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Deconvolution: A Wavelet Frame Approach

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Abstract This paper devotes to analyzing deconvolution algorithms based on wavelet frame approaches, which has already appeared in [6,8,9] as wavelet frame based high resolution image reconstruction methods. We first give a complete formulation of deconvolution in terms of multiresolution analysis and its approximation, which completes the formulation given in [6,8,9]. This formulation converts deconvolution to a problem of filling the missing coefficients of wavelet frames which satisfy certain minimization properties. These missing coefficients are recovered iteratively together with a built-in denoising scheme that removes noise in the data set such that noise in the data will not blow up while iterating. This approach has already been proven to be efficient in solving various problems in high resolution image reconstructions as shown by the simulation results given in [6,8,9]. However, an analysis of convergence as well as the stability of algorithms and the minimization properties of solutions were absent in those papers. This paper is to establish the theoretical foundation of this wavelet frame approach. In particular, a proof of convergence, an analysis of the stability of algorithms and a study of the minimization property of solutions are given.

Keywords deconvolution · denoising · framelets · quasi-affine system · unitary extension principle

Mathematics Subject Classification (2000) 42C40 · 65T60 · 68U99

1 Introduction

1.1 General

This paper is to construct a solution of the convolution equation

$$\mathbf{h}_0 * \mathbf{v} = \mathbf{b} + \varepsilon := \mathbf{c} \tag{1.1}$$

where $\mathbf{h}_0$ is a low pass filter (i.e. $\sum_{k\in\mathbb{Z}} \mathbf{h}_0[k] = 1$), $\mathbf{b}$, $\mathbf{c}$ and ε are in $\ell_2(\mathbb{Z})$. The sequence ε is the error term satisfying $\|\varepsilon\|_{\ell_2(\mathbb{Z})} \leq \epsilon$.

There are many real life problems which can be modeled by a deconvolution process. For example, measurement devices and signal communication can introduce distortions and add noise to original signals. Inverting the degradation is often modeled by a deconvolution process, i.e. a process of finding a solution in (1.1). In fact, the deconvolution problem is a critical factor in many applications, especially visual-communication related applications including remote sensing, military imaging, surveillance, medical imaging, and high resolution image reconstructions.

Solving equation (1.1) is an inverting process, which is often numerically unstable and thus amplifies the noise considerably. Hence, an efficient process of noise removal must be built in numerical algorithms. The

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earlier formulation of the problem was proposed in [34] using linear algorithm and in [25] and [33] applying the regularization idea to solve a system of linear equations the coefficient matrix of which is ill-conditioned. Since then, there are many papers devoted to this method in the literature. Because this approach is not the focus of this paper, instead of a detailed count, we simply refer readers to [24] and [28] and the references there for a complete reference.

The focus of this paper is to use wavelet (more generally, wavelet frame) to solve (1.1). Recently, there are several papers on solving inverse problems by using wavelet methods, and in particular, deconvolution problems. One of the main ideas is to construct a wavelet or "wavelet inspired" basis that can almost diagonalize the given operator and the underlying solution has a sparse expansion with respect to the chosen basis. The Wavelet-Vaguelette decomposition proposed in [20], [22] and [23] and the deconvolution in mirror wavelet bases in [28] and [29] can be both viewed as examples of succuss of this strategy. Another approach is to apply Galerkin-type methods to inverse problems using an appropriate, but fixed wavelet basis (see e.g. [1] and [16]). The underlying intuition is that if the given operator has a close to sparse representation in wavelets and the solution has a sparse expansion with respect to the wavelet basis, then the inversion is reduced approximately to the inversion of a truncated operator. The method is adaptive, so that the finer-scale wavelets are used where lower-scales indicate the presence of singularities. A few new iterative thresholding algorithms different from the above approaches and developed simultaneously and independently are proposed in [6,8, 9,18,20]. It only requires that the underlying solution has a sparse expansion with respect to a given system without any attempt to "almost diagonalize" or "sparsely representing" the convolution operators.

The main idea of [18,20] is to expand each iteration with respect to the chosen orthonormal basis for a given algorithm such as the Landweber method, then a thresholding algorithm is applied to the coefficients of this expansion. The result is then used to derive the next iteration. The algorithm is shown to converge to the minimizer of certain cost functional.

In the studies of high resolution image reconstructions, the wavelet based (in fact the frame-based) reconstruction algorithms are developed in [5–7], and later [8,9] through the perfect reconstruction formula of a bi-frame or tight frame system which has $\mathbf{h}_0$ as its primary low pass filter. The algorithms approximate iteratively the coefficients of wavelet frame folded by the given low pass filter. By this approach, many available techniques developed in the wavelet literatures, such as wavelet-based denoising schemes, can be built in the iteration. When there are no displacement errors, the high resolution image reconstruction is exactly a deconvolution problem. Here, we extend the algorithms in the papers mentioned above to solve equation (1.1). Algorithm 4.3 is used in the papers mentioned above, in particular in [6,8]. This method has been extended to algorithms for high resolution image reconstructions with displacement errors in [8] and [9]. Algorithm 4.1 is given in [9] as one of the options which is motivated by the approaches taken by [18,20]. Algorithms given in [10,11] are based on Algorithm 4.2 where high resolution images are constructed from a series of video clips. The main ideas of all three algorithms are the same. i.e. an iterative process combined with a denoising scheme applied to each iteration. The differences lay in the different denoising schemes applied to different algorithms which in turn minimize different cost functionals. Finally, we remark that converting a deblurring problem of recovering $\mathbf{v}$ from $\mathbf{c}$ to a problem of inpainting of lost wavelet coefficients can be also found in [12] with a different approach. The interested reader should consult [12] for details.

The convergence analysis of Algorithm 2.1 (the iteration without built-in denoising scheme) has already been established in [6] and [9]. However, the convergence of Algorithm 2.2, 2.3, 4.1, 4.2, 4.3 has not been discussed so far. The current paper aims to build up a complete theory for these algorithms. We will first give a solid and complete formulation of reconstructions of a solution to equation (1.1) in terms of multiresolution analysis and its associated frame system. Then the convergence of all algorithms will be given. A complete analysis of minimization properties, i.e. in which sense the solution derived from the algorithms attains its optimal property, will be given. Finally, the stability of the algorithms is also discussed, which shows that numerical solution approaches to the exact solution when the noise level decreases to zero. As it has already been shown in the papers [6,8–11], algorithms are numerically efficient, easy to implement and adaptive to different applications such as high resolution image reconstructions with displacement errors (see e.g. [8] and [9]). In this paper, a theoretical foundation of the underlying algorithms used in those papers is fully laid out.

The paper is organized as follows: Section 2 is devoted to giving a formulation of the deconvolution problem in terms of multiresolution analysis and its associated tight wavelet frame. Algorithms are also derived from this formulation. Section 3 gives a complete analysis of the algorithms, including the convergence and minimization properties of the algorithms. Section 4 focuses on the finite dimensional data set, i.e. the data set has only finitely many entries. Algorithms 2.2 and 2.3 for infinite dimensional data set can be converted by imposing proper boundary conditions. Since any numerical solution of deconvolution ultimately deals with finite dimensional data sets, such conversion is necessary. As we will see, in many cases, the discussion will be simpler and we are able to obtain better results. Since the numerical implementations and simulations, together with comparison of algorithms provided here with other algorithms, e.g. regularization method, are discussed in details in [6,8,9], and since our focus here is to lay the foundation of the algorithms, we omit the detailed discussions of numerical implementations here. Instead, we give a numerical comparison between Algorithms 4.1, 4.2 and 4.3 in Section 4. Finally, Several proofs are left over to the appendices.

In the remaining part of this section, we give the notation and collect basic results of tight frame system that will be used in this paper. Readers whose main interests are the major part of this paper may only need to briefly go through this part to get familiar with the notations.

1.2 Tight Wavelet Frames

We give here a brief introduction of the tight wavelet frame and its quasi-affine counterpart. The decompositions and reconstructions for the affine tight frame system are known (e.g. [19]); however, the analysis of decomposition and reconstruction of quasi-affine systems is not systematically given. Since these results are crucial for our analysis, we introduce them here in details and leave the proofs to the appendix. At the same time, we set the notations used in this paper.

The space $L_p(\mathbb{R})$ is the set of all the functions $f(x)$ satisfying

$$\|f\|_{L_p(\mathbb{R})} := \begin{cases} (\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x)^{\frac{1}{p}} < \infty, & 1 \le p < \infty; \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty; \end{cases}$$

and $\ell_p(\mathbb{Z})$ is the set of all sequences defined on $\mathbb{Z}$ which satisfy that

$$\|\mathbf{h}\|_{\ell_p(\mathbb{Z})} := \begin{cases} (\sum_{k\in\mathbb{Z}} |\mathbf{h}[k]|^p)^{\frac{1}{p}} < \infty, & 1 \le p < \infty; \\ \sup_{k\in\mathbb{Z}} |\mathbf{h}[k]| < \infty, & p = \infty. \end{cases}$$

The Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined as usual by:

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(x) \mathrm{e}^{-i\omega x} \,\mathrm{d}x, \quad \omega \in \mathbb{R},$$

and its inverse is

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) \mathrm{e}^{i\omega x} \,\mathrm{d}\omega, \quad x \in \mathbb{R}.$$

They can be extended to more general functions, i.e. the functions in $L_2(\mathbb{R})$. Similarly, we can define the Fourier series for a sequence $\mathbf{h} \in \ell_2(\mathbb{Z})$ by

$$\widehat{\mathbf{h}}(\omega) := \sum_{k\in\mathbb{Z}} \mathbf{h}[k] e^{-ik\omega}, \quad \omega \in \mathbb{R}.$$

For any function $f \in L_2(\mathbb{R})$, the dyadic dilation operator D is defined by $Df(x) := \sqrt{2} f(2x)$ and the translation operator T is defined by $T_a f(x) := f(x-a)$ for $a \in \mathbb{R}$. Given $j \in \mathbb{Z}$, we have $T_a D^j = D^j T_{2^j a}$. Further, a space V is said to be integer-shift invariant if given any function $f \in V$, $T_j f \in V$ for $j \in \mathbb{Z}$.

A system $X \subset L_2(\mathbb{R})$ is called a tight frame of $L_2(\mathbb{R})$ if

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{g\in X} |\langle f, g\rangle|^2,$$

holds for all $f \in L_2(\mathbb{R})$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\mathbb{R})$ and $\|\cdot\|_{L_2(\mathbb{R})} = \sqrt{\langle\cdot,\cdot\rangle}$. This is equivalent to

$$f = \sum_{g\in X} \langle f, g\rangle g, \ f \in L_2(\mathbb{R}).$$

It is clear that an orthonormal basis is a tight frame.

For given $\Psi := \{\psi_1, \ldots, \psi_r\} \subset L_2(\mathbb{R})$, define the affine system

$$X(\Psi) := \{\psi_{\ell,j,k} : 1 \le \ell \le r;\ j,k \in \mathbb{Z}\},$$

where $\psi_{\ell,j,k} = D^j T_k \psi_\ell = 2^{j/2}\psi_\ell(2^j \cdot -k)$. When $X(\Psi)$ forms an orthonormal basis of $L_2(\mathbb{R})$, then ψ_ℓ, $\ell = 1,\ldots,r$, are called the orthonormal wavelets. When $X(\Psi)$ forms a tight frame of $L_2(\mathbb{R})$, then it is called a tight wavelet frame and ψ_ℓ, $\ell = 1,\ldots,r$, are called the tight framelets.

The tight framelets can be constructed by the unitary extension principle (UEP) given in [31], which uses the multiresolution analysis (MRA). The MRA starts from a refinable function ϕ. A compactly supported function ϕ is refinable if it satisfies a refinement equation

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} \mathbf{h}_0[k]\phi(2x-k), \tag{1.2}$$

for some sequence $\mathbf{h}_0 \in \ell_2(\mathbb{Z})$. By the Fourier transform, the refinable equation (1.2) can be given as

$$\widehat{\phi}(\omega) = \widehat{\mathbf{h}_0}(\omega/2)\widehat{\phi}(\omega/2), \quad \text{a.e.} \quad \omega \in \mathbb{R}.$$

We call the sequence $\mathbf{h}_0$ the refinement mask of ϕ and $\widehat{\mathbf{h}_0}(\omega)$ the refinement symbol of ϕ.

For given finitely supported $\mathbf{h}_0$ with $\widehat{\mathbf{h}_0}(0) = 1$, the refinement equation (1.2) always has distribution solution which can be written in the Fourier domain as

$$\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} \widehat{\mathbf{h}_0}(2^{-j}\omega), \quad \text{a.e.} \quad \omega \in \mathbb{R}.$$

In this paper, we require $\mathbf{h}_0$ being finitely supported. Then the corresponding refinable function ϕ satisfies that

$$\operatorname{ess\,sup}_{\omega\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\widehat{\phi}(\omega+2k\pi)|^2 < \infty, \tag{1.3}$$

whenever $\phi \in L_2(\mathbb{R})$ (see [27]).

For a compactly supported refinable function $\phi \in L_2(\mathbb{R})$, let V_0 be the closed shift invariant space generated by $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ and $V_j := \{f(2^j\cdot) : f \in V_0\}$, $j \in \mathbb{Z}$. It is known that when ϕ is compactly supported, then $\{V_j\}_{j\in\mathbb{Z}}$ forms a multiresolution analysis. Recall that a multiresolution analysis is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$ that satisfies: (i) $V_j \subset V_{j+1}$, (ii) $\bigcup_j V_j$ is dense in $L_2(\mathbb{R})$, and (iii) $\bigcap_j V_j = \{0\}$ (see [2] and [26]).

For given MRA of nested spaces V_j, $j \in \mathbb{Z}$ with the underlying refinable function ϕ and the refinement mask $\mathbf{h}_0$, it is well known that (e.g. see [2]) for any $\psi \in V_1$, there exists a 2π periodic function ϑ, such that

$$\widehat{\psi}(2\cdot) = \vartheta\widehat{\phi}.$$

Let $\Psi := \{\psi_1,\ldots,\psi_r\} \subset V_1$, then

$$\widehat{\psi_\ell}(2\cdot) = \widehat{\mathbf{h}_\ell}\widehat{\phi}, \quad \ell = 1,\ldots,r, \tag{1.4}$$

where $\widehat{\mathbf{h}_1},\ldots,\widehat{\mathbf{h}_r}$ are 2π periodic functions and are called framelet symbols. In the time domain, (1.4) can be written as

$$\psi_\ell(x) = 2\sum_{k\in\mathbb{Z}} \mathbf{h}_\ell[k]\phi(2x-k). \tag{1.5}$$

We call $\mathbf{h}_1,\ldots,\mathbf{h}_r$ framelet masks. We also call the refinement mask $\mathbf{h}_0$ the low pass filter and $\mathbf{h}_1,\ldots,\mathbf{h}_r$ the high pass filters of the system. The UEP says when Ψ becomes a set of tight framelets with $X(\Psi)$ being a tight frame of $L_2(\mathbb{R})$.

Theorem 1.1 (Unitary Extension Principle, [31]) *Let $\phi \in L_2(\mathbb{R})$ be the refinable function with refinement mask $\mathbf{h}_0$ satisfying $\widehat{\mathbf{h}_0}(0) = 1$ that generates an MRA $\{V_j\}_{j\in\mathbb{Z}}$. Let $(\mathbf{h}_1,\ldots,\mathbf{h}_r)$ be a set of sequences with $\left(\widehat{\mathbf{h}_1},\ldots,\widehat{\mathbf{h}_r}\right)$ being a set of 2π-periodic measurable functions in $L_\infty[0,2\pi]$. If the equalities*

$$\sum_{\ell=0}^{r} |\widehat{\mathbf{h}_\ell}(\omega)|^2 = 1 \quad \textit{and} \quad \sum_{\ell=0}^{r} \widehat{\mathbf{h}_\ell}(\omega)\overline{\widehat{\mathbf{h}_\ell}(\omega+\pi)} = 0 \tag{1.6}$$

hold for almost all $\omega \in [-\pi,\pi]$, then the system $X(\Psi)$ where $\Psi = \{\psi_1,\ldots,\psi_r\}$ defined in (1.5) *by $(\mathbf{h}_1,\ldots,\mathbf{h}_r)$ and ϕ forms a tight frame in $L_2(\mathbb{R})$.*

We will use (1.6) in terms of sequences $\mathbf{h}_0,\dots,\mathbf{h}_r$. The first condition $\sum_{\ell=0}^{r}|\widehat{\mathbf{h}}_\ell(\omega)|^2=1$ in terms of corresponding sequences is

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}\overline{\mathbf{h}_\ell[k]}\mathbf{h}_\ell[k-p]=\delta_{0,p},\quad p\in\mathbb{Z}, \tag{1.7}$$

where $\delta_{0,p}=1$ when $p=0$ and 0 otherwise. The second condition $\sum_{\ell=0}^{r}\widehat{\mathbf{h}}_\ell(\omega)\overline{\widehat{\mathbf{h}}_\ell(\omega+\pi)}=0$ can be written as

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}(-1)^{k-p}\overline{\mathbf{h}_\ell[k]}\mathbf{h}_\ell[k-p]=0,\quad p\in\mathbb{Z}. \tag{1.8}$$

With the UEP, the construction of tight framelets become painless. For example, one can construct tight framelets from spline easily. Next, we give two examples of spline tight framelets.

Example 1.1 Let $\mathbf{h}_0=[\frac{1}{4},\frac{1}{2},\frac{1}{4}]$ be the refinement mask of the piecewise linear function $\phi(x)=\max(1-|x|,0)$. Define $\mathbf{h}_1=[-\frac{1}{4},\frac{1}{2},-\frac{1}{4}]$ and $\mathbf{h}_2=[\frac{\sqrt{2}}{4},0,-\frac{\sqrt{2}}{4}]$. Then $\widehat{\mathbf{h}_0}$, $\widehat{\mathbf{h}_1}$ and $\widehat{\mathbf{h}_2}$ satisfy (1.6). Hence, the system $X(\Psi)$ where $\Psi=\{\psi_1,\psi_2\}$ defined in (1.5) by using $\mathbf{h}_1$, $\mathbf{h}_2$ and ϕ is a tight frame of $L_2(\mathbb{R})$. This is the first example constructed via the UEP in [31].

Example 1.2 Let $\mathbf{h}_0=[\frac{1}{16},\frac{1}{4},\frac{3}{8},\frac{1}{4},\frac{1}{16}]$ be the refinement mask of ϕ. Then ϕ is the piecewise cubic B-spline. Define $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$, $\mathbf{h}_4$ as follows:

$$\begin{aligned}\mathbf{h}_1&=[\tfrac{1}{16},-\tfrac{1}{4},\tfrac{3}{8},-\tfrac{1}{4},\tfrac{1}{16}],\quad \mathbf{h}_2=[-\tfrac{1}{8},\tfrac{1}{4},0,-\tfrac{1}{4},\tfrac{1}{8}],\\ \mathbf{h}_3&=[\tfrac{\sqrt{6}}{16},0,-\tfrac{\sqrt{6}}{8},0,\tfrac{\sqrt{6}}{16}],\quad \mathbf{h}_4=[-\tfrac{1}{8},-\tfrac{1}{4},0,\tfrac{1}{4},\tfrac{1}{8}].\end{aligned}$$

Then $\widehat{\mathbf{h}_0},\widehat{\mathbf{h}_1},\widehat{\mathbf{h}_2},\widehat{\mathbf{h}_3},\widehat{\mathbf{h}_4}$ satisfy (1.6) and hence the system $X(\Psi)$ where $\Psi=\{\psi_1,\psi_2,\psi_3,\psi_4\}$ defined in (1.5) by $\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3,\mathbf{h}_4$ and ϕ is a tight frame of $L_2(\mathbb{R})$. This is also first constructed in [31].

The deconvolution processing has to be formulated by quasi-affine systems that were first introduced in [31]. A quasi-affine system from level J is defined as

Definition 1.1 Let $\Psi=\{\psi_1,\dots,\psi_r\}$ be a set of functions. A quasi-affine system from level J is defined as

$$X_J^q(\Psi)=\{\psi_{\ell,j,k}^q:\ 1\le\ell\le r;\ j,k\in\mathbb{Z}\},$$

where $\psi_{\ell,j,k}^q$ is defined by

$$\psi_{\ell,j,k}^q:=\begin{cases}D^jT_k\psi_\ell, & j\ge J;\\ 2^{\frac{j-J}{2}}T_{2^{-J}k}D^j\psi_\ell, & j<J.\end{cases}$$

The quasi-affine system is obtained by over sampling the affine system. More precisely, we over sample the affine system starting from level $J-1$ and downward to a 2^{-J}-shift invariant system. Hence, the whole quasi-affine system is a 2^{-J}-shift invariant system. The quasi-affine system from level 0 was first introduced in [31] to convert a non-shift invariant affine system to a shift invariant system. Further, it was shown in [31, Theorem 5.5] that the affine system $X(\Psi)$ is a tight frame of $L_2(\mathbb{R})$ if and only if the quasi-affine counterpart $X_J^q(\Psi)$ is a tight frame of $L_2(\mathbb{R})$.

In our analysis, we use the quasi-interpolatory operator. Let $\{V_j\}$, $j\in\mathbb{Z}$ be a given MRA with underlying refinable function ϕ and $\Psi=\{\psi_1,\dots,\psi_r\}$ be the set of corresponding tight framelets derived from the UEP. The quasi-interpolatory operator in the affine system $X(\Psi)$ generated by Ψ is defined, for $f\in L_2(\mathbb{R})$,

$$P_j:\ f\mapsto\sum_{k\in\mathbb{Z}}\langle f,\phi_{j,k}\rangle\phi_{j,k}.$$

It is clear that $P_jf\in V_j$. As shown in [19, Lemma 2.4], this quasi-interpolatory operator is the same as *truncated representation*

$$Q_j:\ f\mapsto\sum_{\ell=1}^{r}\sum_{j'<j,k\in\mathbb{Z}}\langle f,\psi_{\ell,j',k}\rangle\psi_{\ell,j',k}.$$

Furthermore, a standard framelet decomposition given in [19] says that

$$P_{j+1}f = P_j f + \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi_{\ell,j,k}\rangle \psi_{\ell,j,k} \quad \text{and} \quad P_j f = Q_j f. \tag{1.9}$$

When we consider the MRA based quasi-affine system $X_J^q(\Psi)$ generated by Ψ, the spaces $V_j,\ j<J$ in the MRA for the affine system are replaced by $V_j^{q,J},\ j<J$, for the quasi-affine system. Note that the space V_j is spanned by functions $\phi_{j,k}$, while the space $V_j^{q,J}$ is spanned by functions $\phi_{j,k}^q$, where $\phi_{j,k}^q$ is defined by

$$\phi_{j,k}^q := \begin{cases} D^j T_k \phi, & j \ge J; \\ 2^{\frac{j-J}{2}} T_{2^{-J}k} D^j \phi, & j < J. \end{cases}$$

The spaces $V_j^{q,J},\ j<J$ are 2^{-J}-shift invariant. We define the quasi-interpolatory operator $P_j^{q,J}$ and the truncated operator $Q_j^{q,J},\ j\in\mathbb{Z}$, for the quasi-affine system by

$$P_j^{q,J}:\ f \mapsto \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}^q\rangle \phi_{j,k}^q \tag{1.10}$$

and

$$Q_j^{q,J}:\ f \mapsto \sum_{\ell=1}^{r}\sum_{j'<j,k\in\mathbb{Z}} \langle f, \psi_{\ell,j',k}^q\rangle \psi_{\ell,j',k}^q. \tag{1.11}$$

The quasi-interpolatory operator $P_j^{q,J}$ maps $f\in L_2(\mathbb{R})$ to $V_j^{q,J}$. From the definition of $\phi_{j,k}^q$, we can see that $P_j^{q,J} = P_j$ when $j \ge J$ and these two operators are different only when $j<J$. Moreover, since for an arbitrary $f\in L_2(\mathbb{R})$ and $j<J$,

$$P_j^{q,J} f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}^q\rangle \phi_{j,k}^q = D^J \sum_{k\in\mathbb{Z}} \langle D^{-J} f, 2^{\frac{j-J-0}{2}} T_k D^{j-J}\phi\rangle 2^{\frac{j-J-0}{2}} T_k D^{j-J}\phi = D^J P_{j-J}^{q,0} D^{-J} f,$$

one only needs to understand the case $J=0$. In this case we simplify our notation by setting

$$P_j^q := P_j^{q,0}, \quad Q_j^q := Q_j^{q,0} \tag{1.12}$$

for the quasi-interpolatory operators and $V_j^q := V_j^{q,0}$, for $j\in\mathbb{Z}$. From now on, we only give the properties for P_j^q and corresponding spaces V_j^q and the associated quasi-affine system $X^q(\Psi) := X_0^q(\Psi)$. The corresponding results for the over sampling rate of $2^{-J}\mathbb{Z}$ can be obtained similarly.

