ is equivalent to $\widehat{a}(0) = 1$. Necessary condition (2) of ϕ to be interpolatory becomes

$$\widehat{a}(\omega) + \widehat{a}(\omega + \pi) = 1. \tag{3}$$

Finitely supported low-pass filters and their corresponding compactly supported interpolatory refinable functions used in this paper are refinement masks of autocorrelation functions of the Daubechies' refinable functions.

Example 1. (Compactly support interpolatory refinable functions).
Let

$$\widehat{a}(\omega) = \cos^{2K}(\omega/2)\sum_{k=0}^{K-1}\binom{K-1-k}{k}\sin^{2k}(\omega/2), \tag{4}$$

where K is a positive integer. The low-pass filters in (4) are finitely supported. Its corresponding refinable function is interpolatory, stable and compactly supported (see [4] and [5]). The mask a is short when K

is small, while the larger K corresponds to longer filter with smoother refinable function.

The mask of band limited (compactly supported in the frequency domain) interpolatory refinable function is derived from the autocorrelation of the Meyer's refinable function (see [12]).

Example 2. (Band limited interpolatory refinable function) Let $\widehat{m}(\omega)$ be the refinement mask of the Meyer's refinable function. Let $\widehat{a}(\omega) = |\widehat{m}(\omega)|^2$. Then, the corresponding refinable function ϕ is:

$$\widehat{\phi}(\omega) = \begin{cases} 1, & |\omega| \le \frac{2\pi}{3}; \\ \cos^2\left[\frac{\pi}{2}\nu(\frac{3}{2\pi}|\omega| - 1)\right], & \frac{2\pi}{3} \le |\omega| \le \frac{4\pi}{3}; \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\nu(\omega) = \omega^4(35 - 84\omega + 70\omega^2 - 20\omega^3), \quad \text{for} \quad 0 \le \omega \le 1.$$

Since ϕ is an autocorrelation of Meyer's refinable function which is band limited and whose shifts form an orthonormal system, the function ϕ is band limited and interpolatory. Furthermore, the refinement mask $\widehat{a}(\omega)$ and its refinable function $\widehat{\phi}$ satisfy

$$\widehat{a}(\omega) = \sum_{\ell\in\mathbb{Z}} \widehat{\phi}(2(\omega + 2\pi\ell)).$$

The low pass filter $a(\alpha)$, $\alpha \in \mathbb{Z}$ is computed by

$$\begin{aligned} a(\alpha) &= \sum_{\alpha\in\mathbb{Z}} \frac{1}{2\pi} \int_{|\omega|\le\pi} \widehat{\phi}(2(\omega + 2\pi\ell)) \exp(i\alpha\omega) d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \widehat{\phi}(\omega) \exp(i\alpha\omega/2) d\omega \\ &= \frac{1}{2\pi} \int_0^{2\pi} \widehat{\phi}(\omega) \cos(i\alpha\omega/2) d\omega \\ &= \frac{1}{\alpha\pi} \sin\frac{\alpha\pi}{3} + \frac{1}{2\pi} \int_{2\pi/3}^{4\pi/3} \cos^2\left[\frac{\pi}{2}\nu(\frac{3}{2\pi}\omega - 1)\right] \cos(\alpha\omega/2) d\omega. \end{aligned}$$

We tabulate the coefficients of the mask $\widehat{a}(\omega)$ in Table 1. The even index coefficients $a_{2k} = 0$, for all $k \in \mathbb{Z}\backslash\{0\}$. Above filter can be used to approximate interpolatory Meyer filter since it has fast decay.

With interpolatory refinable function ϕ and its mask a in hands, one needs a dual low-pass filter a^d of a to have a perfect reconstruction. The filter a^d is called a dual filter of the filter a, whenever the filters a and a^d satisfy

$$\widehat{a}(\omega)\overline{\widehat{a}^d(\omega)} + \widehat{a}(\omega + \pi)\overline{\widehat{a}^d(\omega + \pi)} = 1. \tag{5}$$

k	$a_k = a_{-k}$	k	$a_k = a_{-k}$
0	0.50000000000000	1	0.31607883497448
3	-0.09958233688813	5	0.05334061494462
7	-0.03208281213473	9	0.01977670561515
11	-0.01203366413871	13	0.00707711245054
15	-0.00396356272075	17	0.00208800928556
19	-0.00102335481607	21	0.00046271420195
23	-0.00019308100082	25	0.00007634982641
27	-0.00003100911800	29	0.00001444689158
31	-0.00000768172523	33	0.00000397208498

Tab. 1. The coefficients of interpolatory Meyer Filter.