For operator $P_j^q,\ j\in\mathbb{Z}$, we also have the decomposition and reconstruction formula similar to (1.9).

Lemma 1.1 *Let $X(\Psi)$, where the framelets $\Psi = \{\psi_1,\ldots,\psi_r\}$, be the affine tight frame system obtained from $\mathbf{h}_0$ and ϕ via the UEP and $X^q(\Psi)$ be the quasi-affine frame derived from $X(\Psi)$. Then we have*

$$P_{j+1}^q f = P_j^q f + \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi_{\ell,j,k}^q\rangle \psi_{\ell,j,k}^q, \quad f\in L_2(\mathbb{R}). \tag{1.13}$$

More general, it was proven in [19, Lemma 2.4] that the identity $P_j f = Q_j f$ holds for all $f\in L_2(\mathbb{R})$. Next result shows that a similar result also holds for the quasi-affine systems.

Proposition 1.1 *Let $X(\Psi)$ with $\Psi = \{\psi_1,\ldots,\psi_r\}$ be the affine tight frame system obtained from $\mathbf{h}_0$ and ϕ via the UEP and $X^q(\Psi)$ be the corresponding quasi-affine frame. Then we have $P_j^q f = Q_j^q f$ for all $f\in L_2(\mathbb{R})$.*

We postpone the proof to Appendix A.

1.3 Discrete Form

The identity (1.13) essentially gives the decomposition and reconstruction of a function in quasi-affine tight frame systems. In the implementation, one needs a completely discrete form of the decomposition and reconstruction and we give such form below.

We introduce the Toeplitz matrix to describe the discrete form of the decomposition and reconstruction procedure. Given a sequence $\mathbf{h}_0 = \{\mathbf{h}_0[k]\}_{k\in\mathbb{Z}}$, the Toeplitz matrix generated by $\mathbf{h}_0$ is a matrix satisfying

$$H_0 = (H_0[l,k]) = (\mathbf{h}_0[l-k]),$$

where the (l,k)th entry in H_0 is fully determined by the $(l-k)$th entry in $\mathbf{h}_0$. The Toeplitz matrix is also called the convolution matrix since it can be viewed as the matrix representation of linear time invariant filter which can be written as convolution. Hence the convolution of two sequences can be expressed in terms of matrix vector multiplication, i.e.

$$\mathbf{h}_0 * \mathbf{v} = H_0\mathbf{v}. \tag{1.14}$$

In the following, we will denote the Toeplitz matrix generated from $\mathbf{h}_0$ by $H_0 = \text{Toeplitz}(\mathbf{h}_0)$. Let H_ℓ denote the infinite dimensional Toeplitz matrix $\text{Toeplitz}(\mathbf{h}_\ell)$ for $\ell = 1,\dots,r$. Using the matrix notation, the UEP condition (1.7) can be written as

$$H_0^*H_0 + H_1^*H_1 + \cdots + H_r^*H_r = I, \tag{1.15}$$

where I is the identity operator. To write the decomposition and reconstruction in convolution form, the filters used in decomposition below the 0th level need to be dilated. In level $j<0$, the dilated filter is denoted by $\mathbf{h}_{\ell,j}$, which is defined by (also see (A.1))

$$\mathbf{h}_{\ell,j}[k] = \begin{cases} \mathbf{h}_\ell[2^{j+1}k], & k \in 2^{-j-1}\mathbb{Z}; \\ 0, & k \in \mathbb{Z}\setminus 2^{-j-1}\mathbb{Z}. \end{cases} \tag{1.16}$$

The corresponding Toeplitz matrix is

$$H_{\ell,j} = \text{Toeplitz}(\mathbf{h}_{\ell,j}). \tag{1.17}$$

By the definition of $\mathbf{h}_{\ell,j}$, we have $\widehat{\mathbf{h}_{\ell,j}} = \widehat{\mathbf{h}}_\ell(2^{-j-1}\cdot)$ and hence $|\widehat{\mathbf{h}_{\ell,j}}| \le 1$ a.e. $\omega\in\mathbb{R}$. Moreover, as a byproduct in the proof of Lemma 1.1, we have a condition similar to (1.15) for dilated filters $\mathbf{h}_{0,j},\dots,\mathbf{h}_{r,j},\ j<0$:

$$H_{0,j}^*H_{0,j} + H_{1,j}^*H_{1,j} + \cdots + H_{r,j}^*H_{r,j} = I. \tag{1.18}$$

We can see that when $j=-1$, (1.15) and (1.18) are the same.

The discrete forms of decomposition and reconstruction from level j_1 to level j_2, where $j_1, j_2 \ge 0$, are the same as those in the affine system, which are given in [19]. We only consider the discrete form of decomposition and reconstruction from level j_1 to level j_2, where $j_1, j_2 < 0$. For a function $f\in L_2(\mathbb{R})$, we decompose f in $X^q(\Psi)$ and collect the coefficients in each level $j<0$ to form an infinite column vector

$$\mathbf{v}_{\ell,j} := [\dots,\langle f,\psi^q_{\ell,j,k}\rangle,\dots]^t,$$

where $\psi_0^q := \phi^q$ and $[\cdots]^t$ is the transpose of a row vector. Set the Toeplitz block matrix

$$\mathscr{H}_j := \left[H_{0,j}, H_{1,j},\dots,H_{r,j}\right]^t.$$

With this, condition (1.18) implies $\mathscr{H}_j^*\mathscr{H}_j = I$. The decomposition process (1.13) can be written in the matrix form as:

$$\mathbf{v}_{\ell,j} = H_{\ell,j}\mathbf{v}_{0,j+1}, \quad \ell = 0,\dots,r,$$

or

$$\left[\mathbf{v}_{0,j},\dots,\mathbf{v}_{r,j}\right]^t = \mathscr{H}_j\mathbf{v}_{0,j+1}. \tag{1.19}$$

Because of (1.18), the reconstruction process of Lemma 1.1 can be interpreted in the discrete form as

$$\begin{aligned} \mathbf{v}_{0,j+1} &= \mathscr{H}_j^*\mathscr{H}_j\mathbf{v}_{0,j+1} \\ &= H_{0,j}^*H_{0,j}\mathbf{v}_{0,j+1} + H_{1,j}^*H_{1,j}\mathbf{v}_{0,j+1} + \cdots + H_{r,j}^*H_{r,j}\mathbf{v}_{0,j+1} \\ &= H_{0,j}^*\mathbf{v}_{0,j} + H_{1,j}^*\mathbf{v}_{1,j} + \cdots + H_{r,j}^*\mathbf{v}_{r,j}. \end{aligned} \tag{1.20}$$

The identities (1.19) and (1.20) together give the equivalent discrete representation of (1.13).

The above discussion essentially is one level decomposition and reconstruction. Next, we introduce the notation of several to infinite levels decomposition and reconstruction. For any sequence $\mathbf{v}$, it is decomposed by $\mathscr{H}_{-1}\mathbf{v}$ first, then the low frequency component $H_0\mathbf{v}$ is further decomposed by the same procedure. The same process goes inductively. To describe this discrete process, we define the decomposition operator $\mathscr{A}_J$, $J<0$ and $\mathscr{A}$. They are composed of matrix block like $H_{\ell,j}\prod_{j'=j}^{-1}H_{0,j'}$ where $\prod_{j'=j}^{-1}H_{0,j'}$ is the composition of $|j|$ Toeplitz matrices $H_{0,j'}$, $j\le j'\le -1$, given by (A.1) and (1.17), and acts on any sequence $\mathbf{v}\in\ell_2(\mathbb{Z})$ in the following order:

$$\prod_{j'=j}^{-1}H_{0,j'}\mathbf{v}=H_{0,j}H_{0,j+1}\cdots H_{0,-1}\mathbf{v}.$$

The decomposition operator $\mathscr{A}_J$ is a (rectangular) block matrix defined as:

$$[(\prod_{j=J}^{-1}H_{0,j});(H_{1,J}\prod_{j=J+1}^{-1}H_{0,j});\ldots;(H_{r,J}\prod_{j=J+1}^{-1}H_{0,j});\ldots;\ H_{1,-1};\ldots;H_{r,-1}]^t \tag{1.21}$$

and $\mathscr{A}$ is defined as

$$\begin{aligned}&[\ldots;(H_{1,J-1}\prod_{j=J-1}^{-1}H_{0,j});\ldots;(H_{r,J-1}\prod_{j=J-1}^{-1}H_{0,j});\ (H_{1,J}\prod_{j=J}^{-1}H_{0,j});\ldots;\ (H_{r,J}\prod_{j=J}^{-1}H_{0,j});\\&(H_{1,J+1}\prod_{j=J+1}^{-1}H_{0,j});\ldots;(H_{r,J+1}\prod_{j=J+1}^{-1}H_{0,j});\ldots;\ H_{1,-1};\ldots;H_{r,-1}]^t.\end{aligned} \tag{1.22}$$

In (1.21) and (1.22), $H_{\ell,-1}=H_\ell$, $\ell=0,1,\ldots,r$ and thus $\mathscr{A}_{-1}=\mathscr{H}_{-1}$.

As we will see that both $\mathscr{A}_J$ and $\mathscr{A}$ are the operators defined on $\ell_2(\mathbb{Z})$ into the tensor product space

$$\bigotimes_{\ell=0,j=1}^{r,|J|}\ell_2^{\ell,j}(\mathbb{Z})\quad\text{and}\quad\bigotimes_{\ell=0,j=1}^{r,\infty}\ell_2^{\ell,j}(\mathbb{Z})$$

respectively, with $\ell_2^{\ell,j}(\mathbb{Z})=\ell_2(\mathbb{Z})$. The reconstruction operators

$$\mathscr{A}_J^*=[(\prod_{j=-1}^{J}H_{0,j}^*);(\prod_{j=-1}^{J+1}H_{0,j}^*H_{1,J}^*);\ldots;(\prod_{j=-1}^{J+1}H_{0,j}^*H_{r,J}^*);\ldots;\ H_{1,-1}^*;\ldots;H_{r,-1}^*] \tag{1.23}$$

and

$$\begin{aligned}\mathscr{A}^*=[&\ldots;(\prod_{j=-1}^{J-1}H_{0,j}^*H_{1,J-1}^*);\ldots;(\prod_{j=-1}^{J-1}H_{0,j}^*H_{r,J-1}^*);(\prod_{j=-1}^{J}H_{0,j}^*H_{1,J}^*);\ \ldots;(\prod_{j=-1}^{J}H_{0,j}^*H_{r,J}^*);\\&(\prod_{j=-1}^{J+1}H_{0,j}^*H_{1,J+1}^*);\ldots;(\prod_{j=-1}^{J+1}H_{0,j}^*H_{r,J+1}^*);\ldots;H_{1,-1}^*;\ldots;H_{r,-1}^*]\end{aligned} \tag{1.24}$$

are the adjoint operators of $\mathscr{A}_J$ and $\mathscr{A}$ respectively.

The operators $\mathscr{A}_J$ and $\mathscr{A}$ are closely related to P_0 and Q_0^q. By Lemma 1.1 we have the identity

$$P_0f=P_J^qf+\sum_{\ell=1}^{r}\sum_{j=J}^{-1}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j,k}^q\rangle,\quad J<0.$$

The corresponding coefficients in the right hand side is $\mathscr{A}_J\mathbf{v}_{0,0}$ with $\mathbf{v}_{0,0}=\{\langle f,\phi_{0,k}\rangle\}$. Similarly, the coefficients in the right hand side of the identity used in analysis

$$P_0f=Q_0^qf$$

can be obtained by $\mathscr{A}\mathbf{v}_{0,0}$. Furthermore, the next proposition shows that the decomposition and reconstruction process is perfect, i.e. $\mathscr{A}_J^*\mathscr{A}_J=I$ and $\mathscr{A}^*\mathscr{A}=I$, which will be proven in Appendix A.

Proposition 1.2 *The decomposition operators $\mathscr{A}_J$ and $\mathscr{A}$, as defined in* (1.21) *and* (1.22) *respectively, satisfy $\mathscr{A}_J^*\mathscr{A}_J=I$ and $\mathscr{A}^*\mathscr{A}=I$ where I is the identity operator.*

2 Formulation and Algorithms

This section is to formulate the deconvolution problem via the multiresolution analysis and the framelet analysis. It converts the deconvolution problem to the problem of filling the missing framelet coefficients. Consider the convolution equation

$$\mathbf{h}_0 * \mathbf{v} = \mathbf{b} + \varepsilon = \mathbf{c}, \tag{2.1}$$

where $\mathbf{h}_0$ is a finitely supported low pass filter and $\mathbf{b}$, $\mathbf{c}$ are the sequences in $\ell_2(\mathbb{Z})$. The error term $\varepsilon \in \ell_2(\mathbb{Z})$ satisfies $\|\varepsilon\|_{\ell_2(\mathbb{Z})} \leq \varepsilon$. To simplify our notation, we use $\|\cdot\| := \|\cdot\|_{\ell_2(\mathbb{Z})}$.

Our approach starts with the refinable function generated by the low pass filter $\mathbf{h}_0$. There are many sufficient conditions on the low pass filter $\mathbf{h}_0$ with $\widehat{\mathbf{h}}_0(0) = 1$ under which ϕ is in $L_2(\mathbb{R})$. Here we assume that $\mathbf{h}_0$ satisfies the following condition

$$|\widehat{\mathbf{h}}_0(\omega)|^2 + |\widehat{\mathbf{h}}_0(\omega+\pi)|^2 \leq 1, \quad \text{a.e.} \quad \omega \in \mathbb{R}. \tag{2.2}$$

As we will show in Appendix A, the corresponding refinable function ϕ is in $L_2(\mathbb{R})$ under assumption (2.2). We further remark that this is not a strong assumption. For example, all refinement masks of B-splines, the refinable functions whose shifts form an orthonormal system derived in [17], the base functions of interpolatory functions, and more general, pseudo-splines introduced by [19] and [21] satisfy this assumption. In fact, many low pass filters used in practical problems satisfy (2.2). For example, the low pass filters used in high resolution image reconstructions satisfy (2.2). Furthermore, with this assumption, we can construct a tight frame system via unitary extension principle of [31] which is used in our algorithm.

To make our ideas work here, the crucial step is to construct a tight frame or a bi-frame system via a multiresolution analysis with underlying refinement mask being the given low pass filter. The assumption (2.2) is a necessary and sufficient condition to construct a tight frame system associated with the given low pass filter. However, assumption (2.2) is not crucial for our idea to work. For example, when the underlying refinable function ϕ is in $L_2(\mathbb{R})$, whose refinement mask is the given low pass filter in (2.1), together with some additional minor conditions, we can always obtain a bi-frame system via the mixed unitary extension principle of [32] and more generally the mixed oblique extension principle of [15] and [19]. For example, let

$$h_0(z) := \sum_{k\in\mathbb{Z}} \mathbf{h}_0[k] z^{-k}.$$

Then (2.2) can be replaced by the condition that $h_0(z)$ and $h_0(-z)$ have no common zeros in complex domain. With this, one can construct a bi-frame system by using the mixed unitary extension principle. This is essentially the approach taken by [6]. Our analysis can be carried out for this case with some efforts. To simplify our discussion here, we only use the tight frame system, hence assume (2.2).

Finally, since our approach is based on denoising schemes with threshold of framelet coefficients, we implicitly assume that the underlying function of the data set has a sparse representation in the tight frame system and the errors are relatively small and spread out uniformly in the frame transform domain.

2.1 Formulation in MRA

This section is to formulate the problem of solving

$$\mathbf{h}_0 * \mathbf{v} = \mathbf{b} + \varepsilon = \mathbf{c} \tag{2.3}$$

via the multiresolution analysis framework. As we will see, the approach here reduces solving equation (2.3) to the problem of filling the missing framelet coefficients. This approach was first taken by [6], however, we give a complete analysis and formulation here.

As we mentioned before, by using

$$P_J f = D^J P_0 D^{-J} f \quad \text{and} \quad P^{q,J}_{J-1} f = D^J P^q_{-1} D^{-J} f,$$

we may assume that data set is given on $\mathbb{Z}$ (i.e. $J = 0$) without loss of generality. In fact, when the data set is given on $2^{-J}\mathbb{Z}$, we consider function $f(2^{-J}\cdot)$ instead of f. The approximation power of a function f in space V_J is the same as that of the function $f(2^{-J}\cdot)$ in space V_0.

Let $\phi \in L_2(\mathbb{R})$ be the refinable function with refinement mask $\mathbf{h}_0$ and $\mathbf{h}_1, \ldots, \mathbf{h}_r$ be high pass filters obtained via the UEP which are the framelet masks of $\psi_1, \ldots, \psi_r$. First we suppose that the given data set contains no error, i.e. $\varepsilon = \mathbf{0}$. The convolution equation $\mathbf{h}_0 * \mathbf{v} = \mathbf{b}$ implies that $\mathbf{b}$ is obtained by passing the original sequence $\mathbf{v}$ through a low pass filter $\mathbf{h}_0$. Assume that $\mathbf{b} = \{\langle S, \phi^q_{-1,k}\rangle\}$, where $S \in L_2(\mathbb{R})$ is the underlying function from which the data set $\mathbf{b}$ is obtained. Then we are given

$$P^q_{-1}S = \sum_{k\in\mathbb{Z}} \langle S, \phi^q_{-1,k}\rangle \phi^q_{-1,k} = \sum_{k\in\mathbb{Z}} \mathbf{b}[k]\phi^q_{-1,k}. \tag{2.4}$$

Let $\mathbf{v}^S = \{\langle S, \phi_{0,k}\rangle\}$, then

$$P_0 S = \sum_{k\in\mathbb{Z}} \langle S, \phi_{0,k}\rangle \phi_{0,k} = \sum_{k\in\mathbb{Z}} \mathbf{v}^S[k]\phi_{0,k}. \tag{2.5}$$

Applying the framelet decomposition algorithm (1.13), one obtains that $\mathbf{h}_0 * \mathbf{v}^S = \mathbf{b}$. This implies that solving equation (2.3) is equivalent to reconstructing the quasi-interpolation $P_0 S \in V_0$ from the quasi-interpolation $P^q_{-1}S \in V^q_{-1}$. Since

$$P_0 S = P^q_{-1}S + \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle S, \psi^q_{\ell,-1,k}\rangle \psi^q_{\ell,-1,k},$$

to recover $\mathbf{v}^S = \{\langle S, \phi_{0,k}\rangle\}$ from given $\mathbf{b}$, we need the framelet coefficients $\{\langle S, \psi^q_{\ell,-1,k}\rangle\}$. This leads to an algorithm that restores $\mathbf{v}^S$ from data $\mathbf{b}$ iteratively by updating the framelet coefficients $\{\langle S, \psi^q_{\ell,-1,k}\rangle\}$ in each iteration. All these have been given in [6] and consequent papers [8,9] in their reconstructions of high resolution images. In fact, it motivates the algorithms developed in [6,8,9].

By this approach, we not only give a solution of (2.3), but also give an interpretation in terms of the underlying function S where we view the data $\mathbf{b} = \{\langle S, \phi^q_{-1,k}\rangle\}$ as the given sample of S. Under this setting, we are given $P^q_{-1}S \in V^q_{-1}$, and the solution of (2.3) leads to $P_0 S \in V_0$, which is a higher resolution subspace in the multiresolution analysis. Although there are more than one function whose quasi-interpolations are $P^q_{-1}S$ and $P_0 S$ given as (2.4) and (2.5), we never get the underlying function S. One can only expect to obtain a better approximation $P_0 S$ of S from $P^q_{-1}S$. The approximation power of $P_0 S$ and $P^q_{-1}S$ and their difference can be established for smooth functions by applying the corresponding results in [19] which depend on the properties of the underlying refinable function; more general for piecewise smooth functions, it can be studied by applying results and ideas from [3] and [4] which depend on the properties of the framelets. We omit the detailed discussion here.

Roughly speaking, the idea of solving equation (2.3) here can be understood as for a given coarse level approximation $P^q_{-1}S$ to find a finer level approximation $P_0 S$ is reduced to finding the coefficients $\mathbf{v}^S = \{\langle S, \phi_{0,k}\rangle\}$. The derivation of $\mathbf{v}^S$ is an iterative process which recovers $P_0 S$ from $P^q_{-1}S$ as discussed before and detailed in the algorithms given in the next section. Then $\mathbf{h}_0 * \mathbf{v}^S = \mathbf{b}$ by the decomposition algorithm (1.13) and we conclude that $\mathbf{v}^S$ is a solution of (2.3).

However, the data given may contain errors, i.e. instead of $\mathbf{b}$, the data is given in the form of $\mathbf{c} = \mathbf{b} + \varepsilon$. Furthermore, the given data set $\mathbf{b}$ may not be necessary of the form of $\{\langle S, \phi^q_{-1,k}\rangle\}$, for some $S \in L_2(\mathbb{R})$. In both cases, the exact $\ell_2(\mathbb{Z})$ solution of $\mathbf{h}_0 * \mathbf{v} = \mathbf{c}$ may not exist or it may not be desirable or possible to get the exact solution.

Nevertheless, there is a need to have

$$\tilde{s} = \sum_{\ell=1}^{r}\sum_{j<0,k\in\mathbb{Z}} \tilde{s}_{\ell,j,k}\psi^q_{\ell,j,k} \in V_0$$

to approximate the underlying function where the sample data set $\mathbf{c}$ comes from. Let

$$\tilde{\mathbf{s}} = \{\tilde{s}_{\ell,j,k}\} \quad \text{and} \quad \mathbf{s} = \mathscr{A}^*\tilde{\mathbf{s}}, \tag{2.6}$$

where $\mathscr{A}^*$ is the reconstruction operator given in (1.24). For the vector $\mathbf{s}$ being a candidate of the solution of (2.3), it requires $\mathbf{h}_0 * \mathbf{s}$ within the ε ball of $\mathbf{c}$ and the function $\tilde{s}$ has some smoothness. The smoothness of the function is reflected by the decay of the framelet coefficients which is measured by the ℓ_p norm of $\tilde{\mathbf{s}}$. Given any sequence $\tilde{\mathbf{v}}$ determined by three indices (ℓ, j, k) with $\ell = 1, \ldots, r$, $j < 0$ and $k \in \mathbb{Z}$, we say $\tilde{\mathbf{v}}$ is in space ℓ_p, for a given p, if $\sum_{\ell=1}^{r}\sum_{j<0,k\in\mathbb{Z}} |\tilde{v}_{\ell,j,k}|^p < \infty$.

Assuming that there exists a function S such that $\tilde{s}_{\ell,j,k} = \langle S, \psi^q_{\ell,j,k}\rangle$, then function $\tilde{s} = Q^q_0 S$. For a given p, $1 \le p \le 2$, we say that the pair $(\mathbf{s}, \tilde{\mathbf{s}})$ defined in (2.6) is the solution of (2.3) (and $\tilde{s}$ is an approximation of the underlying function of the data set) if for an arbitrary $g \in L_2(\mathbb{R})$, the pair $(\mathbf{g}, \tilde{\mathbf{g}})$ where $\tilde{\mathbf{g}} = \{\langle g, \psi^q_{\ell,j,k}\rangle\}$ and $\mathbf{g} = \mathscr{A}^*\tilde{\mathbf{g}}$, satisfies the following inequality

$$\|\mathbf{h}_0 * \mathbf{g} - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\langle g, \psi^q_{\ell,j,k}\rangle|^p \ge \|\mathbf{h}_0 * \mathbf{s} - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\langle S, \psi^q_{\ell,j,k}\rangle|^p. \quad (2.7)$$

Here $\gamma \le \lambda_j \le \gamma'$, $j \in \mathbb{Z}$, where $0 < \gamma \le \gamma' \le \infty$, are parameters which will be determined by the error level.

The function $\tilde{s}$ is considered as an approximation of the underlying function whose sample is given by $\mathbf{c}$. The first term measures the residue of the solution $\mathbf{s}$ and the given data set $\mathbf{c}$. The second term is a penalization term using a weighted (with weights λ_j) ℓ_p-norm of the coefficients of framelets. Since the framelet coefficients are closely related to the smoothness of the underlying function (see [3,4]), minimization problem (2.7) balances the fitness of the solution and the smoothness of the solution function $\tilde{s}$.

The minimization condition (2.7) can be stated as following: for a fixed p, $1 \le p \le 2$, the pair $(\mathbf{s}, \tilde{\mathbf{s}})$ defined in (2.6) is a solution of (2.3) (the function $\tilde{s}$ is an approximation of the underlying function of the data) if for an arbitrary $\eta \in L_2(\mathbb{R})$ satisfying $\{\langle \eta, \psi^q_{\ell,j,k}\rangle\} = \{\tilde{\eta}_{\ell,j,k}\} = \tilde{\eta} \in \ell_p$, the pair $(\eta, \tilde{\eta})$, where $\eta = \mathscr{A}^*\tilde{\eta}$, satisfies the following inequality

$$\|\mathbf{h}_0 * (\mathbf{s} + \eta) - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p \ge \|\mathbf{h}_0 * \mathbf{s} - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}_{\ell,j,k}|^p. \quad (2.8)$$

However, as we will see that the sequence $\tilde{\mathbf{s}}$ is uniquely determined by algorithms, it may not be of the form $\{\langle S, \psi^q_{\ell,j,k}\rangle\}$ for any $S \in L_2(\mathbb{R})$, since $\{\psi^q_{\ell,j,k}\}_{j<0}$ is redundant which implies that the representation of $\tilde{s}$ is not unique. Nevertheless, the pair $(\mathbf{s}, \tilde{\mathbf{s}})$ still can be considered as a solution of equation (2.3) if (2.8) holds for an arbitrary pair $(\eta, \tilde{\eta})$, where $\tilde{\eta} = \{\tilde{\eta}_{\ell,j,k}\} = \{\langle \eta, \psi^q_{\ell,j,k}\rangle\} \in \ell_p$, and $\eta = \mathscr{A}^*\tilde{\eta}$ with $\eta \in L_2(\mathbb{R})$. Here, we note that since $\tilde{\eta} = \{\langle \eta, \psi^q_{\ell,j,k}\rangle\}$, $\eta = \mathscr{A}^*\tilde{\eta}$ implies that $\tilde{\eta} = \mathscr{A}\eta$ by the decomposition algorithm.

The function $\tilde{s}$ enters into the discussion that gives an analysis in the function form of the underlying solution. The underlying function $\tilde{s}$ plays a role in analysis, but does not enter the algorithm. Next, we link the formulation to a discrete form of minimization problem (2.8). The minimization problem (2.8) can be stated as follows: for a given p, $1 \le p \le 2$, a pair of sequences $(\mathbf{s}, \tilde{\mathbf{s}})$, satisfying $\tilde{\mathbf{s}} \in \ell_p$ and $\mathbf{s} = \mathscr{A}^*\tilde{\mathbf{s}}$, is the solution of (2.3) if for an arbitrary pair $(\eta, \tilde{\eta})$ satisfying $\tilde{\eta} = \mathscr{A}\eta \in \ell_p$, the following inequality holds:

$$\|\mathbf{h}_0 * (\mathbf{s} + \eta) - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p \ge \|\mathbf{h}_0 * \mathbf{s} - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}_{\ell,j,k}|^p. \quad (2.9)$$

We note that (2.8) and (2.9) look similar, but they are derived in a different setting. For example, sequences in (2.8) are derived from the analysis sequences of functions under the given wavelet frame system, while sequences (2.9) are obtained in a 'purely' discrete sense via filters of the given wavelet frame system. We should also remark here the condition $\mathbf{s} = \mathscr{A}^*\tilde{\mathbf{s}}$ on the pair $(\mathbf{s}, \tilde{\mathbf{s}})$ is different from the condition $\tilde{\eta} = \mathscr{A}\eta$ on the pair $(\eta, \tilde{\eta})$. The condition $\tilde{\eta} = \mathscr{A}\eta$ implies $\eta = \mathscr{A}^*\tilde{\eta}$, since $\mathscr{A}^*\tilde{\eta} = \mathscr{A}^*\mathscr{A}\eta = \eta$ by $\mathscr{A}^*\mathscr{A} = I$. However, the condition $\mathbf{s} = \mathscr{A}^*\tilde{\mathbf{s}}$, in general, does not implies $\tilde{\mathbf{s}} = \mathscr{A}\mathbf{s}$, unless $\mathscr{A}\mathscr{A}^* = I$ or $\tilde{\mathbf{s}}$ happens to be $\mathscr{A}\mathbf{s}$. Note that the identity $\mathscr{A}\mathscr{A}^* = I$ does not hold for any redundant system. The reasons for imposing the different conditions are due to that $(\mathbf{s}, \tilde{\mathbf{s}})$ is obtained by the algorithm which only satisfies $\mathbf{s} = \mathscr{A}^*\tilde{\mathbf{s}}$, while for given η, there is more than one $\tilde{\eta}$ such that $\mathscr{A}^*\tilde{\eta} = \eta$. We choose the canonical pair $(\eta, \tilde{\eta})$ with $\tilde{\eta} = \mathscr{A}\eta$.

2.2 Algorithms

We give algorithms to solve (2.3) with the formulation in MRA. In our approach, the algorithms iteratively improve the framelet coefficients using the result in previous iteration. Let $\mathbf{h}_1, \ldots, \mathbf{h}_r$ be the sequences derived from $\mathbf{h}_0$ via the UEP and $H_0, H_1, \ldots, H_r$ be the corresponding Toeplitz matrices. Our algorithm based on the UEP condition

$$H_0^* H_0 + \sum_{\ell=1}^{r} H_\ell^* H_\ell = I. \quad (2.10)$$

Let $\mathbf{v}_n$ be the solution for the nth iteration, then

$$H_0^* H_0 \mathbf{v}_n + \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{v}_n = \mathbf{v}_n. \tag{2.11}$$

First, we consider the case that $\mathbf{b} = \{\langle S, \phi_{-1,k}^q \rangle\}$, where S is the underlying function and $\mathbf{b}$ is the given data as a set of the samples of S, and $\varepsilon = \mathbf{0}$. Then by $\mathbf{h}_0 * \mathbf{v}^S = \mathbf{b}$ with $\mathbf{v}^S = \{\langle S, \phi_{0,k} \rangle\}$, we have $\mathbf{v}^S$ is a solution to equation (2.3). In each iteration, we can replace $H_0 \mathbf{v}_n$ by the known data $\mathbf{b}$ to improve the approximation. This can also be viewed as that we use the framelet coefficients of the n^{th} iteration to approximate the framelet coefficients of the underlying function S. We summarize the algorithm as follows:

Algorithm 2.1

(i) Choose an initial approximation $\mathbf{v}_0$ (e.g. $\mathbf{v}_0 = \mathbf{b}$);
(ii) Iterate on n until convergence:

$$\mathbf{v}_{n+1} = H_0^* \mathbf{b} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{v}_n. \tag{2.12}$$

As we will see in the next section, Algorithm 2.1 converges, but it converges slowly. We need to adjust the iteration in Algorithm 2.1 to quicken the convergence, which motivates us to introduce the acceleration factor $0 < \beta < 1$ into the above algorithm. The new iteration with β is given below:

$$\mathbf{v}_{n+1} = \beta (H_0^* \mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{v}_n) = H_0^* \beta \mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \beta \mathbf{v}_n. \tag{2.13}$$

This scheme can be viewed as the traditional regularization method used in noise removal, the solution of which satisfies the matrix equation

$$\Big(H_0^* H_0 + (1-\beta) \sum_{\ell=1}^{r} H_\ell^* H_\ell \Big) \mathbf{v} = H_0^* \beta \mathbf{c}.$$

Here β is a regularization parameter. The solution of the original convolution equation (2.3) is $\mathbf{v} = \mathbf{v}^\beta / \beta$ with $\mathbf{v}^\beta$ the solution to the above matrix equation. The solution $\mathbf{v}$ minimizes the following functional:

$$\| H_0 \mathbf{v} - \mathbf{c} \|^2 + \frac{1-\beta}{\beta} \| \mathbf{v} \|^2.$$

This is the standard regularization form with a special regularization operator, which was more or less the [6, Algorithm 2] given to us. The parameter β has to be carefully chosen to balance the error and smoothness of the solution. It plays a role in both convergence acceleration and error removal. However, when a different penalty functional instead of ℓ_2 norm of the solution (e.g. the one given in the formulation), which is desirable in many applications, is used, we need a different approach. In our new algorithms, the acceleration factor β is mainly used to accelerate the convergence and leave the "regularization" part to a threshold process. Finally, we remark that, as will see in §4, in the numerical implementation, when proper boundary conditions (e.g. periodic boundary condition with some modifications) are used, the matrix H_0 becomes a nonsingular finite order matrix. The iteration in Algorithm 2.1 converges with rate $1 - \lambda$, where λ is the minimum eigenvalue of $H_0^* H_0$. Hence, we do not need to introduce the acceleration factor β.

Next, we introduce the following denoising operators to the iteration (2.13).

Denoising Operator When data are contaminated with errors, we need to remove the errors from each iteration before putting it into the next iteration. The denoising scheme is needed to prevent the limit of iteration (2.13) from following the noise residing in $\mathbf{c}$. For any vector $\mathbf{v}$ and given p, $1 \le p \le 2$, let threshold operator be

$$\mathscr{D}_\lambda^p(\mathbf{v}) := \left[t_\lambda^p(\mathbf{v}[0]), t_\lambda^p(\mathbf{v}[1]), \ldots \right]^t, \tag{2.14}$$

where $t_\lambda^p(x)$ is the threshold function. When $p = 1$, $t_\lambda(x) := t_\lambda^1(x) = \operatorname{sgn}(x) \max(|x| - \lambda/2, 0)$ is the soft-threshold function; when $1 < p \le 2$, the threshold function is defined by the inverse of function

$$\mathrm{F}_\lambda^p(x) := x + \frac{p\lambda}{2} \operatorname{sgn}(x) |x|^{p-1}. \tag{2.15}$$

Function $\mathrm{F}_\lambda^p(x)$ is a one-to-one differentiable function with unique inverse. For $1 < p \le 2$, the explicit formula of the inverse of function F_λ^p is not always available. Numerical method may be needed to calculate the value of $t_\lambda^p(x) := (\mathrm{F}_\lambda^p)^{-1}(x)$. As we will see, the difference of the threshold operators $\mathscr{D}_\lambda^p$ according to different p is that the limit of the algorithm has different minimization properties.

When a signal $\mathbf{v}$ is given, the normal procedure is first transforming $\mathbf{v}$ to the framelet domain via the decomposition operator $\mathscr{A}$ to decorrelate the signal, and then applying the threshold operator $\mathscr{D}_{\lambda_j}^p$ with the threshold parameter λ_j depending on the decomposition level j. For a given sequence $\mathbf{v} \in \ell_2(\mathbb{Z})$, the denoising operator $\mathscr{T}^p$ which applies the threshold operator $\mathscr{D}_{\lambda_j}^p$ on $\mathscr{A}\mathbf{v}$ with the threshold parameters $\{\lambda_j\}$ is defined as:

$$\mathscr{T}^p \mathscr{A}(\mathbf{v}) = [\mathscr{D}_{\lambda_j}^p (H_{\ell,j} \prod_{j'=j+1}^{-1} H_{0,j'} \mathbf{v})]_{\ell,j}^t, \tag{2.16}$$

where $1 \le p \le 2,\ \ell = 1,2,\ldots,r,\ j < 0$. This noise removal scheme will then be applied at each iteration before applying the next iteration in Algorithm 2.1.

Algorithm 2.2 is given in [9] which was motivated by [18]. At the nth step, the threshold operator is applied to the framelet decomposition of $H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_n$. The parameters λ_j are fixed during the iteration.

Algorithm 2.2

(i) Choose an initial approximation $\mathbf{v}_0$ (e.g. $\mathbf{v}_0 = \mathbf{c}$);
(ii) Iterate on n until convergence:

$$\mathbf{v}_{n+1} = \mathscr{A}^* \mathscr{T}^p \mathscr{A}(H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_n); \tag{2.17}$$

(iii) Suppose the limit of step (ii) is $\mathbf{v}^\beta$. Then the final solution is

$$\mathbf{s}^\beta = \mathbf{v}^\beta / \beta.$$

We will prove that the pair $(\mathbf{s}^\beta, \tilde{\mathbf{s}}^\beta)$ where $\tilde{\mathbf{s}}^\beta = \frac{1}{\beta} \mathscr{T}^p \mathscr{A}(H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}^\beta)$ obtained from step (iii) of Algorithm 2.2 satisfies inequality (2.9) (up to an arbitrary small ε). Next algorithm has a different denoising scheme from Algorithm 2.2. Instead of applying the denoising operator to each iteration before it is put in the next iteration, the denoising operator only acts on the approximation of the missing framelet coefficients. This is the process suggested by [6,8,9].

Algorithm 2.3

(i) Choose an initial approximation $\mathbf{v}_0$ (e.g. $\mathbf{v}_0 = \mathbf{c}$);
(ii) Iterate on n until convergence:

$$\mathbf{v}_{n+1} = H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* (\mathscr{A}^* \mathscr{T}^p \mathscr{A})(\beta H_\ell \mathbf{v}_n); \tag{2.18}$$

(iii) Let $\mathbf{v}^\beta$ be the final iterative solution from (ii). Then the solution to the algorithm is

$$\mathbf{s}^\beta = \mathbf{v}^\beta / \beta.$$

For better denoising effect, we may apply the denoising scheme to the final result $\mathbf{s}^\beta$, i.e. we take an additional step

(iv) $\upsilon = \mathscr{A}^* \mathscr{T}^p \mathscr{A}(\mathbf{s}^\beta)$

to further remove the error effect arose by $\mathbf{c}$, which is used in [6,8,9].

3 Analysis of Algorithms

This section focuses on the analysis of the algorithms given in §2.2. We first show that all algorithms converge. Secondly, we prove that the solutions of Algorithm 2.2 and 2.3 satisfy some minimization property.

3.1 Convergence

In this section, we will show the convergence of Algorithm 2.1, 2.2 and 2.3. The proof of the convergence of Algorithm 2.1 was given in [6] and [9]. We include the proof here for the sake of the self completeness of the paper. However, the convergence of Algorithm 2.2 and 2.3 is new. This is important, since both algorithms are the ones used in practice.

Proposition 3.1 *Let* $\mathbf{h}_1,\ldots,\mathbf{h}_r$ *be the high pass filters of a tight frame system derived by the UEP with finitely supported* $\mathbf{h}_0$ *being the given low pass filter which satisfies* (2.2). *Suppose there exists a function S such that* $\mathbf{c}=\{\langle S,\phi^q_{-1,k}\rangle\}$. *Then for arbitrary* $\mathbf{v}_0\in\ell_2(\mathbb{Z})$, *the sequence* $\mathbf{v}_n$ *defined by* (2.12) *converges to* $\mathbf{v}=\{\langle S,\phi^q_{0,k}\rangle\}$. *Especially,* $\mathbf{h}_0*\mathbf{v}=\mathbf{c}$.

Proof The proof was given in [6]. Writing (2.12) in frequency domain, one obtains

$$\widehat{\mathbf{v}}_{n+1}=\overline{\widehat{\mathbf{h}}_0}\widehat{\mathbf{c}}+\sum_{\ell=1}^{r}\overline{\widehat{\mathbf{h}}_\ell}\widehat{\mathbf{h}}_\ell\widehat{\mathbf{v}}_n.$$

Let $\mathbf{v}=\{\langle S,\phi_{0,k}\rangle\}$. Since $\mathbf{c}=\{\langle S,\phi^q_{-1,k}\rangle\}$, $\mathbf{v}$ is the solution to (2.3). Using the UEP condition, we have

$$\widehat{\mathbf{v}}=\overline{\widehat{\mathbf{h}}_0}\widehat{\mathbf{c}}+\sum_{\ell=1}^{r}\overline{\widehat{\mathbf{h}}_\ell}\widehat{\mathbf{h}}_\ell\widehat{\mathbf{v}}.$$

For arbitrary $\mathbf{v}_0\in\ell_2(\mathbb{Z})$, applying the iteration n times, we have

$$\widehat{\mathbf{v}}_n-\widehat{\mathbf{v}}=\left(\sum_{\ell=1}^{r}\overline{\widehat{\mathbf{h}}_\ell}\widehat{\mathbf{h}}_\ell\right)^n(\widehat{\mathbf{v}}_0-\widehat{\mathbf{v}}).$$

From (2.2), we have $0\le|\widehat{\mathbf{h}}_0(\omega)|\le 1$ a.e. $\omega\in\mathbb{R}$ and $|\widehat{\mathbf{h}}_0(\omega)|=0$ only holds on a zero measure set since $\widehat{\mathbf{h}}_0(\omega)$ is a polynomial the zero points of which are finite. Because $\mathbf{h}_1,\ldots,\mathbf{h}_r$ satisfy (1.6), it follows that

$$\sum_{\ell=1}^{r}|\widehat{\mathbf{h}}_\ell(\omega)|^2\le 1,\quad \text{a.e.}\quad \omega\in\mathbb{R}$$

and the equality only holds on a zero measure set. Thus we have $|\widehat{\mathbf{v}}_n-\widehat{\mathbf{v}}|\le|\widehat{\mathbf{v}}_0-\widehat{\mathbf{v}}|$ and $\widehat{\mathbf{v}}_n-\widehat{\mathbf{v}}\to 0$ a.e. $\omega\in\mathbb{R}$ as $n\to\infty$. Then by Dominated Convergence Theorem, $\|\mathbf{v}_n-\mathbf{v}\|_{\ell_2(\mathbb{Z})}=\frac{1}{\sqrt{2\pi}}\|\widehat{\mathbf{v}}_n-\widehat{\mathbf{v}}\|_{L_2[-\pi,\pi]}\to 0$, i.e. $\mathbf{v}_n$ converges to $\mathbf{v}$ as $n\to\infty$. □

Since $|\sum_{\ell=1}^r\overline{\widehat{\mathbf{h}}_\ell}\widehat{\mathbf{h}}_\ell|=1$ at π, the convergence of the algorithm is slow. That is the reason why we introduce the acceleration factor β into iteration. The convergence of iteration (2.13) can be proved similarly. Next we show the convergence of the iterations in Algorithm 2.2 and Algorithm 2.3. The following lemma is needed, the proof of which is given in [18, Lemma 2.2].

Proposition 3.2 *The denoising operator* $\mathscr{D}^p_\lambda$, $1\le p\le 2$ *is non-expansive, i.e. for any two sequences* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *in* $\ell_2(\mathbb{Z})$,

$$\|\mathscr{D}^p_\lambda(\mathbf{v}_1)-\mathscr{D}^p_\lambda(\mathbf{v}_2)\|\le\|\mathbf{v}_1-\mathbf{v}_2\|.$$

Furthermore, let $\mathscr{T}^p$ *be the denoising operator defined by* (2.16), *it also satisfies that*

$$\|\mathscr{T}^p\mathscr{A}(\mathbf{v}_1)-\mathscr{T}^p\mathscr{A}(\mathbf{v}_2)\|\le\|\mathbf{v}_1-\mathbf{v}_2\|.$$

In particular, $\mathscr{T}^p\mathscr{A}$ *is continuous and* $\|\mathscr{T}^p\mathscr{A}\mathbf{v}_1\|\le\|\mathbf{v}_1\|$.

Now we are ready to show the convergence of Algorithm 2.2.

Theorem 3.1 *Let* $\mathbf{h}_1,\ldots,\mathbf{h}_r$ *be the high pass filters of a tight frame system derived by the UEP with* $\mathbf{h}_0$ *being the given low pass filter which satisfies* (2.2). *Then the sequence* $\mathbf{v}_n$ *defined by* (2.17) *in Algorithm 2.2 converges for arbitrary initial seed* $\mathbf{v}_0\in\ell_2(\mathbb{Z})$ *to* $\mathbf{v}^\beta$ *which satisfies*

$$\mathbf{v}^\beta=\mathscr{A}^*\mathscr{T}^p\mathscr{A}(H_0^*\beta\mathbf{c}+\sum_{\ell=1}^{r}H_\ell^*H_\ell\beta\mathbf{v}^\beta).\tag{3.1}$$

Proof The idea of the proof is to show that the sequence $\{\mathbf{v}_n\}$ is a Cauchy sequence. We first note that $\|\mathscr{A}^*\| \le 1$. Let

$$\mathbf{v}_n = \mathscr{A}^* \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_{n-1})$$

and for $m > 0$

$$\mathbf{v}_{n+m} = \mathscr{A}^* \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_{n+m-1}).$$

For convenience, denote $\mathbf{u} = H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_{n-1}$ and $\mathbf{u}' = H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}_{n+m-1}$. Then using Proposition 3.2 we have:

$$\|\mathbf{v}_{n+m} - \mathbf{v}_n\| = \|\mathscr{A}^*(\mathscr{T}^p \mathscr{A}\mathbf{u}' - \mathscr{T}^p \mathscr{A}\mathbf{u})\| \le \|\mathscr{T}^p \mathscr{A}\mathbf{u}' - \mathscr{T}^p \mathscr{A}\mathbf{u}\| \le \|\mathbf{u}' - \mathbf{u}\| \le \beta \|\mathbf{v}_{n+m-1} - \mathbf{v}_{n-1}\|.$$

Inductively, we finally obtain that

$$\|\mathbf{v}_{n+m} - \mathbf{v}_n\| \le \beta^n \|\mathbf{v}_m - \mathbf{v}_0\|. \tag{3.2}$$

Then sequence $\{\mathbf{v}_n\}$ is a Cauchy sequence if $\{\mathbf{v}_n\}$ is bounded. Since $0 < \beta < 1$, indeed due to Proposition 3.2 we have

$$\|\mathbf{v}_n\| = \|\mathscr{A}^* \mathscr{T}^p \mathscr{A}\mathbf{u}\| \le \|\mathscr{T}^p \mathscr{A}\mathbf{u}\| \le \|\mathbf{u}\| \le \beta\|\mathbf{c}\| + \beta\|\mathbf{v}_{n-1}\| \le \frac{\beta}{1-\beta}\|\mathbf{c}\| + \|\mathbf{v}_0\|. \tag{3.3}$$

Hence the limit of the iteration (2.17) exists. The limit $\mathbf{v}^\beta$ satisfying $\mathbf{v}^\beta = \mathscr{A}^* \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}^\beta)$ follows the continuity of $\mathscr{T}^p \mathscr{A}$. □

Here we note that the limit $\mathbf{v}^\beta$ of iteration (2.17) satisfies (3.1). Let $\tilde{\mathbf{v}}^\beta = \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}^\beta)$, then the pair $(\mathbf{v}^\beta, \tilde{\mathbf{v}}^\beta)$ satisfies $\mathbf{v}^\beta = \mathscr{A}^* \tilde{\mathbf{v}}^\beta$. As a consequence, the pair $(\mathbf{s}^\beta, \tilde{\mathbf{s}}^\beta)$ with

$$\mathbf{s}^\beta = \frac{1}{\beta}\mathbf{v}^\beta \quad \text{and} \quad \tilde{\mathbf{s}}^\beta = \frac{1}{\beta} \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}^\beta)$$

also satisfies $\mathbf{s}^\beta = \mathscr{A}^* \tilde{\mathbf{s}}^\beta$. We will prove in the next subsection that the pair $(\mathbf{s}^\beta, \tilde{\mathbf{s}}^\beta)$ satisfies the inequality (2.9) up to a small $\varepsilon > 0$ when β is close to 1.