The following Proposition provides a wide choice of dual filters from a given low pass filter of an interpolatory function.

Proposition 1. . *Let a be a low-pass filter that satisfies (2). Assume that $\widehat{a}$ is real and continuous. Then for each N the filter,*

$$\widehat{a^d} := \binom{2N}{N}\widehat{a}^N(1-\widehat{a})^N + \left(\sum_{j=0}^{N-1}\binom{2N}{j}\widehat{a}^{2N-1-j}(1-\widehat{a})^j\right) \tag{6}$$

is a dual filter of a with $\widehat{a}^d(0) = 1$, i.e. a and a^d satisfy 5. Furthermore, a^d is finitely supported, whenever a is so.

Proof: To show that a^d is a dual filter of a, we note the following facts:

$$1-\widehat{a}(\omega+\pi) = \widehat{a}(\omega), \quad 1-\widehat{a}(\omega) = \widehat{a}(\omega+\pi) \quad \text{and} \quad \binom{n}{i} = \binom{n}{n-i}.$$

We write

$$\widehat{a}(\omega)\overline{\widehat{a}^d(\omega)} + \widehat{a}(\omega+\pi)\overline{\widehat{a}^d(\omega+\pi)} = I_1 + I_2 + I_3$$

where

$$I_1 = \binom{2N}{N}\widehat{a}(\omega)^{N+1}(1-\widehat{a}(\omega))^N + \binom{2N}{N}\widehat{a}(\omega+\pi)^{N+1}(1-\widehat{a}(\omega+\pi))^N,$$

$$I_2 = \sum_{j=0}^{N-1}\binom{2N}{j}\widehat{a}(\omega)^{2N-j}(1-\widehat{a}(\omega))^j$$

and

$$I_3 = \sum_{j=0}^{N-1}\binom{2N}{j}\widehat{a}(\omega+\pi)^{2N-j}(1-\widehat{a}(\omega+\pi))^j.$$

Since

$$\begin{aligned} I_1 &= \binom{2N}{N}\widehat{a}(\omega)^{N+1}(1-\widehat{a}(\omega))^N + \binom{2N}{N}\widehat{a}(\omega+\pi)(1-\widehat{a}(\omega))^N\widehat{a}(\omega)^N \\ &= \binom{2N}{N}\widehat{a}(\omega)^N(1-\widehat{a}(\omega))^N. \end{aligned}$$

and

$$\begin{aligned} I_3 &= \sum_{j=0}^{N-1}\binom{2N}{j}(1-\widehat{a}(\omega))^{2N-j}\widehat{a}(\omega)^j \\ &= \sum_{i=N+1}^{2N}\binom{2N}{2N-i}(1-\widehat{a}(\omega))^i\widehat{a}(\omega)^{2N-i} \\ &= \sum_{i=N+1}^{2N}\binom{2N}{i}(1-\widehat{a}(\omega))^i\widehat{a}(\omega)^{2N-i}, \end{aligned}$$

we conclude that

$$\widehat{a}(\omega)\overline{\widehat{a}^d(\omega)} + \widehat{a}(\omega+\pi)\overline{\widehat{a}^d(\omega+\pi)} = (\widehat{a}(\omega) + (1-\widehat{a}(\omega)))^{2N} = 1.$$

Finally, since $\widehat{a}(0) = 1$, we conclude $\widehat{a}(0)^d = 1$ by the definition of $\widehat{a}^d$.
□

Let a be a mask of a interpolatory refinable function from Examples 1 or 2. Let a^d be given by (6) in terms of a. Then, the dual mask $a^d \in \ell_1(\mathbb{Z})$ satisfies

$$\widehat{a}^d(\omega) = 1 + O(|\omega|).$$

Therefore, the infinite product

$$\widehat{\phi^d}(\omega) := \prod_{j=1}^{\infty}\widehat{a}^d(\omega/2^j),$$

converges uniformly on any compact set, hence is continuous (see e.g. [10]). This implies that the Fourier transform of ϕ^d corresponding to the mask a^d is well defined.