A similar proof shows the convergence of iteration (2.18) in Algorithm 2.3 as stated below.

Theorem 3.2 *Let $\mathbf{h}_1, \dots, \mathbf{h}_r$ be the high pass filters of a tight frame system derived by the UEP with $\mathbf{h}_0$ being the given low pass filter which satisfies* (2.2). *Then the sequence $\mathbf{v}_n$ defined by* (2.18) *in Algorithm 2.3 converges for arbitrary initial seed $\mathbf{v}_0 \in \ell_2(\mathbb{Z})$ to $\mathbf{v}^\beta$ which satisfies*

$$\mathbf{v}^\beta = H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* \mathscr{A}^* \mathscr{T}^p \mathscr{A} (H_\ell \beta \mathbf{v}^\beta). \tag{3.4}$$

3.2 Minimization Property of Algorithm 2.2

In this section, we discuss to what extend that the solution $\mathbf{s}^\beta$ obtained from Algorithm 2.2 satisfies (2.9). Without further clarification, we fixed $p \in [1,2]$ in the following discussion.

By Algorithm 2.2,

$$\mathbf{s}^\beta = \frac{1}{\beta}\mathbf{v}^\beta \quad \text{and} \quad \tilde{\mathbf{s}}^\beta = \frac{1}{\beta}\tilde{\mathbf{v}}^\beta,$$

where

$$\tilde{\mathbf{v}}^\beta = \mathscr{T}^p \mathscr{A} (H_0^* \beta \mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \beta \mathbf{v}^\beta) \quad \text{and} \quad \mathbf{v}^\beta = \mathscr{A}^* \tilde{\mathbf{v}}^\beta \tag{3.5}$$

are obtained from the limit of iteration (2.17). First, if $\tilde{\mathbf{s}}^\beta \notin \ell_p$, then for any pair $(\eta, \tilde{\eta})$ with $\tilde{\eta} = \mathscr{A}\eta \in \ell_p$, the values of both sides in (2.9) are infinite and the inequality holds. For the case $\tilde{\mathbf{s}}^\beta \in \ell_p$, what we will prove is a slightly weaker result than (2.9) for the pair $(\mathbf{s}^\beta, \tilde{\mathbf{s}}^\beta)$ as stated below.

For the given constant $C > 0$ and arbitrary $\varepsilon > 0$, the pair $(\mathbf{s}^\beta, \tilde{\mathbf{s}}^\beta)$ satisfies the following inequality

$$\|\mathbf{h}_0 * (\mathbf{s}^\beta + \eta) - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}^\beta_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p \geq \|\mathbf{h}_0 * \mathbf{s}^\beta - \mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \lambda_j |\tilde{s}^\beta_{\ell,j,k}|^p - \varepsilon, \tag{3.6}$$

for any pair $(\eta, \tilde{\eta})$ satisfying $\tilde{\eta} = \mathscr{A}\eta \in \ell_p$ with $\|\eta\| \leq C$, as long as the acceleration factor β is close enough to 1.

As we will see in the next section, when certain boundary conditions are imposed in numerical implementations, the solution will satisfy (2.9).

We first prove the following statement: for given $\varepsilon > 0$ and $C > 0$, the pair $(\mathbf{v}^\beta, \tilde{\mathbf{v}}^\beta)$ given in (3.5) satisfies the following inequality

$$\begin{aligned} &\|H_0(\mathbf{v}^\beta + \eta) - \beta\mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |\tilde{v}^\beta_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p + (1-\beta)^2 \sum_{\ell=1}^{r} \|H_\ell(\mathbf{v}^\beta + \eta)\|^2 \\ &\quad \geq \|H_0\mathbf{v}^\beta - \beta\mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |\tilde{v}^\beta_{\ell,j,k}|^p + (1-\beta)^2 \sum_{\ell=1}^{r} \|H_\ell\mathbf{v}^\beta\|^2 - \varepsilon, \end{aligned} \tag{3.7}$$

whenever the pair $(\eta, \tilde{\eta})$ satisfying $\tilde{\eta} = \mathscr{A}\eta \in \ell_p$ with $\|\eta\| \leq C$ and the acceleration factor β is close enough to 1. Note that the threshold parameters $\beta^{2-p}\lambda_j$ are smaller than those in (3.6). It is reasonable because the use of acceleration factor β helps to damp out the noise residing in $\mathbf{c}$.

To show (3.7), we introduce the following functionals. For a given pair of sequences $(\mathbf{v}, \tilde{\mathbf{v}})$ satisfying $\mathbf{v} = \mathscr{A}^*\tilde{\mathbf{v}}$ and a sequence $\mathbf{a}$, define

$$\Phi(\mathbf{v}) := \|H_0\mathbf{v} - \beta\mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |\tilde{v}_{\ell,j,k}|^p + (1-\beta)^2 \sum_{\ell=1}^{r} \|H_\ell\mathbf{v}\|^2 \tag{3.8}$$

and

$$\tilde{\Phi}(\mathbf{v};\mathbf{a}) := \|H_0\mathbf{v} - \beta\mathbf{c}\|^2 + \sum_{\ell=1}^{r} \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |\tilde{v}_{\ell,j,k}|^p + \sum_{\ell=1}^{r} \|H_\ell(\mathbf{v} - \beta\mathbf{a})\|^2. \tag{3.9}$$

It is clear that when $\mathbf{a} = \mathbf{v}$, we have $\tilde{\Phi}(\mathbf{v};\mathbf{v}) = \Phi(\mathbf{v})$. Furthermore, the following result on $\tilde{\Phi}(\mathbf{v};\mathbf{a})$ holds.

Proposition 3.3 *Suppose $\tilde{\mathbf{c}} = \mathscr{A}\mathbf{c}$ is in ℓ_p. Let $\mathbf{h}_1, \ldots, \mathbf{h}_r$ be the high pass filters obtained from $\mathbf{h}_0$ by the UEP and $H_0, H_1, \ldots, H_r$ be the corresponding matrix counterparts of these filters as defined in* (1.14). *Given a pair $(\mathbf{a}, \tilde{\mathbf{a}})$ satisfying $\mathbf{a} = \mathscr{A}^*\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}} \in \ell_p$, let*

$$\tilde{\mathbf{v}}^\star_\beta = \mathscr{T}^p\mathscr{A}(H_0^*\beta\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \beta\mathbf{a}) = \mathscr{T}^p\mathscr{A}(\beta\mathbf{a} - (H_0^*\beta(\mathbf{c} - H_0\mathbf{a}))) \tag{3.10}$$

and $\mathbf{v}^\star_\beta = \mathscr{A}^*\tilde{\mathbf{v}}^\star_\beta$. Then the pair $(\mathbf{v}^\star_\beta, \tilde{\mathbf{v}}^\star_\beta)$ satisfies that for any pair $(\eta, \tilde{\eta})$ with $\tilde{\eta} = \mathscr{A}\eta \in \ell_p$,

$$\tilde{\Phi}(\mathbf{v}^\star_\beta + \eta; \mathbf{a}) \geq \tilde{\Phi}(\mathbf{v}^\star_\beta; \mathbf{a}) + \|\eta\|^2. \tag{3.11}$$

A similar proposition is proved in [18], where the underlying system used in denoising is orthonormal basis. The proof depends on the fact $\mathscr{A}^*\mathscr{A} = \mathscr{A}\mathscr{A}^* = I$. However, for the tight frame system, one only has $\mathscr{A}^*\mathscr{A} = I$, while $\mathscr{A}\mathscr{A}^* \neq I$. This adds difficulties to the proof and it also leads to the conditions on the pairs $(\mathbf{v}^\star_\beta, \tilde{\mathbf{v}}^\star_\beta)$ and $(\eta, \tilde{\eta})$. We provide a proof of Proposition 3.3 in Appendix B.

To give the minimization property of $(\mathbf{v}^\beta, \tilde{\mathbf{v}}^\beta)$, we need that $\mathbf{v}^\beta$ is uniformly bounded regardless of β. This requires the assumption that the threshold parameters λ_j are independent of β and $\inf_j \lambda_j \geq \gamma > 0$, $j < 0$. This condition is natural in applications. Indeed, this assumption requires to discard the framelet coefficients when $|j|$ is sufficiently large, because for a given signal, when $|j|$ is large enough, the coefficients of the low frequency subband are very small and can be discarded anyway. We first prove the following lemma:

Lemma 3.1 *Let $\mathbf{h}_1,\ldots,\mathbf{h}_r$ be the high pass filters of a tight frame system derived by the UEP with $\mathbf{h}_0$ being the given low pass filter. Suppose the threshold parameters $\lambda > 0$, then there exists a constant $0<\rho<1$ such that for any sequence $\mathbf{v}\in\ell_2(\mathbb{Z})$*

$$\|\mathscr{D}^p_\lambda(\mathbf{v})\| \le \rho\|\mathbf{v}\|,$$

where $\mathscr{D}^p_\lambda$ is the threshold operator defined in (2.14)*. Further, let $\mathscr{T}^p$ be the denoising operator. Assuming that* $\inf_j \lambda_j \ge \gamma > 0$, *we have*

$$\|\mathscr{T}^p\mathscr{A}(\mathbf{v})\| \le \rho\|\mathbf{v}\|, \quad 0<\rho<1.$$

Proof By (2.14), we have

$$\|\mathscr{D}^p_\lambda(\mathbf{v})\|^2 = \sum_{k\in\mathbb{Z}} |t^p_\lambda(\mathbf{v}[k])|^2.$$

When $p=1$, it is the soft-threshold function: $t_\lambda(x) = \mathrm{sgn}(x)\max(|x|-\lambda/2,0)$. If $\lambda \ge 2\sup_{k\in\mathbb{Z}}|\mathbf{v}[k]|$, then $\mathscr{D}^p_\lambda(\mathbf{v}) = \mathbf{0}$ and hence the inequality $|t_\lambda(\mathbf{v}[k])| \le \rho|\mathbf{v}[k]|$ holds for any $0<\rho<1$. If $\lambda < 2\sup_{k\in\mathbb{Z}}|\mathbf{v}[k]|$, then for a given $k\in\mathbb{Z}$, we have

$$\left|\frac{t_\lambda(\mathbf{v}[k])}{\mathbf{v}[k]}\right| \le 1-\frac{\lambda}{2|\mathbf{v}[k]|} \le 1-\frac{\lambda}{2\|\mathbf{v}\|}.$$

Since $\mathbf{v}\in\ell_2(\mathbb{Z})$, we have $\rho = \sup_{k\in\mathbb{Z}}\left|\frac{t_\lambda(\mathbf{v}[k])}{\mathbf{v}[k]}\right| \le 1-\frac{\lambda}{2\|\mathbf{v}\|} < 1$.

Next, when $1<p\le 2$, by (2.15), we have $t^p_\lambda(x) = (F^p_\lambda)^{-1}(x)$ where $F^p_\lambda(x) = x+\frac{p\lambda}{2}\mathrm{sgn}(x)|x|^{p-1}$. Given $\mathbf{v}[k]$ for a fixed $k\in\mathbb{Z}$, if $(F^p_\lambda)^{-1}(\mathbf{v}[k]) \ne 0$, then let $y=(F^p_\lambda)^{-1}(\mathbf{v}[k])$ and we have

$$\left|\frac{(F^p_\lambda)^{-1}(\mathbf{v}[k])}{\mathbf{v}[k]}\right| = \left|\frac{y}{y+\frac{p\lambda}{2}\mathrm{sgn}(y)|y|^{p-1}}\right| = \frac{1}{1+\frac{p\lambda}{2}|y|^{p-2}} \le \frac{1}{1+\frac{p\lambda}{2}\|\mathbf{v}\|^{p-2}} < 1.$$

When $(F^p_\lambda)^{-1}(\mathbf{v}[k]) = 0$, it is clear that

$$|(F^p_\lambda)^{-1}(\mathbf{v}[k])| \le \frac{1}{1+\frac{p\lambda}{2}\|\mathbf{v}\|^{p-2}}|\mathbf{v}[k]|.$$

Thus when $1<p\le 2$, we take

$$\rho = \sup_{k\in\mathbb{Z}}\left|\frac{(F^p_{\lambda_j})^{-1}(\mathbf{v}[k])}{\mathbf{v}[k]}\right| \le \frac{1}{1+\frac{p\lambda}{2}\|\mathbf{v}\|^{p-2}} < 1.$$

Thus threshold operator $\mathscr{D}^p_\lambda$ satisfies

$$\|\mathscr{D}^p_\lambda(\mathbf{v})\| \le \rho\|\mathbf{v}\|, \quad 1\le p\le 2.$$

For the denoising operator, by (2.16),

$$\|\mathscr{T}^p\mathscr{A}(\mathbf{v})\|^2 = \sum_{\ell=1}^{r}\sum_{j=-\infty}^{-1}\|\mathscr{D}^p_{\lambda_j}(H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v})\|^2,$$

then for each sequence $H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v}$, there exists $\rho_{\ell,j}$ such that

$$\|\mathscr{D}^p_{\lambda_j}(H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v})\| \le \rho_{\ell,j}\|H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v}\|.$$

Since $\|H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v}\| \le \|\mathbf{v}\|$ and $\inf_j\lambda_j \ge \gamma > 0$, we can take

$$\rho = \sup_{\ell,j}\rho_{\ell,j} \le \begin{cases} 1-\frac{\gamma}{2\|\mathbf{v}\|} < 1, & \text{when } p=1; \\ \frac{1}{1+\frac{p\gamma}{2}\|\mathbf{v}\|^{p-2}} < 1, & \text{when } 1<p\le 2. \end{cases}$$

Hence,

$$\|\mathcal{T}^p\mathcal{A}(\mathbf{v})\|^2 \le \sum_{\ell=1}^{r}\sum_{j=-\infty}^{-1}\rho_{\ell,j}^2\|H_{\ell,j}\prod_{j'=j+1}^{-1}H_{0,j'}\mathbf{v}\|^2 \le \rho^2\|\mathbf{v}\|^2,$$

which completes our proof. □

Note that since $\mathcal{D}_\lambda^p$ is not linear, we do not have

$$\|\mathcal{T}^p\mathcal{A}(\mathbf{v}_1)-\mathcal{T}^p\mathcal{A}(\mathbf{v}_2)\| \le \rho\|\mathbf{v}_1-\mathbf{v}_2\|,$$

although we have

$$\|\mathcal{T}^p\mathcal{A}(\mathbf{v}_1)-\mathcal{T}^p\mathcal{A}(\mathbf{v}_2)\| \le \|\mathbf{v}_1-\mathbf{v}_2\|$$

by Proposition 3.2.

Based on Lemma 3.1, we can derive that the iterative sequence is uniformly bounded. More precisely, we have the following proposition.

Proposition 3.4 *Let* $\mathbf{h}_1,\ldots,\mathbf{h}_r$ *be the high pass filters of a tight frame system derived by the UEP with* $\mathbf{h}_0$ *being the given low pass filter and* $\mathbf{v}^\beta$ *be the limit of iteration* (2.17) *for* $0<\beta<1$. *Assume that the threshold parameters* λ_j, $j<0$ *are independent of iteration and* β *and* $\inf_j\lambda_j \ge \gamma > 0$. *Then there exists* $C>0$, *such that* $\|\mathbf{v}^\beta\|\le C$, *for all* $0<\beta<1$.

Proof For any given initial value $\mathbf{v}_0\in\ell_2(\mathbb{Z})$ and a fixed $\beta\in(0,1)$, let $\{\mathbf{v}_n^\beta\}$ be the sequence obtained by iteration (2.17) in Algorithm 2.2. Applying Lemma 3.1 and the argument used in (3.3) lead to

$$\|\mathbf{v}_n^\beta\| \le \rho(\|\mathbf{c}\|+\|\mathbf{v}_{n-1}^\beta\|) \le \frac{\rho}{1-\rho}\|\mathbf{c}\|+\|\mathbf{v}_0\|.$$

Let $C=\frac{\rho}{1-\rho}\|\mathbf{c}\|+\|\mathbf{v}_0\|$, then $\|\mathbf{v}_n^\beta\|\le C$. Hence, the limit $\mathbf{v}^\beta$ to $\mathbf{v}_n^\beta$ also satisfies that $\|\mathbf{v}^\beta\|\le C$.

With this, a consequence of Proposition 3.3 is the minimization property of $\mathbf{v}^\beta$.

Proposition 3.5 *Suppose* $\tilde{\mathbf{c}}=\mathcal{A}\mathbf{c}\in\ell_p$. *For given* $\varepsilon>0$ *and* $C>\sup_\beta\|\mathbf{v}^\beta\|$, *there exists* $\delta>0$, *which only depends on* ε *and* C, *such that for all* $\beta\in(1-\delta,1)$, *the corresponding limit* $(\mathbf{v}^\beta,\tilde{\mathbf{v}}^\beta)$ *of iteration* (2.17) *in Algorithm 2.2 satisfies the inequality* (3.7) *for an arbitrary pair* $(\eta,\tilde{\eta})$ *with* $\tilde{\eta}=\mathcal{A}\eta\in\ell_p$ *and* $\|\eta\|\le C$.

Proof Note inequality (3.7) for an arbitrary β is equivalent to

$$\Phi(\mathbf{v}^\beta+\eta)\ge\Phi(\mathbf{v}^\beta)-\varepsilon$$

for all $(\eta,\tilde{\eta})$, satisfying $\tilde{\eta}=\mathcal{A}\eta\in\ell_p$ and $\|\eta\|\le C$.

Applying Proposition 3.3 by letting $\mathbf{a}=\mathbf{v}^\beta$, we have inequality

$$\tilde{\Phi}(\mathbf{v}_\beta^\star+\eta;\mathbf{v}^\beta)\ge\tilde{\Phi}(\mathbf{v}_\beta^\star;\mathbf{v}^\beta)+\|\eta\|^2 \tag{3.12}$$

for any pair $(\eta,\tilde{\eta})$ satisfying $\tilde{\eta}=\mathcal{A}\eta$. Since limit $\mathbf{v}^\beta$ satisfies $\mathbf{v}^\beta=\mathcal{A}^*\mathcal{T}^p\mathcal{A}(H_0^*\beta\mathbf{c}+\sum_{\ell=1}^r H_\ell^*H_\ell\beta\mathbf{v}^\beta)$ by Theorem 3.1 and since $\mathbf{v}_\beta^\star$ satisfies $\mathbf{v}_\beta^\star=\mathcal{A}^*\mathcal{T}^p\mathcal{A}(H_0^*\beta\mathbf{c}+\sum_{\ell=1}^r H_\ell^*H_\ell\beta\mathbf{v}^\beta)$ by (3.10) of Proposition 3.3, we conclude that $\mathbf{v}_\beta^\star=\mathbf{v}^\beta$. Hence, $\tilde{\Phi}(\mathbf{v}_\beta^\star;\mathbf{v}^\beta)=\tilde{\Phi}(\mathbf{v}^\beta;\mathbf{v}^\beta)=\Phi(\mathbf{v}^\beta)$.

By the definition of $\tilde{\Phi}(\mathbf{v};\mathbf{a})$, one obtains that

$$\begin{aligned}\tilde{\Phi}(\mathbf{v}_\beta^\star+\eta;\mathbf{v}^\beta) &= \|H_0(\mathbf{v}^\beta+\eta)-\beta\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\beta^{2-p}\lambda_j|\tilde{v}_{\ell,j,k}^\beta+\tilde{\eta}_{\ell,j,k}|^p\\ &\quad+\sum_{\ell=1}^r\|(1-\beta)H_\ell(\mathbf{v}^\beta+\eta)+\beta H_\ell\eta\|^2.\end{aligned}$$

Since

$$\|(1-\beta)H_\ell(\mathbf{v}^\beta+\eta)+\beta H_\ell\eta\|^2\le(1-\beta)^2\|H_\ell(\mathbf{v}^\beta+\eta)\|^2+\beta^2\|H_\ell\eta\|^2+2\beta(1-\beta)\|H_\ell\eta\|\|H_\ell(\mathbf{v}^\beta+\eta)\|,$$

this leads to

$$\tilde{\Phi}(\mathbf{v}^\star_\beta+\eta;\mathbf{v}^\beta) \le \|H_0(\mathbf{v}^\beta+\eta)-\beta\mathbf{c}\|^2+\sum_{\ell=1}^{r}\sum_{j<0,k\in\mathbb{Z}}\beta^{2-p}\lambda_j|\tilde{v}^\beta_{\ell,j,k}+\tilde{\eta}_{\ell,j,k}|^p+(1-\beta)^2\sum_{\ell=1}^{r}\|H_\ell(\mathbf{v}^\beta+\eta)\|^2$$
$$+\beta^2\sum_{\ell=1}^{r}\|H_\ell\eta\|^2+2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell(\mathbf{v}^\beta+\eta)\|.$$

Note that

$$\Phi(\mathbf{v}^\beta+\eta)=\|H_0(\mathbf{v}^\beta+\eta)-\beta\mathbf{c}\|^2+\sum_{\ell=1}^{r}\sum_{j<0,k\in\mathbb{Z}}\beta^{2-p}\lambda_j|\tilde{v}^\beta_{\ell,j,k}+\tilde{\eta}_{\ell,j,k}|^p+(1-\beta)^2\sum_{\ell=1}^{r}\|H_\ell(\mathbf{v}^\beta+\eta)\|^2.$$

So we have

$$\Phi(\mathbf{v}^\beta+\eta)+\beta^2\sum_{\ell=1}^{r}\|H_\ell\eta\|^2+2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell(\mathbf{v}^\beta+\eta)\| \ge \tilde{\Phi}(\mathbf{v}^\beta+\eta;\mathbf{v}^\beta)\ge\Phi(\mathbf{v}^\beta)+\|\eta\|^2.$$

This leads to the following equality

$$\Phi(\mathbf{v}^\beta+\eta)\ge\Phi(\mathbf{v}^\beta)+\|\eta\|^2-\beta^2\sum_{\ell=1}^{r}\|H_\ell\eta\|^2-2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell(\mathbf{v}^\beta+\eta)\|.$$

Using

$$\|\eta\|^2=\|H_0\eta\|^2+\sum_{\ell=1}^{r}\|H_\ell\eta\|^2,$$

one obtains

$$\begin{aligned}
&\|\eta\|^2-\beta^2\sum_{\ell=1}^{r}\|H_\ell\eta\|^2-2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell(\mathbf{v}^\beta+\eta)\|\\
&=\|H_0\eta\|^2+(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\Big((1+\beta)\|H_\ell\eta\|-2\beta\|H_\ell(\mathbf{v}^\beta+\eta)\|\Big)\\
&\ge\|H_0\eta\|^2+2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\Big(\|H_\ell\eta\|-\|H_\ell\mathbf{v}^\beta\|-\|H_\ell\eta\|\Big)\\
&=\|H_0\eta\|^2-2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell\mathbf{v}^\beta\|.
\end{aligned}$$