To have numerically stable decomposition and reconstruction algorithms, the shifts of ϕ and ϕ^d must be a pair of dual Riesz systems. In other words, ϕ and ϕ^d must be stable and satisfy the following biorthogonal conditions:

$$\langle \phi, \phi^d(\cdot - \alpha)\rangle = \delta_\alpha, \quad \alpha \in \mathbb{Z}. \tag{7}$$

We say that ϕ and ϕ^d are dual functions, if both ϕ and ϕ^d are stable and satisfy (7). It was proven in [3] (also see [10]) that if ϕ and ϕ^d are stable and their masks satisfy (5), then they are dual functions. However, (5) is only a necessary condition for ϕ and ϕ^d to be a dual pair as shown in the following example:

Example 3.
Let

$$\widehat{a}(\omega) = \widehat{a}^d(\omega) = \frac{(1+\exp(-3i\omega))}{2}.$$

Then, a and a^d satisfy (5). However, the corresponding refinable functions

$$\phi(x) = \phi^d(x) = \begin{cases} 1, & 0 \le x < 3; \\ 0, & \text{otherwise} \end{cases}$$

are not stable.

Remark. Consider the low pass filter $\widehat{h} := |\widehat{a}|^2$ where the mask a is defined in the above example. Then, the low pass filter $\widehat{h}$ satisfies (3), while the corresponding refinable function $\phi * \phi(-\cdot)$ of $\widehat{h}$ is not interpolatory. This shows that (2) or (3) is only a necessary condition on the mask whose corresponding refinable function is interpolatory.

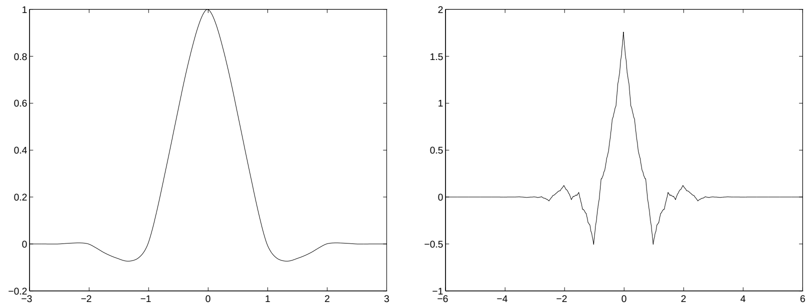
Suppose that ϕ is a compactly supported interpolatory refinable function given by Example 1. Then, it was shown in [9] and [8] that the compactly supported refinable function ϕ^d corresponding the mask a^d defined by (6) is stable and is a dual of ϕ, as long as ϕ^d is continuous. It was further shown in [9] and [8] for each refinable function given in Example 1, the smoothness order of ϕ^d increases as N does. For a given interpolatory refinable function ϕ from Example 1, the remain question is how large N one should pick in (6) to obtain good ϕ^d. The following example from [9] shows that N can be as small as 1.

Example 4.
Let ϕ be the refinable function of mask a given in 4 with $K = 2$ from Example 1. The mask a^d is a dual mask obtained from (6) with $N = 1$. The refinable functions ϕ and ϕ^d are shown in Figure 1.

Let ϕ be the band limited interpolatory refinable function given in Example 2. The next example shows that the dual masks given in (6) generate good dual functions of ϕ.

Example 5.
Let ϕ be the stable interpolatory refinable function given in Example 2 with the corresponding refinement mask a. Let a^d be an arbitrary dual mask derived from (6) for a fixed N and ϕ^d be the corresponding refinable function. Since $\widehat{\phi}^d$ (the Fourier transform of ϕ^d) is continuous and compactly supported, there is a $C < \infty$ such that

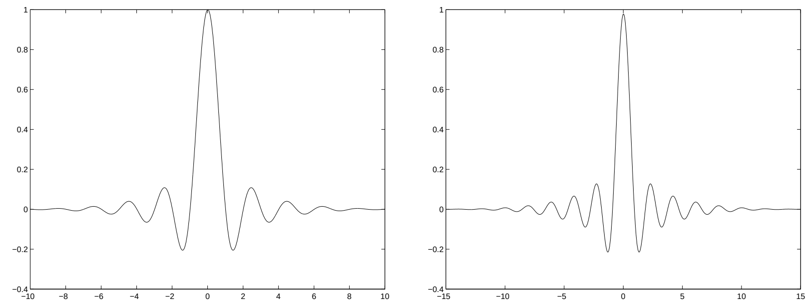
$$\sum_{\alpha\in\mathbb{Z}} |\widehat{\phi}^d(\omega + 2\pi\alpha)|^2 \le C, \quad \omega \in \mathbb{R}.$$



(a) (b)

Fig. 1. (a) Interpolatory refinable functions ϕ and (b) its dual ϕ^d of Example 4

This leads to that $\phi^d \in L_2(\mathbb{R})$ and its shifts form a Bessel system. To prove that ϕ^d is stable, it remains to show ϕ^d is pre-stable. For this, we note that the zero set of $\widehat{a}^d$ is the same as that of $\widehat{a}$. This implies that $\widehat{\phi^d}$ and $\widehat{\phi}$ have the same zero set. Therefore, ϕ^d is pre-stable by the stability of ϕ (see [8]). Altogether, we have that ϕ^d is stable and is a dual function of ϕ. The refinable function ϕ and its dual ϕ^d corresponding to the mask $\widehat{a}^d = \widehat{a}(3 - 2\widehat{a})$ are shown in Figure 2.



(a) (b)

Fig. 2. (a) Interpolatory refinable function ϕ from Example 2 and (b) its dual from Example 5.

For a given pair of the dual refinable functions ϕ and ϕ^d with masks $\widehat{a}$ and $\widehat{a}^d$, the corresponding bi-orthogonal wavelets is defined by

$$\widehat{\psi}(\omega) := \exp(-i\omega/2)\overline{\widehat{a}^d(\omega/2+\pi)}\widehat{\phi}(\omega/2)$$

and

$$\widehat{\psi}^d(\omega) := \exp(-i\omega/2)\overline{\widehat{a}(\omega/2+\pi)}\widehat{\phi}^d(\omega/2).$$

The wavelet masks

$$\widehat{b}(\omega) := \exp(-i\omega)\overline{\widehat{a}^d(\omega+\pi)}; \quad \widehat{b}^d(\omega) := \exp(-i\omega)\overline{\widehat{a}(\omega+\pi)}$$

are a pair of dual wavelet masks. Wavelet masks are also called high pass filters.

It was shown in [15] that as long as the functions

$$R_E(\omega) := \sum_{\alpha\in 2\pi\mathbb{Z}} |\widehat{\psi}(\omega+\alpha)|, \quad R_E^d(\omega) := \sum_{\alpha\in 2\pi\mathbb{Z}} |\widehat{\psi}^d(\omega+\alpha)|,$$

and

$$R_D(\omega) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(2^k\omega)|; \quad R_D^d(\omega) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}^d(2^k\omega)|$$

are in L_∞, the functions

$$2^{k/2}\psi(2^k\cdot -\alpha); \quad k\in\mathbb{Z}, \alpha\in\mathbb{Z}$$

and

$$2^{k/2}\psi^d(2^k\cdot -\alpha); \quad k\in\mathbb{Z}, \alpha\in\mathbb{Z}$$

form a biorthogonal Riesz basis of $L_2(\mathbb{R})$

The conditions R_E and R_E^d in $L_\infty(\mathbb{R})$ are clearly satisfied, since ψ and ψ^d are stable. The conditions R_D and R_D^d in $L_\infty(\mathbb{R})$ will be satisfied, provided the functions $\widehat{\psi}$ and $\widehat{\psi}^d$ have certain decay at the infinity and have a zero of certain order at origin. Both conditions are satisfied in our examples.

The main features of bi-filter scheme are the following: (i) The filters at high resolution levels and low levels are different. At high levels, we use low-high pass filters that have a good localization in the frequency domain. After a coupe of steps, we change them to short support filters. At high level, we use either band limited filters from Examples 2 and its dual filters from Example 5, or very smooth filters from Example 1 and the duals. At low levels, we use the short supported filter and its duals given in Example 4. (ii) The sampling space $S_k(\phi)$ is generated by interpolatory refinable functions.

Next, we analyze how data sequences flow under the bi-filter procedure. Let ϕ and ϕ^d be a pair of stable dual refinable functions with the low-high pass filters a_ϕ, a_ϕ^d, b_ϕ and b_ϕ^d. Let φ, φ^d with low-high filters a_φ, a_φ^d, b_φ, b_φ^d be the another pair. Both ϕ and φ are interpolatory.