This leads to

$$\Phi(\mathbf{v}^\beta+\eta)\ge\Phi(\mathbf{v}^\beta)+\|H_0\eta\|^2-2\beta(1-\beta)\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell\mathbf{v}^\beta\|. \tag{3.13}$$

Because $\mathbf{v}^\beta$ is bounded by C according to Lemma 3.1 and η is also bounded, the term $\sum_{\ell=1}^{r}\|H_\ell\eta\|\|H_\ell\mathbf{v}^\beta\|$ is bounded by rC^2. So given arbitrary $\varepsilon>0$, we can take $\delta\le\frac{\varepsilon}{2rC^2}$ and then for any $\beta\in(1-\delta,1)$,

$$\Phi(\mathbf{v}^\beta+\eta)\ge\Phi(\mathbf{v}^\beta)+\|H_0\eta\|^2-\varepsilon\ge\Phi(\mathbf{v}^\beta)-\varepsilon, \tag{3.14}$$

which completes the proof. □

Based on the minimization of $\mathbf{v}^\beta$, the minimization property of $\mathbf{s}^\beta$ is straightforward as given below:

Theorem 3.3 *Suppose $\tilde{\mathbf{c}}=\mathscr{A}\mathbf{c}\in\ell_p$. Then given $\varepsilon>0$ and $C>\sup_\beta\|\mathbf{v}^\beta\|$, there exists $\delta>0$, which only depends on ε and C, such that for all $\beta\in(1-\delta,1)$, the solution $(\mathbf{s}^\beta,\tilde{\mathbf{s}}^\beta)$ of Algorithm 2.2 satisfies the inequality* (3.6) *for any pair $(\eta,\tilde{\eta})$ with $\tilde{\eta}=\mathscr{A}\eta\in\ell_p$ and $\|\eta\|\le C$.*

Proof For given $(\eta,\tilde{\eta})$, set $\eta_1=\beta\eta$ and $\tilde{\eta}_1=\beta\tilde{\eta}$ which satisfy that $\|\eta_1\|\le C$ and $\tilde{\eta}_1\in\ell_p$. Then, for arbitrary $\varepsilon>0$, applying Proposition 3.5, there exists $\delta_1>0$ such that for any $\beta\in(1-\delta_1,1)$ the pair $(\mathbf{v}^\beta,\tilde{\mathbf{v}}^\beta)$ satisfies

$$\Phi(\mathbf{v}^\beta+\eta_1)\ge\Phi(\mathbf{v}^\beta)-\frac{\varepsilon}{8}, \tag{3.15}$$

as long as $(\eta_1,\tilde{\eta}_1)$ satisfies $\tilde{\eta}_1=\mathscr{A}\eta_1\in\ell_p$ and $\|\eta_1\|\le C$. From Algorithm 2.2, we have $\mathbf{s}^\beta=\frac{\mathbf{v}^\beta}{\beta}$ and $\tilde{\mathbf{s}}^\beta=\frac{\tilde{\mathbf{v}}^\beta}{\beta}$. Dividing β^2 on both sides of (3.15), we have

$$\begin{aligned}\|H_0(\mathbf{s}^\beta+\eta)-\mathbf{c}\|^2&+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}+\frac{(\tilde{\eta}_1)_{\ell,j,k}}{\beta}|^p+\frac{(1-\beta)^2}{\beta^2}\sum_{\ell=1}^r\|H_\ell(\mathbf{v}^\beta+\eta_1)\|^2\\ &\ge\|H_0\mathbf{s}^\beta-\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}|^p+\frac{(1-\beta)^2}{\beta^2}\sum_{\ell=1}^r\|H_\ell\mathbf{v}^\beta\|^2-\frac{\varepsilon}{8\beta^2}.\end{aligned} \tag{3.16}$$

Because $\mathbf{v}^\beta$ and η_1 are bounded, for any given $\varepsilon>0$, we can take $\delta_2\le\frac{1}{5C}\sqrt{\frac{\varepsilon}{r}}$ and then any $\beta\in(1-\delta_2,1)$ satisfies $(1-\beta)^2\sum_{\ell=1}^r(\|H_\ell(\mathbf{v}^\beta+\eta_1)\|^2-\|H_\ell\mathbf{v}^\beta\|^2)<\frac{\varepsilon}{8}$. Taking $\delta=\min(\delta_1,\delta_2,\frac{1}{2})$ and combining with (3.16), the pair $(\mathbf{s}^\beta,\tilde{\mathbf{s}}^\beta)$ satisfies for any $\beta\in(1-\delta,1)$

$$\begin{aligned}\|H_0(\mathbf{s}^\beta+\eta)-\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}+\tilde{\eta}_{\ell,j,k}|^p&\ge\|H_0\mathbf{s}^\beta-\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}|^p-\frac{\varepsilon}{4\beta^2}\\ &\ge\|H_0\mathbf{s}^\beta-\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}|^p-\varepsilon,\end{aligned}$$

as long as the pair $(\eta,\tilde{\eta})$ satisfies $\tilde{\eta}=\mathscr{A}\eta\in\ell_p$ and $\|\eta\|\le C$. □

Remark 3.1 We note that since in each iteration solution pair $(\mathbf{v}^\beta_n,\tilde{\mathbf{v}}^\beta_n)$ satisfy $\mathbf{v}^\beta_n\in\ell_2(\mathbb{Z})$ and $\tilde{\mathbf{v}}^\beta_n\in\ell_2$, and $(\mathbf{v}^\beta,\tilde{\mathbf{v}}^\beta)$ is the limit to the iteration pair, it leads to $\mathbf{v}^\beta\in\ell_2(\mathbb{Z})$ and $\tilde{\mathbf{v}}^\beta\in\ell_2$, and furthermore $\mathbf{s}^\beta\in\ell_2(\mathbb{Z})$ and $\tilde{\mathbf{s}}^\beta\in\ell_2$. The minimization property (3.6) holds with finite value on both sides whenever $p=2$. For $1\le p<2$, as we have already proved that when $\tilde{\mathbf{s}}^\beta\in\ell_p$, the solution satisfies the minimization inequality (3.6). In fact, the values on the both sides of inequality (3.6) are finite.

In the proof of Theorem 3.3 (See (3.16)), when β is chosen to be small (say smaller than $1/2$) instead of closing to 1, we have

$$\begin{aligned}\|H_0(\mathbf{s}^\beta+\eta)-\mathbf{c}\|^2&+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}+\tilde{\eta}_{\ell,j,k}|^p+\lambda\sum_{\ell=1}^r\|H_\ell(\mathbf{s}^\beta+\eta)\|^2\\ &\ge\|H_0\mathbf{s}^\beta-\mathbf{c}\|^2+\sum_{\ell=1}^r\sum_{j<0,k\in\mathbb{Z}}\lambda_j|\tilde{s}^\beta_{\ell,j,k}|^p+\lambda\sum_{\ell=1}^r\|H_\ell\mathbf{s}^\beta\|^2-\varepsilon.\end{aligned}$$

In this case, in addition to penalize the functional in (2.9) we also penalize

$$\sum_{\ell=1}^r\|H_\ell\mathbf{s}^\beta\|^2, \tag{3.17}$$

the high frequency information of the solution. However, as we discussed in the formulation, since the deconvolution processing is essentially to recover the term $\sum_{\ell=1}^r H_\ell\mathbf{s}^\beta$, we do not want to over penalize (3.17). This motivates us to suggest that β to be chosen close to 1, although smaller β will give a fast convergence rate. Our numerical simulation also shows that when smaller β is chosen, the corresponding solution is over smoothed. This leads to inefficient deconvolution. We summarize the numerical results in Table 3.1 where the filters in Example 1.2 are used and the original signal is given in Figure 4.1 (a).

As can be seen from the PSNR, when β is small, peak value of the signal is not well recovered. The algorithm only removes the noise from the data but does not deconvolve the signal significantly. When β

Table 3.1 Numerical results of Algorithm 2.2 when β changes from 0.1 to 0.9

	$\beta=0.1$	$\beta=0.2$	$\beta=0.3$	$\beta=0.4$	$\beta=0.5$	$\beta=0.6$	$\beta=0.7$	$\beta=0.8$	$\beta=0.9$
RE	0.0616	0.0605	0.0593	0.0581	0.0567	0.0553	0.0537	0.0520	0.0501
PSNR	31.3874	31.5849	31.7971	32.0320	32.2860	32.5634	32.8647	33.1873	33.6070
SNR	24.2051	24.3643	24.5341	24.7209	24.9238	25.1485	25.3958	25.6772	26.0016

becomes close to 1, the peak value of the signal is recovered better and relative error becomes smaller and peak signal-to-noise ratio is much better. These numerical data coincide with our analysis because smaller β penalizes more high frequency components which are needed to be recovered from the algorithms.

In practice, only the finite data set is available. As will see in §4, we can make the finite dimensional matrix $H_0^*H_0$ nonsingular and hence the iteration (2.17) will converges without the acceleration factor β. In such a case, we can directly prove inequality (2.9).

3.3 Minimization Property of Algorithm 2.3

In this section, we discuss the minimization property of the solution $\mathbf{s}^\beta$ obtained in Algorithm 2.3. We use the similar approach to that used in the last section.

We characterize the minimization property of solution $\mathbf{s}^\beta$ paralleled to that of Algorithm 2.2. From the iteration (2.18), we obtain the limit $\mathbf{v}^\beta$ which satisfies

$$\mathbf{v}^\beta = H_0^*\beta\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^*\mathscr{A}^*\mathscr{T}^p\mathscr{A}(H_\ell\beta\mathbf{v}^\beta). \tag{3.18}$$

Define

$$\tilde{\mathbf{v}}_\ell^\beta = \mathscr{T}^p\mathscr{A}(H_\ell\beta\mathbf{v}^\beta) \quad \text{and} \quad \mathbf{v}_\ell^\beta = \mathscr{A}^*\tilde{\mathbf{v}}_\ell^\beta, \quad \ell = 1,\dots,r. \tag{3.19}$$

If we further denote $\beta\mathbf{c}$ by $\mathbf{v}_0^\beta$, the limit of iteration (2.18) satisfies $\mathbf{v}^\beta = \mathscr{A}_{-1}^*\{\mathbf{v}_\ell^\beta\}_{\ell=0}^r$ where $\mathscr{A}_{-1}^*$ is given by (1.23). We denote the quantities that determine the limit $\mathbf{v}^\beta$ by the $(r+1)$-tuple be $(\mathbf{v}^\beta,\tilde{\mathbf{v}}_1^\beta,\dots,\tilde{\mathbf{v}}_r^\beta)$.

The solution of Algorithm 2.3 is given by another $(r+1)$-tuple $(\mathbf{s}^\beta,\tilde{\mathbf{s}}_1^\beta,\dots,\tilde{\mathbf{s}}_r^\beta)$ with

$$\tilde{\mathbf{s}}_\ell^\beta = \frac{\tilde{\mathbf{v}}_\ell^\beta}{\beta} \quad \text{and} \quad \mathbf{s}_\ell^\beta = \mathscr{A}^*\tilde{\mathbf{s}}_\ell^\beta, \quad \ell = 1,\dots,r. \tag{3.20}$$

Since $\mathbf{v}^\beta$ satisfies (3.18), we have

$$\mathbf{s}^\beta = \mathscr{A}_{-1}^*\{\mathbf{s}_\ell^\beta\}_{\ell=0}^r = H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^*\mathscr{A}^*\tilde{\mathbf{s}}_\ell^\beta, \tag{3.21}$$

where $\mathbf{s}_0^\beta := \mathbf{c}$ and $\mathbf{v}^\beta$, $\tilde{\mathbf{v}}_\ell^\beta$ are given in (3.18) and (3.19). In the following, we denote the (ℓ', j, k)th entries in $\tilde{\mathbf{s}}_\ell^\beta$, $\ell = 1,\dots,r$, by $(\tilde{s}_\ell^\beta)_{\ell',j,k}$ where $\ell' = 1,\dots,r$, $j<0$ and $k \in \mathbb{Z}$.

The solution of Algorithm 2.3 has different minimization property from the solution of Algorithm 2.2. Given any $\varepsilon > 0$ and $C > 0$, the $(r+1)$-tuple $(\mathbf{s}^\beta,\tilde{\mathbf{s}}_1^\beta,\dots,\tilde{\mathbf{s}}_r^\beta)$ satisfies the following inequality

$$\begin{aligned} &\|\mathbf{h}_0 * (\mathbf{s}^\beta + \eta) - \mathbf{c}\|^2 + \sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j<0,k\in\mathbb{Z}} \lambda_j |(\tilde{s}_\ell^\beta)_{\ell',j,k} + (\tilde{\eta}_\ell)_{\ell',j,k}|^p \\ &\geq \|\mathbf{h}_0 * \mathbf{s}^\beta - \mathbf{c}\|^2 + \sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j<0,k\in\mathbb{Z}} \lambda_j |(\tilde{s}_\ell^\beta)_{\ell',j,k}|^p - \varepsilon. \end{aligned} \tag{3.22}$$

for any $(r+1)$-tuple $(\eta,\tilde{\eta}_1,\dots,\tilde{\eta}_r)$ satisfying $\|\eta\| \le C$ and $\tilde{\eta}_\ell = \mathscr{A}(H_\ell\eta)$ where $\tilde{\eta}_1,\dots,\tilde{\eta}_r \in \ell_p$.

Note that $\tilde{\eta}_\ell = \mathscr{A}(H_\ell\eta)$ implies that $H_\ell\eta = \mathscr{A}^*\mathscr{A}(H_\ell\eta) = \mathscr{A}^*\tilde{\eta}_\ell$ for $\ell = 1,\dots,r$. The high frequency components $H_\ell\eta$, $\ell = 1,\dots,r$, are further decomposed by decomposition operator $\mathscr{A}$. More precisely, $\mathscr{A}(H_\ell\eta)$ is the set of coefficients of framelet packet in canonical form (see [13] and [30]). From the penalty terms in

(3.22), we can also see that the terms $\tilde{\mathbf{s}}_\ell^\beta$, $\ell = 1,\dots,r$ are no longer framelet coefficients in (3.6) but coefficients of framelet packet decomposition of the high frequency component $\mathbf{s}_\ell^\beta = \mathscr{A}^*\tilde{\mathbf{s}}_\ell^\beta$, $\ell = 1,\dots r$, which also reflect certain smoothness of the underlying functions. It is nature to penalize the ℓ_p-norm of framelet packet coefficients of each high frequency component $H_\ell \mathbf{s}^\beta$, $\ell = 1,\dots r$, since as pointed out in the formulation that the deconvolution is essentially to put back the missing components $H_\ell \mathbf{s}^\beta$, $\ell = 1,\dots r$ and we do not want them too rough. In fact, we can put it into a similar formulation as Algorithm 2.2 in terms of the framelet packets. However, we omit the details.

As we did for Algorithm 2.2, we can derive the following result on the minimization property of $(r+1)$-tuple $(\mathbf{s}^\beta, \tilde{\mathbf{s}}_1^\beta, \dots, \tilde{\mathbf{s}}_r^\beta)$. Since the proof is similar to that of Theorem 3.3, and since we will give a full proof of this result for the finite data set, we omit it here.

Theorem 3.4 *For given $\varepsilon > 0$ and $C > \sup_\beta \|\mathbf{v}^\beta\|$, there exists $\delta > 0$, which only depends on ε and C, such that for all $\beta \in (1-\delta, 1)$, the corresponding $(r+1)$-tuple $(\mathbf{s}^\beta, \tilde{\mathbf{s}}_1^\beta, \dots, \tilde{\mathbf{s}}_r^\beta)$ of iteration* (2.18) *in Algorithm 2.3 satisfies inequality* (3.22) *for any $(r+1)$-tuple $(\eta, \tilde{\eta}_1, \dots, \tilde{\eta}_r)$ satisfying $\tilde{\eta}_\ell = \mathscr{A}(H_\ell \eta)$, $\tilde{\eta}_1, \dots, \tilde{\eta}_r \in \ell_p$ and $\|\eta\| \le C$.*

4 Deconvolution of Finite Data Set

In the previous sections, our algorithms and analysis are given for the infinite data set which is of theoretic interests, provides the understanding and insight of algorithms, and connects to multiresolution analysis. However, in application, given data sets are always finite, e.g. a vector in $\mathbb{R}^{N_0}$. Thus it is necessary to adjust our approach for these cases. This is achieved by extrapolating the data out of the boundary. The numerical simulation shows that the algorithms work well under different boundary conditions as shown in [6,8,9].

4.1 Algorithms for Finite Data

In this section, we convert the algorithms given in previous sections to the ones which deal with the finite data set. The convolution equation becomes

$$\mathbf{h}_0 * \mathbf{v} = \mathbf{b} + \varepsilon = \mathbf{c}$$

with the finite given data set $\mathbf{c}$ and $\|\varepsilon\|_2 = \varepsilon < \infty$. Since our data are no longer infinite, the boundary conditions are needed to extend the data beyond their original domain. Basically, there are three types of boundary conditions: zero-padding, periodic and symmetric. Since the zero-padding boundary condition simply adds zeros out of the original domain, it is more or less reduced to the case discussed in the previous section and it normally gives boundary artifacts, we omit discussion on this case. We focus on more detailed discussion on periodic boundary condition and the discussion of symmetric boundary condition can be carried out similarly.

When the given data set α is extended using the periodic boundary condition, i.e.

$$\alpha[n] = \alpha[n \bmod N_0], \quad n \in \mathbb{Z},$$

where N_0 is the length of data α, the convolution of data α with given filter $\mathbf{h}_0$ then becomes a special kind of convolution, *circular convolution*. We denote such circular convolution by

$$\mathbf{h}_0 \circledast \alpha.$$

The circular convolution can also be written as a matrix-vector multiplication where the matrix is a circulant matrix, a special kind of Toeplitz matrix, i.e. the entries of matrix H_0 generated from $\mathbf{h}_0$ are

$$H_0[l,k] = \mathbf{h}_0[(l-k) \bmod N_0], \quad 0 \le l,k < N_0. \tag{4.1}$$

Using periodic boundary condition to extend data implies that the matrices $H_0, H_1, \dots, H_r$ used in convolution are now circulant matrices of finite order generated from the filters $\mathbf{h}_0, \mathbf{h}_1, \dots, \mathbf{h}_r$. Further, we have dilated filters $\mathbf{h}_{0,j}, \dots, \mathbf{h}_{r,j}$ for the jth level decomposition, where $\mathbf{h}_{\ell,j}$ is obtained by inserting $2^{-j-1} - 1$ zeros between every two entries in $\mathbf{h}_\ell$ as defined in (1.16). With these, we define the discrete decomposition and reconstruction operators A_J and A_J^* analog to (1.21) and (1.23) by

$$A_J = [(\prod_{j=J}^{-1} H_{0,j});\ (H_{1,J}\prod_{j=J+1}^{-1} H_{0,j});\ldots;(H_{r,J}\prod_{j=J+1}^{-1} H_{0,j});\ldots;H_1;\ldots;H_r]^t \tag{4.2}$$

and

$$A_J^* = [(\prod_{j=-1}^{J} H_{0,j}^*);\ (\prod_{j=-1}^{J+1} H_{0,j}^* H_{1,J}^*);\ldots;(\prod_{j=-1}^{J+1} H_{0,j}^* H_{r,J}^*);\ldots;H_1^*;\ldots;H_r^*]. \tag{4.3}$$

Each block in A_J and A_J^* is the product of a series of circulant matrices $H_{\ell,j}$ generated from filter $\mathbf{h}_{\ell,j}$. Similar to Proposition 1.2, it can be proved that A_J^* is the adjoint of A_J and $A_J^*A_J = I$, where I is the identity matrix and this identity is due to the condition similar to (1.18).

For the finite data set, we decompose to a finite level to denoise. Hence in the iteration, operators A_J and A_J^* are used instead of $\mathscr{A}$ and $\mathscr{A}^*$. Moreover, we need in the algorithms the following denoising operator for data of finite dimension: given a finite sequence $\mathbf{v}$, define

$$\mathscr{T}^p A_J(\mathbf{v}) = [(\prod_{j=J}^{-1} H_{0,j}\mathbf{v});\ \mathscr{D}_{\lambda_J}^p(H_{1,J}\prod_{j=J+1}^{-1} H_{0,j}\mathbf{v});\ldots;\mathscr{D}_{\lambda_J}^p(H_{r,J}\prod_{j=J+1}^{-1} H_{0,j}\mathbf{v});\ldots;\mathscr{D}_{\lambda_{-1}}^p(H_1\mathbf{v});\ldots;\mathscr{D}_{\lambda_{-1}}^p(H_r\mathbf{v})]^t,$$

where the threshold operator $\mathscr{D}_\lambda^p$ is given in (2.14). With these notations, we can convert Algorithm 2.2 and 2.3 to the finite data set. The first one is given in [9]

Algorithm 4.1 (Algorithm 2.2 for finite data)

(i) Choose an initial approximation $\mathbf{v}_0$ (e.g. $\mathbf{v}_0 = \mathbf{c}$);
(ii) Iterate on n until convergence:

$$\mathbf{v}_{n+1} = A_J^* \mathscr{T}^p A_J (H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{v}_n). \tag{4.4}$$

The following algorithm is the basic algorithm from which the algorithms in [10, 11] are based on.

Algorithm 4.2 (Algorithm 2.3 for finite data)

(i) Choose an initial approximation $\mathbf{v}_0$ (e.g. $\mathbf{v}_0 = \mathbf{c}$);
(ii) Iterate on n until convergence:

$$\mathbf{v}_{n+1} = H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* (A_J^* \mathscr{T}^p A_J)(H_\ell \mathbf{v}_n). \tag{4.5}$$

Since $\mathbf{c}$ contains noise, it was suggested by the numerical simulations in [6, 8, 9], one needs to take an additional step of denoising from the final iteration:

(iii) $\mathbf{v} = A_J^* \mathscr{T}^p A_J(\mathbf{v}_{n_0})$.