Let $f\in L_2(\mathbb{R})$ be a given signal. Choose a sufficiently large k, such that the best approximation f_k^ϕ of f from space $S_k(\phi)$ provides an accurate approximation to f. The function f is identified with f_k^ϕ without much loss. After j step decompositions, we obtain

$$f_{k-j}^\phi(x) = \sum_{\alpha\in\mathbb{Z}} c(\alpha)_{k-j} 2^{(k-j)/2}\phi(2^{k-j}x-\alpha) \in S_{k-j}(\phi)$$

and

$$g_{k-i}^\phi(x) = \sum_{\alpha\in\mathbb{Z}} d(\alpha)_{k-i} 2^{(k-i)/2}\psi_\phi(2^{k-i}x-\alpha), \quad 1\le i\le j,$$

where ψ_ϕ is wavelet function from ϕ. When changing the low-high pass filters to a_φ^d and b_φ^d (say in the decomposition procedure), we essentially get into the wavelet packet decompositions. The details of this wavelet packet analysis can be found in [14].

Changing the low-high filters at a critical point is the only difference between the bi-filter procedure and the normal wavelet decomposition. However, this dose not make any changes in applying the zerotree compression scheme. In fact, the standard zerotree compression scheme ([16]) combined with our bi-filter scheme is used in our numerical examples in the next section. In the reconstruction procedure, the reconstruction filters are changed at the level where the decomposition filters were changed.

The bi-filter scheme gives a clear separation between high and low frequencies at high resolution levels. For texture image, this clear separation provides better compression result as shown in our numerical experiment.

§3. Comparison by Examples

To assess the performance of the proposed bi-filter method, we give a specific set of filters in our implementation as follows.

(1.) Fixed filter method ($D(9/7)$ filters): The well known Daubechies' nine seven filters from [5] are used. The filters are NOT changed as the resolution level changes. The decomposition and reconstruction algorithm together with the zerotree wavelet compression scheme are used here as a benchmark for comparisons with our bi-filter compression scheme. This scheme with $D(9/7)$ filters has been applied to a variety of problems in wavelet literature and proven to be powerful (see e.g., [20]).

(2.) Bi-filter method ($BF(j)$ filters): This is the bi-filter scheme as proposed in Section 2. In this scheme, band limited low pass filters from Example 2 together with its dual filters from Example 5 and the corresponding high pass filters are used at high resolution levels, after j step, they are changed to the dual pair of the low pass filters from Example 4 and the corresponding high pass filters. The number j is either chosen to be 1 or 2.

(3.) Bi-filter method ($LS(j)$ filters): The band limited filters in above (i.e., $BF(j)$ filters) are replaced by long filters whose corresponding refinable functions and wavelets decay fast in the Fourier domain. Long filters are chosen from Example 1 with a suitable K and their duals derived from (6) with $N = 1$.

When $LS(j)$ filters are used, all filter coefficients are of the form $i/2^j$, $i,\, j \in \mathbb{Z}$. Hence the coding scheme suggested by [6] and [2] can be applied.

To evaluate the objective quality of compressed images, the mean squared error, MSE, between the $N \times N$ original image I and the compressed image I_c defined by

$$\text{MSE} = \frac{1}{N^2} \sum_{i,j=0}^{N-1} |I(i,j) - I_c(i,j)|^2$$

is used. The peak signal-to-noise ratio (PSNR) of the compressed image is defined by

$$\text{PSNR} = 10 \log_{10} \frac{255^2}{\text{MSE}}.$$

The performance of the bi-filter scheme is evaluated by comparing the compression results of four original test images "Lena", "Barbara", "Boat", and "Fingerprint" with fixed filter compression scheme using nine seven filters. The signal-to-noise ratio, PSNR, by different methods are tabulated in Table 2.

When the bi-filter compression scheme applying to smooth images as "Lena" and "Boat", we see from Table 2 that the bi-filter scheme is compatible to the fixed filter compression schemes using $D(9/7)$ filters with some improvements in performance. The advantage of the bi-filter compression scheme is clearly shown in Table 2 when applying to texture image as "Barbara" and "Fingerprint". In both cases, clear separation between frequencies at high resolution levels provided by the bi-filter scheme is very useful.

It is important to note that the signal-to-noise ratio, PSNR, is only one summary measure of performance. It does not tell the whole story. It is evident form Figure 3 that bi-filter scheme preserves more details of texture in the compressed image.

§4. Concluding Remarks

We consider a bi-filter decomposition and reconstruction scheme for the compression. Filters with good frequency localization are used at the high resolution levels and they are replaced by good time localization filters at low levels. This provides a means to better adapt to the resolution levels dependent analysis and compression. This also brings out the value of

Image	Filter	Bitrate				
		1	0.5	0.25	0.2	0.125
Lena	D(9/7)	41.01	37.83	34.74	33.76	31.75
	LS(1)	41.02	37.93	34.80	33.80	31.73
	BF(1)	41.02	37.90	34.86	33.83	31.77
	LS(2)	41.04	37.97	34.86	33.85	31.81
	BF(2)	40.98	37.83	34.78	33.78	31.73
Barbara	D(9/7)	37.45	32.11	28.13	27.22	25.38
	LS(1)	37.94	32.42	28.08	27.01	25.16
	BF(1)	38.38	32.98	28.64	27.49	25.19
	LS(2)	38.02	32.54	28.22	27.11	25.21
	BF(2)	38.34	33.09	28.76	27.69	25.37
Boat	D(9/7)	39.11	34.45	30.97	29.97	28.16
	LS(1)	39.13	34.59	31.02	29.90	28.16
	BF(1)	38.98	34.51	30.97	29.91	28.10
	LS(2)	39.13	34.56	30.97	29.88	28.13
	BF(2)	38.82	34.28	30.76	29.75	27.99
Fingerprint	D(9/7)	35.71	31.38	27.77	26.61	24.59
	LS(1)	36.26	31.79	27.80	26.76	24.51
	BF(1)	36.47	32.01	27.79	26.69	24.47
	LS(2)	36.33	31.93	28.07	27.01	24.90
	BF(2)	36.54	32.24	28.38	27.27	25.20

Tab. 2. PSNR values of fixed filter compression scheme using $D(9/7)$, and bi-filter scheme using $LS(1)$, $BF(1)$, $LS(2)$ and $BF(2)$.

wavelets derived from the interpolatory refinable functions. The procedure is computationally simple and stable.

Comparisons are made with the fixed filter method. As we have shown in this paper, the proposed bi-filter method provides somewhat better compression results for texture images. It even shows some improvement for smooth images. But this by no means implies inferiority of other methods. The fixed filter schemes keep the same filters with short support at all resolution levels. Each method of filter design has its distinctive features that are useful for specific purpose. We hope that the present paper would support this line of our reasoning.

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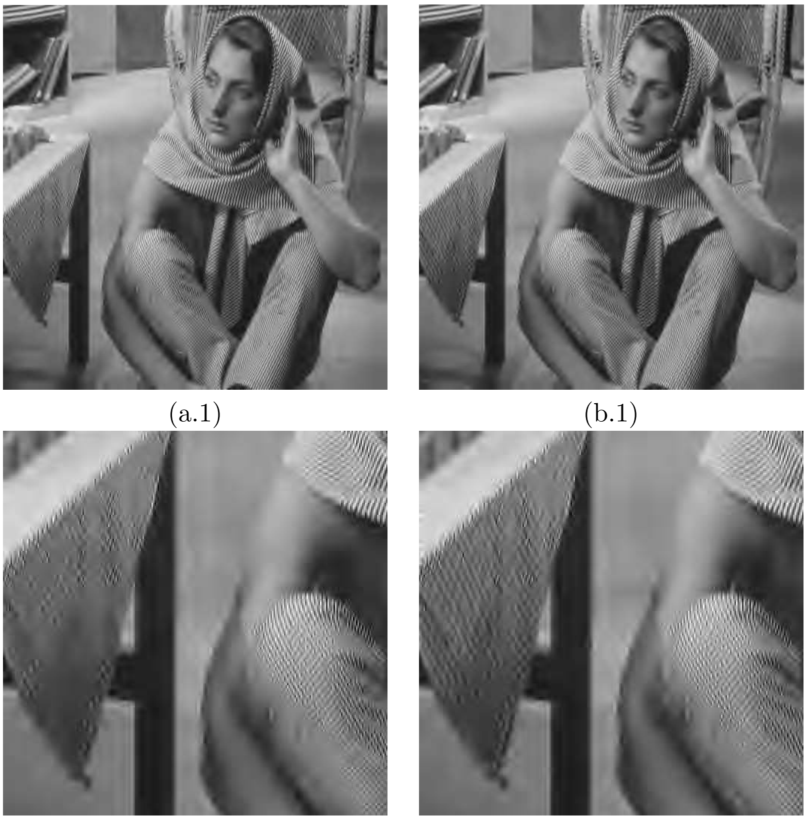

Fig. 3. Compression rate 1:40 (a.1) and (a.2) using fixed filter scheme with $D(9/7)$ filters; (b.1) and (b.2) using $LS(2)$ bi-filter scheme

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