As we can see, the difference between Algorithm 4.1 and 4.2 is the different denoising schemes used in each iteration. Next, we discuss the algorithm which is mainly used in numerical implementation of high resolution image reconstructions in [6, 8, 9]. The algorithm applies a different decomposition operator in denoising scheme. The decomposition operator used is defined as:

$$B_J = [H_0^{|J|};\ H_1 H_0^{|J|-1};\ldots;H_r H_0^{|J|-1};\ldots;H_1;\ldots;H_r]^t \tag{4.6}$$

and the reconstruction operator is its adjoint operator

$$B_J^* = [(H_0^{|J|})^*;\ (H_0^{|J|-1})^* H_1^*;\ldots;(H_0^{|J|-1})^* H_r^*;\ldots;H_1^*;\ldots;H_r^*]. \tag{4.7}$$

It can be easily seen that the difference between A_J and B_J is in the blocks. In A_J, each block is of form

$$H_{\ell,j}\prod_{j'=j+1}^{-1} H_{0,j'},$$

which is a product of matrices generated from up sampled filters $\mathbf{h}_{\ell,j}$, $\ell = 0,\dots,r$; while in B_J, each block is of form

$$H_\ell H_0^{|j|},$$

which is a product of matrices generated from filters $\mathbf{h}_\ell$, $\ell = 0,\dots,r$, without up sampling. This difference implies the filters in decomposition B_J are stationary without up sampling process. Nevertheless, the identity $B_J^* B_J = I$ still hold. In fact, one can prove this identity easily by modifying the proof of Proposition 1.2. The denoising scheme is formed by applying the threshold operator $\mathscr{T}^p$ to $B_J\mathbf{v}$, i.e.

$$[H_0^{|J|}\mathbf{v};\ \mathscr{D}^p_{\lambda_J}(H_1 H_0^{|J|-1}\mathbf{v});\dots;\mathscr{D}^p_{\lambda_J}(H_r H_0^{|J|-1}\mathbf{v});\dots;\mathscr{D}^p_{\lambda_1}(H_1\mathbf{v});\dots;\mathscr{D}^p_{\lambda_1}(H_r\mathbf{v})]^t.$$

The algorithm used in high resolution image reconstructions (see [6,8,9]) is essentially the same as Algorithm 4.2 except the A_J and A_J^* are replaced by B_J and B_J^*, where B_J and B_J^* are defined in (4.6) and (4.7), i.e. step (ii) in above algorithm is replaced by

Algorithm 4.3 Iterate on n until convergence:

$$\mathbf{v}_{n+1} = H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* (B_J^* \mathscr{T}^p B_J)(H_\ell \mathbf{v}_n). \tag{4.8}$$

Here we remark that in most numerical implementation of [6,8–11], the hard threshold is used; though, here we use soft threshold. We also note that in the finite dimensional data case, the acceleration factor is removed from the iterations in both Algorithm 2.2 and 2.3. From the proof of the convergence of Algorithm 2.2 and 2.3, to remove the acceleration factor, we need that the largest eigenvalues of the matrix

$$\sum_{\ell=1}^{r} H_\ell^* H_\ell$$

are less than 1. Since

$$\sum_{\ell=1}^{r} H_\ell^* H_\ell = I - H_0^* H_0,$$

the convergence of the iteration depends on the nonsingularity of matrix $H_0^* H_0$. As we will prove in Proposition 4.1, the matrix $H_0^* H_0$ can be nonsingular by a proper extension of the data before imposing periodic boundary condition.

The underlying framelet analysis for all three algorithms can be carried out by using the framelets on intervals, e.g. periodic framelets when the periodic boundary conditions are imposed. We omit the discussion here. On the other hand, the above algorithms can be also viewed as algorithms to solve the equation:

$$H_0\mathbf{v} = \mathbf{b} + \varepsilon = \mathbf{c}, \tag{4.9}$$

where H_0 is the matrix depends on the boundary condition imposed, e.g. H_0 is a circulant matrix generated by $\mathbf{h}_0$ when the periodic boundary conditions are imposed. Since H_0 is nonsingular in this case by Proposition 4.1, the linear system always has a unique solution.

4.2 Convergence and Minimization Properties

In this section, we discuss the convergence of Algorithm 4.1, 4.2 and 4.3 and the minimization properties of their limits.

The analysis is based on the nonsingularity of matrix H_0. We consider the finite data sets with periodic boundary condition. The eigenvalues of the circulant matrix H_0 generated from $\mathbf{h}_0$ can be given out explicitly as follows:

$$\lambda_p[H_0] = \sum_{k=0}^{K-1} \mathbf{h}_0[k] \exp\left(-\frac{i2kp\pi}{N_0}\right), \quad p = 0,1,\cdots,N_0-1, \tag{4.10}$$

where N_0 is the length of given data and K is the length of filter $\mathbf{h}_0$. Here we assume, without loss of generality, $K < N_0$. The eigenvalues of the matrix H_0 are the values of polynomial $\widehat{\mathbf{h}}_0(\omega)$ at $\omega = \frac{2p\pi}{N_0}$, $p = 0,\dots,N_0-1$. The matrix H_0 is nonsingular, whenever $\widehat{\mathbf{h}}_0(\frac{2p\pi}{N_0})$ is not equal to zero for each $p = 0,\dots,N_0-1$. Since $\widehat{\mathbf{h}}_0$ only has finitely many zeros, we can extend the data set to increase the length of the data from N_0 to N_1 (before making a periodical extension of the data) to avoid zero eigenvalues of H_0. This observation is summarized in the following result, the proof of which is given in Appendix C.

Proposition 4.1 *Let* $\mathbf{h}_0$ *be the given low pass filter with length* K *and the given data having length* $N_0 > K$. *Then the data set can always be extended to have the length* $N_1 > N_0$ *such that the corresponding circulant matrix* H_0 *generated from* $\mathbf{h}_0$ *with the data set of length* N_1 *is nonsingular. Consequently, the matrices* H_0^* *and* $H_0^*H_0$ *are nonsingular.*

As shown in the proof, the number N_1 can be very close to N_0. In fact, for many cases, e.g. $\mathbf{h}_0$ is a refinement mask of splines, pseudo-splines or those used in high resolution image reconstructions (e.g. see [6,8,9]), whenever the length of the data is odd, the corresponding circulant matrix H_0 is nonsingular. This means that $N_1 - N_0$ is at most 1 for those cases. In the following, we assume that the length of data is N_1 such that the corresponding circulant matrix is nonsingular. As we will see, the nonsingularity of H_0 ensures the convergence of iterations without using the acceleration factor β. Furthermore, the threshold parameters λ_j no longer need to satisfy the additional condition $\inf_j \lambda_j > 0$ imposed in the last section.

The convergence of iteration (4.4) in Algorithm 4.1 and iteration (4.5) in Algorithm 4.2 can be proved based on the nonsingularity of the circulant matrix H_0. We give the proof for the second iteration and only list the relevant result of the first iteration.

Theorem 4.1 *Let* $\mathbf{h}_0$ *be the low pass filter in the convolution equation and* $\mathbf{h}_1,\ldots,\mathbf{h}_r$ *be the high pass filters generated from* $\mathbf{h}_0$ *via the UEP. The corresponding circulant matrices are* $H_0,\ldots,H_r$ *with* $H_0^*H_0$ *being nonsingular (by applying Proposition 4.1). Then iteration* (4.4) *in Algorithm 4.1 converges for any initial seed* $\mathbf{v}_0$ *and the limit satisfies*

$$\mathbf{s} = A_J^* \mathscr{T}^p A_J (H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{s}). \tag{4.11}$$

Similarly, iteration (4.5) *in Algorithm 4.2 converges for any initial seed* $\mathbf{v}_0$ *and the limit satisfies*

$$\mathbf{s} = H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell^* A_J^* \mathscr{T}^p A_J (H_\ell \mathbf{s}). \tag{4.12}$$

Proof We only give the proof of the convergence of iteration (4.5). The convergence of iteration (4.4) can be proved similarly. Because H_0 is a nonsingular circulant matrix and $I = H_0^*H_0 + \sum_{\ell=1}^r H_\ell^* H_\ell$, there exists a constant $\mu < 1$ such that $\|\sum_{\ell=1}^r H_\ell^* H_\ell\|_2 = \|I - H_0^*H_0\|_2 \le \mu$. Let $H = [H_1,\ldots,H_r]^t$, then we have

$$\|H\|_2^2 = \max_{\|\mathbf{u}\|_2=1} \|H\mathbf{u}\|_2^2 = \max_{\|\mathbf{u}\|_2=1} \mathbf{u}^* \sum_{\ell=1}^{r} H_\ell^* H_\ell \mathbf{u} = \|\sum_{\ell=1}^{r} H_\ell^* H_\ell\|_2 \le \mu.$$

Denote $\mathbf{g}_{(v,v')} = [(\mathscr{T}^p A_J H_1 \mathbf{v} - \mathscr{T}^p A_J H_1 \mathbf{v}'),\ldots,(\mathscr{T}^p A_J H_r \mathbf{v} - \mathscr{T}^p A_J H_r \mathbf{v}')]^t$ for any two vectors $\mathbf{v}$ and $\mathbf{v}'$. Following the proof of Theorem 3.1, given any $\mathbf{v}_0$, for any positive integers m and n,

$$\begin{aligned}
\|\mathbf{v}_{n+m} - \mathbf{v}_n\|_2 &= \|\sum_{\ell=1}^{r} H_\ell^* A_J^* (\mathscr{T}^p A_J H_\ell \mathbf{v}_{n+m-1} - \mathscr{T}^p A_J H_\ell \mathbf{v}_{n-1})\|_2 = \|H^* A_J^* \mathbf{g}_{(\mathbf{v}_{n+m-1},\mathbf{v}_{n-1})}\|_2 \\
&\le \|H^*\|_2 \|\mathbf{g}_{(\mathbf{v}_{n+m-1},\mathbf{v}_{n-1})}\|_2 \le \|H^*\|_2 \|H\|_2 \|\mathbf{v}_{n+m-1} - \mathbf{v}_{n-1}\|_2 \\
&\le \mu \|\mathbf{v}_{n+m-1} - \mathbf{v}_{n-1}\|_2.
\end{aligned}$$

Similarly, one can prove by using (4.5)

$$\|\mathbf{v}_n\|_2 \le \|\mathbf{c}\|_2 + \mu \|\mathbf{v}_{n-1}\|_2 \le \frac{1}{1-\mu} \|\mathbf{c}\|_2 + \|\mathbf{v}_0\|_2. \tag{4.13}$$

Thus the iteration sequence $\{\mathbf{v}_n\}$ is a Cauchy sequence and the limit exists and satisfies (4.12). □

Note that solution $\mathbf{s}$ to Algorithm 4.1 satisfies (4.11). Let $\tilde{\mathbf{s}} = \mathscr{T}^p A_J (H_0^*\mathbf{c} + \sum_{\ell=1}^r H_\ell^* H_\ell \mathbf{s})$, then we have the pair $(\mathbf{s},\tilde{\mathbf{s}})$ satisfies $\mathbf{s} = A_J^* \tilde{\mathbf{s}}$. As we discussed in the formulation, $\tilde{\mathbf{s}}$ can be viewed as coefficients of certain representation of the underlying function in a frame system. The following theorem states that pair $(\mathbf{s},\tilde{\mathbf{s}})$ is a solution of (2.3) with finite data set and satisfies minimization property (2.9) given by formulation.

Theorem 4.2 *Let $\mathbf{h}_1,\ldots,\mathbf{h}_r$ be the high pass filters obtained from $\mathbf{h}_0$ by the UEP and $H_0,H_1,\ldots,H_r$ be the corresponding circulant matrices of these filters. Then for fixed p, $1\le p\le 2$, the solution pair $(\mathbf{s},\tilde{\mathbf{s}})$ satisfies*

$$\|H_0(\mathbf{s}+\eta)-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|\tilde{s}_{\ell,j,k}+\tilde{\eta}_{\ell,j,k}|^p\ge\|H_0\mathbf{s}-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|\tilde{s}_{\ell,j,k}|^p,$$

for any pair $(\eta,\tilde{\eta})$ with $\tilde{\eta}=A_J\eta$, where A_J is given in (4.2).

The counterpart of this theorem has been proved in §3.2. In fact, the current case is simpler because we do not have the acceleration factor β in the iteration. Instead of giving the proof of this, we prove Theorem 4.3 which states the minimization property of the limit to iteration (4.5). Let

$$\tilde{\mathbf{s}}_\ell=\mathscr{T}^pA_J(H_\ell\mathbf{s}),\ \ell=1,\ldots,r.$$

Since solution $\mathbf{s}$ of iteration (4.5) satisfies (4.12), we have the $(r+1)$-tuple $(\mathbf{s},\tilde{\mathbf{s}}_1,\ldots,\tilde{\mathbf{s}}_r)$ satisfies $\mathbf{s}=H_0^*\mathbf{c}+\sum_{\ell=1}^r H_\ell^*A_J^*\tilde{\mathbf{s}}_\ell$. We will prove this tuple of finite data sets satisfies a similar property to its infinite counterpart and hence $\mathbf{s}$ is a solution to (2.3).

Theorem 4.3 *Let $\mathbf{h}_1,\ldots,\mathbf{h}_r$ be the high pass filters obtained from $\mathbf{h}_0$ by the UEP and $H_0,H_1,\ldots,H_r$ be the corresponding circulant matrices of these filters. Then for fixed p, $1\le p\le 2$, the $(r+1)$-tuple $(\mathbf{s},\tilde{\mathbf{s}}_1,\ldots,\tilde{\mathbf{s}}_r)$ satisfies the following inequality*

$$\|H_0(\mathbf{s}+\eta)-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_0-1}\lambda_j|(\tilde{s}_\ell)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p\ge\|H_0\mathbf{s}-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_0-1}\lambda_j|(\tilde{s}_\ell)_{\ell',j,k}|^p, \tag{4.14}$$

for any $(r+1)$-tuple $(\eta,\tilde{\eta}_1,\ldots,\tilde{\eta}_r)$ satisfying $\tilde{\eta}_\ell=A_J(H_\ell\eta)$ for $\ell=1,\ldots,r$, where A_J is given in (4.2).

Proof We prove this theorem by proving a more general inequality. For a given sequence $\mathbf{a}$, we define $\tilde{\mathbf{v}}_\ell^\star=\mathscr{T}^pA_J(H_\ell\mathbf{a})$ and $\mathbf{v}_\ell^\star=A_J^*(\tilde{\mathbf{v}}_\ell^\star)$ for $\ell=1,\ldots,r$. Next we denote $\mathbf{c}$ by $\mathbf{v}_0^\star$ and define $\mathbf{v}^\star$ by

$$\mathbf{v}^\star=A_{-1}^*\{\mathbf{v}_\ell^\star\}_{\ell=0}^r=H_0^*\mathbf{c}+\sum_{\ell=1}^r H_\ell^*A_J^*\mathscr{T}^pA_J(H_\ell\mathbf{a}).$$

With this set up, we show that the inequality

$$\begin{aligned}\|H_0(\mathbf{v}^\star+\eta)-\mathbf{c}\|_2^2&+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p+\sum_{\ell=1}^r\|H_\ell(\mathbf{v}^\star+\eta)-H_\ell\mathbf{a}\|_2^2\\&\ge\|H_0\mathbf{v}^\star-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|(\tilde{v}_\ell^\star)_{\ell',j,k}|^p+\sum_{\ell=1}^r\|H_\ell\mathbf{v}^\star-H_\ell\mathbf{a}\|_2^2+\|\eta\|_2^2,\end{aligned} \tag{4.15}$$

holds for any $(r+1)$-tuple $(\eta,\tilde{\eta}_1,\ldots,\tilde{\eta}_r)$ with $\tilde{\eta}_\ell=A_J(H_\ell\eta)$ for $\ell=1,\ldots,r$. Note that if we take $\mathbf{a}=\mathbf{s}$, where $\mathbf{s}$ is the limit to iteration (4.5) satisfying (4.12), then $\mathbf{v}^\star=H_0^*\mathbf{c}+\sum_{\ell=1}^r H_\ell^*A_J^*\mathscr{T}^pA_J(H_\ell\mathbf{s})=\mathbf{s}$, and inequality (4.14) can be easily deduced from (4.15). In the following we give the proof of (4.15), which is similar to that of Proposition 3.3.

Given the $(r+1)$-tuple $(\eta,\tilde{\eta}_1,\ldots,\tilde{\eta}_r)$, we expand the left hand side of (4.15) as follows:

$$\begin{aligned}&\|H_0(\mathbf{v}^\star+\eta)-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p+\sum_{\ell=1}^r\|H_\ell(\mathbf{v}^\star+\eta)-H_\ell\mathbf{a}\|_2^2\\&=\|H_0\mathbf{v}^\star-\mathbf{c}\|_2^2+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j|(\tilde{v}_\ell^\star)_{\ell',j,k}|^p+\sum_{\ell=1}^r\|H_\ell\mathbf{v}^\star-H_\ell\mathbf{a}\|_2^2+\|\eta\|_2^2\\&\quad+\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j\left(|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p-|(\tilde{v}_\ell^\star)_{\ell',j,k}|^p\right)+2\langle\eta,\mathbf{v}^\star-H_0^*\mathbf{c}-\sum_{\ell=1}^r H_\ell^*H_\ell\mathbf{a}\rangle.\end{aligned} \tag{4.16}$$

Compare with (4.15), we only need to show

$$\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j\left(|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p-|(\tilde{v}_\ell^\star)_{\ell',j,k}|^p\right)+2\langle\eta,\mathbf{v}^\star-H_0^*\mathbf{c}-\sum_{\ell=1}^{r}H_\ell^*H_\ell\mathbf{a}\rangle\geq 0. \quad (4.17)$$

Denote $\tilde{\mathbf{a}}_\ell=A_J(H_\ell\mathbf{a})$, $\ell=1,\ldots,r$. Using the definition of $\mathbf{v}^\star$ and $A_J^*A_J=I$, we can simplify the inner product in (4.16) as

$$\begin{aligned}\langle\eta,\mathbf{v}^\star-H_0^*\mathbf{c}-\sum_{\ell=1}^{r}H_\ell^*H_\ell\mathbf{a}\rangle&=\langle\eta,\sum_{\ell=1}^{r}H_\ell^*(A_J^*\mathscr{T}^pA_J(H_\ell\mathbf{a})-H_\ell\mathbf{a})\rangle\\&=\sum_{\ell=1}^{r}\langle A_J(H_\ell\eta),\mathscr{T}^pA_J(H_\ell\mathbf{a})-A_J(H_\ell\mathbf{a})\rangle\\&=\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}(\tilde{\eta}_\ell)_{\ell',j,k}((\tilde{v}_\ell^\star)_{\ell',j,k}-(\tilde{a}_\ell)_{\ell',j,k}).\end{aligned} \quad (4.18)$$

The last equality holds because the denoising operator $\mathscr{T}^p$ does not apply to the low frequency component of the vector. With this, (4.17) becomes

$$\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}\lambda_j\left(|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p-|(\tilde{\mathbf{v}}_\ell^\star)_{\ell',j,k}|^p\right)+2\sum_{\ell=1}^{r}\sum_{\ell'=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}(\tilde{\eta}_\ell)_{\ell',j,k}((\tilde{v}_\ell^\star)_{\ell',j,k}-(\tilde{a}_\ell)_{\ell',j,k})\geq 0,$$

which we will be proven by showing each summand is nonnegative, i.e.

$$\lambda_j\left(|(\tilde{v}_\ell^\star)_{\ell',j,k}+(\tilde{\eta}_\ell)_{\ell',j,k}|^p-|(\tilde{v}_\ell^\star)_{\ell',j,k}|^p\right)+2(\tilde{\eta}_\ell)_{\ell',j,k}((\tilde{v}_\ell^\star)_{\ell',j,k}-(\tilde{a}_\ell)_{\ell',j,k})\geq 0.$$

The rest of the proof follows from the exact the same discussion in the proof of Proposition 3.3. □

In the proof of Theorem 4.3, we do not need to consider the convergence of the sum (4.15) since unlike the proof of Proposition 3.3, there are only finite terms in the sum.

Remark 4.1 Roughly speaking, Algorithm 4.3 is obtained by replacing decomposition and reconstruction operators A_J and A_J^* by B_J and B_J^* in the iteration (4.5). The convergence of iteration (4.8) and the minimization property of the corresponding limit can be obtained by simply changing those results for Algorithm 4.2 and the proofs could be carried out with little effort.

Remark 4.2 When boundary conditions other than periodic ones are used, if the corresponding matrix $H_0^*H_0$ is nonsingular, we still have the convergence of Algorithm 4.1, 4.2 and 4.3. If the matrix is singular, we can solve it by introducing the acceleration factor β and hence reduce the problem to the cases discussed in §3. We need to choose β close to 1 in order for better data recovering as discussed in §3.2. We omit the detailed discussion here.

Remark 4.3 Finally, we remark that the algorithms can be applied to higher dimensions, especially the two dimensional images. This is achieved by using tensor product filters and its underlying wavelets. The details can found in [6,8,9].

4.3 Stability Analysis

In this section, we discuss the stability of the algorithms given in §4.1. An algorithm of solving $H_0\mathbf{v}=\mathbf{b}+\varepsilon=\mathbf{c}$ is stable if the result of the algorithm approaches to the exact solution of the equation $H_0\mathbf{v}=\mathbf{b}$, as $\|\varepsilon\|_2=\varepsilon\to 0$. We give the stability analysis of Algorithm 4.1, and the analysis of Algorithms 4.2 and 4.3 is similar.

Let the threshold $\lambda_j=C_j\varepsilon$ for some constant C_j, $J\leq j<0$. Let $C=\max_{J\leq j<0}C_j$ and without loss of generality, we take $C=1$ below. For a given pair $(\mathbf{v},\tilde{\mathbf{v}})$ with $\mathbf{v}=A_J^*\tilde{\mathbf{v}}$, let

$$|\tilde{\mathbf{v}}|_p:=\sum_{\ell=1}^{r}\sum_{j=J}^{-1}\sum_{k=0}^{N_1-1}C_j|\tilde{v}_{\ell,j,k}|^p.$$

Let the pair $(\mathbf{s}^\varepsilon,\tilde{\mathbf{s}}^\varepsilon)$ be the limit of iteration (4.4) associated with the error bound ε. Let ν be the exact solution to linear system $H_0\mathbf{v}=\mathbf{b}$, then $\|H_0\nu-\mathbf{c}\|_2=\|\varepsilon\|_2=\varepsilon$. Here, we note that the existence of ν follows from the nonsingularity of H_0.

Proposition 4.2 *Let* $\mathbf{s}^{\varepsilon}$ *be the limit of iteration* (4.4) *associated with error bound* $\|\varepsilon\|_2 = \varepsilon$ *and* v *be the exact solution to* $H_0\mathbf{v} = \mathbf{b}$. *Then we have*

$$\lim_{\varepsilon \to 0} \|\mathbf{s}^{\varepsilon} - v\|_2 = 0.$$

Proof We only need to consider the case when $\varepsilon \le 1$. Based on a similar proof of (4.13) in Theorem 4.1, one can show that $\mathbf{s}^{\varepsilon}$ is bounded by a constant independent of ε once the initial seed $\mathbf{v}_0$ is fixed. Since $\tilde{\mathbf{s}}^{\varepsilon} = \mathscr{T}^{p} A_J(H_0^*\mathbf{c} + \sum_{\ell=1}^{r} H_\ell H_\ell^* \mathbf{s}^{\varepsilon})$, we have $\|\tilde{\mathbf{s}}^{\varepsilon}\|_{\ell_2} \le \|\mathbf{c}\|_2 + \|\mathbf{s}^{\varepsilon}\|_2$, i.e. its ℓ_2 norm of $\tilde{\mathbf{s}}^{\varepsilon}$ is bounded independent of ε by Proposition 3.2. This leads to $|\tilde{\mathbf{s}}^{\varepsilon}|_p \le B$ with B dependent of neither ε nor p, $1 \le p \le 2$, because $\tilde{\mathbf{s}}^{\varepsilon}$ is a finite dimensional vector. From the boundedness of $\mathbf{s}^{\varepsilon}$ and $\tilde{\mathbf{s}}^{\varepsilon}$ and vector v, for any ε there is a pair $(\eta^{\varepsilon}, \tilde{\eta}^{\varepsilon})$ with $\tilde{\eta}^{\varepsilon} = A_J \eta^{\varepsilon}$ such that $\|\mathbf{s}^{\varepsilon} + \eta^{\varepsilon} - v\|_2 < \varepsilon$ and $|\tilde{\mathbf{s}}^{\varepsilon} + \tilde{\eta}^{\varepsilon}|_p \le B'$. The pair $(\eta^{\varepsilon}, \tilde{\eta}^{\varepsilon})$ depends on ε; however, the constant B' can be chosen to be independent of ε, since bounds of $\mathbf{s}^{\varepsilon}$, $\tilde{\mathbf{s}}^{\varepsilon}$ and v are independent of ε. By minimization property of $\mathbf{s}^{\varepsilon}$ given in Theorem 4.2, we have

$$\begin{aligned}
\|H_0\mathbf{s}^{\varepsilon} - \mathbf{c}\|_2^2 &\le \|H_0\mathbf{s}^{\varepsilon} - \mathbf{c}\|_2^2 + \varepsilon|\tilde{\mathbf{s}}^{\varepsilon}|_p \\
&\le \|H_0(\mathbf{s}^{\varepsilon} + \eta^{\varepsilon}) - \mathbf{c}\|_2^2 + \varepsilon|\tilde{\mathbf{s}}^{\varepsilon} + \tilde{\eta}^{\varepsilon}|_p \\
&\le 2\|H_0(\mathbf{s}^{\varepsilon} + \eta^{\varepsilon}) - H_0 v\|_2^2 + 2\|H_0 v - \mathbf{c}\|_2^2 + \varepsilon|\tilde{\mathbf{s}}^{\varepsilon} + \tilde{\eta}^{\varepsilon}|_p \\
&< 4\varepsilon^2 + \varepsilon B'.
\end{aligned}$$

Thus, $\|H_0\mathbf{s}^{\varepsilon} - \mathbf{c}\|_2 < \sqrt{4\varepsilon^2 + \varepsilon B'}$. Since matrix H_0 is nonsingular, the conclusion of this proposition follows from $\|H_0(\mathbf{s}^{\varepsilon} - v)\|_2 \to 0$ as $\varepsilon \to 0$. For arbitrary ε, we have

$$\|H_0(\mathbf{s}^{\varepsilon} - v)\|_2 \le \|H_0\mathbf{s}^{\varepsilon} - \mathbf{c}\|_2 + \|H_0 v - \mathbf{c}\|_2 < \sqrt{4\varepsilon^2 + \varepsilon B'} + \varepsilon.$$

Then the stability holds by letting $\varepsilon \to 0$. □

4.4 Comparison of Algorithm 4.1, 4.2 and 4.3

As we mentioned before that the methods given here have been implemented successfully in high resolution reconstruction and comparison with some other numerical methods has been given (see [6,8–11]). The focus of this paper is to build up the theory of the methods used in [6,8–11]. Here, we give some numerical results for 1D signal to illustrate the different performances between Algorithm 4.1, 4.2 and 4.3. The methods are evaluated by the relative error (RE), signal-to-noise ratio (SNR) and the peak signal-to-noise ratio (PSNR). They are defined by

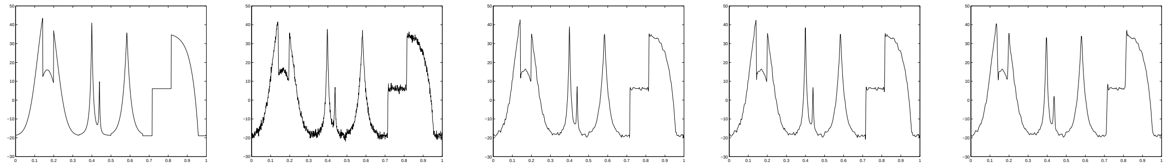
$$\mathrm{RE} = \frac{\|\mathbf{v}_n - \mathbf{v}\|_2}{\|\mathbf{v}\|_2}, \quad \mathrm{SNR} = 10\log_{10}\frac{\|\mathbf{v}_n\|_2^2}{\|\mathbf{v}_n - \mathbf{v}\|_2^2}, \quad \text{and} \quad \mathrm{PSNR} = 10\log_{10}\frac{N_0 \max_{k\in\mathbb{Z}}(\mathbf{v}_n)^2}{\|\mathbf{v}_n - \mathbf{v}\|_2^2},$$

where $\mathbf{v}_n$ is the iterative solution, $\mathbf{v}$ is the original data and N_0 stands for the length of signal.

We choose the piecewise smooth signals in the *WaveLab Toolbox* developed by Donoho's research group. Each signal first passes the low pass filter from the cubic spline, then it is contaminated by white noise at SNR = 25. Applying Algorithm 4.1, 4.2 and 4.3 to recover the original signal where the soft threshold is used. The periodic boundary conditions with proper pre-extension of the data to ensure the convergence of the iterations (4.4), (4.5) and (4.8) are used. The numerical results after 12 iterations with periodic boundary conditions are listed in the following table.

Type of Signal	Algorithm 4.1			Algorithm 4.2			Algorithm 4.3		
	Rel. Err.	PSNR	SNR	Rel. Err.	PSNR	SNR	Rel. Err.	PSNR	SNR
HeaviSine	0.028460	33.217871	30.915279	0.028670	33.156742	30.851414	0.069625	25.455757	23.144756
Bumps	0.069254	39.577640	23.191044	0.075530	38.491668	22.437622	0.618064	17.970758	4.179333
Blocks	0.062193	30.895391	24.125186	0.067464	30.187947	23.418590	0.258222	19.087661	11.760150
Doppler	0.049995	30.580579	26.021500	0.048172	30.905828	26.344157	0.277670	15.688367	11.129433
Ramp	0.031575	32.332058	30.013089	0.039259	30.610982	28.121127	0.157946	17.146609	16.029819
Cusp	0.024914	36.003742	32.070983	0.024833	36.054969	32.099584	0.031825	33.847170	29.944680
Sing	0.089109	46.888308	21.001552	0.089637	46.860294	20.950295	0.805165	22.317274	1.882298
Piece-Polynomial	0.067718	32.882431	23.385962	0.070475	32.443317	23.039271	0.270130	20.086519	11.368543
Piece-Regular	0.048856	34.113632	26.221678	0.049799	33.855047	26.055584	0.221711	20.171143	13.084247

The results show that the performance of Algorithm 4.1 and 4.2 are close, while Algorithm 4.3 seems keeping the smoothness of the signal better as shown in the figures for "Piece-Regular", but it does not perform as good as other two in terms of relative error, PSNR and SNR.



(a) (b) (c) (d) (e)

Fig. 4.1 (a) Original signal; (b) Signal blurred by filter in Example 1.2 and contaminated by white noise at SNR=25; (c) Reconstructed signal by Algorithm 4.1; (d) Reconstructed signal by Algorithm 4.2; (e) Reconstructed signal by Algorithm 4.3.

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Appendix

A Proofs in §1 and §2

We give proofs of Lemma 1.1, Proposition 1.1 and Proposition 1.2.

Proof ***(Proof of Lemma 1.1)*** When $j \geq 0$, we have $\phi^q_{j,k} = D^j T_k \phi = \phi_{j,k}$ and $\psi^q_{j,k} = D^j T_k \psi = \psi_{j,k}$, which imply that

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \phi_{j,k} = \sum_{k\in\mathbb{Z}} \langle f, \phi^q_{j,k}\rangle \phi^q_{j,k} = P^q_j f, \quad \text{and} \quad \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi^q_{\ell,j,k}\rangle = \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi_{\ell,j,k}\rangle.$$

Since in [19, Lemma 2.4], it has already been proved that $P_{j+1} f = P_j f + \sum_{\ell=1}^r \sum_{k\in\mathbb{Z}} \langle f, \psi_{\ell,j,k}\rangle \psi_{\ell,j,k}$, we have

$$P^q_{j+1} f = P_{j+1} f = P_j f + \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi_{\ell,j,k}\rangle \psi_{\ell,j,k} = P^q_j f + \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi^q_{\ell,j,k}\rangle \psi^q_{\ell,j,k},$$

i.e. the identity (1.13) holds when $j \geq 0$. Next we show (1.13) also holds for $j < 0$. We first denote ϕ as ψ_0.

By the definitions of refinable equation (1.2) and framelet (1.5), one obtains that for $\ell = 0, 1, \ldots, r$,

$$\psi_\ell(x) = 2\sum_{k\in\mathbb{Z}} \mathbf{h}_\ell[k]\phi(2x-k).$$

This leads to

$$\begin{aligned}
\psi^q_{\ell,j,k} &= 2^j T_k \psi_\ell(2^j \cdot) = 2^{j+1} T_k\Big(\sum_{k'\in\mathbb{Z}} \mathbf{h}_\ell[k']\psi_0(2^{j+1}\cdot - k')\Big)\\
&= \sum_{k'\in\mathbb{Z}} \mathbf{h}_\ell[k'] 2^{j+1}\psi_0(2^{j+1}(\cdot - k - 2^{-j-1}k'))\\
&= \sum_{k'\in 2^{-j-1}\mathbb{Z}} \mathbf{h}_\ell[2^{j+1}k'] 2^{j+1}\psi_0(2^{j+1}(\cdot - k - k')).
\end{aligned}$$

We define the dilated sequence $\mathbf{h}_{\ell,j}$ by

$$\mathbf{h}_{\ell,j}[k] = \begin{cases} \mathbf{h}_\ell[2^{j+1}k], & k\in 2^{-j-1}\mathbb{Z};\\ 0, & k\notin 2^{-j-1}\mathbb{Z}. \end{cases} \tag{A.1}$$

The sequence $\mathbf{h}_{\ell,j}$ is obtained inductively by inserting 0 between every two entries in $\mathbf{h}_{\ell,j+1}$ with $\mathbf{h}_{\ell,-1} = \mathbf{h}_\ell$. With the dilated sequence, we have $\psi^q_{\ell,j,k} = \sum_{k'\in\mathbb{Z}} \mathbf{h}_{\ell,j}[k']\psi^q_{0,j+1,k+k'}$, and moreover, the right hand side of (1.13) can be written as follows:

$$\begin{aligned}
\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \langle f, \psi^q_{\ell,j,k}\rangle \psi^q_{\ell,j,k} &= \sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \left(\sum_{k'\in\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k']}\langle f, \psi^q_{0,j+1,k'+k}\rangle\right)\left(\sum_{k''\in\mathbb{Z}} \mathbf{h}_{\ell,j}[k'']\psi^q_{0,j+1,k''+k}\right)\\
&= \sum_{k'\in\mathbb{Z}}\sum_{k''\in\mathbb{Z}} \left(\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k+k''-k']\right)\langle f, \psi^q_{0,j+1,k'}\rangle \psi^q_{0,j+1,k''}.
\end{aligned}$$

Next, we check that $\sum_{\ell=0}^r \sum_{k\in\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k+k''-k'] = \delta_{0,k'-k''}$. When $k-k''\in 2^{-j-1}\mathbb{Z}$, there exists $p\in\mathbb{Z}$ such that $k'-k'' = 2^{-j-1}p$ and we have

$$\begin{aligned}
\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k+k''-k'] &= \sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k-2^{-j-1}p]\\
&= \sum_{\ell=0}^{r}\sum_{k\in 2^{-j-1}\mathbb{Z}} \overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k-2^{-j-1}p]\\
&= \sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \overline{\mathbf{h}_{\ell}[k]}\mathbf{h}_{\ell}[k-p] = \delta_{0,p}.
\end{aligned}$$

The last identity follows by (1.7). The sum is nonzero if and only if $p=0$, which is exactly $k'=k''$. When $k'-k''\notin 2^{-j-1}\mathbb{Z}$, there exist $p_1,p_2\in\mathbb{Z}$ and $p_2\notin 2^{-j-1}\mathbb{Z}$ such that $k'-k''=2^{-j-1}p_1+p_2$. Then we have

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}\overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k+k''-k']=\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}\overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k-2^{-j-1}p_1-p_2]=\sum_{\ell=0}^{r}\sum_{k\in 2^{-j-1}\mathbb{Z}}\overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k-2^{-j-1}p_1-p_2].$$

Since $k-2^{-j-1}p_1-p_2\notin 2^{-j-1}\mathbb{Z}$ when $k\in 2^{-j-1}\mathbb{Z}$, we have $\mathbf{h}_{\ell,j}[k-2^{-j-1}p_1-p_2]=0$ for any $k\in 2^{-j-1}\mathbb{Z}$ and the last identity is equal to 0. In conclusion, for the dilated filters $\mathbf{h}_{0,j},\mathbf{h}_{1,j},\ldots,\mathbf{h}_{r,j}$, we still have a similar result as (1.7)

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}\overline{\mathbf{h}_{\ell,j}[k]}\mathbf{h}_{\ell,j}[k-p]=\delta_{0,p},\quad p\in\mathbb{Z}. \tag{A.2}$$

Thus we have

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}\langle f,\psi^q_{\ell,j,k}\rangle\psi^q_{\ell,j,k}=\sum_{k\in\mathbb{Z}}\langle f,\psi^q_{0,j+1,k}\rangle\psi^q_{0,j+1,k}=P^q_{j+1}f.$$

This is the identity we need to prove when $j<0$. In all, identity (1.13) holds for any $j\in\mathbb{Z}$. □

Remark A.1 We note here that in the proof of identity (1.9) for the affine system, one needs both conditions in (1.6); while in the proof of identity (1.13), when the quasi-affine system is used, one only needs condition (1.7).

Proof (***Proof of Proposition 1.1***) First we consider the case $j\geq 0$. In this case, since $\phi^q_{j,k}=D^jT_k\phi=\phi_{j,k}$, we have

$$P_jf=\sum_{k\in\mathbb{Z}}\langle f,\phi_{j,k}\rangle\phi_{j,k}=\sum_{k\in\mathbb{Z}}\langle f,\phi^q_{j,k}\rangle\phi^q_{j,k}=P^q_jf.$$

Next, we show that $Q_jf=Q^q_jf$ when $j\geq 0$. Since $X(\Psi)$ is a tight frame, $X^q(\Psi)$ is also a tight frame by [31, Theorem 5.5]. On the other hand, $j\geq 0$ implies $\psi^q_{\ell,j,k}=D^jT_k\psi_\ell=\psi_{\ell,j,k}$. Thus we have

$$\sum_{\ell=1}^{r}\sum_{j<0}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j,k}\rangle\psi_{\ell,j,k}=f-\sum_{\ell=1}^{r}\sum_{j\geq 0}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j,k}\rangle\psi_{\ell,j,k}=\sum_{\ell=1}^{r}\sum_{j<0}\sum_{k\in\mathbb{Z}}\langle f,\psi^q_{\ell,j,k}\rangle\psi^q_{\ell,j,k}.$$

Hence, when $j\geq 0$,

$$\begin{aligned}Q^q_jf&=\sum_{\ell=1}^{r}\sum_{j'<0,k\in\mathbb{Z}}\langle f,\psi^q_{\ell,j',k}\rangle\psi^q_{\ell,j',k}+\sum_{\ell=1}^{r}\sum_{j'=0}^{j}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j',k}\rangle\psi_{\ell,j',k}\\&=\sum_{\ell=1}^{r}\sum_{j'<0,k\in\mathbb{Z}}\langle f,\psi_{\ell,j',k}\rangle\psi_{\ell,j',k}+\sum_{\ell=1}^{r}\sum_{j'=0}^{j}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j',k}\rangle\psi_{\ell,j',k}\\&=Q_jf.\end{aligned}$$

Since $P_jf=Q_jf$ by [19, Lemma 2.4], we have $P^q_jf=Q^q_jf$ for $j\geq 0$.

Next we show that $P^q_jf=Q^q_jf$ holds when $j<0$. Applying Lemma 1.1 inductively for any $f\in L_2(\mathbb{R})$ and $j<0$, we have

$$P^q_jf=P^q_{j''}f+\sum_{\ell=1}^{r}\sum_{j'=j''}^{j}\sum_{k\in\mathbb{Z}}\langle f,\psi^q_{\ell,j',k}\rangle\psi^q_{\ell,j',k}. \tag{A.3}$$

Thus the proof of $P^q_jf=Q^q_jf$ is transferred to the proof of $P^q_{j''}f\to 0$ as $j''\to-\infty$. The proof below is essentially the same as that of [26, Theorem 2.2].

Since $\mathbf{h}_0$ is finitely supported, the refinable function ϕ derived from $\mathbf{h}_0$ satisfies (1.3), which implies that the integer shifts of $\phi^q_{j'',0}$ is a Bessel sequence. Because

$$P^q_{j''}f=\sum_{k\in\mathbb{Z}}\langle f,\phi^q_{j'',k}\rangle\phi^q_{j'',k},$$

the norm of $P^q_{j''}f$ satisfies

$$\|P^q_{j''}f\|^2_{L_2(\mathbb{R})}\leq C\sum_{k\in\mathbb{Z}}|\langle f,\phi^q_{j'',k}\rangle|^2, \tag{A.4}$$

where the constant C is independent of j''. Based on the result in approximation theory, we only need to check the value of $\|P^q_{j''}f\|_{L_2(\mathbb{R})}$ when f is supported on an interval $[-R,R]$ for an arbitrary given $R>0$. Applying the Cauchy-Schwartz inequality to (A.4), we have for $j''<0$ and $|j''|$ sufficiently large,

$$\|P^q_{j''}f\|^2_{L_2(\mathbb{R})}\leq 2^{j''}C\|f\|^2_{L_2}\sum_{k\in\mathbb{Z}}\int_{E_{j'',k}}|\phi(x)|^2\,\mathrm{d}x, \tag{A.5}$$

where $E_{j'',k} = 2^{j''}(k+[-R,R])$. By monotone convergence theorem, we have

$$\sum_{k\in\mathbb{Z}} \int_{E_{j'',k}} |\phi(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} \chi_{E_{j'',k}}(x) \Big) |\phi(x)|^2 \,\mathrm{d}x.$$

Note that the sets $E_{j'',k}$ are not mutually disjoint; however, we have

$$\sum_{k\in\mathbb{Z}} \chi_{E_{j'',k}}(x) \le 2\lceil R\rceil \chi_{\mathbb{R}}(x)$$

where $\lceil\cdot\rceil$ is the smallest integer which is greater that R. Therefore, we have the following estimate

$$\|P^q_{j''} f\|^2_{L_2(\mathbb{R})} \le 2^{j''+1} C\lceil R\rceil \|f\|^2_{L_2} \|\phi\|^2_{L_2}.$$

Now $P^q_{j''} f \to 0$ follows by letting $j'' \to -\infty$ which is true due to dominated convergence theorem. Then (A.3) becomes

$$P^q_j f = \sum_{\ell=1}^{r} \sum_{j'<j} \sum_{k\in\mathbb{Z}} \langle f, \psi^q_{\ell,j',k}\rangle \psi^q_{\ell,j',k} = Q^q_j f.$$

Thus we complete our proof of $P^q_j f = Q^q_j f$ for any $j\in\mathbb{Z}$. □

Proof ***(Proof of Proposition 1.2)*** The result on $\mathscr{A}_J$ can be proved by induction. When $J=-1$, this follows from (1.15). For arbitrary $J<0$, we start from the definition of $\mathscr{A}_J$. By (1.21), we have

$$\begin{aligned}
\mathscr{A}_J^*\mathscr{A}_J =& \Big(\prod_{j=-1}^{J} H^*_{0,j}\Big)\Big(\prod_{j=J}^{-1} H_{0,j}\Big) + \Big(\prod_{j=-1}^{J+1} H^*_{0,j} H^*_{1,J}\Big)\Big(H_{1,J}\prod_{j=J+1}^{-1} H_{0,j}\Big) + \cdots + \Big(\prod_{j=-1}^{J+1} H^*_{0,j} H^*_{r,J}\Big)\Big(H_{r,J}\prod_{j=J+1}^{-1} H_{0,j}\Big)\\
&+ \sum_{j=J+1}^{-1}\sum_{\ell=1}^{r}\Big(\prod_{j'=-1}^{j+1} H^*_{0,j'} H^*_{\ell,j}\Big)\Big(H_{\ell,j}\prod_{j'=j+1}^{-1} H_{0,j'}\Big)\\
=& \Big(\prod_{j=-1}^{J+1} H^*_{0,j}\Big)\Big(\prod_{j=J+1}^{-1} H_{0,j}\Big) + \sum_{j=J+1}^{-1}\sum_{\ell=1}^{r}\Big(\prod_{j'=-1}^{j+1} H^*_{0,j'} H^*_{\ell,j}\Big)\Big(H_{\ell,j}\prod_{j'=j+1}^{-1} H_{0,j'}\Big)\\
=& \mathscr{A}^*_{J+1}\mathscr{A}_{J+1}.
\end{aligned}$$

In the above, $\prod_{j'=j}^{-1} H_{0,j'} = H_{0,j}H_{0,j+1}\cdots H_{0,-1}$ and $\prod_{j'=-1}^{j} H^*_{0,j'} = H^*_{0,-1}H^*_{0,-2}\cdots H^*_{0,j}$. The last equality is obtained by applying (1.18), which reflects the reconstruction process from Jth level to $(J+1)$th level. Hence, $\mathscr{A}_J^*\mathscr{A}_J = I$ holds for $J<0$ by induction.

For operator $\mathscr{A}$, we note that proving $\mathscr{A}^*\mathscr{A} = I$ is equivalent to proving $\langle\mathscr{A}\mathbf{v},\mathscr{A}\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{v}\rangle$ holds for any sequence $\mathbf{v}\in\ell_2(\mathbb{Z})$. We next note that

$$\langle\mathbf{v},\mathbf{v}\rangle = \langle\mathscr{A}_J\mathbf{v},\mathscr{A}_J\mathbf{v}\rangle = \Big(\mathbf{v}^*\prod_{j=-1}^{J} H^*_{0,j}\Big)\Big(\prod_{j=J}^{-1} H_{0,j}\mathbf{v}\Big) + \sum_{j=J}^{-1}\sum_{\ell=1}^{r}\Big(\mathbf{v}^*\prod_{j'=-1}^{j+1} H^*_{0,j'}H^*_{\ell,j}\Big)\Big(H_{\ell,j}\prod_{j'=j+1}^{-1} H_{0,j'}\mathbf{v}\Big) \tag{A.6}$$

and

$$\langle\mathscr{A}\mathbf{v},\mathscr{A}\mathbf{v}\rangle = \sum_{j=-\infty}^{-1}\sum_{\ell=1}^{r}\Big(\mathbf{v}^*\prod_{j'=-1}^{j+1} H^*_{0,j'}H^*_{\ell,j}\Big)\Big(H_{\ell,j}\prod_{j'=j+1}^{-1} H_{0,j'}\mathbf{v}\Big). \tag{A.7}$$

Thus to show $\langle\mathscr{A}\mathbf{v},\mathscr{A}\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{v}\rangle$ we only need to prove that $\prod_{j=J}^{-1} H_{0,j}\mathbf{v}$ approaches $\mathbf{0}$ as $J\to-\infty$.

Since the matrix $H_{0,j}$ are Toeplitz matrices generated by filters $\mathbf{h}_{0,j}$, we have

$$\widehat{\prod_{j=J}^{-1} H_{0,j}\mathbf{v}} = \prod_{j=J}^{-1}\widehat{\mathbf{h}_{0,j}}\widehat{\mathbf{v}}.$$

Since $|\widehat{\mathbf{h}_{0,j}}| \le 1$,

$$\Big|\widehat{\prod_{j=J}^{-1} H_{0,j}\mathbf{v}}\Big| \le |\widehat{\mathbf{v}}|, \quad \text{a.e.} \quad \omega\in\mathbb{R}.$$

Note that the compactly supported refinable function ϕ obtained from the finite-length low pass filter $\mathbf{h}_0$ can be written as $\widehat{\phi}(\omega) = \prod_{j=0}^{\infty}\widehat{\mathbf{h}}_0(2^{-j-1}\omega)$. Since $\phi\in L_2(\mathbb{R})$ is compactly supported, we have $\phi\in L_1(\mathbb{R})$ and $\widehat{\phi}\neq 0$ a.e. $\omega\in\mathbb{R}$ satisfying that $\widehat{\phi}\to 0$ as $\omega\to\pm\infty$. Let the zero set of $\widehat{\phi}$ be $\mathscr{Z}$, then, it is a zero measure set. Next we consider any $\omega\in\mathbb{R}\backslash\mathscr{Z}$ such that $\widehat{\phi}(\omega)\neq 0$. Because $\widehat{\mathbf{h}_{0,j}}(\omega) = \widehat{\mathbf{h}}_0(2^{-j-1}\omega)$, we have

$$\widehat{\prod_{j=J}^{-1} H_{0,j}\mathbf{v}} = \prod_{j=J}^{-1}\widehat{\mathbf{h}_{0,j}}\widehat{\mathbf{v}} = \prod_{j=J}^{-1}\widehat{\mathbf{h}}_0(2^{-j-1}\cdot)\widehat{\mathbf{v}} = \frac{1}{\prod_{j=0}^{\infty}\widehat{\mathbf{h}}_0(2^{-j-1}\cdot)}\prod_{j=J}^{\infty}\widehat{\mathbf{h}}_0(2^{-j-1}\cdot)\widehat{\mathbf{v}} = \frac{1}{\widehat{\phi}}\widehat{\phi}(2^{-J}\cdot)\widehat{\mathbf{v}}.$$

Thus

$$\lim_{J\to-\infty}\prod_{j=J}^{-1}\widehat{\mathbf{h}_{0,j}}\widehat{\mathbf{v}}=\frac{\widehat{\mathbf{v}}}{\widehat{\phi}}\lim_{J\to-\infty}\widehat{\phi}(2^{-J}\cdot)=0.$$

So for any $\omega\in\mathbb{R}$, $|\prod_{j=J}^{-1}\widehat{\mathbf{h}_{0,j}}||\widehat{\mathbf{v}}|\to 0$ a.e. as $J\to-\infty$. Applying the Dominated Convergence Theorem, we obtain

$$\|\prod_{j=J}^{-1}H_{0,j}\mathbf{v}\|_{\ell_2(\mathbb{Z})}=\frac{1}{\sqrt{2\pi}}\|\prod_{j=J}^{-1}\widehat{\mathbf{h}_{0,j}}\widehat{\mathbf{v}}\|_{L_2[-\pi,\pi]}\to 0,\quad J\to-\infty.$$

Let $J\to-\infty$ in (A.6), we have $\langle\mathbf{v},\mathbf{v}\rangle=\lim_{J\to-\infty}\langle\mathscr{A}_J\mathbf{v},\mathscr{A}_J\mathbf{v}\rangle=\langle\mathscr{A}\mathbf{v},\mathscr{A}\mathbf{v}\rangle$, which completes our proof. □

Next we give the following result which demonstrates that once the low pass filter $\mathbf{h}_0$ satisfies assumption (2.2), the refinable function ϕ with refinable mask $\mathbf{h}_0$ as given in (1.2) exists and hence a tight frame system can be constructed from $\mathbf{h}_0$ via the UEP.

Proposition A.1 *Suppose* $\mathbf{h}_0$ *is finitely supported and satisfies the following condition:*

$$\begin{cases}|\widehat{\mathbf{h}}_0(\omega)|^2+|\widehat{\mathbf{h}}_0(\omega+\pi)|^2\le 1, & a.e.\quad \omega\in\mathbb{R};\\ \widehat{\mathbf{h}}_0(0)=1.\end{cases}\tag{A.8}$$

The solution ϕ *of the refinement equation*

$$\phi(x)=2\sum_{k\in\mathbb{Z}}\mathbf{h}_0[k]\phi(2x-k)$$

is in $L_2(\mathbb{R})$.

Proof Since $\mathbf{h}_0$ is finitely supported and $\widehat{\mathbf{h}}_0(0)=1$, the compactly supported refinable function ϕ exists in the sense of distribution with the Fourier transform of ϕ given by

$$\widehat{\phi}(\omega)=\prod_{j=1}^{\infty}\widehat{\mathbf{h}}_0(\frac{\omega}{2^j}),\tag{A.9}$$

satisfying $\widehat{\phi}(0)=1$. Further, the distribution solution ϕ is unique. In the following, we will prove $\phi\in L_2(\mathbb{R})$ whenever $\mathbf{h}_0$ satisfies (A.8).

Our proof uses the cascade algorithm defined by

$$\widehat{\phi_n}(\omega)=\widehat{\mathbf{h}}_0(\frac{\omega}{2})\widehat{\phi_{n-1}}(\frac{\omega}{2})=\prod_{j=1}^{n}\widehat{\mathbf{h}}_0(\frac{\omega}{2^j})\widehat{\phi}_0(\frac{\omega}{2^n}),\quad n>0,\tag{A.10}$$

with initial function ϕ_0 satisfying $\widehat{\phi}_0(\omega)=\chi_{[-\pi,\pi]}(\omega)$. It is known that the cascade algorithm always converges to ϕ as a distribution. Since $\widehat{\phi}_0(\omega)$ satisfies

$$\sum_{k\in\mathbb{Z}}|\widehat{\phi}_0(\omega+2k\pi)|^2=1,\quad \text{a.e.}\quad \omega\in\mathbb{R},$$

it can be proven inductively that for any ϕ_n, $n>0$,

$$\sum_{k\in\mathbb{Z}}|\widehat{\phi}_n(\omega+2k\pi)|^2\le 1,\quad \text{a.e.}\quad \omega\in\mathbb{R}.$$

Thus we have

$$\|\widehat{\phi}_n\|_{L_2(\mathbb{R})}^2=\int_{\mathbb{R}}|\widehat{\phi}_n(\omega)|^2\,\mathrm{d}\omega=\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}|\widehat{\phi}_n(\omega+2k\pi)|^2\,\mathrm{d}\omega\le 2\pi.$$

Since the sequence $\{\|\widehat{\phi}_n\|_{L_2(\mathbb{R})}\}$ is bounded for each n, there exists a subsequence $\{\widehat{\phi_{n_j}}\}$ which converges weakly to some function $\widehat{g}\in L_2(\mathbb{R})$. As shown in [17], when $\mathbf{h}_0$ is finitely supported, $\widehat{\phi}_n$ in (A.10) converges absolutely and uniformly on compact sets. Thus the function $\widehat{\phi}$ is uniformly continuous on compact sets. Since $\widehat{\phi}(0)=1$, in a neighborhood of 0, we have $\widehat{\phi}\ne 0$. Thus each $\widehat{\phi_{n_j}}\ne 0$ in such a neighborhood. It leads to the weak limit $\widehat{g}\ne 0$ in this neighborhood. On the other hand, because the sequence $\{\phi_n\}$ converges to the function ϕ in the sense of distribution, which is weaker than the weak convergence, we have $\phi=g\in L_2(\mathbb{R})$. □

Remark A.2 It was shown in [14] that if $\mathbf{h}_0$ satisfies (A.8) and if the corresponding refinable function ϕ is in $L_2(\mathbb{R})$, then there is constructive way to derive a set of tight framelets. Further, if ϕ is symmetric, the framelets are symmetric or antisymmetric. Constructions of tight frames when the refinement mask $\mathbf{h}_0$ satisfies (A.8) are also given in [19] in their construction of tight frames from pseudo-splines (also available in [21]). The above proposition shows that condition (A.8) on $\mathbf{h}_0$ implies the corresponding refinable function $\phi\in L_2(\mathbb{R})$.

B Proofs of Minimization Properties

Here, we give a proof of Proposition 3.3.

Proof ***(Proof of Proposition 3.3)*** The (ℓ, j, k)th entries of sequences $\tilde{\mathbf{v}}_\beta^\star$ and $\tilde{\eta} = \mathscr{A}\eta$ are denoted by $(\tilde{v}_\beta^\star)_{\ell,j,k}$ and $\tilde{\eta}_{\ell,j,k}$ respectively.

From the definition of $\tilde{\Phi}(\mathbf{v};\mathbf{a})$ by (3.9), we have

$$\begin{aligned}
\tilde{\Phi}(\mathbf{v}_\beta^\star + \eta;\mathbf{a}) &= \|H_0(\mathbf{v}_\beta^\star+\eta) - \beta\mathbf{c}\|^2 + \|\mathbf{v}_\beta^\star + \eta - \beta\mathbf{a}\|^2 - \|H_0(\mathbf{v}_\beta^\star+\eta) - \beta H_0\mathbf{a}\|^2 + \sum_{\ell=1}^r \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p \\
&= \|H_0\mathbf{v}_\beta^\star - \beta\mathbf{c}\|^2 + 2\langle H_0\eta, H_0\mathbf{v}_\beta^\star - \beta\mathbf{c}\rangle + \|H_0\eta\|^2 + \|\mathbf{v}_\beta^\star - \beta\mathbf{a}\|^2 + \|\eta\|^2 + 2\langle \eta, \mathbf{v}_\beta^\star - \beta\mathbf{a}\rangle \\
&\quad - \|H_0\mathbf{v}_\beta^\star - \beta H_0\mathbf{a}\|^2 - \|H_0\eta\|^2 - 2\langle H_0\eta, H_0\mathbf{v}_\beta^\star - \beta H_0\mathbf{a}\rangle + \sum_{\ell=1}^r \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p \\
&= \|H_0\mathbf{v}_\beta^\star - \beta\mathbf{c}\|^2 + \|\mathbf{v}_\beta^\star - \beta\mathbf{a}\|^2 - \|H_0\mathbf{v}_\beta^\star - \beta H_0\mathbf{a}\|^2 + \sum_{\ell=1}^r \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p \\
&\quad + \|\eta\|^2 + 2\langle \eta, H_0^*(H_0\mathbf{v}_\beta^\star - \beta\mathbf{c})\rangle + 2\langle \eta, \mathbf{v}_\beta^\star - \beta\mathbf{a}\rangle - 2\langle \eta, H_0^*(H_0\mathbf{v}_\beta^\star - \beta H_0\mathbf{a})\rangle \\
&\quad + \sum_{\ell=1}^r \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j (|(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p) \\
&= \tilde{\Phi}(\mathbf{v}_\beta^\star;\mathbf{a}) + \|\eta\|^2 + 2\langle \eta, \mathbf{v}_\beta^\star - \beta\mathbf{a} - H_0^*\beta\mathbf{c} + H_0^*H_0\beta\mathbf{a}\rangle + \sum_{\ell=1}^r \sum_{j<0,k\in\mathbb{Z}} \beta^{2-p}\lambda_j (|(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p).
\end{aligned} \tag{B.1}$$

Since $\mathscr{A}^*\mathscr{A} = I$ by Lemma 1.2, the inner product in (B.1) can be written as

$$\begin{aligned}
\langle \eta, \mathbf{v}_\beta^\star - \beta\mathbf{a} - H_0^*\beta\mathbf{c} + H_0^*H_0\beta\mathbf{a}\rangle &= \langle \eta, \mathscr{A}^*\tilde{\mathbf{v}}_\beta^\star - \beta\mathscr{A}^*\tilde{\mathbf{a}} - \mathscr{A}^*\mathscr{A}H_0^*\beta\mathbf{c} + \mathscr{A}^*\mathscr{A}H_0^*H_0\beta\mathbf{a}\rangle \\
&= \langle \mathscr{A}\eta, \tilde{\mathbf{v}}_\beta^\star - \beta\tilde{\mathbf{a}} - \mathscr{A}(H_0^*(\beta\mathbf{c} - H_0\beta\mathbf{a}))\rangle.
\end{aligned} \tag{B.2}$$

Next we denote the (ℓ, j, k)th entry of $\tilde{\mathbf{a}}$ and $\mathscr{A}(H_0^*(\mathbf{c} - H_0\mathbf{a}))$ by $\tilde{a}_{\ell,j,k}$ and $\tilde{w}_{\ell,j,k}$. Together with the simplified notation $\sum_{\ell,j,k} := \sum_{\ell=1}^r \sum_{j<0} \sum_{k\in\mathbb{Z}}$ and $\lambda_j^\beta := \beta^{2-p}\lambda_j$, $\tilde{\Phi}(\mathbf{v}_\beta^\star + \eta;\mathbf{a})$ becomes:

$$\tilde{\Phi}(\mathbf{v}_\beta^\star + \eta;\mathbf{a}) = \tilde{\Phi}(\mathbf{v}_\beta^\star;\mathbf{a}) + \|\eta\|^2 + \sum_{\ell,j,k} \lambda_j^\beta (|(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p) + \sum_{\ell,j,k} 2\tilde{\eta}_{\ell,j,k}((\tilde{v}_\beta^\star)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k}). \tag{B.3}$$

Next we prove the inequality

$$\Phi(\mathbf{v}_\beta^\star + \eta;\mathbf{a}) \geq \Phi(\mathbf{v}_\beta^\star;\mathbf{a}) + \|\eta\|^2$$

for p, $1 \leq p \leq 2$. We only need to show

$$\sum_{\ell,j,k} \lambda_j^\beta (|(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p) + \sum_{\ell,j,k} 2\tilde{\eta}_{\ell,j,k}((\tilde{v}_\beta^\star)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k}) \geq 0. \tag{B.4}$$

Since $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{c}}$ are in ℓ_p, if further assuming $\tilde{\mathbf{v}}_\beta^\star \in \ell_p$, by applying the Minkowski's and Young's inequalities as well as the nonexpansive property of the threshold function $t_\lambda^p(x)$, we have $\{(\tilde{v}_\beta^\star)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k}\} \in \ell_p$. Because $\tilde{\eta} \in \ell_p$, we have $\tilde{\eta} \in \ell_q$ for $q = \frac{p}{p-1} \geq p$, and then by Hölder inequality, we can derive that

$$\{\tilde{\eta}_{\ell,j,k}((\tilde{v}_\beta^\star)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k})\} \in \ell_1(\mathbb{Z}).$$

Thus the sequences in (B.4) are absolutely convergent, hence we only need to prove (B.4) term by term, i.e.

$$\lambda_j^\beta |(\tilde{v}_\beta^\star)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - \lambda_j^\beta |(\tilde{v}_\beta^\star)_{\ell,j,k}|^p + 2\tilde{\eta}_{\ell,j,k}((\tilde{v}_\beta^\star)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k}) \geq 0. \tag{B.5}$$

First we consider the case $p = 1$. The threshold function is the soft-threshold function and the (ℓ, j, k)th entry of $\tilde{\mathbf{v}}_\beta^\star$ satisfies that $(\tilde{v}_\beta^\star)_{\ell,j,k} = t_{\lambda_j^\beta}(\beta a_{\ell,j,k} + \beta\tilde{w}_{\ell,j,k})$. We show (B.5) case by case.

1. $(\tilde{v}_\beta^\star)_{\ell,j,k} = 0$, then $|\beta\tilde{a}_{\ell,j,k} + \beta\tilde{w}_{\ell,j,k}| \leq \lambda_j^\beta/2$.

$$\lambda_j^\beta |\tilde{\eta}_{\ell,j,k}| + 2\tilde{\eta}_{\ell,j,k}(-\beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k}) \geq \lambda_j^\beta (|\tilde{\eta}_{\ell,j,k}| - \tilde{\eta}_{\ell,j,k}) \geq 0;$$

2. $(\tilde{v}^\star_\beta)_{\ell,j,k} > 0$, then $(\tilde{v}^\star_\beta)_{\ell,j,k} = \beta\tilde{a}_{\ell,j,k} + \beta\tilde{w}_{\ell,j,k} - \lambda_j^\beta/2$.

$$\begin{aligned}&\lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| - \lambda_j^\beta(\tilde{v}^\star_\beta)_{\ell,j,k} + 2\tilde{\eta}_{\ell,j,k}((\tilde{v}^\star_\beta)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k})\\ &= \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| - \lambda_j^\beta(\tilde{v}^\star_\beta)_{\ell,j,k} + 2\tilde{\eta}_{\ell,j,k}(-\lambda_j^\beta/2) = \lambda_j^\beta\left(|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| - ((\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k})\right) \geq 0;\end{aligned}$$

3. $(\tilde{v}^\star_\beta)_{\ell,j,k} < 0$, then $(\tilde{v}^\star_\beta)_{\ell,j,k} = \beta\tilde{a}_{\ell,j,k} + \beta\tilde{w}_{\ell,j,k} + \lambda_j^\beta/2$.

$$\begin{aligned}&\lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| + \lambda_j^\beta(\tilde{v}^\star_\beta)_{\ell,j,k} + 2\tilde{\eta}_{\ell,j,k}((\tilde{v}^\star_\beta)_{\ell,j,k} - \beta\tilde{a}_{\ell,j,k} - \beta\tilde{w}_{\ell,j,k})\\ &= \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| + \lambda_j^\beta(\tilde{v}^\star_\beta)_{\ell,j,k} + 2\tilde{\eta}_{\ell,j,k}(\lambda_j^\beta/2) = \lambda_j^\beta\left(|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}| + ((\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k})\right) \geq 0.\end{aligned}$$

Thus when $p = 1$ the sum in (B.3) is nonnegative and hence the inequality holds.

Next we consider the case of the fixed p, $1 < p \leq 2$. When $1 < p \leq 2$, the value of $\mathbf{v}^\star_\beta$ is given by $(\tilde{v}^\star_\beta)_{\ell,j,k} = (\mathrm{F}^p_{\lambda_j^\beta})^{-1}(\beta\tilde{a}_{\ell,j,k} + \beta\tilde{w}_{\ell,j,k})$ and we have

$$\begin{aligned}&\lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k}|^p + 2\tilde{\eta}_{\ell,j,k}((\tilde{v}^\star_\beta)_{\ell,j,k} - \mathrm{F}^p_{\lambda_j^\beta}((\tilde{v}^\star_\beta)_{\ell,j,k}))\\ &= \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k}|^p - \tilde{\eta}_{\ell,j,k}p\lambda_j\,\mathrm{sgn}((\tilde{v}^\star_\beta)_{\ell,j,k})|(\tilde{v}^\star_\beta)_{\ell,j,k}|^{p-1}.\end{aligned}$$

If $(\tilde{v}^\star_\beta)_{\ell,j,k} = 0$, then (B.5) holds clearly. If $(\tilde{v}^\star_\beta)_{\ell,j,k} \neq 0$, we check it using function $\theta(t) = |t|^p$ where $p > 1$. The second order derivative is $\theta''(t) = p(p-1)|t|^{p-2}$, which is nonnegative for any value of t except 0. By Taylor expansion,

$$\lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}|^p - \lambda_j^\beta|(\tilde{v}^\star_\beta)_{\ell,j,k}|^p - 2\tilde{\eta}_{\ell,j,k}((\tilde{v}^\star_\beta)_{\ell,j,k} - \mathrm{F}^p_{\lambda_j^\beta}(\tilde{v}^\star_{\ell,j,k})) = \frac{1}{2}\lambda_j^\beta p(p-1)|\xi|^{p-2}\tilde{\eta}^2_{\ell,j,k} \geq 0,$$

where ξ is between $(\tilde{v}^\star_\beta)_{\ell,j,k}$ and $(\tilde{v}^\star_\beta)_{\ell,j,k} + \tilde{\eta}_{\ell,j,k}$. Thus (B.5) still holds when $1 < p \leq 2$.

In conclusion, when $1 \leq p \leq 2$, we always have (B.5) and hence (B.4). Therefore, the inequality $\tilde{\Phi}(\mathbf{v}^\star_\beta + \eta; \mathbf{a}) \geq \tilde{\Phi}(\mathbf{v}^\star_\beta; \mathbf{a}) + \|\eta\|^2$ holds for $1 \leq p \leq 2$. □

C Proof in §4

This is to give the proof of Proposition 4.1.

Proof ***(Proof of Proposition 4.1)*** We start the proof from the explicit form of the eigenvalues of the circulant matrix H_0 generated from filter $\mathbf{h}_0$ with the data of length N. The eigenvalues of the N-by-N circulant matrix H_0 are:

$$\lambda_p[H_0] = \sum_{k=0}^{K-1}\mathbf{h}_0[k]\exp\left(-\frac{i2\pi kp}{N}\right) = \widehat{\mathbf{h}}_0(\frac{2p\pi}{N}), \quad p = 0, 1, \cdots, N-1.$$

If $\widehat{\mathbf{h}}_0(2\pi\omega) \neq 0$, for $\omega \in \mathbb{Q}$ with $\mathbb{Q}$ the set of rational numbers, then $\lambda_p[H_0] \neq 0$ for $p = 0, 1, \ldots, N-1$. Since $\mathbf{h}_0$ is finitely supported, the polynomial $\widehat{\mathbf{h}}_0(\omega)$ has finitely many zeros. Suppose those zeros of $\widehat{\mathbf{h}}_0$ in terms of rational multiples of 2π are

$$\left\{2\pi\frac{q_i}{p_i} : i = 1, 2, \cdots, n\right\}, \tag{C.1}$$

where for each i, $\gcd(q_i, p_i) = 1$. Because $\widehat{\mathbf{h}}_0$ is 2π-periodic, we can take the rationales being proper fractions, i.e. $q_i < p_i$. It is not necessary to consider the case $p_i = q_i$ since $\widehat{\mathbf{h}}_0(2\pi) = \widehat{\mathbf{h}}_0(0) = 1$. To make the matrix H_0 nonsingular, the value of N should satisfy

$$\frac{p}{N} \notin \left\{\frac{q_i}{p_i} : i = 1, 2, \cdots, n\right\},\ p = 0, 1, \cdots, N-1. \tag{C.2}$$

One sufficient condition on N such that (C.2) holds is

$$p_i \nmid N, \quad i = 1, 2, \cdots, n. \tag{C.3}$$

This is because, assuming (C.2) is not true, i.e. there exist p_{i_0} and q_{i_0} in the set given in (C.2) such that $\frac{p}{N} = \frac{q_{i_0}}{p_{i_0}}$, for some $0 < p < N$, then $pp_{i_0} = Nq_{i_0}$. Since $\gcd(p_{i_0}, q_{i_0}) = 1$, it leads to $p_{i_0}|N$, which is a contradiction of $p_{i_0} \nmid N$. Hence, for a given filter $\mathbf{h}_0$, there are infinitely many N such that as long as the data length N satisfies (C.3), the corresponding circulant matrix H_0 is nonsingular. For a given data with length N_0, if N_0 dose not satisfies (C.2), we just simply extend the data to the length $N_1 > N_0$ satisfying (C.2), for example, we can take N_1 prime to each p_i. Then the circulant matrix H_0 generated from $\mathbf{h}_0$ with respect to the extended data of length N_1 is nonsingular. Since $\det(H_0^*) = \det(H_0)$ and $\det(H_0^*H_0) = \det(H_0)^2$, the matrices H_0^* and $H_0^*H_0$ are nonsingular once H_0 is. □

Remark C.1 In practice, if the data are needed to be extended, small N_1 is appreciated. This processing is constructive once all the zeros in terms of rational multiples of 2π as those in (C.1) are available. Based on the sufficient condition (C.3), we first factorize p_i, $i = 1, \cdots, n$, in (C.1) as:

$$\begin{cases} p_1 = m_1^{e_{11}} m_2^{e_{12}} \cdots m_l^{e_{1l}}; \\ p_2 = m_1^{e_{21}} m_2^{e_{22}} \cdots m_l^{e_{2l}}; \\ \vdots \quad \vdots \\ p_n = m_1^{e_{n1}} m_2^{e_{n2}} \cdots m_l^{e_{nl}}, \end{cases}$$

where m_i, $i = 1, \ldots, l$ are prime numbers. Then $N_1 \geq N_0$ satisfying (C.3) means that

$$m_i \nmid N_1, \; i = 1, \cdots, l.$$

Starting from above criterion, we can find the minimum N_1 by directly computation, e.g. using sieve of Eratosthenes.

After we calculate the value of N_1, we need to extend the data by $N_1 - N_0$ entries. To make the extension meaningful, a possible way is to repeat the entries in the original data set. For instance, we can append the first $N_1 - N_0$ entries to the end of the data set. If $\mathbf{h}_0$ is a refinement mask of a spline, pseudo-spline or one of those used in high resolution image reconstructions, then $N_1 - N_0 \leq 1$, since $\widehat{\mathbf{h}_0}(2\omega\pi) = 0$ with $\omega \in \mathbb{Q}$ only when $\omega = \frac{1}{2}$. Thus, as long as N_0 is odd, the corresponding circulant matrix H_0 is nonsingular. This implies that we can simply append at most the first entry in the data set to guarantee the nonsingularity of H_0.